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Zero-Free Intervals for Flow Polynomials of Near-Cubic Graphs

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Let $P(G,t)$ and $F(G,t)$ denote the chromatic and flow polynomials of a graph G. G. D. Birkhoff and D C. Lewis showed that, if G is a plane near-triangulation, then the only zeros of $P(G,t)$ in $(-\infty,2]$ are 0, 1 and 2. We will extend their theorem by showing that a stronger result to the dual statement holds for both planar and non-planar graphs: if G is a bridgeless graph with at most one vertex of degree other than three, then the only zeros of $F(G,t)$ in $(-\infty,\alpha]$ are 1 and 2, where $\alpha \approx 2.225\cdots$ is the real zero in $(2,3)$ of the polynomial $t^4-8t^3+22t^2-28t+17$. In addition we construct a sequence of 'near-cubic' graphs whose flow polynomials have zeros converging to α from above.

1. Introduction

All graphs considered are finite and may contain loops and multiple edges. We shall refer to graphs without loops or multiple edges as simple graphs. We use $P(G,t)$ to denote the chromatic polynomial of a graph G, and $F(G,t)$ to denote its flow polynomial. We shall use the terms *chromatic root* and *flow root* of G to refer to the zeros of $P(G,t)$ and $F(G,t)$, respectively. The study of the distribution of chromatic roots was initiated by G. D. Birkhoff and D. C. Lewis in [1]. Inspired by the 4-Colour Conjecture, they showed that for all plane near-triangulations G, $P(G,t)$ has no zeros in the intervals $(-\infty,0)$, (0,1), (1,2) and $[5,\infty)$. In addition, D. R. Woodall [15] has shown that 2 is a simple zero of $P(G,t)$ if G is 3-connected.

For an arbitrary loopless graph G, it is known (see Tutte [12] or Woodall [15]) that $P(G,t)$ has no zeros in the intervals $(-\infty,0)$ and $(0,1)$, that 0 is a zero of $P(G,t)$ of multiplicity equal to the number of components of G, and that 1 is a zero of $P(G,t)$ of multiplicity equal to the number of non-trivial blocks of G, where a block is *non-trivial* if it has at least one edge. In addition, in [3] it is shown that $P(G,t)$ has no zeros in the interval $(1,\frac{32}{27}]$. Wakelin [13] showed that the dual statements also hold for an arbitrary bridgeless graph G: $F(G,t)$ has no zeros in the intervals $(-\infty,1)$ and $(1,\frac{32}{27}]$ and 1 is a

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zero of $F(G,t)$ of multiplicity equal to the number of non-trivial blocks of G. (A common extension of these results on chromatic and flow polynomials to matroids is obtained in [2].)

In this paper we show that the dual statements to the above-mentioned results on plane near-triangulations hold for both planar and non-planar graphs. We will consider *near-cubic graphs*: graphs with minimum degree at least two and at most one vertex of degree greater then three. We will show that if G is bridgeless and near-cubic, then $F(G,t)$ has no zeros in $(1,2)$; 2 is a simple zero of $F(G,t)$ when G is 3-connected; and $F(G,t)$ has no zeros in $(2,\alpha]$, where $\alpha \approx 2.225\cdots$ is the real zero in $(2,3)$ of the polynomial $t^4 - 8t^3 + 22t^2 - 28t + 17$. We give an example of a sequence of near-cubic graphs whose flow polynomials have zeros converging to α from above. These results extend the above-mentioned results on chromatic roots of plane near-triangulations in $(\infty,2]$ by planar duality. We refer the reader to [4] for a more detailed survey of chromatic and flow roots of graphs.

2. Definitions and preliminary results

Let G be a graph. An *edge-cut* of G is the set of edges, S, from U to $V(G) - U$ for some proper subset $U \subset V(G)$. We say that the subgraphs of G induced by U and $V(G) - U$ are the *sides* of S and that S is a k-edge-cut if $|S| = k$. A *bridge* of G is a 1-edge-cut. The graph G is *k-edge-connected* if G has no r-edge-cuts for $r < k$. An edge-cut S of G is *cyclic* if both sides of S contain circuits, and G is *cyclically k-edge-connected* if each cyclic edge-cut of G has at least k edges.

We say that G is *k-connected* for some integer $k \geqslant 2$ if G is loopless, $|V(G)| \geqslant k+1$, and $G - U$ is connected for all $U \subset V(G)$ with $|U| < k$. A vertex x is a *cut-vertex* of a graph G if $G - x$ has more components than G. We say that G is *non-separable* if either $|V(G)| = 1$ and $|E(G)| \leqslant 1$, or G is loopless, connected and has no cut-vertices. Thus a graph on at least three vertices is non-separable if and only if it is 2-connected. A *block* of G is a maximal non-separable subgraph of G. (Thus every loop of G is the edge set of a block of G.)

Given an edge e of G, we shall use G/e and $G - e$ to denote the graphs obtained from G by contracting e and deleting e, respectively. Note that, if e is a loop, then $G/e = G - e$ and that, if e belongs to a set of parallel edges, then every edge in the set other than e becomes a loop in G/e. Throughout this paper we use n and m to denote the numbers of vertices and edges, respectively, in a graph G. We shall extend this notation by using subscripts and superscripts. Thus, for example, the number of edges in a graph G_1^+ will be denoted by m_1^+.

Flow polynomials

Let Γ be an additive abelian group and let G be a graph. Suppose we construct a digraph $\vec{G}$ by giving the edges of G an arbitrary orientation. For $U \subseteq V(G)$ and $\bar{U} = V(G) - U$, let $E^+(U)$ be the set of arcs from U to $\bar{U}$ in $\vec{G}$ and $E^-(U) = E^+(\bar{U})$. Let $f : E(\vec{G}) \to \Gamma$ and put $f^+(U) = \sum_{e \in E^+(U)} f(e)$ and $f^-(U) = \sum_{e \in E^-(U)} f(e)$. For $v \in V(G)$ let $f^+(v) = f^+(\{v\})$ and $f^-(v) = f^-(\{v\})$. Then f is a *Γ-flow* for G, with respect to $\vec{G}$, if $f^+(v) = f^-(v)$ for all

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$v \in V(G)$. If, in addition, $f(e) \neq 0$ for all $e \in E(G)$, then we say that f is a *nowhere-zero* Γ-*flow* for G. It can be seen that the condition $f^+(v) = f^-(v)$ for all $v \in V(G)$ is equivalent to the apparently stronger condition that $f^+(U) = f^-(U)$ for all $U \subseteq V(G)$. Thus, if G has a nowhere-zero Γ-flow, then G is bridgeless. Since reversing the orientation on an edge e of $\vec{G}$ is equivalent to replacing $f(e)$ by $-f(e)$, the number of distinct nowhere-zero Γ-flows for G is independent of the chosen orientation $\vec{G}$ of G.

A *nowhere-zero* t-*flow* for G is a nowhere-zero $\mathbb{Z}$-flow, f, such that $|f(e)| \leqslant t-1$ for all $e \in E(G)$. Tutte [9] has shown that G has a nowhere-zero t-flow if and only if G has a nowhere-zero $\mathbb{Z}_t$-flow. Furthermore, the number of distinct nowhere-zero Γ-flows for G is the same for all abelian groups Γ of the same order. Note, however, that the number of nowhere-zero $\mathbb{Z}_t$-flows for G may differ from the number of nowhere-zero t-flows for G. Nowhere-zero flows were introduced by Tutte [9] as a dual concept to proper colourings. He showed that a connected plane graph G has a proper t-colouring if and only if its planar dual G^* has a nowhere-zero t-flow. The two concepts differ for non-planar graphs, however. Indeed, whereas there exist loopless graphs which are not t-colourable for arbitrarily large integers t, Tutte [10] conjectured that every bridgeless graph has a nowhere-zero 5-flow. Jaeger [6] obtained a partial solution by showing that every bridgeless graph has a nowhere-zero 8-flow. His result was subsequently improved by Seymour.

Theorem 2.1. (Seymour [8]) *Every bridgeless graph has a nowhere-zero* 6*-flow.*

Following Tutte [10] we define the *flow polynomial* $F(G,t)$ of G as the number of distinct nowhere-zero $\mathbb{Z}_t$-flows for G for any positive integer t. Thus $F(G,t) \equiv 1$ if $E(G) = \emptyset$ and $F(G,t) \equiv 0$ if G has a bridge.

By the above remarks, $F(G,t)$ is independent of the chosen orientation of G. Note also that since the existence of a nowhere-zero t-flow for G implies the existence of a nowhere-zero $(t+1)$-flow by definition, and is equivalent to the existence of a nowhere-zero $\mathbb{Z}_t$-flow as noted above, we may deduce that if $F(G,t_0) \neq 0$ for some positive integer t_0, then $F(G,t) \neq 0$ for all integers $t \geqslant t_0$.

We could also consider the polynomial $I(G,t)$ defined to be the number of distinct nowhere-zero t-flows for G. Kochal [7] gives relationships between $I(G,t)$ and $F(G,t)$, but these seem to be the only results on $I(G,t)$ in the literature. Attention has concentrated on $F(G,t)$ because it is dual to $P(G,t)$ for plane graphs: if G is a connected plane graph and G^* its planar dual, then there is a surjection from the t-vertex-colourings of G^* to the nowhere-zero $\mathbb{Z}_t$-flows for G such that each nowhere-zero $\mathbb{Z}_t$-flow for G has exactly t pre-images: see [10]. Thus

$$F(G,t) = t^{-1}P(G^*,t).$$

We may use this identity to restate results and conjectures on chromatic roots of families of plane graphs in terms of flow roots of the dual families. For non-planar graphs, however, the zero distributions of chromatic and flow polynomials are very different. Indeed there is a tendency for the zero distribution of flow polynomials to be similar in both the planar and non-planar case. The results of this paper are an example of this.

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The following conjectured strengthening of Tutte's 5-flow conjecture due to Welsh [14] would be another example.

Conjecture 2.2. *Let G be a bridgeless graph. Then $F(G,t) > 0$ for all $t \in (4,\infty)$.*

Although Theorem 2.1 implies that $F(G,t) > 0$ for all bridgeless graphs G and all integers $t \geqslant 6$, it is not even known whether there exists a finite t_0 such that $F(G,t) > 0$ for all $t \in (t_0,\infty)$. This contrasts with the result of Birkhoff and Lewis [1] that $P(G,t) > 0$ for all loopless planar graphs G and all $t \in [5,\infty)$, and their conjecture that $P(G,t) > 0$ for all $t \in [4,\infty)$.

Some recurrence relations for flow polynomials

Our inductive proof technique for working with flow polynomials is based on the following elementary recurrence relations.

Lemma 2.3. *Let G be a graph and e be an edge of G.*

(a) *If e is a loop then $F(G,t) = (t-1)F(G/e,t)$.*
(b) *If e is not a loop then $F(G,t) = F(G/e,t) - F(G-e,t)$.*

Proof. We may deduce that the lemma holds for each integer $t \geqslant 2$, and hence for all t, as follows. If e is a loop then each nowhere-zero $\mathbb{Z}_t$-flow for $G-e$ corresponds to $t-1$ nowhere-zero $\mathbb{Z}_t$-flows for G. If e is not a loop then each nowhere-zero $\mathbb{Z}_t$-flow for G/e corresponds to either a nowhere-zero $\mathbb{Z}_t$-flow for G or a nowhere-zero $\mathbb{Z}_t$-flow for $G-e$. □

We may use Lemma 2.3 inductively to obtain the following reduction lemmas for flow polynomials of graphs of low connectivity.

Lemma 2.4. *The flow polynomials of two homeomorphic graphs are identical.*

Lemma 2.5. *Let G be a graph and G_1 and G_2 be edge-disjoint subgraphs of G such that $G_1 \cup G_2 = G$ and $|V(G_1) \cap V(G_2)| \leqslant 1$. Then*

$$F(G,t) = F(G_1,t)F(G_2,t).$$

Lemma 2.6. *Let G be a graph, v be a vertex of G, $e = u_1u_2$ be an edge of G, and H_1 and H_2 be edge-disjoint subgraphs of G such that $H_1 \cup H_2 = G-e$, $H_1 \cap H_2 = \{v\}$, $u_1 \in V(H_1)$ and $u_2 \in V(H_2)$. Let G_i be obtained from G by contracting $E(H_{3-i})$, for $i \in \{1,2\}$. Then*

$$F(G,t) = \frac{F(G_1,t)F(G_2,t)}{(t-1)}.$$

As a special case of Lemma 2.6 we have the following.

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Lemma 2.7. *Let G be a graph, S be a 2-edge-cut of G, and H_1 and H_2 be the sides of S. Let G_i be obtained from G by contracting $E(H_{3-i})$, for $i \in \{1,2\}$. Then*

$$F(G,t) = \frac{F(G_1,t)F(G_2,t)}{(t-1)}.$$

Lemma 2.8. *Let G be a graph, S be a 3-edge-cut of G, and H_1 and H_2 be the sides of S. Let G_i be obtained from G by contracting $E(H_{3-i})$, for $i \in \{1,2\}$. Then*

$$F(G,t) = \frac{F(G_1,t)F(G_2,t)}{(t-1)(t-2)}.$$

Lemma 2.9. *Let G be a graph, x be a vertex of G, $e = u_1u_2, f = v_1v_2$ be edges of G, and H_1 and H_2 be edge-disjoint subgraphs of G such that $H_1 \cup H_2 = G - \{e,f\}$, $H_1 \cap H_2 = \{x\}$, $u_1, v_1 \in V(H_1)$ and $u_2, v_2 \in V(H_2)$. Let G^+ be the graph obtained from $G - x$ by adding two new vertices, x_1, x_2, an edge x_1x_2, and an edge from x_i to each of the neighbours of x in H_i, for $1 \leqslant i \leqslant 2$. For $i = 1,2$, let G_i^+ be the graph obtained from $G^+[V(H_i) - x + x_i]$ by adding a new vertex y_i and edges y_ix_i, y_iu_i, y_iv_i. Let $G_i^- = G_i^+ - x_iy_i$. Then*

$$(t-1)(t-2)F(G,t) = F(G_1^+,t)F(G_2^+,t) + (t-2)F(G_1^-,t)F(G_2^-,t).$$

Proof. Let $G^- = G^+ - x_1x_2$. By Lemma 2.3, $F(G,t) = F(G^+,t) + F(G^-,t)$. The lemma now follows by applying Lemmas 2.7 and 2.8 to G^- and G^+, respectively. □

3. Near-cubic graphs

The family of cubic graphs has special significance for nowhere-zero flows since many problems on flows in general graphs can be reduced to the special case of cubic graphs. In particular, the truth of Tutte's 5-flow conjecture and Conjecture 2.2 would follow from their special cases for cubic graphs. Since the family of cubic graphs is not closed under the operations of edge contraction or deletion, it is often helpful to consider the larger family of near-cubic graphs. We shall use the ordered pair (G,x) to represent a near-cubic graph G together with a distinguished vertex x such that each vertex of G other than x has degree two or three. A *k-bond* is a loopless graph with two vertices and k edges. We denote the isomorphism class of all k-bonds by K_2^k.

3.1. The interval $(-\infty, 2)$

We show in this subsection that a bridgeless near-cubic graph (G,x) has no flow roots in $(-\infty,2)$ apart from the integer flow root at 1. We shall in fact prove a stronger result concerning the derivatives of the quotient polynomial $q_1(G,t) = F(G,t)/(t-1)$. Since G is near-cubic, it has at least one edge and hence 1 is a zero of $F(G,t)$. Since $F(G,t)$ is a polynomial in t of degree $m-n+1$, it follows that $q_1(G,t)$ is a polynomial of degree $m-n$. Let $q_1^{(i)}(G,t)$ be the ith derivative of $q_1(G,t)$.

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Theorem 3.1. *Let (G,x) be a non-separable near-cubic graph with n vertices and m edges. Then $q_1^{(i)}(G,t)$ is non-zero with sign $(-1)^{m-n+i}$ for all $0 \leqslant i \leqslant m-n$ and all $t \in (-\infty, 2)$.*

Proof. We proceed by contradiction. Suppose the theorem is false and let (G,x) be a counter-example chosen such that m is as small as possible. If $n=1$ then $m=1$, $F(G,t) = t-1$ and $q_1(G,t)=1$. If $n=2$ then either $G=K_2^2$, $F(G,t)=t-1$ and $q_1(G,t)=1$, or $G=K_2^3$, $F(G,t)=(t-1)(t-2)$ and $q_1(G,t)=t-2$. Since the theorem holds in all three cases we have $n \geqslant 3$. Thus G is 2-connected. Using Lemma 2.4 we may also deduce that G has minimum degree three.

Claim 3.2. *$G-x$ is 2-edge-connected.*

Proof. Since G is non-separable, $G-x$ is connected. Suppose $G-x$ has a bridge e. Adopting the notation of Lemma 2.6, we have $F(G,t) = F(G_1,t)F(G_2,t)/(t-1)$, and hence $q_1(G,t) = q_1(G_1,t)q_1(G_2,t)$. Thus

$$q_1^{(i)}(G,t) = \sum_{j=0}^{i} \binom{i}{j} q_1^{(j)}(G_1,t) q_1^{(i-j)}(G_2,t).$$

The graphs (G_1,x) and (G_2,x) are both non-separable near-cubic graphs with fewer edges than G. Since $m = m_1 + m_2 - 1$ and $n = n_1 + n_2 - 1$, we may deduce that the theorem holds for G by applying induction to G_1 and G_2. □

Let e be an edge of G incident to x. The facts that (G,x) is near-cubic, $G-x$ is 2-edge-connected and $n \geqslant 3$ imply that $(G/e,x)$ and $(G-e,x)$ are non-separable near-cubic graphs. Applying Lemma 2.3 to $G-e$ and G/e, and differentiating i times, we may deduce inductively that the theorem holds for G. □

Note that if (G,x) is a separable bridgeless near-cubic graph, then x is the unique cut-vertex of G and we may apply Theorem 3.1 to each block of G using Lemma 2.5 to deduce that $q_1^i(G,t) \neq 0$ for all $0 \leqslant i \leqslant m-n$ and all $t \in (-\infty, 2)$.

3.2. The multiplicity of the zero at 2

It is easy to see that a graph G has a nowhere-zero $\mathbb{Z}_2$-flow if and only if every vertex of G has even degree. Thus 2 is a zero of $F(G,t)$ if and only if G has at least one vertex of odd degree. We shall show in this section that 2 is a simple zero of a cubic graph G if and only if G is 3-connected. As in the previous section, our inductive proof requires us to work with near-cubic graphs.

We say that a graph is *essentially 3-connected* if it is a subdivision of a 3-connected graph or a k-bond for some $k \geqslant 3$. We shall use the following 'reduction' lemma due to Tutte [11, Theorem 12.65].

Lemma 3.3. *Let G be a 3-connected graph and $e \in E(G)$. Then either $G-e$ is essentially 3-connected or G/e is 3-connected.*

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Theorem 3.4. *Let (G, x) be a non-separable near-cubic graph with m edges, n vertices, and at least one vertex of degree three. Let*

$$q_2(G, t) = \frac{F(G, t)}{(t-1)(t-2)}.$$

Then $q_2(G, t)$ is a polynomial in t. Furthermore:

(a) *if G is not essentially 3-connected then $q_2(G, 2) = 0$,*
(b) *if G is essentially 3-connected then $q_2(G, 2)$ is non-zero with sign $(-1)^{m-n+1}$.*

Proof. The fact that $q_2(G, t)$ is a polynomial in t follows from the fact that $F(G, t)$ is a polynomial in t with a zero at 1 and a zero at 2 (since G has at least one vertex of degree three).

(a) Suppose that G is not essentially 3-connected. Since (G, x) is near-cubic it follows that we can find a vertex v of G, an edge e of G, and edge-disjoint subgraphs H_1 and H_2 of G such that $H_1 \cup H_2 = G - e$, $H_1 \cap H_2 = \{v\}$, and both H_1 and H_2 contain a vertex of degree three in G. Without loss of generality $x \in V(H_1)$. Let G_1 be obtained from G by contracting $E(H_2)$ to v, and define G_2 analogously. By Lemma 2.6, $F(G, t) = F(G_1, t)F(G_2, t)/(t-1)$, and hence

$$q_2(G, t) = q_2(G_1, t)q_2(G_2, t)(t-2). \tag{3.1}$$

It can be seen that (G_1, x) and (G_2, v) are both non-separable near-cubic graphs with fewer edges than G and both contain a vertex of degree three. Thus $q_2(G_1, t)$ and $q_2(G_2, t)$ are both polynomials in t and $q_2(G, 2) = 0$ by (3.1). This completes the proof of (a).

(b) Suppose that G is essentially 3-connected. We prove (b) by induction on m. Let G' be the graph of minimum degree three which is homeomorphic to G. If $G' \neq G$ then since $q_2(G', t) = q_2(G, t)$ by Lemma 2.4, and $m' - n' = m - n$, we may deduce inductively that the theorem holds for G. Thus we may suppose that $G' = G$. If $G = K_2^3$, then $F(G, t) = (t-1)(t-2)$, $q_2(G, t) \equiv 1$ and the theorem holds for G. Hence we may assume that G is 3-connected.

Let e be an edge of G incident to x. By Lemma 3.3, either $G - e$ or G/e is essentially 3-connected. If $G - e$ is essentially 3-connected, then we may use Lemma 2.3 to deduce that (b) holds for G by applying (b) inductively to $G - e$, applying (a) to G/e if G/e is not essentially 3-connected, and applying (b) inductively to G/e if G/e is essentially 3-connected. A similar proof holds if G/e is essentially 3-connected □

3.3. Cleavage units

Tutte [11] developed a theory by which a 2-connected graph G can be uniquely decomposed into pieces called *cleavage units* which are either 3-connected simple graphs or bonds or circuits. We will use this decomposition extensively in the next subsection to show that the interval $(2, \alpha]$ contains no flow roots of near-cubic graphs. We give a brief description of Tutte's theory below for the sake of completeness.

We shall adopt the notation and terminology of Tutte [11] with the exception that we will refer to 'polygons', 'simple paths', and 'valency' as circuits, paths and degree

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respectively. Let x and y be vertices of a 2-connected graph G such that $G-\{x,y\}$ has components $G_1, G_2, \ldots, G_r$, with $r \geqslant 2$ if xy is not a multiple edge of G, and $r \geqslant 1$ otherwise. Each of the subgraphs of G induced by $V(G_i) \cup \{x,y\}$, but with edges joining x and y deleted, is called an $\{x,y\}$-*component* of G. In addition, if $xy \in E(G)$, the subgraph induced by $\{x,y\}$ is a *trivial* $\{x,y\}$-*component* of G. Let H be an $\{x,y\}$-component of G and put $H' = G-(H-\{x,y\})$. We say that H is *excisable* if H is nontrivial and either H or H' is a 2-connected graph or a k-bond for some $k \geqslant 2$. If an excisable $\{x,y\}$-component H of G exists, we say that $\{x,y\}$ is a *hinge* of G and H is called a *hinge component* of G. A hinge $\{x,y\}$ of G is of *Type I* if G has exactly two $\{x,y\}$-components and of *Type II* otherwise.

For each hinge $\{x,y\}$ of G we adjoin to G a *virtual edge* with ends x and y for each excisable $\{x,y\}$-component of G to form the *augmented graph* G^a. Two distinct hinge components of G give rise to the same virtual edge if and only if they are the two $\{x,y\}$-components of some hinge $\{x,y\}$ of Type I. The virtual edges are required to be distinct from the true edges of G (but may be parallel to them). If H is an excisable $\{x,y\}$-component of G , the two graphs D_1 and D_2 derived from H and H' by adjoining to each of H and H' the virtual edge e associated with H are called the *cleavage graphs* of G at e.

The *cleavage units* of G are the minimal cleavage graphs obtained by recursively constructing cleavage graphs from cleavage graphs. Thus no cleavage unit of G can have a hinge, and each virtual edge of G belongs to exactly two cleavage units. The *cleavage unit tree* T of G is the graph whose vertices and edges are the cleavage units and virtual edges, respectively, of G, in which a cleavage unit D and an edge e are incident in T if and only if e is an edge of D. We say that G is a *cleavage unit path* if its cleavage unit tree is a path on at least two vertices.

These definitions are illustrated for the graph G in Figure 1. We form G^a by adding the five virtual edges e_1, e_2, e_3, e_4 and e_5 to G. The hinges of G are the pairs of end vertices of the virtual edges. The cleavage units of G, $X_1, X_2, X_3, X_4, X_5, X_6$, and its cleavage unit tree, T, are shown in Figure 2.

Tutte shows in Chapter 11 of [11] that the cleavage unit tree of a 2-connected graph G is indeed a tree and that each cleavage unit of G is either a 3-connected simple graph or a circuit of length at least three, or a bond with at least three edges. By an *end-cleavage unit* of G we shall mean a cleavage unit which has degree one in the cleavage unit tree. By an *internal vertex* of G we shall mean a vertex which does not belong to any hinge of G. Thus v is an internal vertex of G if either $d(v) = 2$, or $G-v$ is non-separable and v is not incident with a parallel edge of G.

We shall need the following observations on the cleavage unit decomposition of a graph.

Lemma 3.5. *Let G be a 2-connected graph and T be the cleavage unit tree of G.*

(a) *The set of circuit cleavage units and the set of bond cleavage units of G are independent sets of vertices in T,*

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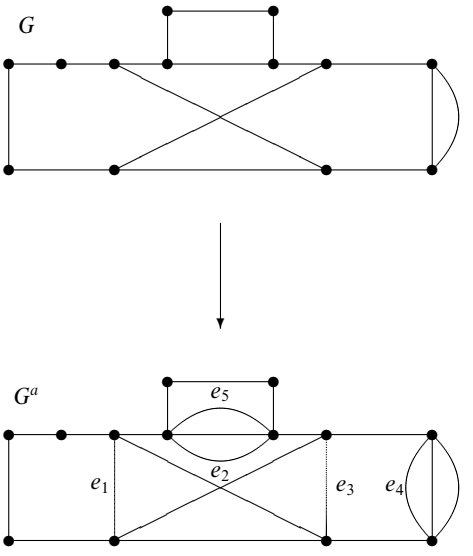


Figure 1. A graph G and its augmented graph G^a.

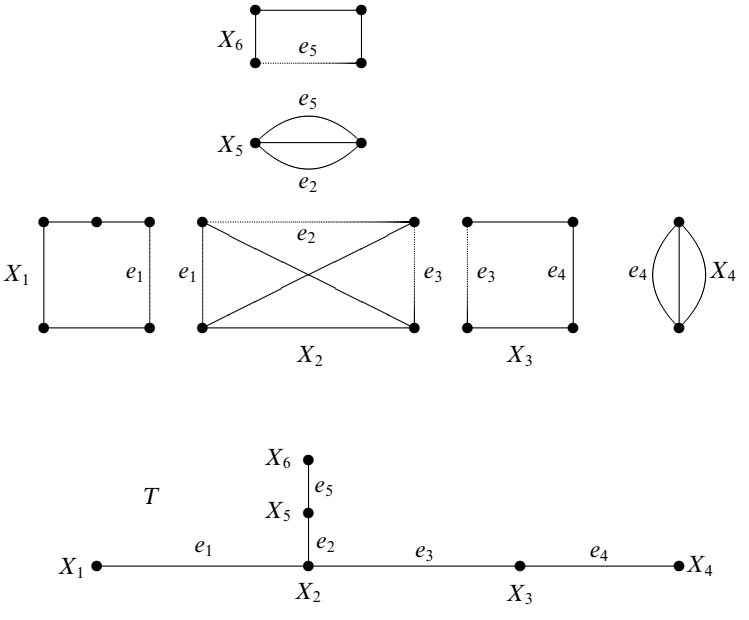


Figure 2. The cleavage units of G and its cleavage unit tree T.

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(b) *Suppose G has at least two end cleavage units. Let G' be obtained from G by adding a new vertex x and an edge from x to an internal vertex in each end cleavage unit of G. Then G' is essentially 3-connected.*

We shall also need the following elementary result on 3-connectivity.

Lemma 3.6. *Let G be a 3-connected graph and $e = uv$ be an edge of G of multiplicity one. Suppose $G - \{u, v\}$ is 2-connected. Then G/e is 3-connected.*

Cleavage units of near-cubic graphs. It can be seen that the degree of a vertex in a cleavage unit of a 2-connected graph G is bounded above by its degree in G. Thus the cleavage units of a near-cubic graph (G, x) are themselves near-cubic. Lemma 2.6 can be used to express $F(G, t)$ in terms of the flow polynomials of its non-circuit cleavage units.

Lemma 3.7. *Let (G, x) be a 2-connected near-cubic graph and $G_1, G_2, \ldots, G_c$ be the cleavage units of G which are not circuits. Then*

$$F(G, t) = (t-1)^{-c+1} \prod_{i=1}^{c} F(G_i, t).$$

Proof. We use induction on the total number of cleavage units of G. The lemma holds trivially if G is a 3-connected simple graph, or a 3-bond, or a circuit (in the case of a circuit we have $c = 0$, the empty product is equal to 1, and $F(G, t) = t - 1$). Thus we may assume that G has at least two cleavage units and hence G has a hinge $\{u, v\}$. Let H be an excisable $\{u, v\}$-component, e be the virtual edge incident with u, v corresponding to H, and G_1, G_2 the cleavage graphs of G at e. Without loss of generality, $u \neq x$. Then $d_G(u) = 3$ and u is incident to a bridge uu' of $G - v$. Relabelling if necessary, we have $u' \in V(G_2)$ and $d_{G_2}(u) = 2$. By Lemmas 2.4 and 2.6, $F(G, t) = (t-1)^{-1} F(G_1, t) F(G_2, t)$. Furthermore, each cleavage unit of G is a cleavage unit of G_1 or G_2 (but not both). The lemma now follows by applying induction to G_1 and G_2. □

As an example, we may apply Lemma 3.7 to the graph G in Figure 1 to deduce that

$$F(G, t) = (t-1)^{-2} F(K_4, t) F(K_2^3, t)^2 = (t-1)(t-2)^3(t-3).$$

Lemma 3.7 and Theorem 3.4 imply the following.

Corollary 3.8. *Let (G, x) be a 2-connected near-cubic graph with at least one vertex of degree three. Then 2 is a zero of $F(G, t)$ of multiplicity c, where c is the number of cleavage units of G which are not circuits.*

We shall use the following property of circuit cleavage units in graphs of maximum degree three.

Lemma 3.9. *Let G be a 2-connected graph of maximum degree three. If X is a circuit cleavage unit of G then the virtual edges of G contained in X are independent.*

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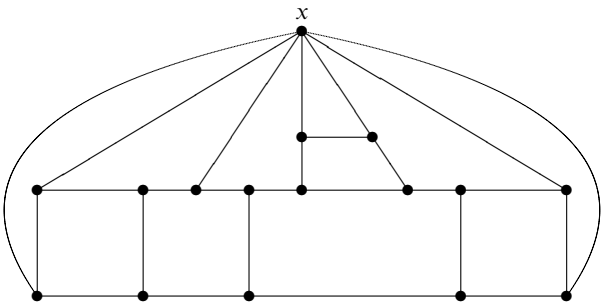


Figure 3. An extension of (K_4, x).

Proof. Let $e = uv$ be a virtual edge of G contained in X and let Y be the other cleavage unit of G which contains e. Let H_1, H_2 be the cleavage graphs of G at e. Relabelling if necessary, we have $V(X) \subset H_1$ and $V(Y) \subseteq H_2$. Lemma 3.5(a) and the fact that X is a circuit cleavage unit imply that Y is not a circuit cleavage unit. Thus $d_{H_2}(u) \geqslant 3$ and $d_{H_2}(v) \geqslant 3$. Since G has maximum degree three, we must have $d_{H_1}(u) = 2 = d_{H_1}(v)$. Thus u and v are both internal vertices of H_1. Hence neither u nor v can be incident to a virtual edge of H_1. Thus e is the only virtual edge of G incident to u and v in X. □

3.4. The interval (2,α]

We shall first construct a sequence of near-cubic graphs with flow roots converging to α from above. We then show that bridgeless near-cubic graphs have no flow roots in $(2, \alpha]$.

3.4.1. A special family of near-cubic graphs. Let (H, x) and (G, x) be near-cubic graphs, and $xu, uv, uw \in E(H)$. We say that G is a *single edge extension* of H if G can be obtained from $H - u$ by adding four new vertices u_1, u_2, u_3, u_4 and eight new edges xu_1, xu_2, u_1u_2, u_2u_3, u_3u_4, u_4u_1, u_3v, u_4w. We say that (G, x) is an *extension* of (H, x) if (G, x) can be obtained from (H, x) by a sequence of single edge extensions. We shall show that near-cubic graphs which are extensions of (K_4, x) have no flow roots in $(2, \alpha]$, but can have flow roots arbitrarily close to α. An example of an extension of (K_4, x) is given in Figure 3. This can be seen by recursively performing the inverse operation to a single edge extension: choosing a pair of edges e_1, e_2 incident to x which belong to both a 3-cycle e_1, e_2, f_0 and a 5-cycle e_1, e_2, f_1, f_2, f_3; contracting f_0, f_1, f_2, f_3 and deleting either e_1 or e_2.

We need to consider one other family of near-cubic graphs. Let (L_0, x) be the loopless near-cubic graph with $V(L_0) = \{x, y, z\}$ in which x has degree two and all other vertices have degree three. Let $\mathscr{L}(x, y)$ be the isomorphism class of all near-cubic graphs which are isomorphic to either (L_0, x) or a graph which can be obtained recursively from (L_0, x) by applying the single edge extension operation to edges which are incident to x but not to y. Let (L_1, x) be the near-cubic graph obtained from (L_0, x) by applying the single edge extension operation to the edge xz. Thus $V(L_1) = \{x, z_1, z_2, z_3, z_4, y\}$ and $E(L_1) = \{xz_1, xz_2, z_1z_2, z_2z_3, z_3z_4, z_4z_1, z_3y, z_4y\}$.

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Lemma 3.10. *Let* $(G,x) \in \mathscr{L}(x,y)$. *Let* $t \in (2,3)$ *and* $g > 0$ *be real numbers satisfying* $f(t,g) := (3-t)g - 2(t-2) \geqslant 0$ *and*

$$h(t,g) := (3-t)^2(g^2+t-2) - (3-t)(t-2)(3-t+2g) - g(t-2)(g^2+t-2) \geqslant 0.$$

Then:

(a) $F(G-xy,t) \geqslant 0$ *with strict inequality when* $G \neq L_0$,
(b) $F(G,t) \geqslant gF(G-xy,t)$,
(c) $F(G,t) > 0$.

Proof. We prove (a), (b) and (c) simultaneously by induction on m. If $m = 4$ then $G = L_0$, $F(G,t) = (t-1)(t-2)$, $F(G-xy,t) \equiv 0$ and the lemma holds. Hence we may suppose that G is either equal to (L_1,x) or can be obtained from (L_1,x) by recursively applying the single edge extension operation to edges incident to x but not y. Thus $d_G(y) = 3$, $yx, yz_3, yz_4, z_3z_4 \in E(G)$, and $G - \{x,y,z_3,z_4\}$ has a bridge $e = u_1u_2$ corresponding to the bridge z_1z_2 of $L_1 - \{x,y,z_3,z_4\}$. Hence there exist $G_1, G_2 \in \mathscr{L}(x,y)$, such that $G - \{y,z_3,z_4,e\} = (G_1 - y) \cup (G_2 - y)$, $(G_1 - y) \cap (G_2 - y) = \{x\}$, $u_1 \in V(G_1 - y)$ and $u_2 \in V(G_2 - y)$. The near-cubic graphs (G_i,x) are obtained by applying the same sequence of single edge extensions to the edge xz of (L_0,x) as were applied to the edge xz_i in the construction of (G,x) from (K_4,x).

We shall express $F(G-xy,t)$ and $F(G,t)$ in terms of $F(G_1,t)$, $F(G_1-xy,t)$, $F(G_2,t)$ and $F(G_2-xy,t)$. Let G^+ be the graph obtained by contracting $\{z_3z_4, yz_3, yz_4\}$ onto y. Using Lemmas 2.7 and 2.9, we have

$$\begin{aligned}(t-1)F(G-xy,t) &= (t-1)(t-2)F(G^+-xy,t)\\ &= F(G_1,t)F(G_2,t) + (t-2)F(G_1-xy,t)F(G_2-xy,t). \qquad (3.2)\end{aligned}$$

Applying (a), (c) inductively to G_1, G_2, G_1-xy, G_2-xy, we deduce that $F(G-xy,t) > 0$. Hence (a) holds for G.

By Lemma 2.8, $(t-1)F(G,t) = (t-1)(t-3)F(G^+,t)$. Applying Lemma 2.3 to the edge xy of G^+ and then using Lemmas 2.6 and 2.9, and the fact that $F(G_i/xy,t) = F(G_i,t) + F(G_i-xy,t)$ by Lemma 2.3, we may deduce that

$$\begin{aligned}(t-1)(t-2)F(G,t) = (3-t)\big[&(3-t)F(G_1,t)F(G_2,t)\\ &-(t-2)F(G_1,t)F(G_2-xy,t) - (t-2)F(G_1-xy,t)F(G_2,t)\big].\end{aligned}$$

Let r be a positive real number such that $3-t-2r \geqslant 0$. Then

$$\begin{aligned}(t-1)(t-2)F(G,t) = (3-t)\big[&(3-t-2r)F(G_1,t)F(G_2,t)\\ &+F(G_1,t)\big(rF(G_2,t) - (t-2)F(G_2-xy,t)\big)\\ &+F(G_2,t)\big(rF(G_1,t) - (t-2)F(G_1-xy,t)\big)\big].\end{aligned}$$

Since $t \in (2,3)$ and $3-t-2r \geqslant 0$, we may apply the inductive hypothesis to G_1, G_2 to obtain

$$\begin{aligned}(t-1)(t-2)F(G,t) &\geqslant (3-t)\\ &\times [(3-t-2r)F(G_1,t)F(G_2,t) + 2g(rg-t+2)F(G_1-xy,t)F(G_2-xy,t)]. \qquad (3.3)\end{aligned}$$

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In order to compare $F(G,t)$ and $F(G-xy,t)$, we choose a value for r so that the ratio of the coefficients in (3.2) and (3.3) are equal. Let $r = \frac{1}{2}(t-2)(3-t+2g)(g^2+t-2)^{-1}$. Then $3-t-2r = g(g^2+t-2)^{-1}f(t,g) \geqslant 0$ by the hypotheses of the lemma. Thus we may use (3.2) and (3.3) to obtain

$$F(G,t) \geqslant \left[\frac{(3-t)^2}{t-2} - \frac{(3-t)(3-t+2g)}{g^2+t-2}\right] F(G-xy,t).$$

Since $h(t,g) \geqslant 0$ we have $F(G,t) \geqslant gF(G,t)$ and (b) holds for G. The truth of (c) now follows from (a) and (b). □

We next choose a value for g in Lemma 3.10 in order to maximize the range of values of t for which the lemma can be applied. It can be seen that the maximum value of $t \in (2,3)$ for which the constraint $h(t,g) \geqslant 0$ holds occurs when $h(t,g) = 0 = \frac{\partial h}{\partial g}$. Solving these simultaneous equations we obtain $t = \alpha$ and $g = \rho$ where $\rho \approx 1.33\cdots$ is the zero in $(1,2)$ of the polynomial $g^4 + 2g^3 - 2g^2 - 4g + 1$. Since $f(t,\rho)$ and $h(t,\rho)$ are decreasing functions of t for $t \in (2,\alpha)$ and $f(\alpha,\rho) > 0 = h(\alpha,\rho)$, we have the following.

Corollary 3.11. *Let $(G,x) \in \mathscr{L}(x,y)$ and $t \in (2,\alpha]$. Then:*

(a) $F(G-xy,t) \geqslant 0$ *with strict inequality when* $G \neq L_0$,
(b) $F(G,t) \geqslant \rho F(G-xy,t)$, *where* $\rho \approx 1.33\cdots$ *is the zero in* $(1,2)$ *of the polynomial* $g^4 + 2g^3 - 2g^2 - 4g + 1$,
(c) $F(G,t) > 0$.

We now use this corollary to prove that no extension of (K_4,x) can have a flow root in $(2,\alpha]$.

Lemma 3.12. *Let (G,x) be an extension of (K_4,x). Then $F(G,t) < 0$ for all $t \in (2,\alpha]$.*

Proof. Let e_1,e_2,e_3 be the edges of (K_4,x) which are not incident to x. Since (G,x) is an extension of (K_4,x), there exist graphs (G_1,x), (G_2,x), and (G_3,x) in $\mathscr{L}(x,y)$ such that $G - \{e_1,e_2,e_3\} = (G_1-y) \cup (G_2-y) \cup (G_3-y)$, $(G_i-y) \cap (G_{i+1}-y) = \{x\}$, and e_i is incident with a vertex of $G_i - x$ and $G_{i+1} - x$ for all $1 \leqslant i \leqslant 3$, reading subscripts modulo three. Applying Lemmas 2.9, 2.8, 2.7 and 2.3 to G, we may deduce that

$$\begin{aligned}(t-1)^2(t-2)^2F(G,t) = {} & (t-3)F(G_1,t)F(G_2,t)F(G_3,t)\\ & + (t-2)F(G_1,t)F(G_2,t)F(G_3-xy,t)\\ & + (t-2)F(G_1,t)F(G_2-xy,t)F(G_3,t)\\ & + (t-2)F(G_1-xy,t)F(G_2,t)F(G_3,t)\\ & + (t-2)^2F(G_1-xy,t)F(G_2-xy,t)F(G_3-xy,t). \qquad (3.4)\end{aligned}$$

If $G_1 = G_2 = G_3 = L_0$ then $G = K_4$, $F(G,t) = (t-1)(t-2)(t-3)$ and the lemma holds for G. Hence we may suppose that $G_1 \neq L_0$.

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Suppose that $G_2 = G_3 = L_0$. Then $F(G_i, t) = (t-1)(t-2)$ and $F(G_i - xy, t) \equiv 0$ for all $2 \leqslant i \leqslant 3$. Substituting into (3.4), we obtain

$$F(G, t) = (t-3)F(G_1, t) + (t-2)F(G_1 - xy, t).$$

Corollary 3.11 and the fact that $t \in (2, \alpha]$ now give

$$F(G, t) \leqslant -\big[\rho(3-\alpha) - (\alpha-2)\big]F(G_1 - xy, t) < 0.$$

Hence we may suppose that $G_1, G_2 \neq L_0$.

Suppose that $G_3 = L_0$. Then $F(G_3) = (t-1)(t-2)$ and $F(G_3 - xy) \equiv 0$. Substituting into (3.4), we obtain

$$\begin{aligned}(t-1)(t-2)F(G, t) &= (t-3)F(G_1, t)F(G_2, t)\\ &\quad + (t-2)\big[F(G_1 - xy, t)F(G_2, t) + F(G_1, t)F(G_2 - xy, t)\big]\\ &= -\tfrac{1}{2}F(G_1, t)\big[(3-t)F(G_2, t) - 2(t-2)F(G_2 - xy, t)\big]\\ &\quad - \tfrac{1}{2}F(G_2, t)\big[(3-t)F(G_1, t) - 2(t-2)F(G_1 - xy, t)\big]. \qquad (3.5)\end{aligned}$$

Corollary 3.11 and the fact that $t \in (2, \alpha]$ now give

$$(t-1)(t-2)F(G, t) \leqslant -\rho\big[\rho(3-\alpha) - 2(\alpha-2)\big]F(G_1 - xy, t)F(G_2 - xy, t) < 0.$$

Hence we may suppose that $G_1, G_2, G_3 \neq L_0$.

We may rewrite (3.4) as

$$\begin{aligned}(t-1)^2(t-2)^2F(G, t) &= -\tfrac{1}{3}F(G_1, t)F(G_2, t)\big[(3-t)F(G_3, t) - 3(t-2)F(G_3 - xy, t)\big]\\ &\quad - \tfrac{1}{3}F(G_1, t)F(G_3, t)\big[(3-t)F(G_2, t) - 3(t-2)F(G_2 - xy, t)\big]\\ &\quad - \tfrac{1}{3}F(G_2, t)F(G_3, t)\big[(3-t)F(G_1, t) - 3(t-2)F(G_1 - xy, t)\big]\\ &\quad + (t-2)^2F(G_1 - xy, t)F(G_2 - xy, t)F(G_3 - xy, t).\end{aligned}$$

Corollary 3.11 and the fact that $t \in (2, \alpha]$ now give

$$\begin{aligned}(t-1)^2(t-2)^2F(G, t) &\leqslant -\big[(3-\alpha)\rho^3 - 3(\alpha-2)\rho^2 - (\alpha-2)^2\big]\\ &\quad \times F(G_1 - xy, t)F(G_2 - xy, t)F(G_3 - xy, t) < 0.\end{aligned}$$

□

Our final result in this subsection is to show that there exists a sequence of extensions of (K_4, x) with flow roots converging to α from above. Let (G_1, x) be a copy of (K_4, x) and e be an edge of G_1 incident to x. For $i \geqslant 2$, let (G_i, x) be the near-cubic graph obtained from (G_{i-1}, x) by applying the single edge extension operation to every edge incident to x other than e.

Theorem 3.13. *For all $\epsilon > 0$ there exists an $n \geqslant 1$ such that G_n has a flow root in $(\alpha, \alpha + \epsilon]$.*

Proof. Let (L_0, x) be the near-cubic graph defined in the paragraph before Lemma 3.10. For $i \geqslant 1$, let (L_i, x) be the near-cubic graph obtained from (L_{i-1}, x) by applying the single edge extension operation to every edge incident to x other than xy. For $i \geqslant 0$, let $a_i(t) = F(L_i, t)$, $b_i(t) = F(L_i - xy, t)$, and $c_{i+1}(t) = F(G_{i+1}, t)$. Applying the argument used

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in the derivation of (3.2), we deduce that for $i \geqslant 0$,

$$(t-1)b_{i+1} = a_i^2 + (t-2)b_i^2. \tag{3.6}$$

Applying the argument used in the derivation of (3.5), we deduce that for $i \geqslant 0$,

$$(t-1)(t-2)c_{i+1} = (t-3)a_i^2 + 2(t-2)a_i b_i. \tag{3.7}$$

Applying Lemma 2.8 to L_i for $i \geqslant 1$, using the 3-edge-cut incident to the triangle containing y, we have

$$a_i = (t-3)c_i. \tag{3.8}$$

For $i \geqslant 1$, let $r_i(t) = c_i(t)/b_i(t)$. Thus $r_1(t) = (t-3)/(t-2)$ and equations (3.6), (3.7) and (3.8) imply that

$$r_{i+1}(t) = \frac{(t-3)[(t-3)^2 r_i^2 + 2(t-2)r_i]}{(t-2)[(t-3)^2 r_i^2 + (t-2)]}. \tag{3.9}$$

Let

$$f(r,t) = \frac{(t-3)[(t-3)^2 r^2 + 2(t-2)r]}{(t-2)[(t-3)^2 r^2 + (t-2)]}.$$

Thus $r_{i+1}(t) = f(r_i(t), t)$ for all $i \geqslant 1$.

Claim 3.14. *If $r < 0$ and $t \in (\alpha, 3)$ then $f(r,t) > r$.*

Proof. We have

$$f(r,t) - r = -r(t-2)^{-1}[(3-t)^2 r^2 + (t-2)]^{-1} g(r,t),$$

where

$$g(r,t) = (3-t)[(3-t)^2 r + 2(t-2)] + (t-2)[(3-t)^2 r^2 + (t-2)].$$

Solving $g(r,t) = 0$ for r, we obtain

$$r = \frac{t^2 - 6t + 9 \pm \sqrt{t^4 - 8t^3 + 22t^2 - 28t + 17}}{2(t-2)(t-3)}.$$

Since $t^4 - 8t^3 + 22t^2 - 28t + 17 < 0$ for $t \in (\alpha, 3)$ we deduce that $g(r,t) > 0$ for all r and all $t \in (\alpha, 3)$. Thus $f(r,t) - r > 0$ for all $r < 0$ and all $t \in (\alpha, 3)$. □

Claim 3.15. *If $t \in (\alpha, 3)$ then $r_n(t) \geqslant 0$ for some $n \geqslant 1$.*

Proof. Suppose $r_n(t) < 0$ for all $n \geqslant 1$. Since $r_{i+1}(t) = f(r_i(t), t)$, Claim 3.14 implies that $\{r_i(t) : i \geqslant 1\}$ is a monotonically increasing sequence bounded above by zero. Hence it must converge to a limit $r \leqslant 0$ satisfying $r = f(r,t)$. Using Claim 3.14, we may deduce that $r = 0$. This is impossible, since for fixed $t \in (\alpha, 3)$ the right-hand side of (3.9) is positive when $r_i(t)$ is negative and sufficiently close to zero. □

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We may now complete the proof of the theorem. By Claim 3.15, we may choose $n \geqslant 1$ such that $r_n(\alpha+\epsilon) \geqslant 0$. Since (3.6) implies that $b_n(\alpha+\epsilon) \geqslant 0$, we must have $c_n(\alpha+\epsilon) \geqslant 0$. On the other hand, Lemma 3.12 implies that $c_n(\alpha) < 0$. Since $F(G_n, t) = c_n(t)$ is continuous, it must have a zero in $(\alpha, \alpha+\epsilon]$. □

3.4.2. The zero-free interval. We show that no near-cubic graph can have a flow root in $(2, \alpha]$. We shall need the following 'characterization' of the family of near-cubic graphs which are extensions of (K_4, x). We only state and prove the direction of the characterization which we require. The reader may find it helps their understanding of the following proof if they first verify that the graph in Figure 3 satisfies conditions (a) and (b) below.

Lemma 3.16. *Let (G, x) be a 3-connected near-cubic graph and T be the cleavage unit tree of $G - x$. Suppose that:*

(a) *each non-circuit cleavage unit of $G - x$ is a K_2^3 and has degree two in T,*
(b) *each circuit cleavage unit of $G - x$ either is a C_4 and has degree one in T, or is a C_5 and has degree two in T, or is a C_6 and has degree three in T.*

Then (G, x) is an extension of (K_4, x).

Proof. Let $P = X_1 X_2 \cdots X_r$ be a longest path in T. Then X_1 has degree one in T, so is a C_4-cleavage unit of $G - x$ by (b). Let $X_1 = u_1u_2u_3u_4u_1$ where u_1, u_2 have degree two in $G - x$. Since G is 3-connected, $xu_1, xu_2 \in E(G)$. By Lemma 3.5(a) and hypothesis (a) of the lemma, X_2 is a K_2^3-cleavage unit of $G - x$ and has degree two in T. Thus $V(X_2) = \{u_3, u_4\}$ and $u_3u_4 \in E(G)$. By Lemma 3.5(a), X_3 is a circuit cleavage unit of G. By (b), X_3 has degree at most three in T. If X_3 has degree one in T then, by hypothesis (b), X_3 is a C_4-cleavage unit of $G - x$ and (G, x) is a single edge extension of (K_4, x). Thus we may assume that X_3 has degree two or three in T.

Suppose X_3 has degree two in T. By (b), X_3 is a C_5-cleavage unit of $G - x$. Let $X_3 = u_3vzwu_4u_3$. By Lemma 3.9 and symmetry, we may suppose that the virtual edge of $G - x$ contained in X_3 and distinct from u_3u_4, is wz. Thus v is a vertex of degree two in $G - x$ and $xv \in E(G)$. Let (G', x) be the near-cubic graph obtained from $G - \{u_1, u_2, u_3, u_4\}$ by adding a new vertex u and edges ux, uv, uw. Then (G, x) is a single edge extension of (G', x). Furthermore G' is 3-connected, the set of cleavage units of G' is obtained from the set of cleavage units of G by deleting X_1, X_2, X_3 and adding a new C_4-cleavage unit $X' = uvzwu$, and the cleavage unit tree T' of $G' - x$ is obtained from $T - \{X_1, X_2, X_3\}$ by adding the new vertex X' and edge $X'X_4$. Thus G' satisfies the hypotheses of the lemma. By induction, (G', x) is an extension of (K_4, x). Thus (G, x) is an extension of (K_4, x).

Suppose X_3 has degree three in T. By (b), X_3 is a C_6-cleavage unit of $G - x$. Let $X_3 = u_3vyzwu_4u_3$. By Lemma 3.9, the virtual edges of $G - x$ contained in X_3 are u_3u_4, vy, zw. By symmetry, we may assume that $vy \in E(X_4)$. Let Y_2 be the neighbour of X_3 in T distinct from X_2, X_4. As above, Y_2 is a K_2^3-cleavage unit of $G - x$ and has degree two in T. Thus $V(Y_2) = \{w, z\}$ and $wz \in E(G)$. Let Y_1 be the neighbour of Y_2 in T distinct from X_3. The maximality of P implies that Y_1 is an end-vertex of T and hence is a C_4-cleavage unit of

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G. Let (G', x) be the near-cubic graph obtained from $G - \{u_1, u_2, u_3, u_4\}$ by adding a new vertex u and edges ux, uv, uw. Then (G, x) is a single edge extension of (G', x). Furthermore G' is 3-connected, the set of cleavage units of G' is obtained from the set of cleavage units of G by deleting X_1, X_2, X_3 and adding a new C_5-cleavage unit $X' = wuvyzw$, and the cleavage unit tree T' of $G' - x$ is obtained from $T - \{X_1, X_2, X_3\}$ by adding the new vertex X' and edges $X'X_4, X'Y_2$. Thus G' satisfies the hypotheses of the lemma. By induction, (G', x) is an extension of (K_4, x). Thus (G, x) is an extension of (K_4, x). □

Let C_k denote the isomorphism class of circuits of length k.

Theorem 3.17. *Let (G, x) be a non-separable near-cubic graph with m edges, n vertices and c non-circuit cleavage units. Then $F(G, t)$ is non-zero with sign $(-1)^{m-n+c}$ for $t \in (2, \alpha]$, where $\alpha \approx 2.225\cdots$ is the real zero in $(2, 3)$ of the polynomial $t^4 - 8t^3 + 22t^2 - 28t + 17$.*

Proof. We proceed by contradiction. Suppose the theorem is false. Then we may choose a $t \in (2, \alpha]$ and a near-cubic graph (G, x) satisfying the hypotheses of the theorem such that $(-1)^{m-n+c}F(G, t) \leqslant 0$ and such that, for this fixed value of t, m is as small as possible. We shall show that G satisfies the hypotheses of Lemma 3.16 and hence is an extension of (K_4, x). This will give the required contradiction by Lemma 3.12. Since the theorem holds if G is a circuit or a 3-bond, we deduce that G is 2-connected, $n \geqslant 3$ and $m \geqslant 4$. Since suppressing vertices of degree two in G leaves $m - n$ and c unchanged and also leaves $F(G, t)$ unchanged by Lemma 2.4, G has minimum degree three.

Claim 3.18. *G is 3-connected.*

Proof. Suppose the claim is false and let G_1 be an end-cleavage unit of G. Since G has minimum degree three, G_1 is not a circuit, and hence G_1 is either a 3-bond or is 3-connected. We may express G as $G = H_1 \cup H_2$ with $H_1 \cap H_2 = \{u, v\}$ for some $u, v \in V(G)$ and $G_1 = H_1 + uv$. Without loss of generality $x \neq v$. Let $G_2 = H_2 + uv$. Since G is near-cubic and $d_{G_1}(v) = 3$, we have $d_{G_2}(v) = 2$. For $i = 1, 2$, let $x_i = x$ if $x \in V(H_i)$ and otherwise relabel an arbitrary vertex of G_i as x_i. Then (G_i, x_i) is near-cubic and non-separable, $m = m_1 + m_2 - 2$, $n = n_1 + n_2 - 2$, $c_1 = 1$ and $c_2 = c - 1$. By Lemma 2.6, $(t-1)F(G, t) = F(G_1, t)F(G_2, t)$. Since $m_1, m_2 \geqslant 3$, we have $m_1, m_2 \leqslant m - 1$. Thus we may apply induction to G_1 and G_2 to deduce that the theorem holds for G. □

Claim 3.19. *G is cyclically 4-edge-connected.*

Proof. Suppose the claim is false. Let S be a cyclic 3-edge-cut in G and H_1, H_2 be the components of $G - S$. For $i = 1, 2$, let G_i be the near-cubic graph obtained from G by contracting $E(H_{3-i})$. Since G is 3-connected, G_1 and G_2 are 3-connected, and hence $c = 1 = c_1 = c_2$. Furthermore $m = m_1 + m_2 + 3$, $n = n_1 + n_2 + 2$ and, by Lemma 2.8, $(t-1)(t-2)F(G, t) = F(G_1, t)F(G_2, t)$. Applying induction to G_1 and G_2 we deduce that the theorem holds for G. □

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Using Claim 3.19 we immediately deduce the following.

Claim 3.20. *$G - e$ is essentially 3-connected for all $e \in E(G)$.*

Claim 3.21. *G/e is a cleavage unit path with an even number of non-circuit cleavage units for all edges e of G incident to x.*

Proof. Let $G_1 = G/e$ and $G_2 = G - e$. By Claim 3.18, G_1 is 2-connected and by Claim 3.20, G_2 is essentially 3-connected. By Lemma 2.3, $F(G,t) = F(G_1,t) - F(G_2,t)$. Since $n = n_1 + 1 = n_2$, $m = m_1 - 1 = m_2 - 1$ and $c = 1 = c_2$, we may deduce that the theorem will hold for G by applying induction to G_1 and G_2 if c_1 is odd. Hence c_1 is even.

Let $e = xv$ and let x_1 be the vertex of G_1 obtained by contracting e. Choose an end cleavage unit H_1 of G_1. Then we may express G_1 as $G_1 = (H_1 - yz) \cup (H_2 - yz)$ where $H_1 \cap H_2 = \{y, z, yz\}$ for some $y, z \in V(G_1)$. Since G is 3-connected by Claim 3.18, $x_1 \in \{y, z\}$, say $x_1 = y$. Then $G - \{x, v, z\}$ is disconnected. Since G is 3-connected, some vertex of $V(H_1) - \{x_1, z\}$ is adjacent to v in G. Thus every end cleavage unit of G_1 has an internal vertex which is a neighbour of v in G distinct from x. Since G is near-cubic, v has exactly two neighbours distinct from x. Thus G_1 has exactly two end cleavage units and hence G_1 is a cleavage unit path. □

Claim 3.22. *Suppose $S = \{u_1u_2, v_1v_2\}$ is a 2-edge-cut of $G - x$. Let $G - S = H_1 \cup H_2$ where $H_1 \cap H_2 = \{x\}$, $u_1, v_1 \in V(H_1)$ and $u_2, v_2 \in V(H_2)$. Suppose further that $d_{H_1}(x) \geqslant 2$ and $d_{H_2}(x) \geqslant 2$. Let G^- be the graph obtained from $G - x$ by adding two new vertices x_1, x_2, and joining x_i to each neighbour of x in H_i for $1 \leqslant i \leqslant 2$. Let G^+ be obtained from G^- by adding a new edge $e = x_1x_2$. For $i = 1, 2$, let G_i^+ be the graph obtained from G^+ by contracting $E(H_{3-i})$ to the single vertex y_i, and $G_i^- = G_i^+ - x_iy_i$. Then:*

(a) *G^- is a cleavage unit path and $c^- = c_1^- + c_2^-$ is odd,*

(b) *if x is adjacent to a vertex in $\{u_i, v_i\}$ for some $i \in \{1, 2\}$, then G_i^- is essentially 3-connected,*

(c) *x is not adjacent to both a vertex in $\{u_1, v_1\}$ and a vertex in $\{u_2, v_2\}$,*

(d) *if c_2^- is odd, then $c(G_2^+/x_2y_2)$ is even and $c_2^- = 1$.*

Proof. **(a)** The fact that G^- is a cleavage unit path follows since the 3-connectivity of G implies that each end cleavage unit of G^- must contain either x_1 or x_2 as an internal vertex. By Lemma 2.9,

$$(t-1)(t-2)F(G,t) = F(G_1^+,t)F(G_2^+,t) + (t-2)F(G_1^-,t)F(G_2^-,t). \tag{3.10}$$

Since G is 3-connected, G_1^+ and G_2^+ are both 3-connected. Thus $c_1^+ = 1 = c_2^+$. Furthermore,

$$m = m_1^+ + m_2^+ - 4 = m_1^- + m_2^- - 2$$

and

$$n = n_1^+ + n_2^+ - 3 = n_1^- + n_2^- - 3.$$

If $c_1^- + c_2^-$ is even, then we may deduce that the theorem holds for G from (3.10) by applying induction to $G_1^+, G_2^+, G_1^-, G_2^-$. Thus $c^- = c_1^- + c_2^-$ is odd.

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(b) Suppose x is adjacent to u_1. Since G_1^+ is 3-connected and G_1^+/x_1y_1 is not 3-connected (u_1 has only two neighbours in G_1^+/x_1y_1), Lemma 3.3 implies that $G_1^- = G_1^+ - x_1y_1$ is essentially 3-connected.

(c) Suppose x is adjacent to both a vertex in $\{u_1, v_1\}$ and a vertex in $\{u_2, v_2\}$. Using (b) we deduce that G_1^- and G_2^- are both essentially 3-connected. Thus $c_1^- = c_2^- = 1$. This is impossible since $c_1^- + c_2^- = c^-$ is odd by (a).

(d) Henceforth we assume that c_2^- is odd. Then c_1^- is even by (a). If $|V(H_2)| = 3$ then $G_2^+ \cong K_4$ and (d) holds. Thus we may assume that $|V(H_2)| \geqslant 4$.

Let $G_2 = G_2^+/x_2y_2$, and suppose that c_2 is odd. By Lemma 2.3, $F(G_2^+, t) = F(G_2, t) - F(G_2^-, t)$. Substitution into (3.10) gives

$$(t-1)(t-2)F(G,t) = F(G_1^+,t)F(G_2,t) - \big[F(G_1^+,t) - (t-2)F(G_1^-,t)\big]F(G_2^-,t). \tag{3.11}$$

Let G_1^* be the graph obtained from $G_1^+ - y_1$ by adding three new vertices z_1, z_2, z_3 and edges $z_1x_1, z_2u_1, z_3v_1, z_1z_2, z_2z_3, z_3z_1$, and let $G_1' = G_1^*/x_1z_1$. Since G_1^+ is 3-connected, G_1^* and G_1' are 3-connected and $c_1' = 1$. Applying induction to G_1', we deduce that $(-1)^{m_1'-n_1'+1}F(G_1', t) > 0$. Applying Lemma 2.3 to G_1^* using the edge x_1z_1, and then using Lemmas 2.6 and 2.8, we deduce that

$$F(G_1', t) = (t-3)F(G_1^+, t) + (t-2)F(G_1^-, t).$$

Thus $(-1)^{m_1'-n_1'+1}[(t-3)F(G_1^+,t) + (t-2)F(G_1^-,t)] > 0$. Since $m_1' = m_1^+ + 2$ and $n_1' = n_1^+ + 1$ we have

$$(3-t)(-1)^{m_1^+-n_1^++1}F(G_1^+,t) - (t-2)(-1)^{m_1^+-n_1^++1}F(G_1^-,t) > 0.$$

Since $0 < 3 - t < 1$, and $(-1)^{m_1^+-n_1^++1}F(G_1^+,t) > 0$ by induction, we have

$$(-1)^{m_1^+-n_1^++1}[F(G_1^+,t) - (t-2)F(G_1^-,t)] > 0.$$

Thus $F(G_1^+,t) - (t-2)F(G_1^-,t)$ is non-zero with sign $(-1)^{m_1^+-n_1^++1}$. Substituting into (3.11), applying induction to G_1^+, G_2, G_2^-, and using $m = m_1^+ + m_2 - 3 = m_1^+ + m_2^- - 3$, $n = n_1^+ + n_2 - 2 = n_1^+ + n_2^- - 3$, $c = 1 = c_1^+$ and c_2, c_2^- are odd, we deduce that $F(G,t)$ is non-zero with sign $(-1)^{m-n+1}$. This contradicts the choice of G. Thus the assumption that c_2 is odd must be false. Hence c_2 is even and, in particular, G_2 is not 3-connected. Lemma 3.3 now implies that $G_2^- = G_2^+ - x_2y_2$ is essentially 3-connected and hence $c_2^- = 1$. □

Claim 3.23. *Let X be an end cleavage unit of $G - x$. Then $X \cong C_4$.*

Proof. Since G is 3-connected, x is adjacent to at least one internal vertex v of X. Thus $d_{G-x}(v) = 2 = d_X(v)$. Hence X is neither 3-connected nor a 3-bond, so $X \cong C_r$ for some $r \geqslant 3$. Let $X = v_1v_2 \cdots v_rv_1$, where v_i is an internal vertex of X for $2 \leqslant i \leqslant r-1$. Then, for $2 \leqslant i \leqslant r-1$, we have $d_X(v_i) = 2 = d_{G-x}(v_i)$ and hence $e_i = xv_i \in E(G)$. Since $d_{G-x}(v_1), d_{G-x}(v_r) \geqslant 3$, we have $xv_1, xv_r \notin E(G)$. If $r = 3$, then $G - \{x, v_2\}$ is 2-connected and Lemma 3.6 implies that G/e_2 is 3-connected. This would contradict Claim 3.21 and hence $r \geqslant 4$.

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Let $S = \{v_1v_2, v_3v_4\}$. Then S is a 2-edge-cut of $G - x$. We have $G - S = H_1 \cup H_2$ where $H_1 \cap H_2 = \{x\}$ and $V(H_1) = \{x, v_2, v_3\}$. Let G^-, G_1^-, G_2^- be as defined in the statement of Claim 3.22. Then G_1^- is homeomorphic to K_2^3 so $c_1^- = 1$.

Suppose $r \geqslant 5$. Then $xv_4 \in E(G)$ so by Claim 3.22(b), G_2^- is essentially 3-connected. Thus $c_2^- = 1$ and $c^- = c_1^- + c_2^- = 2$. This contradicts Claim 3.22(a) and hence $r = 4$. □

Claim 3.24. *Let X be a non-circuit cleavage unit of $G - x$ and u_2v_2 be a virtual edge of $G - x$ contained in X. Let $S = \{u_1u_2, v_1v_2\}$ be the 2-edge-cut of $G - x$ incident to $\{u_2, v_2\}$. Let $G^-, G_1^+, G_2^+, G_1^-, G_2^-$ be as defined in the statement of Claim 3.22. Then:*

(a) *c_2^- is even and $c_1^- = 1$,*

(b) *X has degree two in the cleavage unit tree T of $G - x$.*

Proof. Note that Claim 3.23 implies that $d_{G_1^-}(x), d_{G_2^-}(x) \geqslant 2$.

(a) Suppose c_2^- is odd. Then Claim 3.22(d) implies that $G_2 = G_2^+/x_2y_2$ is not 3-connected. Since G_2^+ is 3-connected and near-cubic, we may use Lemma 3.6 to deduce that $G_2^+ - \{x_2, y_2\}$ has a cut edge f. Thus u_2 and v_2 belong to different components of $G_2^- - \{x_2, y_2, f\}$. This contradicts the fact that u_2 and v_2 belong to the same non-circuit cleavage unit X of $G - x$ and hence are joined by three internally disjoint paths in $G - x$. Thus c_2^- is even, and by Claim 3.22(a), (d), $c_1^- = 1$.

(b) Suppose X had degree at least three in T. Let e^* be the edge of T corresponding to the virtual edge u_2v_2 of $G - x$, and T_2 be the component of $T - e^*$ which contains X. Then T_2 is the cleavage unit tree of the graph H_2 obtained by suppressing y_2 in $G_2^- - x_2$. Since X has degree at least three in T, it has degree at least two in T_2. Thus X is not an end cleavage unit of H_2. Since G is 3-connected, x is adjacent to an internal vertex in each end cleavage unit of $G - x$. Thus x_2 is adjacent to an internal vertex in each end cleavage unit of H_2. By Lemma 3.5(b), G_2^- is essentially 3-connected. Hence $c_2^- = 1$. This contradicts (a) and completes the proof of (b). □

Claim 3.25. *Let X be a 3-connected cleavage unit of $G - x$ and uv, wz be the virtual edges of $G - x$ contained in X. Let X^* be obtained from $X - \{uv, wz\}$ by adding a new vertex y and new edges yu, yv, yw, yz. Then X^* is 3-connected.*

Proof. Since X is 3-connected, no vertex of X is adjacent to x in G. Suppose X^* is not 3-connected. Then y is contained in a 2-vertex-cut of X^*. Since each vertex of X^* other than y has degree three, this implies that $X^* - \{y, e\}$ is disconnected for some $e \in E(X^*)$. Thus $S = \{uv, wz, e\}$ is a 3-edge-cut of X. Let X_1 and X_2 be the components of $X - S$. Relabelling if necessary, we may suppose that $u, w \in V(X_1)$ and $v, z \in V(X_2)$. Let $S_1 = \{uu_1, vv_1\}$ and $S_2 = \{ww_1, zz_1\}$ be the 2-edge-cuts of $G - x$ incident to the hinges $\{u, v\}$ and $\{w, z\}$, respectively. Then $S_1' = \{uu_1, ww_1, e\}$ is a 3-edge-cut of G and X_1 is a component of $G - S_1'$. Using Claim 3.19, we deduce that $X_1 \cong K_1$. Arguing similarly using $S_2' = \{vv_1, zz_1, e\}$, we may deduce that $X_2 \cong K_1$. Thus $X \cong K_2^3$, contradicting the hypothesis that X is 3-connected. □

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Claim 3.26. *Let X be a non-circuit cleavage unit of $G-x$. Then X is a 3-bond.*

Proof. Suppose X is 3-connected. Let T be the cleavage unit tree of $G-x$. By Claim 3.24(b), X has degree two in T and hence X contains exactly two virtual edges u_1v_1, w_1z_1 of $G-x$. Let $S=\{u_1u_2, v_1v_2\}$ and $S'=\{w_1w_0, z_1z_0\}$ be the 2-edge-cuts of $G-x$ incident to the hinges $\{u_1, v_1\}$ and $\{w_1, z_1\}$, respectively. Since X is a 3-connected cleavage unit of $G-x$, no vertex of X is adjacent to x in G.

Let x_1, x_2, G^-, G^+ be as defined in the statement of Claim 3.23 with respect to S. By Lemma 2.3,

$$F(G,t)=F(G^+,t)+F(G^-,t). \tag{3.12}$$

Let $G-S=H_1\cup H_2$ where $H_1\cap H_2=\{x\}$ and $V(X)\subseteq V(H_1)$. Let G_2^+ be the graph obtained from G^+ by contracting $E(H_1)$ to the single vertex y_2, and $G_2^-=G_2^+-x_2y_2$. Let $G-S'=H_0\cup H_1'$ where $H_0\cap H_1'=\{x\}$ and $V(X)\subseteq V(H_1')$. Define G_0^+, G_0^-, y_0 in a similar way to G_2^+, G_2^-, y_2 by contracting $E(H_1')$. Finally let G_1^- be obtained from G^- by contracting $E(H_2)$ to a single vertex y_2 and then contracting $E(H_0)$ to a single vertex y_0. Let $G_1^+=G_1^-+y_2y_0$. The 3-connectivity of G implies that G_0^+, G_1^+, G_2^+ are 3-connected, Claim 3.24(a) implies that G_2^-, G_0^- are essentially 3-connected, and the fact that G_1^- is homeomorphic to X implies that G_1^- is essentially 3-connected.

Applying Lemma 2.8 twice to G^+ using the 3-edge-cuts

$$\{u_1u_2, v_1v_2, x_1x_2\} \quad \text{and} \quad \{w_1w_0, z_1z_0, x_1x_2\},$$

we obtain

$$(t-1)^2(t-2)^2F(G^+,t)=F(G_0^+,t)F(G_1^+,t)F(G_2^+,t).$$

Similarly, applying Lemma 2.6 twice to G^- using the 2-edge-cuts

$$\{u_1u_2, v_1v_2\} \quad \text{and} \quad \{w_1w_0, z_1z_0\},$$

we obtain

$$(t-1)^2F(G^-,t)=F(G_0^-,t)F(G_1^-,t)F(G_2^-,t).$$

Substituting into (3.12) gives

$$\begin{aligned}(t-1)^2(t-2)^2F(G,t)&=F(G_0^+,t)F(G_1^+,t)F(G_2^+,t)\\&\quad+(t-2)^2F(G_0^-,t)F(G_1^-,t)F(G_2^-,t).\end{aligned} \tag{3.13}$$

Let G_1 be obtained from G_1^+ by contracting the edge y_0y_2. Then G_1 is 3-connected by Claim 3.25. By Lemma 2.3, $F(G_1^+,t)=F(G_1,t)-F(G_1^-,t)$. Substituting into (3.13) we obtain

$$\begin{aligned}(t-1)^2(t-2)^2F(G,t)&=F(G_0^+,t)F(G_2^+,t)F(G_1,t)\\&\quad-F(G_1^-,t)[F(G_0^+,t)F(G_2^+,t)-(t-2)^2F(G_0^-,t)F(G_2^-,t)].\end{aligned} \tag{3.14}$$

We shall show that both of the terms on the right-hand side of (3.14) are non-zero with sign $(-1)^{m-n+1}$.

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Let G_3 be obtained from $G - V(X)$ by adding two new vertices u, v and new edges uv, uw_0, vz_0, uu_2, vv_2. The 3-connectivity of G implies that G_3 is 3-connected. We may apply the reduction used in the derivation of (3.13) to G_3. In this analysis the graphs X, G_1^+, and G_1^- are replaced by graphs isomorphic to K_2^3, K_4, and K_4^-, respectively, where K_4^- is obtained from K_4 by deleting an edge. Using the fact that $F(K_4, t) = (t-1)(t-2)(t-3)$ and $F(K_4^-, t) = (t-1)(t-2)$, we deduce that

$$(t-1)^2(t-2)^2F(G_3,t) = F(G_0^+,t)(t-1)(t-2)(t-3)F(G_2^+,t) + (t-2)^2F(G_0^-,t)(t-1)(t-2)F(G_2^-,t).$$

Applying induction to G_3 we have

$$(-1)^{m_3-n_3+1}[(t-3)F(G_0^+,t)F(G_2^+,t) + (t-2)^2F(G_0^-,t)F(G_2^-,t)] > 0.$$

Thus

$$(-1)^{m_3-n_3}[(3-t)F(G_0^+,t)F(G_2^+,t) - (t-2)^2F(G_0^-,t)F(G_2^-,t)] > 0.$$

By induction $(-1)^{m_0^+-n_0^++m_2^+-n_2^+}F(G_0^+,t)F(G_2^+,t) > 0$. Since $m_3 = m_0^+ + m_2^+ - 1$, $n_3 = n_0^+ + n_2^+ - 1$, and $3 - t > 0$, we have

$$(-1)^{m_3-n_3}(3-t)F(G_0^+,t)F(G_2^+,t) > 0.$$

Since $0 < 3 - t < 1$, it follows that

$$(-1)^{m_3-n_3}[F(G_0^+,t)F(G_2^+,t) - (t-2)^2F(G_0^-,t)F(G_2^-,t)] > 0. \tag{3.15}$$

Applying induction to G_1^-, we obtain $(-1)^{m_1^- - n_1^- + 1}F(G_1^-, t) > 0$. Using (3.15), and the facts that $m = m_3 + m_1^- - 5$, and $n = n_3 + n_1^- - 4$, we may deduce that

$$(-1)^{m-n}F(G_1^-,t)[(F(G_0^+,t)F(G_2^+,t) - (t-2)^2F(G_0^-,t)F(G_2^-,t)] > 0. \tag{3.16}$$

We may also deduce by induction that

$$(-1)^{m_0^+-n_0^++m_2^+-n_2^++m_1-n_1+1}F(G_0^+,t)F(G_2^+,t)F(G_1,t) > 0.$$

Since $m = m_0^+ + m_2^+ + m_1 - 6$, and $n = n_0^+ + n_2^+ + n_1 - 4$, it follows that

$$(-1)^{m-n+1}F(G_0^+,t)F(G_2^+,t)F(G_1,t) > 0. \tag{3.17}$$

Using (3.14), (3.16) and (3.17), we deduce that $(-1)^{m-n+1}F(G,t) > 0$. This contradicts the choice of G as a counter-example to the theorem and completes the proof of the claim. □

Claim 3.27. *Let $X = v_1v_2\cdots v_rv_1$ be a circuit cleavage unit of $G - x$ which is not an end cleavage unit and T be the cleavage unit tree of $G - x$.*

(a) *Each neighbour of X in T is a 3-bond cleavage unit of $G - x$.*

(b) *At most one vertex of X is adjacent to x in G.*

(c) *If a vertex of X is adjacent to x in G then $X \cong C_5$ and X has degree two in T.*

(d) *If no vertex of X is adjacent to x in G then $X \cong C_6$ and X has degree three in T.*

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Proof. **(a)** follows from Lemma 3.5(a) and Claim 3.26. Note that (a) implies $v_iv_{i+1} \in E(G)$ for all $1 \leqslant i \leqslant N$.

(b) Suppose $xv_i, xv_j \in E(G)$ for some $1 \leqslant i < j \leqslant r$. If $v_j \notin \{v_{i-1}, v_{i+1}\}$ then $S = \{v_{i-1}v_i, v_{j-1}v_j\}$ is a 2-edge-cut of $G - x$ which contradicts Claim 3.22(c). Hence we may assume without loss of generality that $xv_1, xv_2 \in E(G)$ and that $xv_s \notin E(G)$ for all $3 \leqslant s \leqslant r$. Since each vertex of G other than x has degree 3, it follows that r is even and $v_{2s-1}v_{2s}$ is a virtual edge of G contained in X for all $2 \leqslant s \leqslant r/2$. Since X has degree at least two in T, it contains at least two virtual edges of G and hence $r \geqslant 6$. Let $S = \{v_2v_3, v_rv_1\}$. Then S is a 2-edge-cut of $G - x$. Let G^-, G_1^-, G_2^- be as defined in the statement of Claim 3.22, where $v_1, v_2 \in V(G_1^-)$. Since G_1^- is homeomorphic to K_2^3, $c_1^- = 1$. Furthermore $c_2^- = 1$ by Lemma 3.5(b). This contradicts Claim 3.22(a).

(c) We assume that x is adjacent to exactly one vertex of X. Relabelling if necessary, we may assume without loss of generality that $xv_1 \in E(G)$ and that $xv_s \notin E(G)$ for all $2 \leqslant s \leqslant r$. Thus r is odd and $v_{2s}v_{2s+1}$ is a virtual edge of G contained in X for all $1 \leqslant s \leqslant (r-1)/2$. Since X has degree at least two in G, it contains at least two virtual edges of G and hence $r \geqslant 5$.

Suppose $r \geqslant 7$. Let $S = \{v_1v_2, v_5v_6\}$. Then S is a 2-edge-cut of $G - x$. Let G^-, G_1^-, G_2^- be as defined in the statement of Claim 3.22, where $v_1, v_6 \in V(G_1^-)$. Then $c_1^- = 1$ by Claim 3.22(b) and $c_2^- = 1$ by Lemma 3.5(b). This contradicts Claim 3.22(a). Hence $r = 5$, X contains exactly two virtual edges of G, and X has degree two in T.

(d) We assume that x is adjacent to no vertex of X. Then r is even and the virtual edges of G in X are a 1-factor of X. Relabelling if necessary, we may assume without loss of generality that $v_{2s}v_{2s+1}$ is a virtual edge of G contained in X for all $1 \leqslant s \leqslant r/2$. Since X has degree at least two in G, it contains at least two virtual edges of G and hence $r \geqslant 4$.

Suppose $r = 4$. Let $S = \{v_1v_2, v_3v_4\}$. Then S is a 2-edge-cut of $G - x$. Let G^-, G_1^-, G_2^- be as defined in the statement of Claim 3.22, where $v_1, v_4 \in V(G_1^-)$. Since the two cleavage units of $G - x$ adjacent to X in T have degree two in T by Claim 3.24(b), we have $c_1^- \geqslant 2$ and $c_2^- \geqslant 2$. This gives a contradiction since Claim 3.22(a) and (d) imply that either $c_1^- = 1$ or $c_2^- = 1$. Hence $r \geqslant 6$.

Suppose $r \geqslant 8$. Let $S = \{v_1v_2, v_5v_6\}$. Then S is a 2-edge-cut of $G - x$. Let G^-, G_1^-, G_2^- be as defined in the statement of Claim 3.22, where $v_1, v_6 \in V(G_1^-)$. Then $c_1^- = 1 = c_2^-$ by Lemma 3.5(b). This contradicts Claim 3.22(a). Hence $r = 6$, X contains exactly three virtual edges of G, and X has degree three in T. □

Lemma 3.16 and Claims 3.18, 3.23, 3.26 and 3.27 imply that G is an extension of (K_4, x). Since each single edge extension adds three vertices and five edges to K_4, we have $m - n$ is even. By Lemma 3.12, $F(G, t) < 0$ for $t \in (2, \alpha]$. This contradicts the choice of G as a counter-example to the theorem. □

Using Theorems 3.1 and 3.17 and planar duality, we have the following.

Corollary 3.28. *Let G be plane near-triangulation. Then the only zeros of $P(G, t)$ in $(-\infty, \alpha]$ are 1 and 2.*

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Since extensions of (K_4, x) are planar, Theorem 3.13 and planar duality show that there exists an infinite family of plane near-triangulations with chromatic roots converging to α from above.

4. Closing remarks

In a subsequent paper [5] we extend the zero-free interval $(2, \alpha]$ for certain families of near-cubic graphs. As a corollary we deduce that if G is a cubic bridgeless graph, then $F(G, t)$ has no zeros in $(2, \gamma)$, where $\gamma \approx 2.54\cdots$ is the root in $(2, 3)$ of the flow polynomial of the cube. The dual statement for plane triangulations had previously been verified by Woodall [16, 17].

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References

[1] Birkhoff, G. D. and Lewis, D. C. (1946) Chromatic polynomials. *Trans. Amer. Math. Soc.* **60** 355–451.
[2] Edwards, H., Hierons, R. and Jackson, B. (1998) The zero-free intervals for characteristic polynomials of matroids. *Combin. Probab. Comput.* **7** 153–165.
[3] Jackson, B. (1993) A zero-free interval for chromatic polynomials of graphs. *Combin. Probab. Comput.* **2** 325–336.
[4] Jackson, B. (2003) Zeros of chromatic and flow polynomials of graphs. *J. Geometry* **76** 95–109.
[5] Jackson, B. A zero-free interval for flow polynomials of cubic graphs. *J. Combin. Theory* Ser. B, in press.
[6] Jaeger, F. (1979) Flows and generalized colouring theorems in graphs. *J. Combin. Theory Ser. B* **26** 205–216.
[7] Kochal, M. (2002) Polynomials associated with nowhere-zero flows. *J. Combin. Theory Ser. B* **84** 260–269.
[8] Seymour, P. D. (1981) Nowhere-zero 6-flows. *J. Combin. Theory Ser. B* **30** 130–135.
[9] Tutte, W. T. (1950) On the imbedding of linear graphs in surfaces. *Proc. London Math. Soc.* **51** 474–483.
[10] Tutte, W. T. (1954) A contribution to the theory of chromatic polynomials. *Canad. J. Math.* **6** 80–91.
[11] Tutte, W. T. (1966) *Connectivity in Graphs*, University of Toronto Press.
[12] Tutte, W. T. (1974) Chromials. In Vol. 411 of *Springer Lecture Notes in Mathematics*, pp. 243–266.
[13] Wakelin, C. D. (1994) Chromatic polynomials. PhD Thesis, University of Nottingham.
[14] Welsh, D. J. A. Personal communication.
[15] Woodall, D. R. (1977) Zeros of chromatic polynomials. In *Combinatorial Surveys* (P. Cameron, ed.), *Proc. Sixth British Combinatorial Conference*, Academic Press, London, pp. 199–223.
[16] Woodall, D. R. (1992) A zero-free interval for chromatic polynomials. *Discrete Math.* **101** 333–341.
[17] Woodall, D. R. (2004) Corrigendum: A zero-free interval for chromatic polynomials. *Discrete Math.*, **275** 385–390.

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