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Scattering of drift waves by drift vortices

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The scattering of free drift waves by drift vortices is investigated analytically in the Born and eikonal approximations. Scattering amplitudes and scattering cross-section are found for a wide range of parameters of the vortex and wave. It is shown that under certain conditions the wave–vortex interaction becomes very strong; that is, the scattering cross-section is much more than the vortex radius.

1. Introduction

It is well known that as a result of drift–wave instabilities (because of the driving forces of density and/or temperature gradients), coherent vortex structures may arise in a magnetized plasma (Meiss and Horton 1983; Shukla 1985; Horton *et al.* 1986; Nycander 1988; Petviashvili and Yan'kov 1989; Shukla and Weiland 1989; Infeld and Rowlands 1992). It is widely believed that a drift turbulent plasma state can be represented as a superposition of such vortex structures and weakly correlated turbulent fluctuations. This reflects the fundamental property of intermittency of plasma turbulence. One possible point of view is that the strong drift turbulence may be considered as a gas of drift plasmons (in fact, linear modes with small nonlinear corrections) and a gas of vortices (strongly nonlinear localized modes). In this approach, the weakly interacting gases of perturbations corresponding to the various kinds of intermittency are considered. Such a phenomenological approach was first suggested by Horton (1989). In particular, within the framework of this theory, the anomalous transport associated with vortex–vortex collisions has been studied (Horton 1989; Nycander and Isichenko 1990; Pavlenko and Weiland 1993). In these works the waves and vortices were assumed to be uncorrelated. Later, Muhm *et al.* (1992) derived concrete equations describing consistently the interaction of a regular strongly nonlinear vortex with small-scale turbulence. They calculated the changes in spatial as well as temporal distributions of the weak drift-wave turbulence due to interaction with regular structures. However, there is still very little analytical information about the vortex–vortex and vortex–wave interactions. One reason for this is surely the mathematical difficulty involved. Nevertheless, theoretical investigation of these problems is necessary in order to obtain a fuller understanding of the basic properties of drift plasma turbulence, and of a number of important physical phenomena (e.g. anomalous plasma transport) that occur in laboratory plasmas. In this paper we present a simple analytical theory describing the scattering of a drift wave by a drift vortex within the framework of the Hasegawa–Mima equation (Hasegawa and Mima 1978; Petviashvili and Yan'kov 1989; Infeld and Rowlands 1992). The Born and eikonal approximations are used to

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calculate the scattering amplitude and scattering cross-section characterizing the efficiency of the wave–vortex interaction.

2. Basic equation

We consider a non-uniform plasma with equilibrium density $n_0(x)$ immersed in a homogeneous magnetic field $\mathbf{B}_0$, which is oriented along the z axis. Our analysis is based on the Hasegawa–Mima equation (Hasegawa and Mima 1978)

$$\frac{\partial}{\partial t}(1-\Delta)\,\Phi + v_d \frac{\partial \Phi}{\partial y} = \{\Phi, \Delta\Phi\}, \tag{1}$$

where dimensionless units have been introduced by

$$(x, y) \to (\rho x, \rho y) = \rho \mathbf{r} = \left(\frac{T_e}{m_e \omega_{ci}^2}\right)^{1/2} \mathbf{r}, \qquad \omega_{ci} t \to t, \qquad \frac{e\Phi}{T_e} \to \Phi,$$

v_d is the electron diamagnetic drift speed,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

and the Poisson bracket is given by

$$\{\Phi, \Delta\Phi\} = \frac{\partial \Phi}{\partial x}\frac{\partial \Delta\Phi}{\partial y} - \frac{\partial \Phi}{\partial y}\frac{\partial \Delta\Phi}{\partial x}.$$

Equation (1) has a well-known solitary dipole vortex solution (modon) propagating with speed u in the y direction (Meiss and Horton 1983):

$$\Phi_v = au\cos\theta \begin{cases} \left(1+\dfrac{\beta^2}{\gamma^2}\right)\dfrac{r}{a} - \dfrac{\beta^2}{\gamma^2}\dfrac{J_1(\gamma r/a)}{J_1(\gamma)} & (r \leqslant a), \\ \dfrac{K_1(\beta r/a)}{K_1(\beta)} & (\rho \geqslant a), \end{cases} \tag{2}$$

where $x = r\cos\theta$ and $y - ut = r\sin\theta$, J_1 and K_1 are Bessel functions, $\beta^2 = a^2(1 - v_d/u)$, and γ is determined by

$$\frac{K_2(\beta)}{\beta K_1(\beta)} = -\frac{J_2(\gamma)}{\gamma J_1(\gamma)} \equiv \delta. \tag{3}$$

The linear version of (1) has a continuous spectrum of linear drift wave modes

$$\omega_{\mathbf{k}} = \frac{k_y v_d}{1+k^2}. \tag{4}$$

As was pointed out by Horton (1989), at a given time the total field may be represented by

$$\Phi = \sum_{i=1}^{N} \Phi_i^v(x - x_i;\, y - y_i) + \sum_{\mathbf{k}} \Phi_{\mathbf{k}}^w \exp(i\mathbf{k}\cdot\mathbf{r}), \tag{5}$$

where the first term corresponds to the vortex field (N is the number of vortices) and the second term is the wave component of the field. Here, to describe an

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'elementary interaction act' between the vortex and surrounding small-amplitude drift wave, we consider the case of a single vortex and represent the electrostatic potential in the form

$$\Phi = \Phi_v(x-x_0,\, y-y_0-ut)+\Phi_w, \tag{6}$$

where Φ_v is given by (2). We also assume that the wave field $\Phi_w \ll \Phi_v$. It is obvious that the combined field is not an exact solution of (1). Moreover, in the presence of the wave field, the vortex can modify its shape, and the parameters x_0, y_0 and u can depend on time. But we focus our attention only on the spatial structure of the wave field (therefore considering the boundary-value problem) and assume that the drift-wave perturbation has a sufficiently small amplitude that all vortex parameters are fixed, i.e. $x_0 = y_0 = 0$, a = const and u = const. In this connection, it is important to note that our analysis implies that the modon solution is sufficiently stable. A detailed discussion of this question was given by Nycander (1992). However, there is still no definite answer, but – at least for vortices with $u < 0$ – a number of numerical simulations indicate that they are rather stable.

Substituting (6) into (1) and linearizing it with respect to Φ_w, we obtain the following equation for Φ_w:

$$\frac{\partial}{\partial t}(1-\Delta)\,\Phi_w + v_d\frac{\partial \Phi_w}{\partial y} = \{\Phi_v, \Delta\Phi_w\}+\{\Phi_w, \Delta\Phi_v\}. \tag{7}$$

The terms on the left-hand side of (7) describe the free drift wave with the dispersion given by (4), while those on the right describe the effect of scattering of the wave by the modon.

3. Scattering of long-wavelength drift waves

It is convenient to write (7) in the frame moving with the vortex speed u in the y direction and to make the substitution $\Phi_w = \phi\exp(-i\omega t)$. Then we have

$$-i\omega(1-\Delta)\,\phi + (v_d-u+u\Delta)\frac{\partial \phi}{\partial y} = \{\Phi_v, \Delta\phi\}+\{\phi, \Delta\Phi_v\}, \tag{8}$$

where $\Phi_v = \Phi_v(x, y)$. By imposing the additional condition

$$\left|\frac{v_d-u}{u}\right| \gg k^2 \tag{9}$$

(or the equivalent form $\beta^2 \gg k^2a^2$), where k is the characteristic wavenumber of the drift-wave perturbations, and introducing a new function Ψ by $\phi = \Psi\exp(iQy)$, where $Q = (v_d-u)/2\omega$, we can reduce (8) to

$$\Delta\Psi+\kappa^2\Psi = \hat{L}\Psi, \tag{10}$$

where $\kappa^2 = Q^2-1$ and the operator $\hat{L}$ is

$$\hat{L} = -\frac{i}{\omega}\left\{\left[\frac{\partial\Phi_v}{\partial x}\left(iQ+\frac{\partial}{\partial y}\right)-\frac{\partial\Phi_v}{\partial y}\frac{\partial}{\partial x}\right]\left[\left(iQ+\frac{\partial}{\partial y}\right)^2+\frac{\partial^2}{\partial x^2}\right]\right.$$
$$\left.+\frac{\partial(\Delta\Phi_v)}{\partial y}\frac{\partial}{\partial x}-\frac{\partial(\Delta\Phi_v)}{\partial x}\left(iQ+\frac{\partial}{\partial y}\right)\right\}. \tag{11}$$

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According to a general theory, (10) can be represented in the integral form

$$\Psi(\mathbf{r}) = w(\mathbf{r}) + \int_{-\infty}^{\infty} G(\mathbf{r},\mathbf{r}')\,\hat{L}\Psi(\mathbf{r}')\,d\mathbf{r}', \tag{12}$$

where $w(\mathbf{r})$ and $G(\mathbf{r},\mathbf{r}')$ are the general solution and the Green function of the equation

$$\Delta\Psi + \kappa^2\Psi = 0. \tag{13}$$

Considering a boundary-value problem, we choose the boundary conditions corresponding to incident and scattered waves. Therefore we have $w = \exp(i\mathbf{q}_0\cdot\mathbf{r})$, with $q_0^2 = \kappa^2$, and the Green function with the appropriate boundary conditions (outgoing waves) is

$$G(\mathbf{r},\mathbf{r}') = -\tfrac{1}{4}iH_0^{(1)}(q_0|\mathbf{r}-\mathbf{r}'|),$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. As a result, (12) takes the form

$$\Psi(\mathbf{r}) = \exp(i\mathbf{q}_0\cdot\mathbf{r}) - \frac{i}{4}\int_{-\infty}^{\infty} H_0^{(1)}(q_0|\mathbf{r}-\mathbf{r}'|)\,\hat{L}\Psi(\mathbf{r}')\,d\mathbf{r}'. \tag{14}$$

We then apply perturbation theory to (14). The last term on the right-hand side of (14) describes the effect of scattering of the drift wave by the vortex. Considering this term as a perturbation, in the Born approximation we have

$$\Psi(\mathbf{r}) = \exp(i\mathbf{q}_0\cdot\mathbf{r}) - \frac{i}{4}\int_{-\infty}^{\infty} H_0^{(1)}(q_0|\mathbf{r}-\mathbf{r}'|)\exp(i\mathbf{q}_0\cdot\mathbf{r}')\,U(\mathbf{r}')\,d\mathbf{r}', \tag{15}$$

where

$$U(\mathbf{r}) = \frac{\hat{L}\exp(i\mathbf{q}_0\cdot\mathbf{r})}{\exp(i\mathbf{q}_0\cdot\mathbf{r})}. \tag{16}$$

In this form, (15) is similar to the equation describing the scattering of a quantum-mechanical particle of momentum $\mathbf{q}_0$ by the two-dimensional scattering potential $U(\mathbf{r})$ (Landau and Lifshitz 1977). Taking into account that the vortex solution satisfies the vorticity–stream-function relationships

$$\Delta\Phi_v = \begin{cases} \dfrac{(\beta^2+\gamma^2)\,u}{a^2}\,x - \dfrac{\gamma^2}{a^2}\Phi_v & (r \leqslant a), \\[2ex] \dfrac{\beta^2}{a^2}\Phi_v & (r \geqslant a), \end{cases}$$

from (11) and (16) we can obtain

$$U(r,\theta) = \frac{1}{\omega}\begin{cases} \left(q_x\dfrac{\partial\Phi_v}{\partial y} - k_0\dfrac{\partial\Phi_v}{\partial x}\right)(k_0^2+q_x^2) + q_x\dfrac{\beta^2}{a^2}\dfrac{\partial\Phi_v}{\partial y} - k_0\dfrac{\beta^2}{a^2}\dfrac{\partial\Phi_v}{\partial x} & (r \geqslant a), \\[2ex] \left(q_x\dfrac{\partial\Phi_v}{\partial y} - k_0\dfrac{\partial\Phi_v}{\partial x}\right)(k_0^2+q_x^2) - q_x\dfrac{\gamma^2}{a^2}\dfrac{\partial\Phi_v}{\partial y} - k_0\left[\dfrac{(\beta^2+\gamma^2)\,u}{a^2} - \dfrac{\gamma^2}{a^2}\dfrac{\partial\Phi_v}{\partial x}\right] & (r \leqslant a), \end{cases} \tag{17}$$

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where

$$k_0 = Q + q, \qquad \mathbf{q}_0 = (q_x, q_y),$$

$$\frac{\partial \Phi_v}{\partial x} = \frac{\partial F}{\partial r}\cos^2\theta + \frac{F}{r}\sin^2\theta, \qquad \frac{\partial \Phi_v}{\partial y} = \left(\frac{\partial F}{\partial r} - \frac{F}{r}\right)\sin\theta\cos\theta$$

($F(r)$ is the radial part of Φ_v, i.e. $\Phi_v = F(r)\cos\theta$). One can see from (2) and (17) that $U(\mathbf{r})$ decays exponentially as $r \to \infty$ and the effective radius of the potential $U(\mathbf{r})$ is of the order of the vortex radius a. At large distances from the origin (centre of the vortex) $r \gg r'$, we can write $|\mathbf{r} - \mathbf{r}'| \approx r - \mathbf{r}' \cdot \mathbf{n}'$, where $\mathbf{n}' = \mathbf{r}/r$. Then, using the asymptotic expression $H_0^{(1)}(z) \underset{z\to\infty}{\sim} (2/\pi z)^{1/2}\exp(iz - \frac{1}{4}i\pi)$, we obtain from (15) that, far from the origin, the total field Ψ is a superposition of incident and scattered waves,

$$\Psi \sim e^{i\mathbf{q}_0\cdot\mathbf{r}} + \frac{f(\mathbf{n}_0, \mathbf{n}')}{r^{1/2}} e^{iq_0 r}, \tag{18}$$

where $\mathbf{n}_0 = \mathbf{q}_0/q_0$, and the scattering amplitude is

$$f(\mathbf{n}_0, \mathbf{n}') = -\frac{e^{i\pi/4}}{2(2\pi q_0)^{1/2}} \int_{-\infty}^{\infty} e^{-i\mathbf{q}\cdot\mathbf{r}'} U(\mathbf{r}')\, d\mathbf{r}', \tag{19}$$

where $\mathbf{q} = \mathbf{q}' - \mathbf{q}_0$, $\mathbf{q}' = q_0\mathbf{n}'$ and $q = 2q_0\sin\frac{1}{2}\alpha$, with α the scattering angle.

For the Born approximation to be valid, the second term on the right-hand side of (15) must be small compared with the first term. In this section we shall consider the case when the wavelength is larger than the vortex radius, i.e. $q_0 a < 1$. Keeping in mind that the potential $U(\mathbf{r})$ decays exponentially at $r > a$ and the region (inside the vortex region) $0 < r \leqslant a$ makes the main contribution, taking into account that $H_0^{(1)}(z) \underset{z\to 0}{\sim} \ln(1/z)$ and replacing $\exp(i\mathbf{q}_0\cdot\mathbf{r}')$ by 1, we can estimate the integral in (15) to obtain the condition

$$a^2|U_m| \ln(1/q_0 a) \ll 1, \tag{20}$$

where U_m is the maximum value of $U(\mathbf{r})$. However, it turns out (as will be seen later) that the potential $U(\mathbf{r})$ is such that, on integration over the angle in (19), the first two terms in the expansion of the exponent in the parameter $q_0 a$ give zero identically. Therefore, in estimating the integral in (15), we may replace $\exp(i\mathbf{q}_0\cdot\mathbf{r}')$ by $q_0^2 a^2$. The condition (20) for the given potential should therefore be corrected on the left-hand side by a factor $q_0^2 a^2$. Evaluating U_m from (2), (17) as $|U_m| \sim (q_0/\omega a)(q_0^2 + \beta^2/a^2)\Phi_{v,\max}$ and taking (9) into account, we can write the condition for the validity of the Born approximation in the form

$$\ln\left(\frac{1}{q_0 a}\right) q_0^2 a^2 \beta^2 \ll \frac{c}{v_0} \lesssim 1, \tag{21}$$

where $c \approx \omega/q_0$ is the phase velocity of the linear wave and $v_0 \approx \Phi_{v,\max}/a$ is the maximum equilibrium fluid velocity in the vortex.

Choosing the angle ψ so that $\psi = 0$ corresponds to head-on incidence (i.e. along the y axis) and denoting $q_x = q_0\sin\psi$ and $q_y = q_0\cos\psi$, from (19) we have

$$f(\alpha) = -\frac{e^{i\pi/4}}{2(2\pi q_0)^{1/2}} \int_0^{\infty}\int_0^{2\pi} U(r,\theta)\exp[-2iq_0 r\sin\tfrac{1}{2}\alpha\cos(\theta + \tfrac{1}{2}\alpha + \psi)]\, r\, dr\, d\theta, \tag{22}$$

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where $U(r,\theta)$ is determined by (17). Assuming that $q_0 a \ll 1$, we expand the exponential in (22) up to order $q_0^2 a^2$ and, after integration, finally obtain for the scattering amplitude

$$f(\alpha) = \frac{\pi e^{i\pi/4}(k_0^2+q_x^2)}{4\omega(2\pi q_0)^{1/2}}\left(1+\frac{\beta^2}{\gamma^2}\right)(4\delta+1)\, q_0^2 a^4 u \times \sin^2\tfrac{1}{2}\alpha\,[2k_0\cos^2(\tfrac{1}{2}\alpha+\psi)-q_x\sin(\alpha+2\psi)]. \quad (23)$$

As one can see, the angular dependence of the scattering is non-monotonic, and essentially depends on the angle of incidence ψ, but the forward scattering is absent in all cases. When $\psi = 0$, the expression for $|f(\alpha)|$ takes the form

$$|f(\alpha)| = \frac{\pi^{1/2}(Q+q_0)^3}{2^{5/2}\omega}\left(1+\frac{\beta^2}{\gamma^2}\right)(4\delta+1)\, q_0^{3/2} a^4 u \sin^2\alpha. \quad (24)$$

So, in this case backward scattering ($\alpha = \pi$) is also absent, and sideward scattering ($\alpha = \frac{1}{2}\pi$) is maximum. The scattering is symmetric with respect to the x and y axes, which is a consequence of the symmetry of the effective potential $U(r,\theta)$ in the case $q_x = 0$. When the incident wave propagates in the x direction ($\psi = \frac{1}{2}\pi$), we get

$$|f(\alpha)| = \frac{\pi^{1/2}(Q^2+q_0^2)}{2^{5/2}\omega}\left(1+\frac{\beta^2}{\gamma^2}\right)(4\delta+1)\, q_0^{3/2} a^4 u \sin^2\tfrac{1}{2}\alpha\,(2Q\sin^2\tfrac{1}{2}\alpha + q_0\sin\alpha). \quad (25)$$

Scattering is absent in the directions $\alpha = 0$ and $\alpha_{\rm cr}$ determined by $\tan\frac{1}{2}\alpha = -q_0(q_0^2+1)^{-1/2}$. For example, if the wavelength is much less than the ion Larmor radius at the electron temperature (that is, $q_0 \gg 1$), the angle $\alpha_{\rm cr}$ is close to $\frac{3}{2}\pi$. In the opposite case ($q_0 \ll 1$) the corresponding angle is close to $-2q_0$. It is of interest to note that in this case (and for the other $\psi \neq 0$) the scattering is asymmetrical with respect to left- and right-hand sides. Namely, one can see from (25) that the scattering at angles $\alpha < \pi$ is much stronger than that at $\alpha+\pi$. This means, in particular, that although in our model we neglect the influence of the incident wave on the vortex, we could expect in this case that the breaking of momentum balance along the y axis leads to an acceleration of the vortex in the case $u > 0$ or to its deceleration if $u < 0$. One can also see that slowing (that is, $\delta u < 0$) leads to decreasing asymmetry, while asymmetry increases if $\delta u > 0$. Such differing behaviour of vortices with $u > 0$ and $u < 0$ is in agreement with the results of Nycander and Isichenko (1990) and Nycander (1992), who showed that the vortex with $u > 0$ ('westward-travelling' in the terminology of Nycander (1992)) becomes unstable under infinitesimal perturbations along the axis perpendicular to its direction of propagation.

The total scattering cross-section σ can be obtained from (23) by integrating $|f(\alpha)|^2$ over the angle α. This yields, after straightforward calculations,

$$\sigma = \frac{\pi^2}{16} a^{12}\left(\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)^2 (4\delta+1)^2\, q_0^3 (1+q_0^2)^4\, B(\psi,\epsilon), \quad (26)$$

where

$$B(\psi,\epsilon) = (1+2\epsilon\sin\psi+\epsilon^2)^2\,[\tfrac{3}{8}+\tfrac{5}{8}\epsilon^2+3(1+\tfrac{1}{4}\epsilon)\sin\psi + 3(1+\tfrac{5}{4}\epsilon)\sin^2\psi + 5\epsilon\sin^3\psi+\sin^4\psi]$$

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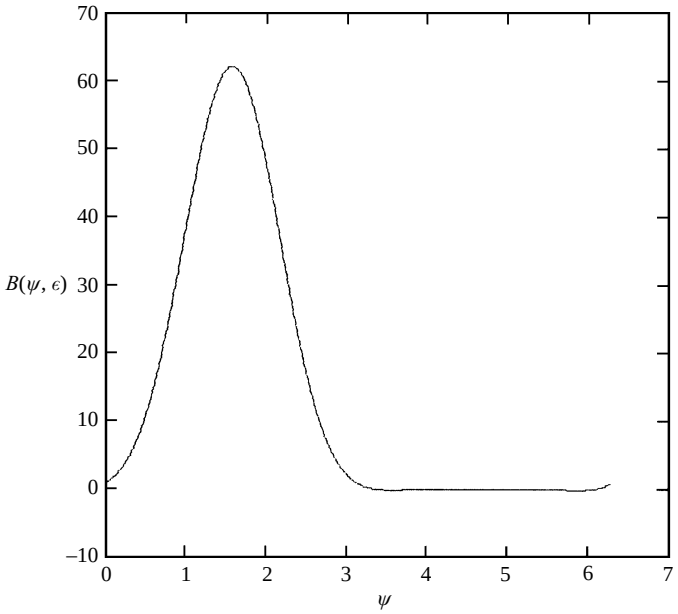


Figure 1. Dependence of $B(\psi, \epsilon)$ on the angle ψ for $\epsilon = 0.5$.

and $\epsilon = q_0(q_0^2+1)^{-1/2}$. One can easily show that $B(\psi, \epsilon) > 0$. The dependence $B(\psi, \epsilon)$ for $\epsilon = 0.5$ is presented in Fig. 1. The picture does not depend qualitatively on q_0. The scattering cross-section is maximum for angles ψ close to $\frac{1}{2}\pi$, and decreases sharply for $\psi \gtrsim \pi$. In the latter case the vortex becomes almost transparent to the incident wave. For angles $\psi \approx \frac{1}{2}\pi$ and $q_0 \ll 1$, the scattering cross-section can be evaluated as

$$\sigma \sim \begin{cases} 4\pi^2 (q_0 a)^3 \left(\dfrac{a^2}{\beta^2}\right)^4 a \ll a \quad (\beta \ll 1), \\ \dfrac{\pi^2}{16} (q_0 a)^3 \left(\dfrac{a^2}{\gamma^2}\right)^2 a^5 \ll a \quad (\beta \gg 1). \end{cases} \tag{27}$$

Thus in all cases the scattering cross-section is much less than the vortex radius, and the interaction between the vortex and the wave is weak.

4. Eikonal approximation

In this section we consider the case where the wavelength of the incident wave is much less than the vortex radius, i.e. $q_0 a \gg 1$. In fact, we shall use the so-called eikonal approximation (Landau and Lifshitz 1977). Choosing the direction of the incidence to be along the y axis, we represent Ψ as $\Psi = \exp(iq_0 y)$ $W(x, y)$, where W is slowly varying function compared to $\exp(iq_0 y)$, and, substituting this into (10), we obtain

$$2iq_0 \frac{\partial W}{\partial y} = U(x, y)\, W, \tag{28}$$

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where $U(x,y)$ is determined by (16). This equation has a solution

$$W = \exp\left(-\frac{i}{2q_0}\int_{-\infty}^{y} U(x,y)\,dy\right).$$

After using the asymptotic expression for $H_0^{(1)}$ at large distances, the second term in (14) (scattered wave) can be written in the form

$$\Psi_{\rm sc} = \frac{f}{r^{1/2}}e^{iq_0 r}, \tag{29}$$

with scattering amplitude given by

$$f = -\frac{e^{i\pi/4}}{2(2\pi q_0)^{1/2}}\int_{-\infty}^{\infty}\exp(-q_0\,\mathbf{n}'\cdot\mathbf{r}')\,\hat{L}\Psi(\mathbf{r}')\,d\mathbf{r}'. \tag{30}$$

Note that under the condition $q_0\,a \gg 1$, it follows from the expansion

$$q|\mathbf{r}-\mathbf{r}'| = qr - q\frac{\mathbf{r}\cdot\mathbf{r}'}{r} + \frac{q}{2}\left[\frac{r'^2}{r} - \frac{(\mathbf{r}\cdot\mathbf{r}')^2}{r^3}\right] + \ldots \quad (r' \ll r)$$

(taking into account that $r' \approx a$ because of the exponential decay of $U(\mathbf{r})$) that the asymptotic expression (29) is valid only at distances $r \gg q_0 a^2 \gg a$. In particular, this requires smallness of $|\Psi_{\rm sc}|$ compared with 1 (i.e. the amplitude of the incident wave), even if $f_{\max} \approx (q_0 a^2)^{1/2}$.

Substituting $\Psi(\mathbf{r})$ determined by (28) into (30) and taking into account the assumptions made above, we obtain

$$f = -\frac{e^{i\pi/4}}{2(2\pi q_0)^{1/2}}\int_{-\infty}^{\infty}\exp(iq_0\,y' - iq_0\,\mathbf{n}'\cdot\mathbf{r}')\,U(x',y')\,W(x',y')\,dx'\,dy'. \tag{31}$$

The scattering with $q_0 a \gg 1$ is mainly at small angles $\Delta\alpha \approx 1/q_0 a$ (outside this cone, the exponential in (31) is a rapidly oscillating function), so the vector of transferred momentum $\mathbf{q}$ is almost perpendicular to the vector of the incident wave $\mathbf{q}_0 = (0, q_0)$, and we can write

$$iq_0\,y' - iq_0\,\mathbf{n}'\cdot\mathbf{r}' \approx iqx'.$$

Then the expression (31) takes the form

$$f = e^{-i\pi/4}\left(\frac{q_0}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty}\exp(iqx')\frac{\partial W}{\partial y'}dx'\,dy', \tag{32}$$

and, after integration over y', we obtain for the scattering amplitude

$$f(\alpha) = e^{-i\pi/4}\left(\frac{q_0}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty}\{\exp[i\mu(x)]-1\}\exp(2iq_0\,x\sin\tfrac{1}{2}\alpha)\,dx, \tag{33}$$

where

$$\mu(x) = -\frac{1}{2q_0}\int_{-\infty}^{\infty}U(x,y)\,dy. \tag{34}$$

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The main contribution in (33), due to the rapidly oscillating factor $\exp(2iq_0 x \sin \frac{1}{2}\alpha)$, will be at $x^2 \ll a^2$. Taking this into account, after straightforward calculations, we obtain

$$\mu(x) = c - dx^2, \tag{35}$$

where

$$c = \left|\frac{uk_0}{\omega}\right| \frac{a}{q_0}\left[k_0^2\left(1+\frac{\beta^2}{\gamma^2}\right) - \left(k_0^2 - \frac{\gamma^2}{a^2}\right)\frac{\beta^2}{\gamma^2 J_1(\gamma)} + \frac{\beta^2}{a^2}\right],$$

$$d = \left|\frac{uk_0}{\omega}\right| \frac{1}{q_0 a}\left[2k_0^2\left(1+\frac{\beta^2}{\gamma^2}\right) - 2\left(k_0^2 - \frac{\gamma^2}{a^2}\right)\frac{\beta^2}{J_1(\gamma)} + \frac{\beta^2}{a^2}\right].$$

Note that $c \gg 1$, $d \gg 1/a^2$, and the Born approximation is not valid. For the scattering amplitude in the eikonal approximation, we have from (33)

$$f(\alpha) = e^{-i\pi/4}\left(\frac{q_0}{2\pi}\right)^{1/2}\left\{\frac{2\sin(q_0 a \sin\frac{1}{2}\alpha)}{q_0 \sin\frac{1}{2}\alpha} - \frac{1}{2}\left(\frac{\pi}{d}\right)^{1/2}\exp\left[i\left(c + \frac{q_0^2}{4d}\sin^2\tfrac{1}{2}\alpha\right)\right]\right\}. \tag{36}$$

As one can see, the scattering is sharply anisotropic. It is mainly within the cone $\Delta\alpha \approx 1/q_0 a \ll 1$, and is maximum at $\alpha = 0$. There is also a small peak at the angle $\alpha = \pi$. The total cross-section σ can be easily found by using the optical theorem (Landau and Lifshitz 1977), which connects σ with the imaginary part of the scattering amplitude at zero angle $\mathrm{Im} f(0)$. As is well known (Landau and Lifshitz 1977), the optical theorem in scattering theory is not applicable for the Born approximation, but it is valid for the eikonal one. In our two-dimensional case the optical theorem can be written in the form (taking into account the constant phase shift $\frac{1}{4}\pi$)

$$\sigma = \left(\frac{8\pi}{q_0}\right)^{1/2} \mathrm{Im}[e^{-i\pi/4} f(0)]. \tag{37}$$

From (36) and (37), we thus obtain for the scattering cross-section in the eikonal approximation

$$\sigma = 4\left[a - \frac{1}{4}\left(\frac{\pi}{d}\right)^{1/2}\cos c\right]. \tag{38}$$

It can be seen that the scattering cross-section is close to twice the vortex diameter, $4a$. This is a quite natural result for the eikonal approximation. Indeed, in the limit $q_0 a \gg 1$ the incident wave may be regarded as a particle beam. Almost all the 'particles' with impact parameter $\rho < a$ are scattered, and in this sense the region close to a may be regarded as an absorbing body. Under this approximation, according to the Babine principle, the total scattering cross-section is equal to twice the absorbing (geometrical) cross-section.

5. Summary

In this paper we have investigated the scattering of drift waves by a dipole vortex in the framework of the Hasegawa–Mima equation. The influence of an incident wave on the vortex was neglected and we considered a boundary-value

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problem. The scattering amplitude and scattering cross-section have been obtained for two limiting cases. When the wavelength of the incident wave is much greater than the vortex radius a, the scattering is almost isotropic and the scattering cross-section is much less than a. In the opposite, short-wavelength case, when the eikonal approximation is applicable, the scattering turns out to be sharply anisotropic (forward scattering mainly takes place) and the scattering cross-section is comparable to the geometrical size of the vortex, which indicates the rather strong interaction.

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