Probability in the Engineering and Informational Sciences, **34**, 2020, 92–111.
doi:10.1017/S0269964818000360

PROPERTIES FOR GENERALIZED CUMULATIVE PAST MEASURES OF INFORMATION

CAMILLA CALÌ
Dipartimento di Matematica e Applicazioni
Università di Napoli Federico II Via Cintia, I-80126 Napoli, Italy
E-mail: camilla.cali@unina.it

MARIA LONGOBARDI
Dipartimento di Matematica e Applicazioni
Università di Napoli Federico II Via Cintia, I-80126 Napoli, Italy
E-mail: maria.longobardi@unina.it

JORGE NAVARRO
Facultad de Matemáticas
Universidad de Murcia, 30100 Murcia,
Spain
E-mail: jorgenav@um.es

The Shannon entropy based on the probability density function is a key information measure with applications in different areas. Some alternative information measures have been proposed in the literature. Two relevant ones are the cumulative residual entropy (based on the survival function) and the cumulative past entropy (based on the distribution function). Recently, some extensions of these measures have been proposed. Here, we obtain some properties for the generalized cumulative past entropy. In particular, we prove that it determines the underlying distribution. We also study this measure in coherent systems and a closely related generalized past cumulative Kerridge inaccuracy measure.

Keywords: applied probability, reliability theory

Mathematics Subject Classifications: 94a17, 62n05, 60e15

1. INTRODUCTION

The differential entropy is a baseline concept in the field of information theory. It was introduced by Shannon [23] and Wiener [25] and it is also invoked to deal with information in the context of probability theory. As is well-known, the classical approach to the description of information related to a non-negative absolutely continuous random variable X, with probability density function f_X, is based on the differential entropy of X (or Shannon information measure) defined by

$$H(X) = -\mathrm{E}[\log f_X(X)] = -\int_0^{+\infty} f_X(x) \log f_X(x)\, dx, \tag{1.1}$$

© Cambridge University Press 2018 0269-9648/18 $25.00

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

where 'log' means natural logarithm and where, by convention, $0 \log 0 = 0$. Entropy $H(X)$ is shift-independent because the integrand function on the right-hand-side of Eq. (1.1) depends on x only via $f_X(x)$. In other terms, $H(X)$ is position-free, in the sense that X possesses the same differential entropy as $X + b$, for any $b > 0$.

Many possible alternative information measures have been defined in the past, in order to reduce various deficiencies of the differential entropy in Eq. (1.1) when it is used as a continuous counterpart of the classical Shannon entropy for discrete random variables.

Rao [20] defined another measure of information called the cumulative residual entropy (CRE) given by

$$\mathcal{E}(X) = -\int_0^{+\infty} \overline{F}_X(x) \log \overline{F}_X(x)\, dx, \tag{1.2}$$

where $\overline{F}_X(x) = \Pr(X > x)$ is the survival (or reliability) function of X. While Shannon differential entropy can be used to measure how far is the distribution of X from a uniform distribution, the CRE can be seen as a dispersion measure (see Toomaj, Sunoj and Navarro [24]). Note that $\mathcal{E}(X) \geq 0$ and that $\mathcal{E}(X) = 0$ if and only if $X = c$ (the more informative case when X has an atom distribution).

Its dynamic version (see Asadi and Zohrevand [1]; Navarro, del Águila and Asadi [15]) was defined as the CRE of the residual lifetime $X_t = (X - t | X > t)$ given by

$$\mathcal{E}(X;t) = \mathcal{E}(X_t) = -\int_t^{+\infty} \frac{\overline{F}_X(x)}{\overline{F}_X(t)} \log \frac{\overline{F}_X(x)}{\overline{F}_X(t)} dx, \quad t > 0 : \ \overline{F}_X(t) > 0.$$

If $\mathcal{E}(X;t)$ is increasing for all $t \geq 0$, then $\mathcal{E}(X;t)$ uniquely determines $\overline{F}_X(t)$ (see Asadi and Zohrevand [1]).

An information measure similar to that in Eq. (1.2) is the cumulative past entropy (CPE) of X, defined by Di Crescenzo and Longobardi [5] as

$$\mathcal{CE}(X) = -\int_0^{+\infty} F_X(x) \log F_X(x) dx,$$

where $F_X(x) = \Pr(X \leq x)$ is the distribution function of X. The CPE is nonnegative and can also be seen as a dispersion measure. As $0 \leq x(1-x) \leq -x \log x$ for all $x \in [0,1]$, then $2\mathcal{CE}(X) \geq D_G(X)$, where $D_G(X)$ is the Gini mean difference dispersion measure defined by

$$D_G(X) = \int_0^{+\infty} 2\ F_X(x) \overline{F}_X(x) dx.$$

A similar property holds for the CRE (see Toomaj *et al.* [24]).

The random variable $X_{(t)} = (X \,|\, X \leq t)$ describes the past lifetime of the system at age t. Its CPE was considered in [4] as a dynamic version of CPE given by

$$\mathcal{CE}(X;t) = \mathcal{CE}(X_{(t)}) = -\int_0^t \frac{F_X(x)}{F_X(t)} \log \frac{F_X(x)}{F_X(t)} dx, \quad t > 0 : \ F_X(t) > 0.$$

Note that $\mathcal{CE}(X;t) = \mathcal{E}(t - X \,|\, X \leq t)$. If $\mathcal{CE}(X;t)$ is increasing for all $t \geq 0$, then $\mathcal{CE}(X;t)$ uniquely determines $F_X(t)$ (see Di Crescenzo and Longobardi [5]).

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

Recently, some extensions of these two measures have been proposed. The generalized CRE (GCRE) was defined in [19] as

$$\mathcal{E}_n(X) = \int_0^{+\infty} \overline{F}_X(x) \frac{[\Lambda_X(x)]^n}{n!} dx \tag{1.3}$$

for $n = 0, 1, 2, \ldots$, where $\Lambda_X(x) = -\log \overline{F}_X(x)$ and where, by convention, $0(\log 0)^n = 0$ for $n = 1, 2, \ldots$. Note that $\mathcal{E}_0(X) = E(X)$ and $\mathcal{E}_1(X) = \mathcal{E}(X)$. Additional properties of $\mathcal{E}_n(X)$ were given in [19].

In analogy with the measure given in Eq. (1.3), Kayal [10] defined the *generalized CPE* (GCPE) of a non-negative random variable X as

$$\mathcal{CE}_n(X) = \int_0^{+\infty} F_X(x) \frac{[T_X(x)]^n}{n!} dx, \quad \text{for } n = 0, 1, 2, \ldots$$

where $T_X(t) = -\log F_X(t)$. Note that $\mathcal{CE}_1(X) = \mathcal{CE}(X)$. However $\mathcal{CE}_0(X) = +\infty$. Kayal [10] also introduced the *dynamic generalized CPE* (DGCPE), as

$$\mathcal{CE}_n(X;t) = \mathcal{CE}_n(X_{(t)}) = \frac{1}{n!} \int_0^t \frac{F_X(x)}{F_X(t)} \left[-\log \frac{F_X(x)}{F_X(t)} \right]^n dx, \quad t > 0 : \ F_X(t) > 0,$$

for $n = 0, 1, 2, \ldots$. Note that $\mathcal{CE}_1(X;t) = \mathcal{CE}(X;t)$ and

$$\mathcal{CE}_0(X;t) = \int_0^t \frac{F_X(x)}{F_X(t)} dx = E(t - X | X \le t).$$

In this paper, we obtain additional properties for $\mathcal{CE}_n(X)$ and $\mathcal{CE}_n(X;t)$. In particular, we prove (Section 3) that $\mathcal{CE}_n(X;t)$, for a fixed n, uniquely determines the underlying distribution. In Section 4, we study the information $\mathcal{CE}_n(T)$ when T is the lifetime of a coherent reliability system with identically distributed components. Moreover, we propose and study (Section 5) a generalized cumulative Kerridge inaccuracy measure closely related to $\mathcal{CE}_n(X)$ and its empirical version (Section 6). The notations and the remaining basic definitions are given in the following Section 2 and in Table 1.

2. NOTATION AND BASIC DEFINITIONS

Let X be a non-negative random variable with absolutely continuous distribution function F_X and probability density function f_X. If X describes the random lifetime of a unit (or a system), such as an organism or a cell, then $X_t = (X - t \,|\, X > t)$ describes the remaining (residual) lifetime of the unit at age t. Its distribution function is

$$F_{X_t}(x) = \frac{F_X(x+t) - F_X(t)}{\overline{F}_X(t)}, \quad x \ge 0,$$

where $\overline{F}_X = 1 - F_X$ is the survival (or reliability) function of X and we assume $\overline{F}_X(t) > 0$. The mean residual lifetime is

$$\mu_X(t) = E(X_t) = E(X - t \,|\, X > t) = \frac{1}{\overline{F}_X(t)} \int_t^{+\infty} \overline{F}_X(x)\, dx.$$

It is well known that the mean residual lifetime uniquely determines F_X.

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

In reliability theory, the duration of the time between an inspection point t and the failure time X, given that at time t the system has been found failed, is called *inactivity time* and is represented by the random variable $(t - X \mid X \leq t)$, for $t > 0$ such that $F_X(t) > 0$. The mean inactivity time is

$$\tilde{\mu}_X(t) = E(t - X \mid X \leq t) = \frac{1}{F_X(t)} \int_0^t F_X(x)\, dx.$$

The mean inactivity time uniquely determines F_X. For further results about these concepts in reliability theory see [14] and the references therein.

The hazard rate function λ_X and the reversed hazard rate function τ_X of X are important functions in this context. They are defined as the ratio of the density of X to the survival function of X or the distribution function of X, respectively, that is,

$$\lambda_X(t) = -\frac{d}{dt} \log \overline{F}_X(t) = \frac{f_X(t)}{\overline{F}_X(t)}$$

and

$$\tau_X(t) = \frac{d}{dt} \log F_X(t) = \frac{f_X(t)}{F_X(t)}$$

for every t such that $\overline{F}_X(t) > 0$ and $F_X(t) > 0$, respectively. Both the hazard rate and the reversed hazard rate functions uniquely determine F_X. For further properties on hazard rate and reversed hazard rate functions see, for example, [2,14,22].

In the last 50 years, stochastic orders have attracted many authors of different areas of probability and statistics. Orders are invoked to give bounds and inequalities and also comparisons among stochastic systems. The basic idea is to establish if a random variable is larger than another one (in different senses). Let us recall the definitions of the stochastic orders used in this paper (for their basic properties see [22]). Let X and Y be continuous random variables with f_X, f_Y their probability density functions, F_X, F_Y their distribution functions and $\overline{F}_X$, $\overline{F}_Y$ their survival functions, respectively. Then:

- $X \leq_{\mathrm{st}} Y$ (in the usual stochastic order) if and only if

$$E[\phi(X)] \leq E[\phi(Y)] \tag{2.1}$$

  for all increasing functions $\phi : \mathbb{R} \to \mathbb{R}$ for which the expectations exist. Equivalently, $X \leq_{\mathrm{st}} Y$ if and only if $F_X(t) \geq F_Y(t)$ for all $t \in \mathbb{R}$.
- $X \leq_{\mathrm{dcx}} Y$ (in the decreasing convex order) if and only if (2.1) is true for all decreasing convex functions $\phi : \mathbb{R} \to \mathbb{R}$ for which the expectations exist.
- $X \leq_{\mathrm{disp}} Y$ (in the dispersive ordering) if and only if

$$F_X^{-1}(v) - F_X^{-1}(u) \leq F_Y^{-1}(v) - F_Y^{-1}(u) \text{ for all } 0 < u < v < 1,$$

  where $F^{-1}(u) = \sup\{x : F(x) \leq u\}$. If F_X and F_Y are absolutely continuous with respective probability density functions f_X and f_Y, then it is characterized by the following property:

$$f_Y(F_Y^{-1}(u)) \leq f_X(F_X^{-1}(u)) \text{ for all } u \in (0, 1).$$

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

3. SOME PROPERTIES OF DGCPE FUNCTIONS

In this section, we prove that, under some assumptions and for a fixed n, the DGCPE $\mathcal{CE}_n(X;t)$ uniquely determines the distribution function F_X. We use the same tools as that used in [18]. To this end, we need some preliminary results. The first one is a property of systems of differential equations (see, e.g., Hairer, Norsett and Wanner [9]).

THEOREM 1: *Let U be an open set in $\mathbb{R}^{n+1}$ and let $f_i : U \to \mathbb{R}$ be continuous functions such that $\partial f_i/\partial y_j$ is continuous on U for all $i,j = 1,\dots,n$. Then, for every $(x_0, z_1, \dots, z_n) \in U$, there exist a unique solution of the following system of differential equations*

$$\begin{cases} y_1' = f_1(x, y_1, \dots, y_n) \\ \quad \dots \\ y_n' = f_n(x, y_1, \dots, y_n) \end{cases}$$

such that $y_i(x_0) = z_i$ for $i = 1,\dots,n$ which can be continued up to the boundary of U.

The second result involves a well-known relationship between the reversed hazard rate function τ_X and the mean inactivity time $\tilde{\mu}_X$ and the inversion formula to determine F_X (see, e.g., Di Crescenzo and Longobardi [5]).

THEOREM 2: *Let X be an absolutely continuous random variable with distribution function F_X, reversed hazard rate function τ_X and the mean inactivity time $\tilde{\mu}_X$. Then*

$$\tau_X(t) = \frac{1 - \tilde{\mu}_X'(t)}{\tilde{\mu}_X(t)}$$

and

$$F_X(t) = \exp\left(- \int_t^{+\infty} \frac{1 - \tilde{\mu}_X'(x)}{\tilde{\mu}_X(x)} dx \right) \tag{3.1}$$

for all t such that $F_X(t) > 0$.

The third one is a property of DGCPE functions stated in Theorem 4.9 of [10].

THEOREM 3: *Let X be a non-negative absolutely continuous random variable, then*

$$\mathcal{CE}_n'(X;t) = \tau_X(t) \left[\mathcal{CE}_{n-1}(X;t) - \mathcal{CE}_n(X;t) \right] \tag{3.2}$$

for $n = 1, 2, \dots$.

Now we can state the following result.

THEOREM 4: *Let X be an absolutely continuous random variable with distribution function F_X and differentiable DGCPE functions $\mathcal{CE}_i(X;t)$ for $i = 0, 1, \dots, n$ and $t \geq 0$. Let us also assume that*

$$\mathcal{CE}_n(X;t) \neq \mathcal{CE}_{n-1}(X;t) \tag{3.3}$$

for all $t \geq 0$ such that $F_X(t) > 0$ and let $t_0 \geq 0$. Then the values $\mathcal{CE}_i(X;t_0)$ for $i = 0, 1, \dots, n-1$ and the function $\mathcal{CE}_n(X;t)$ uniquely determine F_X.

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

PROOF: From Eq. (3.2), for all $t \geq 0$ such that $F_X(t) > 0$, we have

$$\begin{cases} \mathcal{CE}_1'(X;t) = \tau_X(t)[\mathcal{CE}_0(X;t) - \mathcal{CE}_1(X;t)] \\ \mathcal{CE}_2'(X;t) = \tau_X(t)[\mathcal{CE}_1(X;t) - \mathcal{CE}_2(X;t)] \\ \dots \\ \mathcal{CE}_n'(X;t) = \tau_X(t)[\mathcal{CE}_{n-1}(X;t) - \mathcal{CE}_n(X;t)]. \end{cases}$$

■

From the last equality and (3.3), we can write

$$\tau_X(t) = \frac{\mathcal{CE}_n'(X;t)}{\mathcal{CE}_{n-1}(X;t) - \mathcal{CE}_n(X;t)}.$$

Substituting in the previous equalities, we obtain

$$\mathcal{CE}_j'(X;t) = \frac{\mathcal{CE}_{j-1}(X;t) - \mathcal{CE}_j(X;t)}{\mathcal{CE}_{n-1}(X;t) - \mathcal{CE}_n(X;t)} \mathcal{CE}_n'(X;t), \quad j = 1, 2, \dots, n-1.$$

Recalling that $\mathcal{CE}_0(X;t) = \tilde{\mu}_X(t)$, we have

$$\mathcal{CE}_0'(X;t) = -\tau_X(t)\mathcal{CE}_0(X;t) + 1$$

and then we get the expression:

$$\mathcal{CE}_0'(X;t) = -\frac{\mathcal{CE}_n'(X;t)\mathcal{CE}_0(X;t)}{\mathcal{CE}_{n-1}(X;t) - \mathcal{CE}_n(X;t)} + 1.$$

If $\mathcal{CE}_n(X;t)$ is a known function, we have to solve the following system of differential equations:

$$\begin{cases} y_0' = f_0(t, y_0, \dots, y_{n-1}) \\ y_1' = f_1(t, y_0, \dots, y_{n-1}) \\ \dots \\ y_{n-1}' = f_{n-1}(t, y_0, \dots, y_{n-1}), \end{cases}$$

where $y_j = \mathcal{CE}_j(X;t)$ for $j = 0, 1, 2, \dots, n-1$,

$$f_0(t, y_0, \dots, y_{n-1}) = -\frac{\mathcal{CE}_n'(X;t) y_0}{y_{n-1} - \mathcal{CE}_n(X;t)} + 1$$

and

$$f_j(t, y_0, \dots, y_{n-1}) = \frac{y_{j-1} - y_j}{y_{n-1} - \mathcal{CE}_n(X;t)} \mathcal{CE}_n'(X;t), \quad j = 1, 2, \dots, n-1.$$

From Theorem 1, this system of differential equations has a unique solution when we fix the initial values $\mathcal{CE}_j(X;t_0)$, with $j = 0, 1, \dots, n-1$. So $\mathcal{CE}_n(X;t)$ uniquely determines $\mathcal{CE}_j(X;t)$ for $j = 0, 1, \dots, n-1$ and, in particular, $\mathcal{CE}_0(X;t) = \tilde{\mu}_X(t)$. Then, from Eq. (3.1), we obtain $F_X(t)$.

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

The following example shows how the preceding theorem can be used to characterize particular models.

Example 1: Let us consider a random variable X with the following Power distribution $F(t) = t^c/b^c$ for $t \in [0, b]$, where $b, c > 0$. Then, a straightforward calculation shows that

$$\mathcal{CE}_0(X;t) = \frac{t}{c+1}$$

and

$$\mathcal{CE}_1(X;t) = \frac{c\ t}{(c+1)^2}$$

for $t \in [0, b]$. Note that

$$\mathcal{CE}_1(X;t) = \frac{c\ t}{(c+1)^2} = \frac{c}{c+1}\ \mathcal{CE}_0(X;t) < \mathcal{CE}_0(X;t)$$

for all $t \in [0, b]$. Therefore, this model can be characterized by using Theorem 4 from the preceding expression for $\mathcal{CE}_1(X;t)$ and the initial value $\mathcal{CE}_0(X;0) = 0$.

In the following remark, we characterize the model in which $\mathcal{CE}_n(X;t)$ does not depend on n. This model cannot be characterized from Theorem 4.

Remark 1: Note that if for a fixed positive integer n

$$\mathcal{CE}_n(X;t) = \mathcal{CE}_{n-1}(X;t) \tag{3.4}$$

for all $t \geq 0$, then, from Eq. (3.2), we have $\mathcal{CE}'_n(X;t) = 0$ for all $t \geq 0$. Hence, from Eq. (3.4), we get $\mathcal{CE}'_{n-1}(X;t) = 0$ for all $t \geq 0$ and, by using again Eq. (3.2), we have

$$\mathcal{CE}_n(X;t) = \mathcal{CE}_{n-1}(X;t) = \mathcal{CE}_{n-2}(X;t)$$

for all $t \geq 0$. Repeating this process we prove that

$$\mathcal{CE}_n(X;t) = \mathcal{CE}_{n-1}(X;t) = \cdots = \mathcal{CE}_1(X;t) = \mathcal{CE}_0(X;t),$$

where $\mathcal{CE}_0(X;t) = E(t - X|X \leq t)$ is the mean inactivity time which uniquely determines F_X. In this case, from Eq. (3.1), we have that the right-end r_X of the support of X should be finite, $r_X = b$ and that $\mathcal{CE}_0(X;t) = c > 0$ for $t \in [0, b]$ and $\mathcal{CE}_0(X;t) = t - b + c > 0$ for $t > b$. Then, from Eq. (3.1), $F_X(t) = \exp((t - b)/c)$ for $t \in [0, b]$.

4. GCPE OF COHERENT SYSTEMS

First, we note that if the distribution function F_X of X is absolutely continuous with probability density function f_X and with support (α, β), with $0 \leq \alpha < \beta \leq +\infty$, then the

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

GCPE can be written as

$$\begin{aligned}\mathcal{CE}_n(X) &= \frac{1}{n!}\int_0^{+\infty} F_X(x)[-\log(F_X(x))]^n dx \\ &= \frac{1}{n!}\int_\alpha^\beta \phi_n(F_X(x))dx \\ &= \frac{1}{n!}\int_0^1 \frac{\phi_n(u)}{f_X(F_X^{-1}(u))}du\end{aligned}$$

for $n = 1, 2, \ldots$, where

$$\phi_n(u) := u[-\log(u)]^n \geq 0, \quad 0 < u < 1$$

and $\phi(0) = \phi(1) = 0$. For example, if X has a uniform distribution in $(0, \beta)$, then

$$\mathcal{CE}_n(X) = \frac{\beta}{n!}\int_0^1 \phi_n(u)du = \frac{\beta}{2^{n+1}}, \quad n = 1, 2, \ldots.$$

If T is the lifetime of a coherent system with m identically distributed (i.d.) components, then its distribution function F_T can be written as

$$F_T(t) = q(F_X(t))$$

(see, e.g., Burkschat and Navarro [3]; Navarro, del Águila and Sordo [16]), where F_X is the common distribution of the component lifetimes and where q is a *distortion function*, that is, $q : [0, 1] \to [0, 1]$ is a continuous increasing function such that $q(0) = 0$ and $q(1) = 1$. The function q depends on the structure of the system and the copula of the component lifetimes. For further results on coherent systems see [12,17,24].

For example, for a parallel system with lifetime $T = \max(X_1, X_2, \ldots, X_m)$, we have

$$F_T(t) = \Pr(T \leq t) = \Pr(X_1 \leq t, X_2 \leq t, \ldots, X_m \leq t) = \delta(F_X(t)),$$

where $\delta(u) = C(u, u, \ldots, u)$ is the diagonal section of the copula C of the random vector $(X_1, X_2, \ldots, X_m)$ with the component lifetimes. In particular, if the components are independent and identically distributed (i.i.d.), then $q(u) = \sum_{i=1}^m b_i u^i$, where the vector $(b_1, b_2, \ldots, b_m)$ is called the *maximal signature* of the system. For a parallel system with i.i.d. components we have $q(u) = u^m$ (as mentioned above). For a 2-out-of-3 system with i.i.d. components, we have $q(u) = 3u^2 - 2u^3$.

Hence, proceeding as above, the GCPE of T can be written as

$$\begin{aligned}\mathcal{CE}_n(T) &= \frac{1}{n!}\int_0^{+\infty} F_T(x)[-\log(F_T(x))]^n dx \\ &= \frac{1}{n!}\int_0^{+\infty} \phi_n(F_T(x))dx \\ &= \frac{1}{n!}\int_0^{+\infty} \phi_n(q(F_X(x)))dx \\ &= \frac{1}{n!}\int_0^1 \frac{\phi_n(q(u))}{f_X(F_X^{-1}(u))}du \qquad \textbf{(4.1)}\end{aligned}$$

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

for $n = 1, 2, \ldots$. For example, for a 2-out-of-3 system with i.i.d. components with a uniform distribution in $(0, 1)$, we have

$$\mathcal{CE}_1(T) = -\int_0^1 (3u^2 - 2u^3)\log(3u^2 - 2u^3)du = \frac{9}{8} - \frac{27}{32}\log 3 \cong 0.198 < \mathcal{CE}_1(X) = 0.25$$

and

$$\mathcal{CE}_2(T) = \frac{1}{2}\int_0^1 (3u^2 - 2u^3)\left(-\log(3u^2 - 2u^3)\right)^2 du \cong 0.111 < \mathcal{CE}_2(X) = 0.125.$$

Thus, we can obtain the following general result. The proof is immediate from the preceding expressions.

Proposition 1: *Let T be the lifetime of a coherent system with m i.d. components and with distortion function q. Let $\phi_n(u) = u[-\log(u)]^n$. If $\phi_n(q(u)) \geq \phi_n(u)$ (resp. $\leq$), then $\mathcal{CE}_n(T) \geq \mathcal{CE}_n(X_1)$ (resp. $\leq$).*

The condition $\phi_n(q(u)) \geq \phi_n(u)$ is not a necessary condition for $\mathcal{CE}_n(T) \geq \mathcal{CE}_n(X_1)$ to hold (e.g. consider a 2-out-of-3 system with i.i.d. components having a uniform distribution as above).

Using the dispersive ordering, we obtain the following result.

Proposition 2: *Let T_1 and T_2 be the lifetimes of two coherent systems with the same structure and with i.d. components having common distributions F_1 and F_2, respectively, and having a common copula function C. If $F_1 \leq_{\text{disp}} F_2$, then $\mathcal{CE}_n(T_1) \leq \mathcal{CE}_n(T_2)$ for $n = 1, 2, \ldots$.*

Proof: As the systems have the same structure and the components a common copula, then both systems have a common distortion function q. Then the stated result is obtained from Eq. (4.1) and the characterization of the dispersive order is given in Section 2. ■

In particular, we obtain that, if $X \leq_{\text{disp}} Y$, then $\mathcal{CE}_n(X) \leq \mathcal{CE}_n(Y)$ for $n = 1, 2, \ldots$. Therefore, these conditions are necessary conditions for the dispersive order to hold. Hence $\mathcal{CE}_n(X)$ can be seen as a dispersion measure.

We can obtain bounds for $\mathcal{CE}_n(T)$ in terms of $\mathcal{CE}_n(X_1)$ in the case of identically distributed components as follows.

Proposition 3: *Let T be the lifetime of a coherent system with i.d. components and with distortion function q. Let $\phi_n(u) = u[-\log(u)]^n$. Then*

$$B_{1,n}\mathcal{CE}_n(X_1) \leq \mathcal{CE}_n(T) \leq B_{2,n}\mathcal{CE}_n(X_1)$$

for $n = 1, 2, \ldots$, where $B_{1,n} = \inf_{u\in(0,1)}(\phi_n(q(u))/\phi_n(u))$ and $B_{2,n} = \sup_{u\in(0,1)}(\phi_n(q(u))/\phi_n(u))$.

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

PROOF: From Eq. (4.1), the upper bound can be obtained as follows

$$\begin{aligned}\mathcal{CE}_n(T) &= \frac{1}{n!}\int_0^1 \frac{\phi_n(q(u))}{f_X(F_X^{-1}(u))}du \\ &= \frac{1}{n!}\int_0^1 \frac{\phi_n(q(u))}{\phi_n(u)}\frac{\phi_n(u)}{f_X(F_X^{-1}(u))}du \\ &\leq \frac{1}{n!}\sup_{u\in(0,1)}\frac{\phi_n(q(u))}{\phi_n(u)}\int_0^1 \frac{\phi_n(u)}{f_X(F_X^{-1}(u))}du \\ &= \left(\sup_{u\in(0,1)}\frac{\phi_n(q(u))}{\phi_n(u)}\right)\mathcal{CE}_n(X_1)\end{aligned}$$

for $n = 1, 2, \ldots$. The lower bound can be obtained in a similar way. ■

This result improves that given in Proposition 1. For example, for a 2-out-of-3 system with i.i.d. components, the distortion function is $q(u) = 3u^2 - 2u^3$ and, from the preceding proposition, we obtain

$$\mathcal{CE}_1(T) \leq 1.01201\ \mathcal{CE}_1(X_1)$$

for any F. The lower bound is zero (a trivial bound). Note that Proposition 1 cannot be applied to this example (because the upper bound is greater than 1 and the lower bound is smaller than 1). Analogously, for $n = 2, 3$, we obtain

$$\mathcal{CE}_2(T) \leq 1.17433\ \mathcal{CE}_2(X_1)$$

and

$$\mathcal{CE}_3(T) \leq 1.50256\ \mathcal{CE}_3(X_1).$$

The preceding proposition can be extended to compare the GCPE of two systems as follows. The proof is similar.

PROPOSITION 4: *Let T_1 and T_2 be the lifetimes of two coherent systems with i.d. components and with distortion functions q_1 and q_2, respectively. Let $\phi_n(u) = u[-\log(u)]^n$. Then*

$$\left(\inf_{u\in(0,1)}\frac{\phi_n(q_2(u))}{\phi_n(q_1(u))}\right)\mathcal{CE}_n(T_1) \leq \mathcal{CE}_n(T_2) \leq \left(\sup_{u\in(0,1)}\frac{\phi_n(q_2(u))}{\phi_n(q_1(u))}\right)\mathcal{CE}_n(T_1)$$

for $n = 1, 2, \ldots$.

In particular, if

$$\sup_{u\in(0,1)}\frac{\phi_n(q_2(u))}{\phi_n(q_1(u))} \leq 1,$$

then $\mathcal{CE}_n(T_2) \leq \mathcal{CE}_n(T_1)$. For example, if $T_1 = X_{2:2} := \max(X_1, X_2)$ is the lifetime of a 2-components parallel system and T_2 is the lifetime of a 2-out-of-3 system both with i.i.d. components, from the preceding proposition, we obtain

$$\mathcal{CE}_1(T_2) \leq 3\ \mathcal{CE}_1(T_1).$$

The lower bound is zero.

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

We can obtain additional bounds for $\mathcal{CE}_n(T)$ when the i.d. components have a bounded density. The result can be stated as follows.

PROPOSITION 5: *Let T be the lifetime of a coherent system with i.d. components and with distortion function q. Let $\phi_n(u) = u[-\log(u)]^n$. Let us assume that the components have an absolutely continuous distribution F_X with probability density function f_X and support S.*

(i) *If $f_X(x) \leq M$ for all $x \in S$, then*

$$\mathcal{CE}_n(T) \geq \frac{1}{M(n!)} \int_0^1 \phi_n(q(u))du$$

for $n = 1, 2, \ldots$.

(ii) *If $f_X(x) \geq L > 0$ for all $x \in S$, then*

$$\mathcal{CE}_n(T) \leq \frac{1}{L(n!)} \int_0^1 \phi_n(q(u))du$$

for $n = 1, 2, \ldots$.

PROOF: If $f_X(x) \leq M$ for all $x \in S$, then

$$\begin{aligned}\mathcal{CE}_n(T) &= \frac{1}{n!} \int_0^1 \frac{\phi_n(q(u))}{f_X(F_X^{-1}(u))}du \\ &\geq \frac{1}{n!} \int_0^1 \frac{\phi_n(q(u))}{M}du \\ &= \frac{1}{M(n!)} \int_0^1 \phi_n(q(u))du\end{aligned}$$

for $n = 1, 2, \ldots$. The other bound is obtained in a similar way. ■

In particular, for X, we obtain

$$\mathcal{CE}_n(X) \geq \frac{1}{M(n!)} \int_0^1 \phi_n(u)du$$

whenever $f_X(x) \leq M$ for all $x \in S$, and

$$\mathcal{CE}_n(X) \leq \frac{1}{L(n!)} \int_0^1 \phi_n(u)du$$

whenever $f_X(x) \geq L > 0$ for all $x \in S$, for $n = 1, 2, \ldots$. Note that the bounds are attained when X has a uniform distribution in $(0, b)$. In this case

$$\begin{aligned}\mathcal{CE}_n(X) &= \frac{1}{n!} \int_0^b \frac{x}{b} \left[-\log \frac{x}{b}\right]^n dx \\ &= \frac{(-1)^n}{n!b} \left[-\int_0^b \frac{x}{2} n \left(\log \frac{x}{b}\right)^{n-1} dx\right] \\ &= \frac{1}{2b} \left[\int_0^b \frac{(-1)^{n-1}}{(n-1)!} x \left(\log \frac{x}{b}\right)^{n-1} dx\right] \\ &= \frac{1}{2}\mathcal{CE}_{n-1}(X)\end{aligned}$$

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

for $n = 1, 2, \ldots$. Hence

$$\mathcal{CE}_n(X) = \frac{1}{2}\mathcal{CE}_{n-1}(X) = \cdots = \frac{1}{2^n}\mathcal{CE}_0(X),$$

where $\mathcal{CE}_0(X) = \int_0^b (x/b)dx = b/2$. Therefore, $\mathcal{CE}_n(X) = b/2^{n+1}$ for $n = 0, 1, \ldots$ and

$$\frac{1}{n!}\int_0^1 \phi_n(u)du = \frac{1}{2^{n+1}}.$$

So the bounds for $\mathcal{CE}_n(X)$ and a general distribution F_X can be written as

$$\mathcal{CE}_n(X) \geq \frac{1}{M2^{n+1}}$$

whenever $f_X(x) \leq M$ for all $x \in S$, and

$$\mathcal{CE}_n(X) \leq \frac{1}{L2^{n+1}}$$

whenever $f_X(x) \geq L > 0$ for all $x \in S$, for $n = 1, 2, \ldots$.

For example, if the i.d. components have an exponential distribution with mean μ, then $f_X(x) \leq 1/\mu = M$ and

$$\mathcal{CE}_n(T) \geq \frac{\mu}{n!}\int_0^1 \phi_n(q(u))du, \quad n = 1, 2, \ldots.$$

In this case, the upper bound cannot be obtained since we have $L = 0$ (i.e. the upper bound in $+\infty$). In particular, for the i.d. exponential components we have

$$\mathcal{CE}_n(X_1) \geq \frac{\mu}{n!}\int_0^1 \phi_n(u)du = \frac{\mu}{2^{n+1}}.$$

The following example shows how to obtain the bounds for a system with dependent and identically distributed (d.i.d.) components.

Example 2: Let us consider the parallel system $T = X_{2:2} = \max(X_1, X_2)$ with d.i.d. components having a common distribution F. Then the joint distribution function of the random vector (X_1, X_2) can be written as

$$\Pr(X_1 \leq x_1, X_2 \leq x_2) = C(F(x_1), F(x_2)),$$

where C is a given copula, that is, a continuous distribution with uniform marginals over the interval $(0, 1)$. Then the system distribution can be written as

$$F_{2:2}(t) = \Pr(X_{2:2} \leq t) = \Pr(X_1 \leq t, X_2 \leq t) = C(F(t), F(t)) = q(F(t)),$$

where $q(u) = C(u, u)$. For example, let us consider the following Clayton–Oakes copula

$$C(u, v) = \frac{1}{1/u + 1/v - 1} = \frac{uv}{u + v - uv}, \quad 0 \leq u, v \leq 1.$$

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

TABLE 1. Distortion functions q for all the coherent systems with 1-4 i.i.d. components.

N	T	$q(u)$
1	X_1	u
2	$X_{1:2} = \min(X_1, X_2)$	$2u - u^2$
3	$X_{2:2} = \max(X_1, X_2)$	u^2
4	$X_{1:3} = \min(X_1, X_2, X_3)$	$3u - 3u^2 + u^3$
5	$\min(X_1, \max(X_2, X_3))$	$u + u^2 - u^3$
6	$X_{2:3}$	$3u^2 - 2u^3$
7	$\max(X_1, \min(X_2, X_3))$	$2u^2 - u^3$
8	$X_{3:3} = \max(X_1, X_2, X_3)$	u^3
9	$X_{1:4} = \min(X_1, X_2, X_3, X_4)$	$4u - 6u^2 + 4u^3 - u^4$
10	$\max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$	$2u - 2u^3 + u^4$
11	$\min(X_{2:3}, X_4))$	$u + 3u^2 - 5u^3 + 2u^4$
12	$\min(X_1, \max(X_2, X_3), \max(X_2, X_4))$	$u + 2u^2 - 3u^3 + u^4$
13	$\min(X_1, \max(X_2, X_3, X_4))$	$u + u^3 - u^4$
14	$X_{2:4}$	$6u^2 - 8u^3 + 3u^4$
15	$\max(\min(X_1, X_2), \min(X_1, X_3, X_4), \min(X_2, X_3, X_4))$	$5u^2 - 6u^3 + 2u^4$
16	$\max(\min(X_1, X_2), \min(X_3, X_4))$	$4u^2 - 4u^3 + u^4$
17	$\max(\min(X_1, X_2), \min(X_1, X_3), \min(X_2, X_3, X_4))$	$4u^2 - 4u^3 + u^4$
18	$\max(\min(X_1, X_2), \min(X_2, X_3), \min(X_3, X_4))$	$3u^2 - 2u^3$
19	$\min(\max(X_1, X_2), \max(X_2, X_3), \max(X_3, X_4))$	$3u^2 - 2u^3$
20	$\min(\max(X_1, X_2), \max(X_1, X_3), \max(X_2, X_3, X_4))$	$2u^2 - u^4$
21	$\min(\max(X_1, X_2), \max(X_3, X_4))$	$2u^2 - u^4$
22	$\min(\max(X_1, X_2), \max(X_1, X_3, X_4), \max(X_2, X_3, X_4))$	$u^2 + 2u^3 - 2u^4$
23	$X_{3:4}$	$4u^3 - 3u^4$
24	$\max(X_1, \min(X_2, X_3, X_4))$	$3u^2 - 3u^3 + u^4$
25	$\max(X_1, \min(X_2, X_3), \min(X_2, X_4))$	$u^2 + u^3 - u^4$
26	$\max(X_{2:3}, X_4)$	$3u^3 - 2u^4$
27	$\max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$	$2u^3 - u^4$
28	$X_{4:4} = \max(X_1, X_2, X_3, X_4)$	u^4

Then $q(u) = u/(2-u)$. Hence, from Proposition 3, we obtain

$$\frac{1}{2}\,\mathcal{CE}_1(X_1) \le \mathcal{CE}_1(T) \le 2\,\mathcal{CE}_1(X_1).$$

Analogously, as

$$\int_0^1 \phi_1(q(u))du = -2\log 2 + \frac{1}{6}\pi^2 = 0.2586397,$$

from Proposition 5, we obtain

$$\frac{0.2586397}{M} \le \mathcal{CE}_1(T) \le \frac{0.2586397}{L}$$

whenever $L \le f_X(x) \le M$ for all $x \in S$.

In Table 1, we give the distortion functions for all the coherent systems with 1-4 i.i.d. components and in Table 2, we give $\mathcal{CE}_n(T)$ for these systems when the components have

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

TABLE 2. $\mathcal{CE}_n(T)$ and bounds for $\mathcal{CE}_n(T)$ obtained from Proposition 3 and Proposition 5 for all the coherent systems given in Table 1 with i.i.d. components having a standard exponential distribution ($\mu = 1$) and for $n = 1, 2, 3$.

N	n	$\mathcal{CE}_n(T)$	$B_{1,n}$	$B_{2,n}$	D_n	$B_{2,n}\mathcal{CE}_n(X_1)$
1	1	0.644934	1	1	0.25	0.644934
	2	0.202057	1	1	0.125	0.202057
	3	0.082323	1	1	0.0625	0.082323
2	1	0.322467	0	2	0.186915	1.289868
	2	0.101028	0	2	0.077728	0.404114
	3	0.041162	0	2	0.035552	0.164646
3	1	0.789868	0	2	0.222222	1.289868
	2	0.308228	0	4	0.148148	0.808228
	3	0.158586	0	8	0.098765	0.658586
4	1	0.214978	0	3	0.146386	1.934802
	2	0.067352	0	3	0.056225	0.606171
	3	0.027441	0	3	0.024833	0.246970
5	1	0.399153	0	1.056342	0.207481	0.681271
	2	0.142395	0	1.015601	0.101309	0.205209
	3	0.063844	0	1.004519	0.052122	0.082695
6	1	0.431768	0	1.012007	0.198046	0.652678
	2	0.172516	0	1.174332	0.111139	0.237282
	3	0.089776	0	1.502561	0.067755	0.123696
7	1	0.641597	0	1.150827	0.221637	0.742207
	2	0.233557	0	1.516010	0.130927	0.306320
	3	0.116690	0	2.132897	0.081244	0.175587
8	1	0.851469	0	3	0.1875	1.934802
	2	0.360179	0	9	0.140625	0.606171
	3	0.201894	0	27	0.105469	0.246970
9	1	0.161234	0	4	0.119811	2.579736
	2	0.050514	0	4	0.044013	0.808228
	3	0.020581	0	4	0.019079	0.329293
10	1	0.258196	0	2	0.165315	1.289868
	2	0.087430	0	2	0.070027	0.404114
	3	0.037390	0	2	0.032899	0.164647
11	1	0.288437	0	1.136607	0.174634	0.733036
	2	0.108046	0	1.041939	0.083079	0.210531
	3	0.050790	0	1.012924	0.043346	0.083387
12	1	0.346483	0	1.098163	0.192391	0.7082429
	2	0.123801	0	1.029188	0.091655	0.2079546
	3	0.056551	0	1.008789	0.047299	0.0830468
13	1	0.445965	0	1.012468	0.219642	0.652975
	2	0.163535	0	1.001239	0.111855	0.202307
	3	0.073025	0	1.000127	0.057992	0.082334
14	1	0.300888	0	1.020121	0.170339	0.657911
	2	0.121017	0	1.021857	0.088125	0.206473
	3	0.063164	0	1.15848	0.051613	0.095370
15	1	0.351058	0	1.005136	0.183012	0.648247
	2	0.136686	0	1.046841	0.095728	0.211521
	3	0.070459	0	1.22319	0.056271	0.100697

Continued

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

TABLE 2. (*continued*)

N	n	$\mathcal{CE}_n(T)$	$B_{1,n}$	$B_{2,n}$	D_n	$B_{2,n}\mathcal{CE}_n(X_1)$
16,17	1	0.394934	0	1.000151	0.192397	0.645032
	2	0.154114	0	1.092515	0.103585	0.220750
	3	0.079293	0	1.326676	0.061669	0.109216
18,19	1	0.431768	0	1.012007	0.198046	0.652678
	2	0.172516	0	1.174332	0.111139	0.237282
	3	0.089776	0	1.502561	0.067755	0.123696
20,21	1	0.460559	0	1.045234	0.199342	0.674107
	2	0.190620	0	1.310963	0.117482	0.264889
	3	0.101742	0	1.807005	0.074229	0.148759
22	1	0.479865	0	1.09667	0.195363	0.707280
	2	0.206149	0	1.510379	0.120958	0.305183
	3	0.114162	0	2.300398	0.080054	0.189376
23	1	0.487245	0	1.159029	0.184466	0.747497
	2	0.213967	0	1.765755	0.11768	0.356783
	3	0.122231	0	3.02405	0.080962	0.248950
24	1	0.623248	0	1.057191	0.224525	0.681818
	2	0.212041	0	1.258253	0.124132	0.254239
	3	0.101572	0	1.618131	0.073492	0.133210
25	1	0.649105	0	1.24926	0.212759	0.805690
	2	0.250910	0	1.858282	0.133867	0.375479
	3	0.132327	0	2.973261	0.088163	0.244768
26	1	0.642545	0	1.343604	0.195741	0.866536
	2	0.257031	0	2.245712	0.127710	0.453762
	3	0.141765	0	4.118911	0.088438	0.339082
27	1	0.766283	0	2	0.197631	1.289868
	2	0.308478	0	4	0.136703	0.808228
	3	0.168417	0	8.197383	0.097174	0.674835
28	1	0.885292	0	4	0.16	2.579736
	2	0.390318	0	16	0.128	3.23291
	3	0.228564	0	64	0.1024	5.268687

a standard exponential distribution ($\mu = 1$). We also give the respective bounds obtained from Proposition 3 and Proposition 5. For the standard exponential, the value M defined in Proposition 5 is $M = 1$ and hence

$$D_n := \frac{1}{n!}\int_0^1 \phi_n(q(u))du$$

is a lower bound for $\mathcal{CE}_n(T)$. Note that $B_{1,n}$, $B_{2,n}$, and D_n do not depend on F_X and so these values can be used for other models (with possible different values for $\mathcal{CE}_n(T)$ and M). Note that $B_{1,n} = 0$ for all the coherent systems with 2–4 components. So the lower bound is D_n. The upper bound is $B_{2,n}\mathcal{CE}_n(X_1)$. This bound is given in the last columns of Table 2 for systems with i.i.d. standard exponential components.

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

5. GENERALIZED CUMULATIVE KERRIDGE INACCURACY

Given two random lifetimes X and Y having distribution functions F_X and F_Y, the cumulative Kerridge inaccuracy defined in [8] is

$$K[F_X, F_Y] = -\int_0^{+\infty} F_X(u) \log F_Y(u)\, du. \tag{5.1}$$

It is the cumulative analog of the measure of inaccuracy due to Kerridge [11].

In analogy with the measure defined in Eq. (5.1), we now introduce the *generalized cumulative Kerridge inaccuracy of order n* defined as

$$K_n[F_X, F_Y] = \frac{1}{n!}\int_0^{+\infty} F_X(u)[-\log F_Y(u)]^n du. \tag{5.2}$$

We now give a probabilistic meaning of the generalized cumulative inaccuracy from Eq. (5.3) and (5.6). First, we note that Kayal [10] introduces the following decreasing convex function:

$$T_{n,X}^{(2)}(x) = \frac{1}{n!}\int_x^{+\infty} [-\log F_X(z)]^n dz. \tag{5.3}$$

Also Lemma 3.12 in [10] provides the following expression of the GCPE:

$$\mathcal{CE}_n(X) = E[T_{n,X}^{(2)}(X)]. \tag{5.4}$$

In the following proposition, we give a similar result for the measure defined in Eq. (5.2).

PROPOSITION 6: *Let X and Y two non-negative absolutely continuous random variables having distribution functions F_X and F_Y, respectively. Then we have*

$$K_n[F_X, F_Y] = E\left[T_{n,Y}^{(2)}(X)\right], \tag{5.5}$$

where

$$T_{n,Y}^{(2)}(x) = \frac{1}{n!}\int_x^{+\infty} [-\log F_Y(z)]^n dz. \tag{5.6}$$

PROOF: From Eq. (5.2) and Fubini's theorem, it follows

$$\begin{aligned} K_n[F_X, F_Y] &= \frac{1}{n!}\int_0^{+\infty} F_X(x)[-\log F_Y(x)]^n dx \\ &= \frac{1}{n!}\int_0^{+\infty}\left[\int_0^x dF_X(t)\right][-\log F_Y(x)]^n dx \\ &= \int_0^{+\infty}\left[\int_t^{+\infty}\frac{[-\log F_Y(x)]^n}{n!}dx\right] dF_X(t) \\ &= \int_0^{+\infty} T_{n,Y}^{(2)}(t) dF_X(t), \end{aligned}$$

which immediately yields the stated result from Eq. (5.6). ■

In the following proposition, we obtain a connection between our measure of discrimination and some stochastic orders.

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

PROPOSITION 7: *Let X and Y be non-negative random variables. If $X \leq_{\mathrm{st}} Y$, then*

$$K_n[F_Y, F_X] \leq \mathcal{CE}_n(X) \leq K_n[F_X, F_Y]$$

for $n = 1, 2, \ldots$.

PROOF: By assumption $F_X(t) \geq F_Y(t)$ for all $t \geq 0$. Then, for $n = 1, 2, \ldots$,

$$K_n[F_X, F_Y] = \frac{1}{n!} \int_0^{+\infty} F_X(u)[-\log F_Y(u)]^n du \geq \frac{1}{n!} \int_0^{+\infty} F_X(u)[-\log F_X(u)]^n du$$

and

$$K_n[F_Y, F_X] = \frac{1}{n!} \int_0^{+\infty} F_Y(u)[-\log F_X(u)]^n du \leq \frac{1}{n!} \int_0^{+\infty} F_X(u)[-\log F_X(u)]^n du.$$

We remark that $X \leq_{\mathrm{st}} Y$ does not imply in general $\mathcal{CE}_n(X) \leq \mathcal{CE}_n(Y)$. ■

PROPOSITION 8: *Let X and Y be non-negative random variables. If $X \leq_{\mathrm{dcx}} Y$, then*

$$\mathcal{CE}_n(X) \leq K_n[F_Y, F_X]$$

for $n = 1, 2, \ldots$.

PROOF: By assumption $E[\phi(X)] \leq E[\phi(Y)]$ for all decreasing convex function $\phi : \mathbb{R} \to \mathbb{R}$. Note that both GCPE and $K_n[F_Y, F_X]$ can be expressed as mean value of $T^{(2)}_{n,X}(\cdot)$. So, from Eqs. (5.4) and (5.5), the proof follows by noting that $T^{(2)}_{n,X}$ is a decreasing convex function. ■

6. EMPIRICAL GENERALIZED CUMULATIVE KERRIDGE INACCURACY

In this section, we study the empirical version of our new inaccuracy measure. In a recent past, several authors studied some applications of the empirical measures of discrimination in various fields (see, for instance, Di Crescenzo and Longobardi [6,7] and Longobardi [13]).

Let $X_1, X_2, \ldots, X_m$ be a random sample of size m from a lifetime distribution with absolutely continuous cumulative distribution function $F(x)$. Kayal [10] defined the empirical GCPE as

$$\mathcal{CE}_n(\hat{F}_m) = \frac{1}{n!} \int_0^{+\infty} \hat{F}_m(x)[-\log \hat{F}_m(x)]^n dx, \quad n = 1, 2, \ldots, \tag{6.1}$$

where

$$\hat{F}_m(x) = \frac{1}{m} \sum_{i=1}^{m} I_{(X_i \leq x)}, \quad x \in \mathbb{R}$$

is the empirical distribution of the sample and I is the indicator function. Denoting $X_{1:m}, X_{2:m}, \ldots, X_{m:m}$ as the order statistics of the sample, Eq. (6.1) can be written as

$$\mathcal{CE}_n(\hat{F}_m) = \sum_{j=1}^{m-1} \frac{1}{n!} \int_{X_{j:m}}^{X_{j+1:m}} \hat{F}_m(x)[-\log \hat{F}_m(x)]^n dx, \quad n = 1, 2, \ldots. \tag{6.2}$$

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

Moreover, if F is continuous, then

$$\hat{F}_m(x) = \begin{cases} 0, & x < X_{1:m} \\ \dfrac{j}{m}, & X_{j:m} \le x < X_{j+1:m}, \quad j = 1, 2, \ldots, m-1 \\ 1, & x \ge X_{m:m}. \end{cases}$$

Hence, Eq. (6.2) can be written as

$$\mathcal{CE}_n(\hat{F}_m) = \frac{1}{n!} \sum_{j=1}^{m-1} U_{j+1} \frac{j}{m} \left(-\log \frac{j}{m} \right)^n, \quad n = 1, 2, \ldots,$$

where

$$U_1 = X_{1:m}, \quad U_i = X_{i:m} - X_{i-1:m}, \quad i = 1, 2, \ldots, m$$

are the sample spacings corresponding to the sample order statistics.

Let us now consider another random sample $Y_1, Y_2, \ldots, Y_m$ of non-negative, absolutely continuous i.i.d. random variables and denote its empirical GCPE by

$$\mathcal{CE}_n(\hat{G}_m) = \frac{1}{n!} \int_0^{+\infty} \hat{G}_m(y)[-\log \hat{G}_m(y)]^n dy,$$

where $\hat{G}_m$ is the empirical distribution of the second sample.

According to Eq. (5.2), we define the *empirical generalized cumulative inaccuracy* as

$$K_n[\hat{F}_m, \hat{G}_m] = \frac{1}{n!} \int_0^{+\infty} \hat{F}_m(u)[-\log \hat{G}_m(u)]^n du.$$

It can be expressed as

$$K_n[\hat{F}_m, \hat{G}_m] = \frac{1}{n!} \sum_{j=1}^{m-1} \left(-\log \frac{j}{m} \right)^n \int_{Y_{j:m}}^{Y_{j+1:m}} \hat{F}_m(u) du,$$

where $Y_{1:m}, Y_{2:m}, \ldots, Y_{m:m}$ are the order statistics of the new sample. Let us denote by

$$N_j = \sum_{i=1}^{m} I_{(X_i \le Y_{j:m})}, \quad j = 1, 2, \ldots, m$$

the number of random variables of the first sample that are less than or equal to the j-th order statistic of the second sample. We rename by $X_{j,1} < X_{j,2} < \cdots$ the random variables of the first sample belonging to the interval $(Y_{j:m}, Y_{j+1:m}]$. From the above position, we

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

have:

$$\int_{Y_{j:m}}^{Y_{j+1:m}} \hat{F}_m(u)du = \frac{N_j}{m}[Y_{j+1:m} - Y_{j:m}] + \frac{1}{m}\sum_{r=1}^{N_{j+1}-N_j}[Y_{j+1:m} - X_{j,r}].$$

Then

$$K_n[\hat{F}_m, \hat{G}_m] = \frac{1}{mn!}\sum_{j=1}^{m-1}\left[N_{j+1}Y_{j+1:m} - N_jY_{j:m} - \sum_{r=1}^{N_{j+1}-N_j} X_{j,r}\right]\left(-\log\frac{j}{m}\right)^n.$$

Let us remember that the well-known theorem of Glivenko–Cantelli states that:

$$\sup_x \mid \hat{F}_m(x) - F(x) \mid \to 0 \quad \text{a.s. as} \quad m \to +\infty.$$

Using this result, we prove the following theorem, which discloses an asymptotic property of the empirical generalized cumulative inaccuracy.

THEOREM 5: *Let X and Y be non-negative random variables in L^p for some $p > 1$; the empirical generalized cumulative inaccuracy converges to the generalized cumulative inaccuracy of X and Y, that is*

$$K_n[\hat{F}_m, \hat{G}_m] \to K_n[F, G] \quad \textit{a.s. as} \quad m \to \infty.$$

PROOF: It goes similarly to the proof of Theorem 9 of [21]. By the dominated convergence theorem, the integral of $\hat{F}_m(u)[-\log\hat{G}_m(u)]^n$ converges to that of $F(u)[-\log G(u)]^n$ on any finite interval. Hence, we only have to show that

$$\left|\int_1^{+\infty} \hat{F}_m(u)[-\log\hat{G}_m(u)]^n du - \int_1^{+\infty} F(u)[-\log G(u)]^n du\right| \to 0 \quad \text{a.s. as} \quad m \to \infty.$$

Note that, for a fixed n, the function $u[-\log(v)]^n$ is continuous in $[a, 1]^2$ for any $a > 0$ and so bounded. Hence, if $K = \min(\hat{F}_m(1), \hat{G}_m(1)) > 0$, we get

$$\left|\hat{F}_m(u)[-\log\hat{G}_m(u)]^n\right| \leq K.$$

Now, by applying the dominated convergence theorem and by virtue of the Glivenko–Cantelli theorem, we prove the theorem. ■

Acknowledgements

We would like to thank the associate editor and three anonymous reviewers for several helpful suggestions. CC and ML were supported by GNAMPA, Istituto Nazionale di Alta Matematica and JN was supported in part by Ministerio de Economía, Industria y Competitividad of Spain under grant MTM2016-79943-P.

References

1. Asadi, M. & Zohrevand, Y. (2007). On the dynamic cumulative residual entropy. *Journal of Statistical Planning and Inference* 137: 1931–1941.
2. Barlow, R.E. & Proschan, F. (1965). *Mathematical Theory of Reliability*. New York: Wiley.
3. Burkschat, M. & Navarro, J. (2018). Stochastic comparisons of systems based on sequential order statistics via properties of distorted distributions. *Probability in the Engineering and Informational Sciences* 32: 246–274.

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press

4. Di Crescenzo, A. & Longobardi, M. (2002). Entropy-based measure of uncertainty in past lifetime distributions. *Journal of Applied Probability* 39: 434–440.
5. Di Crescenzo, A. & Longobardi, M. (2009). On cumulative entropies. *Journal of Statistical Planning and Inference* 139: 4072–4087.
6. Di Crescenzo, A. & Longobardi, M. (2009). On cumulative entropies and lifetime estimations. In J. Mira, J.M. Ferrandez, J.R. Alvarez Sanchez, F. Paz & J. Toledo, (eds.), *Methods and Models in Artificial and Natural Computation*. IWINAC 2009, Lecture Notes in Computer Science Vol. 5601, Berlin: Springer-Verlag, pp. 132–141.
7. Di Crescenzo, A. & Longobardi, M. (2010). More on cumulative entropy. In R. Trappl, (eds.), *Cybernetics and Systems 2010*. Vienna: Austrian Society for Cybernetic Studies, pp. 181–186.
8. Di Crescenzo, A. & Longobardi, M. (2013). Stochastic comparisons of cumulative entropies. In H. Li & X. Li, (eds.), *Stochastic Orders in Reliability and Risk, In Honor of Professor Moshe Shaked*. Lecture Notes in Statistics 208, New York: Springer, pp. 167–182.
9. Hairer, E., Norsett, P., & Wanner, G (2008). *Solving Ordinary Differential Equations I*. Berlin, Heildelberg: Springer-Verlag.
10. Kayal, S. (2016). On generalized cumulative entropies. *Probability in the Engineering and Informational Sciences* 30: 640–662.
11. Kerridge, D.F. (1961). Inaccuracy and Inference. *Journal of the Royal Statistical Society, Series B* 23: 184–194.
12. Li, X. & Lu, J. (2003). Stochastic comparisons on residual life and inactivity time of series and parallel systems. *Probability in the Engineering and Informational Sciences* 17: 267–275.
13. Longobardi, M. (2014). Cumulative measures of information and stochastic orders. *Ricerche di Matematica* 63: 209–223.
14. Nanda, A.K., Singh, H., Misra, N., & Paul, P. (2003). Reliability properties of reversed residual lifetime. *Communication in Statistics Theory and Methods* 32: 2031–2042. (with correction in *Communication in Statistics Theory and Methods* 33: 991–992, 2004).
15. Navarro, J., del Águila, Y. & Asadi, M. (2010). Some new results on the cumulative residual entropy. *Journal of Statistical Planning and Inference* 140: 310–322.
16. Navarro, J., del Águila, Y., Sordo, M.A., & Suárez-Llorens, A. (2014). Preservation of reliability classes under the formation of coherent systems. *Applied Stochastic Models in Business and Industry* 30: 444–454.
17. Navarro, J., del Águila, Y., Sordo, M.A., & Suárez-Llorens, A. (2016). Preservation of stochastic orders under the formation of generalized distorted distributions. *Applications to coherent systems. Methodology and Computing in Applied Probability* 18: 529–545.
18. Navarro, J. & Psarrakos, G. (2017). Characterization based on generalized cumulative residual entropy functions. *Communication in Statistics Theory and Methods* 46: 1247–1260.
19. Psarrakos, G. & Navarro, J. (2013). Generalized cumulative residual entropy and record values. *Metrika* 76: 623–640.
20. Rao, M. (2005). More on a new concept of entropy and information. *Journal of Theoretical Probability* 18: 967–981.
21. Rao, M., Chen, Y., Vemuri, B.C., & Wang, F. (2004). Cumulative residual entropy: a new measure of information. *IEEE Transactions on Information Theory* 50: 1220–1228.
22. Shaked, M. & Shanthikumar, J.G. (2007). *Stochastic Orders*. New York: Springer.
23. Shannon, C.E. (1948). A mathematical theory of communication. *Bell System Technical Journal* 27: 279–423.
24. Toomaj, S., Sunoj, S.M., & Navarro, J. (2017). Some properties of the cumulative residual entropy of coherent and mixed systems. *Journal of Applied Probability* 54: 379–393.
25. Wiener, N. (1961). *Cybernetics* (2nd Ed.). New York: The MIT Press and Wiley.

https://doi.org/10.1017/S0269964818000360 Published online by Cambridge University Press