A STATE SPACE SOLUTION APPROACH FOR PROBLEMS OF CYLINDRICAL TUBES AND CIRCULAR PLATES

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ABSTRACT

We present a general solution approach for analysis of transversely isotropic cylindrical tubes and circular plates. On the basis of Hamiltonian state space formalism in a systematic way, rigorous solutions of the twisting problems are determined by means of separation of variables and symplectic eigenfunction expansion.

Keywords: Cylindrical tubes, Circular plates, Hamiltonian, State space.

1. INTRODUCTION

While numerous publications on analysis of elastic cylinders [1-4] exist, solutions to this class of problems were obtained, in general, by relaxing the end conditions. To our knowledge, it is difficult, if not impossible, to obtain these exact solutions following the classical approach of elasticity. For example, Flugge and Kelkar [1] developed a general method of solution which satisfies exactly any arbitrary boundary conditions prescribed on the curved and end surfaces of a cylinder of any length. But since the governing equations are not self-adjoint, the eigensolutions cannot be proved to form an orthogonal and complete set. Thus they introduced certain auxiliary functions to get the extended eigenvectors and then expanded the stresses or displacements prescribed at the end of the cylinder in a series of the extended eigensolutions. Knowles and Horgan [2] applied methods involving energy-decay inequalities to the end problem for a circular elastic cylinder and obtained explicit lower bounds for the rate of exponential decay of stresses. But the loads at the ends of cylinder are limited to axisymmetric and self-equilibrated. Power and Childs [3] solved the problem of a finite circular bar with the lateral surface assumed to be traction free and the end sections subjected to prescribed tractions or displacements. The problem is formally reduced to an infinite system of linear algebraic equations which can be satisfactorily approximated by truncation. But a family of biorthogonal eigenfunctions via Love's function has to be generated for the axisymmetric case. Moore [4] examined the response of long thick elastic tubes subjected to 3D load systems and gave exact solutions for isolated and periodic disturbances. Infinitely long structures have been investigated. However, the described procedure cannot be used to study the disturbances near the ends of tubes with finite lengths.

The Hamiltonian state space formalism [5-8] puts anisotropic elasticity in a broad prospect. It makes three-dimensional problems of anisotropic elastic media less formidable. With the symplectic orthogonality of the system, it is possible to express an arbitrary integrable state vector in terms of orthogonal eigenvectors by using eigenfunction expansion in the state space. Determination of the eigensolution is critical to the explicit solution for a problem. Recently, we have succeeded in applying the state space approach to exact analysis of finite circular elastic cylinders [9-14] and plates [15], the focus therein was on developing a general solution approach for three-dimensional analysis of anisotropic cylindrical tubes and circular plates. The state equations and the output equations, with r and z as the distinct coordinate, respectively, are derived via the Hamiltonian variational formulation and provide an essential basis for the systematic solution approach. The choice of the state equation for analysis depends on the solution approach for solving problem.

For delineating the solution approach, we consider relatively simple problems of twisting deformation in cylindrical tubes and circular plates under prescribed boundary conditions (BC). For twisting problems of cylindrical bars, both Lekhnitskii [16] and Tarn & Chang [9] had obtained the solutions of homogeneous materials and functionally graded materials as well. Lekhnitskii solved this kind of problem by using the stress function approach. The earlier version of Tarn and Chang was formulated on the basis of grouping the field variables and partitioning the constitutive matrices accordingly. As a result, the state equation and output equation do not exhibit permutable structure so that choice of the state vector must be done judiciously.

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Besides, the undetermined constants in the end conditions were solved by means of the Fourier-Bessel expansions, not by the sympletic orthogonality. So we have the necessity to re-visit this problem. As for twisting problem of plates, we believe that such solutions had not appeared in the literature. The approach for tubes bears a remarkable resemblance to its counterpart for plates.

When a cylindrical body is subjected to torque independent of θ, the problem is axially symmetric if the material is isotropic or transversely isotropic with the z axis being the axis of rotational symmetry. Since the expressions are simple, explicit, and much clearer than that of the other complicate problems, exact and closed-form solutions, each in their own right, can be derived in a systematic and self-contained manner. As such, correspondence between twisting problems and the other problems emerges in the state space setting. The state equation and the output equation bear a remarkable resemblance, differing only in the dimensions and entities of the corresponding matrices, which enables us to determine the solutions for various problems of tubes and plates by extension of the corresponding approach of twisting problems.

2. BASIC EQUATIONS FOR THE PROBLEMS

Consider an elastic cylindrical body and locate the origin of the cylindrical coordinates (r, θ, z) at the center of the body. With z axis being the axis of symmetry, the constitutive equations of transversely isotropic elastic materials are

$$\begin{bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\\ \sigma_{\theta z}\\ \sigma_{rz}\\ \sigma_{r\theta}\end{bmatrix}=\begin{bmatrix}c_{11} & c_{12} & c_{13} & 0 & 0 & 0\\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0\\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{zz}\\ 2\varepsilon_{\theta z}\\ 2\varepsilon_{rz}\\ 2\varepsilon_{r\theta}\end{bmatrix}, \tag{1}$$

where σ_{ij} and ε_{ij} are the stress and strain components; c_{ij} are the elastic constants.

The strain-displacement relations are

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\quad \varepsilon_{\theta\theta}=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r},\quad \varepsilon_{zz}=\frac{\partial u_z}{\partial z},$$
$$\varepsilon_{\theta z}=\frac{1}{2}\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right),\quad \varepsilon_{rz}=\frac{1}{2}\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right),$$
$$\varepsilon_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right). \tag{2}$$

The equilibrium equations in the absence of the body force are

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0,$$
$$\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\partial\sigma_{\theta z}}{\partial z}+\frac{2\sigma_{r\theta}}{r}=0,$$
$$\frac{\partial\sigma_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{\sigma_{rz}}{r}=0. \tag{3}$$

when the torque is the only external force acting on the circular cylinder with appropriate boundary conditions, the deformation and stress fields are independent of θ and we can consider only the displacement u_θ and the stresses $\sigma_{r\theta}$ and $\sigma_{\theta z}$ which are really related to this kind of problem. Decoupling from Eqs. (1) ~ (3), we have

$$\begin{bmatrix}\sigma_{r\theta}\\ \sigma_{\theta z}\end{bmatrix}=\begin{bmatrix}c_{66} & 0\\ 0 & c_{44}\end{bmatrix}\begin{bmatrix}2\varepsilon_{r\theta}\\ 2\varepsilon_{\theta z}\end{bmatrix}, \tag{4}$$

$$\varepsilon_{\theta z}=\frac{1}{2}\frac{\partial u_\theta}{\partial z},\quad \varepsilon_{r\theta}=\frac{1}{2}\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right). \tag{5}$$

$$\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{\partial\sigma_{\theta z}}{\partial z}+\frac{2\sigma_{r\theta}}{r}=0. \tag{6}$$

with $c_{44}=(c_{11}-c_{12})/2$, the independent elastic constant reduce from four to two for isotropic materials.

3. HAMILTONIAN FORMULATION

The formulation begins with the potential energy of the elastic body [8]

$$\Pi=\int_\Omega Ur\,dr\,dz-\int_\Gamma \mathbf{t}^T\mathbf{u}\,d\Gamma, \tag{7}$$

where and henceforth the superscript T denotes the transpose, the body force is not considered, $\mathbf{u}$ is the displacement vector, $\mathbf{t}$ denotes the prescribed traction on the boundary Γ of the body Ω, $U=U(\mathbf{u}, \mathbf{u}_{,z}, \mathbf{u}_{,r})$ is the strain energy density function in terms of the displacement vector $\mathbf{u}=u_\theta$ and its derivatives, given by

$$U=\frac{1}{2}\begin{bmatrix}\mathbf{u}_{,r}\\ r^{-1}\mathbf{K}\mathbf{u}\\ \mathbf{u}_{,z}\end{bmatrix}^T\begin{bmatrix}\mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13}\\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23}\\ \mathbf{C}_{31} & \mathbf{C}_{32} & \mathbf{C}_{33}\end{bmatrix}\begin{bmatrix}\mathbf{u}_{,r}\\ r^{-1}\mathbf{K}\mathbf{u}\\ \mathbf{u}_{,z}\end{bmatrix}, \tag{8}$$

where a comma in the subscript denotes partial differentiation, the matrices $\mathbf{C}_{ij}$ of the transversely isotropic material are

$$\mathbf{C}_{21}=\mathbf{C}_{12}^T,\ \mathbf{C}_{31}=\mathbf{C}_{13}^T,\ \mathbf{C}_{32}=\mathbf{C}_{23}^T,\ \mathbf{K}=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$\mathbf{C}_{11}=\begin{bmatrix}0 & 0 & 0\\ 0 & c_{66} & 0\\ 0 & 0 & c_{44}\end{bmatrix},\ \mathbf{C}_{12}=\begin{bmatrix}0 & 0 & 0\\ c_{66} & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\ \mathbf{C}_{13}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ c_{44} & 0 & 0\end{bmatrix},$$

$$\mathbf{C}_{22}=\begin{bmatrix}c_{66} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & c_{44}\end{bmatrix},\ \mathbf{C}_{23}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & c_{44} & 0\end{bmatrix},$$

$$\mathbf{C}_{33}=\begin{bmatrix}c_{44} & 0 & 0\\ 0 & c_{44} & 0\\ 0 & 0 & 0\end{bmatrix}.$$

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Minimizing $\prod$ yields three second-order partial differential equations in terms of the displacement components according to the Lagrangian formulation. The equations are rather complicated and very difficult to work with.

In formulating the basic equations into the state space setting in which the system matrix is a Hamiltonian matrix, it is essential that the state vector be composed of the dual variables. Duality of the state variables arises as a natural result of the Hamiltonian formulation.

Let the displacement vector $\mathbf{u}$ play the role of the dynamic variable and the coordinate z play the role of the time variable in a dynamic system, *i.e.* $\mathbf{q} \equiv \mathbf{u}$, $\dot{\mathbf{q}} \equiv \mathbf{u}_{,z}$, according to the notation in classical mechanics. The Hamiltonian of a conservative system is derived from the Lagrangian by introducing a conjugate variable $\mathbf{p}$, dual with $\mathbf{u}$, through Legendre's transformation

$$H(\mathbf{u},\ \mathbf{p},\ \mathbf{u}_{,r}) = \mathbf{p}^T \mathbf{u}_{,z} - L(\mathbf{u},\ \mathbf{u}_{,z},\ \mathbf{u}_{,r}), \tag{9}$$

such that

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{u}_{,z}}. \tag{10}$$

The Lagrangian of the system in terms of the displacement vector is given by $L = rU(\mathbf{u}, \mathbf{u}_{,z}, \mathbf{u}_{,r})$.

Replacing L in $\prod$ by H via Eq. (9), performing $\delta \prod = 0$ and integrating by parts, we obtain

$$\int_{\Omega} \left[\left(\frac{\partial \mathbf{u}}{\partial z} - \frac{\partial H}{\partial \mathbf{p}} \right)^T \delta \mathbf{p} - \left(\frac{\partial \mathbf{p}}{\partial z} + \frac{\partial H}{\partial \mathbf{u}} - \frac{\partial}{\partial r} \frac{\partial H}{\partial \mathbf{u}_{,r}} \right)^T \delta \mathbf{u} \right] dr dz = 0. \tag{11}$$

There follows the canonical equations:

$$\frac{\partial \mathbf{u}}{\partial z} = \frac{\partial H}{\partial \mathbf{p}}, \quad \frac{\partial \mathbf{p}}{\partial z} = -\frac{\partial H}{\partial \mathbf{u}} + \frac{\partial}{\partial r} \frac{\partial H}{\partial \mathbf{u}_{,r}}. \tag{12}$$

The state equation is obtained by expressing Eq. (12) in terms of the state vector. To this end, a substitution of $L = rU(\mathbf{u}, \mathbf{u}_{,z}, \mathbf{u}_{,r})$ into Eq. (10) gives

$$\mathbf{p} = p_\theta = rc_{44} u_{\theta,z} = r\sigma_{\theta z}, \tag{13}$$

which reveals that, when Z is taken to be the distinguished coordinate, the conjugate vector dual with $\mathbf{u} = u_\theta$ is $\mathbf{p} = r\sigma_{\theta z}$, rather than the traction $\sigma_{\theta z}$ on the plane normal to Z axis.

Expressing $\mathbf{u}_{,z}$ in terms of $\mathbf{u}$ and $\mathbf{p}$, then substituting it into Eq. (9) to express H in terms of $\mathbf{u}$ and $\mathbf{p}$ and using it in Eq. (12), we obtain

$$\frac{\partial}{\partial z} \begin{bmatrix} u_\theta \\ r\sigma_{\theta z} \end{bmatrix} = \begin{bmatrix} 0 & c_{44}^{-1} r^{-1} \\ -c_{66}(\partial_r + r^{-1})(r\partial_r - 1) & 0 \end{bmatrix} \begin{bmatrix} u_\theta \\ r\sigma_{\theta z} \end{bmatrix}, \tag{14}$$

Expressing the remaining stress components in terms of the state vector through Eqs. (4) ~ (5), we obtain

$$r\sigma_{r\theta} = c_{66}(r\partial_r - 1)\, u_\theta, \tag{15}$$

where ∂_r and ∂_z denote partial differentiation with respect to r and z, respectively.

Equations (14) and (15) are the state equation and the output equation for 3D elasticity of transversely isotropic materials with z axis being the distinguished coordinate, in which the state vector is composed of the displacement vector and the conjugate stress vector as the dual variables. It will be shown that the system possesses Hamiltonian characteristics, which are essential in determining the solution for the problem.

Following the same line, one can derive the state equation and the output equation with r as the distinguished coordinate, as follows:

$$r\frac{\partial}{\partial r} \begin{bmatrix} u_\theta \\ r\sigma_{r\theta} \end{bmatrix} = \begin{bmatrix} 1 & c_{66}^{-1} \\ -c_{44} r^2 \partial_{zz} & -1 \end{bmatrix} \begin{bmatrix} u_\theta \\ r\sigma_{r\theta} \end{bmatrix}, \tag{16}$$

$$r\sigma_{\theta z} = c_{44} r \partial_z u_\theta. \tag{17}$$

The choice of the state equation for analysis depends on the solution approach for solving problem.

4. PROBLEM STATEMENT

4.1 z as the Distinguishing Coordinate

For various problems such as finite cylindrical bars and tubes, whose lengths are great in comparison with their radii, it is essential to take z as the distinguishing coordinate and determine the eigenvector that satisfies the state equation and the homogeneous BC in the radial direction (together with the interfacial continuity conditions for laminated medium). The symplectic orthogonality in the r axis will be used to determine the unknown constants in the series solution by satisfying the BC on the end sections (end conditions on the flat planes in finite cylinders).

Consider an elastic cylindrical tube of inner radius a, outer radius b and length l (Fig. 1). Let us locate the origin of the cylindrical coordinates (r, θ, z) at one end of the tube. The cylindrical surfaces at $r = a$ and $r = b$ are traction-free:

$$[\sigma_{r\theta}]_{r=a} = [\sigma_{r\theta}]_{r=b} = 0. \tag{18}$$

The end conditions at $z = 0$ is assumed to be perfectly bonded:

$$[u_\theta]_{z=0} = 0, \tag{19}$$

and at $z = l$ is subjected to distributed load:

$$[\sigma_{\theta z}]_{z=l} = p(r). \tag{20}$$

Given the state space formulation of the problem which takes z as the distinguishing coordinate, our task is to solve the state equation Eq. (14) under the boundary conditions at $r = a, b$, Eqs. (18) ~ (20). Once the state vector $[u_\theta\ r\sigma_{\theta z}]^T$ is determined, the remaining unknown stress component $r\sigma_{r\theta}$ is derived from the output equation, Eq. (15).

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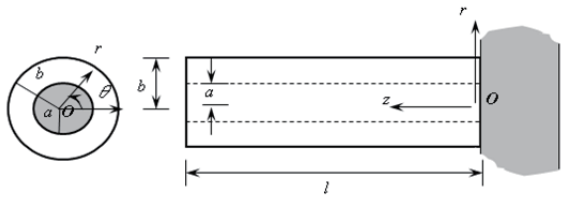


Fig. 1 An elastic cylindrical tube of inner radius a, outer radius b and length l.

4.2 r as the Distinguishing Coordinate

For various problems such as circular plates with holes at their center, whose radii are great in comparison with their thicknesses, it is essential to take r as the distinguishing coordinate and determine the eigensolution (eigenvector) that satisfies the state equation and the homogeneous BC in the thickness direction (together with the interfacial continuity conditions for laminated system or layered medium). The symplectic orthogonality in the z axis will be used to determine the unknown constants in the series solution by satisfying the BC on the cylindrical boundary surfaces (edge BC in circular plates).

Consider an elastic circular plate of inner radius a, outer radius b and thickness $2h$ (Fig. 2). Let us locate the origin of the cylindrical coordinates (r, θ, z) at the center of the circular plate. The boundary conditions on the top and bottom surfaces at $z = \pm h$ are traction-free:

$$[\sigma_{\theta z}]_{z=h} = [\sigma_{\theta z}]_{z=-h} = 0\,. \tag{21}$$

The edge boundary conditions at $r = a$ is considered to be the clamped edge:

$$[u_\theta]_{r=a} = 0\,, \tag{22}$$

and at $r = b$ is subjected to distributed load :

$$[\sigma_{r\theta}]_{r=b} = p(z)\,. \tag{23}$$

Given the state space formulation of the problem which takes z as the distinguishing coordinate, our task is to solve the state equation Eq. (16) under the boundary conditions at $z = \pm h$, Eqs. (21) ~ (23). Once the state vector $[u_\theta\ \sigma_{r\theta}]^T$ is determined, the remaining unknown stress component $\sigma_{\theta z}$ is derived from the output equation, Eq. (17).

5. SOLUTION OF THE STATE EQUATION

To determine the eigenvector that satisfies the state equation, we have to assume the solution forms in the axial direction for tubes and in the radial direction for plates, respectively.

5.1 Cylindrical Tube

Guided by our experience with the problems in the cylindrical coordinate system [9-14], we seek the solution to Eq. (14) in the form

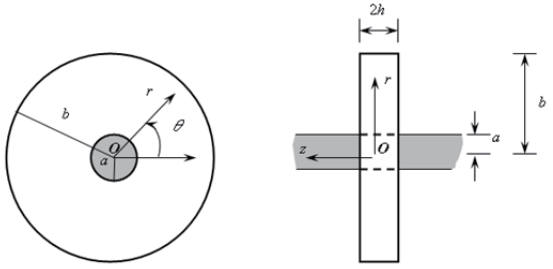


Fig. 2 An elastic plate of inner radius a, outer radius b and thickness $2h$.

$$\begin{bmatrix} u_\theta \\ r\sigma_{\theta z} \end{bmatrix} = \begin{bmatrix} U(r) \\ S(r) \end{bmatrix} \exp(\mu z)\,, \tag{24}$$

where μ is an undetermined parameter.

Substituting Eq. (24) in Eq. (14), after manipulation we arrive at

$$\mathbf{H}\boldsymbol{\psi} = \mu\boldsymbol{\psi}, \tag{25}$$

where

$$\mathbf{H} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{11}^* \end{bmatrix}, \quad \boldsymbol{\psi} = \begin{bmatrix} U \\ S \end{bmatrix},$$

$$L_{11} = 0\,,\ L_{11}^* = 0\,,\ L_{12} = c_{44}^{-1} r^{-1}\,,$$
$$L_{21} = -c_{66}(d_r + r^{-1})(rd_r - 1)\,;$$

d_r denotes ordinary differentiation with respect to r.

We shall show in the subsequent section that Eq. (25) under the traction-free boundary conditions at $r = a$, b is a Hamiltonian eigensystem. At this point, we carry on solving Eq. (25) by letting

$$\boldsymbol{\psi}_1 = [c_1 \quad c_2]^T J_1(\lambda r)\,,\ \boldsymbol{\psi}_2 = [d_1 \quad d_2]^T Y_1(\lambda r)\,, \tag{26}$$

then Eq. (25) reduces to

$$\begin{bmatrix} -\mu & c_{44}^{-1} \\ c_{66}\lambda^2 & -\mu \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -\mu & c_{44}^{-1} \\ c_{66}\lambda^2 & -\mu \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{27}$$

which have non-trivial solutions if and only if the determinants of the coefficient matrices equal to zero. This condition yields

$$c_{66}\kappa^2 - c_{44} = 0\,, \quad (\kappa = \lambda / \mu) \tag{28}$$

which has the roots

$$\kappa_1 = -\kappa_2 = \kappa = \sqrt{c_{66}^{-1} c_{44}}\,. \tag{29}$$

To each κ_j there corresponds a solution of Eq. (27) determined within a constant. As a result,

$$\begin{bmatrix} u_\theta \\ r\sigma_{\theta z} \end{bmatrix} = \begin{bmatrix} AJ_1(\kappa\mu r) + BY_1(\kappa\mu r) \\ Ac_{44}\mu r J_1(\kappa\mu r) + Bc_{44}\mu r Y_1(\kappa\mu r) \end{bmatrix} \exp(\mu z)\,, \tag{30}$$

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$$r\sigma_{r\theta} = -c_{66}\kappa\mu r\left[AJ_2(\kappa\mu r) + BY_2(\kappa\mu r)\right]\exp(\mu z), \quad (31)$$

where A and B are undetermined constants.

For isotropy we have $c_{44} = (c_{11} - c_{12})/2$, Eq. (29) yields $\kappa = 1$.

Another solution can be obtained by letting

$$\begin{bmatrix} u_\theta \\ r\sigma_{\theta z} \end{bmatrix} = \begin{bmatrix} U(r) \\ S(r) \end{bmatrix} \exp(-\mu z). \quad (32)$$

As a result,

$$\begin{bmatrix} u_\theta \\ r\sigma_{\theta z} \end{bmatrix} = \begin{bmatrix} AJ_1(\kappa\mu r) + BY_1(\kappa\mu r) \\ -Ac_{44}\mu r J_1(\kappa\mu r) - Bc_{44}\mu r Y_1(\kappa\mu r) \end{bmatrix} \exp(-\mu z), \quad (33)$$

$$r\sigma_{r\theta} = -c_{66}\kappa\mu r\left[AJ_2(\kappa\mu r) + BY_2(\kappa\mu r)\right]\exp(-\mu z), \quad (34)$$

5.2 Circular Plate

Guided by our experience with the problems in the cylindrical coordinate system [15], we seek the solution to Eq. (16) in the form

$$\begin{bmatrix} u_\theta \\ r\sigma_{r\theta} \end{bmatrix} = \begin{bmatrix} U(z)\, I_1(\lambda r) \\ S(z)\, rI_2(\lambda r) \end{bmatrix}, \quad (35)$$

where $I_n(\lambda r)$ is the modified Bessel functions of the first kind of order n, which is a monotonically decaying function, λ is an undetermined parameter.

Substituting Eq. (35) in Eq. (16), after manipulation using the derivative formulas of Bessel functions [17], we arrive at

$$\mathbf{H}\boldsymbol{\psi} = \lambda\boldsymbol{\psi}, \quad (36)$$

where

$$\mathbf{H} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{11}^* \end{bmatrix}, \quad \boldsymbol{\psi} = \begin{bmatrix} U \\ r^{-1}S \end{bmatrix},$$

$$L_{11} = 0, \; L_{11}^* = 0, \; L_{12} = c_{66}^{-1}, \; L_{21} = -c_{44}\frac{d^2}{dz^2}.$$

We shall show in the subsequent section that Eq. (36) under the traction-free boundary conditions at $z = \pm h$ is a Hamiltonian eigensystem. At this point, we carry on solving Eq. (36) by letting

$$\boldsymbol{\psi}_1 = [c_1 \;\; c_2]^T \cos\mu z, \; \boldsymbol{\psi}_2 = [d_1 \;\; d_2]^T \sin\mu z, \quad (37)$$

then Eq. (36) reduces to

$$\begin{bmatrix} \lambda & -c_{66}^{-1} \\ -c_{44}\mu^2 & \lambda \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \lambda & -c_{66}^{-1} \\ -c_{44}\mu^2 & \lambda \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (38)$$

which have non-trivial solutions if and only if the determinants of the coefficient matrices equal to zero. This condition yields

$$c_{44}\kappa^2 - c_{66} = 0, \quad (\kappa = \mu/\lambda) \quad (39)$$

which has the roots

$$\kappa_1 = -\kappa_2 = \kappa = \sqrt{c_{44}^{-1}c_{66}}. \quad (40)$$

To each κ_j there corresponds a solution of Eq. (38) determined within a constant. As a result,

$$\begin{bmatrix} u_\theta \\ r\sigma_{r\theta} \end{bmatrix} = \begin{bmatrix} \left[A\cos(\kappa\lambda z) + B\sin(\kappa\lambda z)\right]I_1(\lambda r) \\ c_{66}\lambda\left[A\cos(\kappa\lambda z) + B\sin(\kappa\lambda z)\right]\, rI_2(\lambda r) \end{bmatrix}, \quad (41)$$

$$r\sigma_{\theta z} = c_{44}\kappa\lambda\left[-A\sin(\kappa\lambda z) + B\cos(\kappa\lambda z)\right]\, rI_1(\lambda r), \quad (42)$$

where A and B are undetermined constants.

For isotropy we have $c_{44} = (c_{11} - c_{12})/2$, Eq. (40) yields $\kappa = 1$.

Another solution can be obtained by letting

$$\begin{bmatrix} u_\theta \\ r\sigma_{r\theta} \end{bmatrix} = \begin{bmatrix} U(z)K_1(\lambda r) \\ S(z)rK_2(\lambda r) \end{bmatrix}, \quad (43)$$

where $K_n(\lambda r)$ is the modified Bessel functions of the second kind of order n, which is also a monotonically decaying function.

As a result,

$$\begin{bmatrix} u_\theta \\ r\sigma_{r\theta} \end{bmatrix} = \begin{bmatrix} \left[A\cos(\kappa\lambda z) + B\sin(\kappa\lambda z)\right]K_1(\lambda r) \\ -c_{66}\lambda\left[A\cos(\kappa\lambda z) + B\sin(\kappa\lambda z)\right]rK_2(\lambda r) \end{bmatrix}, \quad (44)$$

$$r\sigma_{\theta z} = c_{44}\kappa\lambda\left[-A\sin(\kappa\lambda z) + B\cos(\kappa\lambda z)\right]rK_1(\lambda r). \quad (45)$$

6. EIGENSOLUTION

Now we have to determine the eigensolutions that satisfy the homogeneous BC in the radial direction for tubes and in the axial direction for plates, respectively.

6.1 Cylindrical Tube

At this stage, we need to derive an eigensolution that satisfies the traction-free boundary conditions at $r = a$, b. To this end, imposing the traction-free BC: $\sigma_{r\theta} = 0$ at $r = a$, b on the relevant stress component in Eq. (31) leads to

$$\begin{bmatrix} \mu J_2(\kappa\mu a) & \mu Y_2(\kappa\mu a) \\ \mu J_2(\kappa\mu b) & \mu Y_2(\kappa\mu b) \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (46)$$

which has non-trivial solutions if the determinant of the coefficient matrices equal to zero:

$$\mu^2\left[J_2(\kappa\mu a)Y_2(\kappa\mu b) - J_2(\kappa\mu b)Y_2(\kappa\mu a)\right] = 0,$$
$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} Y_2(\kappa\mu b) \\ -J_2(\kappa\mu b) \end{bmatrix}. \quad (47)$$

Equation (47) allows us to determine the eigenvalue μ_n, subsequently the associated eigenvectors that satisfy the state equation and the traction-free boundary conditions

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on the inner and outer cylindrical surfaces $r = a, b$. By inspection, Eq. (47) is satisfied if μ_n is replaced by $-\mu_n$. So, if μ_n is an eigenvalue, so is $-\mu_n$. Moreover, $\mu_0 = 0$ is a common factor of power 2 in Eq. (47). Thus, $\mu_0 = 0$ is a repeated eigenvalue. The result is in accord with the symplectic characteristics, but hardly derivable from a conventional approach.

Since $\mu_0 = 0$ is a repeated root of Eq. (63), linearly independent eigenvectors associated with $\mu_0 = 0$ can be determined by considering the Jordan-chain solution given as

$$\begin{cases} \boldsymbol{\psi}_{00} = \boldsymbol{\psi}_0^{(0)}, \\ \boldsymbol{\psi}_{01} = \boldsymbol{\psi}_0^{(0)} z + \boldsymbol{\psi}_0^{(1)}, \end{cases} \tag{48}$$

where $\boldsymbol{\psi}_0^{(0)}$ and $\boldsymbol{\psi}_0^{(1)}$ are determined from the Jordan chain:

$$\begin{cases} \mathbf{H}\boldsymbol{\psi}_0^{(0)} = \mathbf{0}, \\ \mathbf{H}\boldsymbol{\psi}_0^{(1)} = \boldsymbol{\psi}_0^{(0)}, \end{cases} \tag{49}$$

with the traction-free BC at $r = a, b$ and we arrive at

$$\boldsymbol{\psi}_0^{(0)} = \begin{bmatrix} r \\ 0 \end{bmatrix}, \quad \boldsymbol{\psi}_0^{(1)} = \begin{bmatrix} 0 \\ c_{44} r^2 \end{bmatrix}, \tag{50}$$

and

$$\boldsymbol{\psi}_{00} = \begin{bmatrix} r \\ 0 \end{bmatrix}, \quad \boldsymbol{\psi}_{01} = \begin{bmatrix} r \\ 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ c_{44} r^2 \end{bmatrix}. \tag{51}$$

6.2 Circular Plate

At this stage, we need to derive an eigensolution that satisfies the traction-free boundary conditions at $z = \pm h$. To this end, imposing the traction-free BC: $\sigma_{\theta z} = 0$ at $z = \pm h$ on the relevant stress component in Eq. (42) leads to

$$\begin{bmatrix} -\lambda \sin(\kappa\lambda h) & \lambda \cos(\kappa\lambda h) \\ \lambda \sin(\kappa\lambda h) & \lambda \cos(\kappa\lambda h) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{52}$$

which has non-trivial solutions if the determinant of the coefficient matrices equal to zero:

$$2\lambda^2 \sin(2\kappa\lambda h) = 0, \quad \lambda h = \frac{n\pi}{2\kappa}, \quad \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \cos(\kappa\lambda h) \\ \sin(\kappa\lambda h) \end{bmatrix}. \tag{53}$$

Equation (53) allows us to determine the eigenvalue λ_n, subsequently the associated eigenvectors that satisfy the state equation and the traction-free boundary conditions on the top and bottom surfaces $z = \pm h$. $\lambda_0 = 0$ is a common factor of power 2 in Eq. (53). Thus, $\lambda_0 = 0$ is a repeated eigenvalue. The result is in accord with the symplectic characteristics, but hardly derivable from a conventional approach.

Determination of the linearly independent eigenvectors associated with the repeated eigenvalue $\lambda_0 = 0$ calls for special treatment. Knowing that the Bessel function $I_p(\lambda r)$ in (35) approaches $(\lambda r)^p / (2^p p!)$ as $\lambda r \to 0$ and $K_p(\lambda r)$ in (43) approaches $2^{p-1}(p-1)!(\lambda r)^{-p}$ as $\lambda r \to 0$ [17], we seek the eigensolution for $\lambda_0 = 0$ in the form

$$\boldsymbol{\psi}_0(r, z) = \sum_{k=0}^{\infty} \mathbf{Z}^{(k)} r^{\alpha - k}, \quad \mathbf{Z}^{(k)} = \begin{bmatrix} U_k(z) & r^{-1} S_k(z) \end{bmatrix}^T, \tag{54}$$

where α is an undetermined parameter, $Z^{(k)}$ are unknown functions of z.

Substituting Eq. (54) in Eq. (16), we obtain

$$\sum_{k=0}^{\infty} \left\{ \mathbf{H}\mathbf{Z}^{(k)} + \left[\mathbf{H}_1 - (\alpha - k + 1)\mathbf{I} \right] \mathbf{Z}^{(k-1)} \right\} r^{\alpha - k} = \mathbf{0}, \tag{55}$$

with the traction-free boundary conditions at $z = \pm h$:

$$\sum_{k=0}^{\infty} \left[\mathbf{B}\mathbf{Z}^{(k)} \right]_{z=\pm h} r^{\alpha - k} = \mathbf{0}, \tag{56}$$

where $Z^{(-1)} = 0$, and

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} c_{44} r \partial_z & 0 \end{bmatrix}. \tag{57}$$

There follow

$$\mathbf{H}\mathbf{Z}^{(0)} = \mathbf{0}, \quad \left[\mathbf{B}_1 \mathbf{Z}^{(0)} \right]_{z=\pm h} = \mathbf{0}, \tag{58}$$

$$\mathbf{H}\mathbf{Z}^{(1)} = -(\mathbf{H}_1 - \alpha\mathbf{I})\mathbf{Z}^{(0)}, \quad \left[\mathbf{B}\mathbf{Z}^{(1)} \right]_{z=\pm h} = \mathbf{0}, \tag{59}$$

$$\mathbf{H}\mathbf{Z}^{(k)} = -\left[\mathbf{H}_1 - (\alpha - k + 1)\mathbf{I} \right] \mathbf{Z}^{(k-1)}, \quad \left[\mathbf{B}\mathbf{Z}^{(k)} \right]_{z=\pm h} = \mathbf{0}, \quad k \geq 2. \tag{60}$$

Let $\alpha = 1$ and -1, respectively, solving Eqs. (58) ~ (60) successively, and substituting the results in (54) yields the eigensolutions for $\lambda_0 = 0$:

$$\boldsymbol{\psi}_{00} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} r \quad \text{for } \alpha = 1, \tag{61}$$

$$\boldsymbol{\psi}_{01} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} r^{-1} + \begin{bmatrix} 0 \\ -2c_{66} \end{bmatrix} r^{-2} \quad \text{for } \alpha = -1. \tag{62}$$

7. SYMPLECTIC CHARACTERISTICS

Having determined the eigenvector that satisfies the state equation, we have to delineate the symplectic characteristics in the radial direction for tubes and in the axial direction for plates, respectively.

7.1 Cylindrical Tube

To facilitate exposition, let us express the eigensystem given by Eq. (25) under the traction-free boundary conditions at $r = a, b$ as

$$\mathbf{H}\boldsymbol{\psi} = \mu\boldsymbol{\psi}, \quad \mathbf{B}(\boldsymbol{\psi}) = \mathbf{0}, \tag{63}$$

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where μ is the eigenvalue, $\boldsymbol{\psi}$ is the associated eigenvector and $\mathbf{B} = [c_{66}(rd_r - 1)\ 0]_{r=a,b}$. The system matrix $\mathbf{H}$ exhibits the feature which is essential for the symplectic characteristics delineated as follows.

To bring out the symplectic orthogonality of the eigenvectors, it is necessary to arrange them as follows:

1. the eigenvectors $\boldsymbol{\psi}_k$ ($k = 1,2,\cdots$) associated with the eigenvalues μ_k.
2. the eigenvectors $\boldsymbol{\psi}_{-k} \equiv \boldsymbol{\phi}$ ($i = 1, 2$; $k = 1,2,\cdots$) associated with the eigenvalues μ_{-k} ($= -\mu_k$).
3. the eigenvectors $\boldsymbol{\psi}_0$ associated with the repeated eigenvalue $\mu_0 = 0$.

Consider two linearly independent eigenvectors $\boldsymbol{\psi}_p$ and $\boldsymbol{\psi}_q$ associated with the eigenvalues μ_p and μ_q, respectively,

$$\mathbf{H}\boldsymbol{\psi}_p = \mu_p \boldsymbol{\psi}_p\ ,\quad \mathbf{B}(\boldsymbol{\psi}_p) = \mathbf{0}\ , \tag{64}$$

$$\mathbf{H}\boldsymbol{\psi}_q = \mu_q \boldsymbol{\psi}_q\ ,\quad \mathbf{B}(\boldsymbol{\psi}_q) = \mathbf{0}\ . \tag{65}$$

Pre-multiplying Eq. (64) by $\boldsymbol{\psi}_q^T\mathbf{J}$ and Eq. (65) by $\boldsymbol{\psi}_p^T\mathbf{J}$, integrating the resulting equations with respect to r over (a, b), we obtain

$$\int_a^b \boldsymbol{\psi}_q^T \mathbf{J}\mathbf{H}\boldsymbol{\psi}_p\, dr = \mu_p \int_a^b \boldsymbol{\psi}_q^T \mathbf{J}\boldsymbol{\psi}_p\, dr, \tag{66}$$

$$\int_a^b \boldsymbol{\psi}_p^T \mathbf{J}\mathbf{H}\boldsymbol{\psi}_q dr = \mu_q \int_a^b \boldsymbol{\psi}_p^T \mathbf{J}\boldsymbol{\psi}_q\, dr\ , \tag{67}$$

where

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Substracting Eqs. (66) and (67) gives

$$(\mu_p + \mu_q)\int_a^b \boldsymbol{\psi}_p^T \mathbf{J}\boldsymbol{\psi}_q\, dr = 0\ . \tag{68}$$

Since $\mu_p + \mu_q \neq 0$ ($p \neq -q$) according to the sequence of the eigenvectors defined before, there follows the orthogonal relation for the eigenvectors associated with μ_p, $\mu_q \neq 0$ ($p, q = 1, 2,\cdots$):

$$\int_a^b \boldsymbol{\psi}_p^T \mathbf{J}\boldsymbol{\psi}_q\, dr \begin{cases} = 0 & \text{for}\quad q \neq -p \\ \neq 0 & \text{for}\quad q = -p \end{cases}. \tag{69}$$

The case of the zero eigenvalue needs special consideration. It has been shown that $\mu_0 = 0$ is a repeated eigenvalue. The $\boldsymbol{\psi}_{00}$ and $\boldsymbol{\psi}_{01}$ containing two linearly independent eigenvectors $\boldsymbol{\psi}_0^{(0)}$ and $\boldsymbol{\psi}_0^{(1)}$ have been determined from the Jordan chain associated with $\mu_0 = 0$ in Eq. (51), and the orthogonal relations of $\boldsymbol{\psi}_{00}$ and $\boldsymbol{\psi}_{01}$ associated with $\mu_0 = 0$ are

$$\int_a^b \boldsymbol{\psi}_{0\alpha}^T \mathbf{J}\boldsymbol{\psi}_{0\beta}\, dr \begin{cases} = 0 & \text{for}\quad \alpha = \beta \\ \neq 0 & \text{for}\quad \alpha \neq \beta \end{cases} (\alpha,\ \beta = 0,\ 1)\ . \tag{70}$$

Since $\mu_i + \mu_0 \neq 0$ ($i \neq 0$), there follows the orthogonal relation for the eigenvectors associated with μ_i ($i \neq 0$), μ_0: $\int_a^b \boldsymbol{\psi}_i^T \mathbf{J}\boldsymbol{\psi}_0^{(\alpha)} dr = 0$, $(\alpha = 0,\ 1)$.

7.2 Circular Plate

To facilitate exposition, let us express the eigensystem given by Eq. (36) under the traction-free boundary conditions at $z = \pm h$ as

$$\mathbf{H}\boldsymbol{\psi} = \lambda\boldsymbol{\psi}\ ,\quad \mathbf{B}(\boldsymbol{\psi}) = \mathbf{0}, \tag{71}$$

where λ is the eigenvalue, $\boldsymbol{\psi}$ is the associated eigenvector and $\mathbf{B} = [c_{44}rd_z\ 0]_{z=\pm h}$ in which d_z denotes ordinary differentiation with respect to z. The system matrix $\mathbf{H}$ exhibits the feature which is essential for the symplectic characteristics delineated as follows.

To bring out the symplectic orthogonality of the eigenvectors, it is necessary to arrange them as follows:

1. the eigenvectors $\boldsymbol{\psi}_k$ ($k = 1,2,\cdots$) associated with the eigenvalues λ_k.
2. the eigenvectors $\boldsymbol{\psi}_0$ associated with the repeated eigenvalue $\lambda_0 = 0$.

Consider two linearly independent eigenvectors $\boldsymbol{\psi}_p$ and $\boldsymbol{\psi}_q$ associated with the eigenvalues λ_p and λ_q, respectively,

$$\mathbf{H}\boldsymbol{\psi}_p = \lambda_p \boldsymbol{\psi}_p\ ,\quad \mathbf{B}(\boldsymbol{\psi}_p) = \mathbf{0}, \tag{72}$$

$$\mathbf{H}\boldsymbol{\psi}_q = \lambda_q \boldsymbol{\psi}_q\ ,\quad \mathbf{B}(\boldsymbol{\psi}_q) = \mathbf{0}\ . \tag{73}$$

Pre-multiplying Eq. (72) by $\boldsymbol{\psi}_q^T\mathbf{J}$ and Eq. (73) by $\boldsymbol{\psi}_p^T\mathbf{J}$, integrating the resulting equations with respect to z over $(-h, h)$, we obtain

$$\int_{-h}^{h} \boldsymbol{\psi}_q^T \mathbf{J}\mathbf{H}\boldsymbol{\psi}_p\, dz = \lambda_p \int_{-h}^{h} \boldsymbol{\psi}_q^T \mathbf{J}\boldsymbol{\psi}_p\, dz\ , \tag{74}$$

$$\int_{-h}^{h} \boldsymbol{\psi}_p^T \mathbf{J}\mathbf{H}\boldsymbol{\psi}_q\, dz = \lambda_q \int_{-h}^{h} \boldsymbol{\psi}_p^T \mathbf{J}\boldsymbol{\psi}_q\, dz\ . \tag{75}$$

Substracting Eqs. (74) and (75) gives

$$(\lambda_p + \lambda_q)\int_{-h}^{h} \boldsymbol{\psi}_p^T\, \mathbf{J}\boldsymbol{\psi}_q\, dz = 0\ . \tag{76}$$

Since $\lambda_p + \lambda_q \neq 0$ ($p \neq -q$) according to the sequence of the eigenvectors defined before, there follows the orthogonal relation for the eigenvectors associated with λ_p, $\lambda_q \neq 0$ ($p, q = 1, 2,\cdots$):

$$\int_{-h}^{h} \boldsymbol{\psi}_p^T \mathbf{J}\boldsymbol{\psi}_q dz \begin{cases} = 0 & \text{for}\quad q \neq -p \\ \neq 0 & \text{for}\quad q = -p \end{cases} \tag{77}$$

The case of the zero eigenvalue needs special consideration. It has been shown that $\lambda_0 = 0$ is a repeated eigenvalue. The $\boldsymbol{\psi}_{00}$ and $\boldsymbol{\psi}_{01}$ containing two linearly independent eigenvectors have been determined from the Jordan chain associated with $\lambda_0 = 0$ in Eqs. (61) ~ (62), and the orthogonal relations of the eigenvectors associated with $\lambda_0 = 0$ are

$$\int_{-h}^{h} \boldsymbol{\psi}_{0\alpha}^T \mathbf{J}\boldsymbol{\psi}_{0\beta}\, dz \begin{cases} = 0 & \text{for}\quad \alpha = \beta \\ \neq 0 & \text{for}\quad \alpha \neq \beta \end{cases} (\alpha, \beta = 0,\ 1) \tag{78}$$

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Since $\lambda_i + \lambda_0 \neq 0$ ($i \neq 0$), there follows the orthogonal relation for the eigenvectors associated with λ_i ($i \neq 0$), $\mu\lambda_0$: $\int_{-h}^{h} \boldsymbol{\psi}_i^T \mathbf{J} \boldsymbol{\psi}_0^{(\alpha)} dz = 0$, $(\alpha = 0,1)$.

8. SYMPLECTIC EIGENFUNCTION EXPANSION

After the combination of eigensolutions, we have to determine the undetermined constants in them by satisfying the end conditions in the axial direction for tubes and in the radial direction for plates, respectively.

8.1 Cylindrical Tube Fixed at $z = 0$ and Subjected to Distributed Load at $z = l$

A combination of the linearly independent eigenvectors in Eqs. (30), (33) and (51) yields the complete solution that satisfies the traction-free boundary conditions at $r = a$, b:

$$[u_\theta \quad r\sigma_{\theta z}]^T = A_{00}\boldsymbol{\psi}_{00} + A_{01}\boldsymbol{\psi}_{01} + \sum_{k=1}^{\infty}\left(A_k \mathbf{X}_k + A_{-k}\mathbf{X}_{-k}\right), \tag{79}$$

where

$$\mathbf{X}_k = \boldsymbol{\psi}_k(r)\exp(\mu_k z), \quad \mathbf{X}_{-k} = \boldsymbol{\psi}_{-k}(r)\exp(-\mu_k z), \tag{80}$$

$$\boldsymbol{\psi}_k(r) = \begin{bmatrix} 1 \\ c_{44}\mu_k r \end{bmatrix}\left[Y_2(\kappa\mu_k b)J_1(\kappa\mu_k r) - J_2(\kappa\mu_k b)Y_1(\kappa\mu_k r)\right],$$

$$\boldsymbol{\psi}_{-k}(r) = \begin{bmatrix} 1 \\ -c_{44}\mu_k r \end{bmatrix}\left[Y_2(\kappa\mu_k b)J_1(\kappa\mu_k r) - J_2(\kappa\mu_k b)Y_1(\kappa\mu_k r)\right].$$

The constants of linear combination A_{00}, A_{01}, A_k and A_{-k} ($k = 1, 2, \cdots$) in Eq. (79) are to be determined by satisfying the end conditions, Eqs. (19) ~ (20).

Imposing the end conditions on Eq. (79), we obtain

$$\mathbf{F}_0(r) = A_{00}(\boldsymbol{\psi}_{00})_{z=0} + A_{01}(\boldsymbol{\psi}_{01})_{z=0} + \sum_{k=1}^{\infty}\left[A_k(\mathbf{X}_k)_{z=0} + A_{-k}(\mathbf{X}_{-k})_{z=0}\right], \tag{81}$$

$$\mathbf{F}_l(r) = A_{00}(\boldsymbol{\psi}_{00})_{z=l} + A_{01}(\boldsymbol{\psi}_{01})_{z=l} + \sum_{k=1}^{\infty}\left[A_k(\mathbf{X}_k)_{z=l} + A_{-k}(\mathbf{X}_{-k})_{z=l}\right], \tag{82}$$

where $\mathbf{F}_0(r)$ and $\mathbf{F}_l(r)$ contain the unknown stress at $z = 0$ and the unknown displacement at $z = l$ respectively, such that

$$\mathbf{F}_0(r) = [0 \quad r\sigma_{\theta z}(r, 0)]^T, \quad \mathbf{F}_l(r) = \left[u_\theta(r, l) \quad rp(r)\right]^T.$$

with the symplectic orthogonality relations, we are able to determine the unknown constants in Eqs. (81) ~ (82) by means of eigenfunction expansion. To this end, multiplying both sides of Eq. (81) by $(\boldsymbol{\psi}_{00})_{z=0}^T\mathbf{J}$, $(\boldsymbol{\psi}_{01})_{z=0}^T\mathbf{J}$, $(\mathbf{X}_k)_{z=0}^T\mathbf{J}$ and $(\mathbf{X}_{-k})_{z=0}^T\mathbf{J}$, respectively, and integrating them over (a, b), making use of the symplectic orthogonality, we obtain

$$A_{01} = \frac{\int_a^b (\boldsymbol{\psi}_{00})_{z=0}^T \mathbf{J}\mathbf{F}_0(r)\,dr}{\int_a^b (\boldsymbol{\psi}_{00})_{z=0}^T \mathbf{J}(\boldsymbol{\psi}_{01})_{z=0}\,dr}, \quad A_{00} = \frac{\int_a^b (\boldsymbol{\psi}_{01})_{z=0}^T \mathbf{J}\mathbf{F}_0(r)\,dr}{\int_a^b (\boldsymbol{\psi}_{01})_{z=0}^T \mathbf{J}(\boldsymbol{\psi}_{00})_{z=0}\,dr}, \tag{83}$$

$$A_{-k} = \frac{\int_a^b (\mathbf{X}_k)_{z=0}^T \mathbf{J}\mathbf{F}_0(r)\,dr}{\int_a^b (\mathbf{X}_k)_{z=0}^T \mathbf{J}(\mathbf{X}_{-k})_{z=0}\,dr}, \quad A_k = \frac{\int_a^b (\mathbf{X}_{-k})_{z=0}^T \mathbf{J}\mathbf{F}_0(r)\,dr}{\int_a^b (\mathbf{X}_{-k})_{z=0}^T \mathbf{J}(\mathbf{X}_k)_{z=0}\,dr}, \tag{84}$$

where the term $(\mathbf{X}_p)^T\mathbf{J}(\mathbf{X}_q)$ can be reduced to an expression containing $\boldsymbol{\psi}_p^T\mathbf{J}\boldsymbol{\psi}_q$ thus the symplectic orthogonality is applicable.

Likewise, there follow from Eq. (82)

$$A_{01} = \frac{\int_a^b (\boldsymbol{\psi}_{00})_{z=l}^T \mathbf{J}\mathbf{F}_l(r)\,dr}{\int_a^b (\boldsymbol{\psi}_{00})_{z=l}^T \mathbf{J}(\boldsymbol{\psi}_{01})_{z=l}\,dr}, \quad A_{00} = \frac{\int_a^b (\boldsymbol{\psi}_{01})_{z=l}^T \mathbf{J}\mathbf{F}_l(r)\,dr}{\int_a^b (\boldsymbol{\psi}_{01})_{z=l}^T \mathbf{J}(\boldsymbol{\psi}_{00})_{z=l}\,dr}, \tag{85}$$

$$A_{-k} = \frac{\int_a^b (\mathbf{X}_k)_{z=l}^T \mathbf{J}\mathbf{F}_l(r)\,dr}{\int_a^b (\mathbf{X}_k)_{z=l}^T \mathbf{J}(\mathbf{X}_{-k})_{z=l}\,dr}, \quad A_k = \frac{\int_a^b (\mathbf{X}_{-k})_{z=l}^T \mathbf{J}\mathbf{F}_l(r)\,dr}{\int_a^b (\mathbf{X}_{-k})_{z=l}^T \mathbf{J}(\mathbf{X}_k)_{z=l}\,dr}. \tag{86}$$

In Eqs. (83) ~ (86) the vectors $\mathbf{F}_0(r)$ and $\mathbf{F}_l(r)$ contain four components each, of which the traction $\sigma_{\theta z}(r, 0)$ is unknown at the fixed end $z = 0$, the displacement $u_\theta(r, l)$ is unknown at the free end $z = l$. Substituting Eqs. (83) ~ (84) in Eqs. (85) ~ (86) gives

$$\int_a^b (\boldsymbol{\psi}_{00})_{z=0}^T \mathbf{J}\mathbf{F}_0(r)\,dr = \int_a^b (\boldsymbol{\psi}_{00})_{z=l}^T \mathbf{J}\mathbf{F}_l(r)\,dr, \tag{87}$$

$$\int_a^b (\boldsymbol{\psi}_{01})_{z=0}^T \mathbf{J}\mathbf{F}_0(r)\,dr = \int_a^b (\boldsymbol{\psi}_{01})_{z=l}^T \mathbf{J}\mathbf{F}_l(r)\,dr, \tag{88}$$

$$\int_a^b (\mathbf{X}_k)_{z=0}^T \mathbf{J}\mathbf{F}_0(r)\,dr = \int_a^b (\mathbf{X}_k)_{z=l}^T \mathbf{J}\mathbf{F}_l(r)\,dr, \tag{89}$$

$$\int_a^b (\mathbf{X}_{-k})_{z=0}^T \mathbf{J}\mathbf{F}_0(r)\,dr = \int_a^b (\mathbf{X}_{-k})_{z=l}^T \mathbf{J}\mathbf{F}_l(r)\,dr. \tag{90}$$

Equations (87) ~ (90) constitute a system of integral equations, solvable by the method of collocation for solving the integral equations [18]. Note that the unknowns $\sigma_{\theta z}(r, 0)$ in $\mathbf{F}_0(r)$ and $u_\theta(r, l)$ in $\mathbf{F}_l(r)$ are related to the constants A_{00}, A_{01}, A_k and A_{-k} by Eq. (79). Thus, substituting the relevant component equations of Eq. (79) in Eqs. (87) ~ (90), taking $2N$ terms of the series for computation, we arrive at a system of $2N + 2$ algebraic equations (2 equations from Eqs. (87) ~ (88); $2N$ equations from Eqs. (89) ~ (90) for $k = 1, 2, \cdots 2N$) for the $2N + 2$ unknowns (A_{00}, A_{01}, A_k and A_{-k} ($k = 1, 2, \cdots$)),

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which can be easily solved by using a standard method for simultaneous linear algebraic equations.

Since this problem are rather simple, its closed-form solution can be derived as in appendix B.

8.2 Circular Plate Clamped at $r = a$ and Subjected to Distributed Load at $r = b$

A combination of the linearly independent eigenvectors in Eqs. (41), (44) and (61) ~ (62) yields the complete solution that satisfies the traction-free boundary conditions at $z = \pm h$:

$$[u_\theta \quad \sigma_{r\theta}]^T = A_{00}\boldsymbol{\psi}_{00} + A_{01}\boldsymbol{\psi}_{01} + \sum_{k=1}^{\infty}(A_k\mathbf{X}_k + A_{-k}\mathbf{X}_{-k}) \ , \quad (91)$$

where

$$\mathbf{X}_k = \mathbf{K}_k(r)\boldsymbol{\psi}_k(z) \ , \quad \mathbf{X}_{-k} = \mathbf{K}_{-k}(r)\boldsymbol{\psi}_{-k}(z) \ , \quad (92)$$

$$\mathbf{K}_k(r) = \begin{bmatrix} I_1(\lambda_k r) & 0 \\ 0 & I_2(\lambda_k r) \end{bmatrix},$$

$$\mathbf{K}_{-k}(r) = \begin{bmatrix} K_1(\lambda_k r) & 0 \\ 0 & K_2(\lambda_k r) \end{bmatrix},$$

$$\boldsymbol{\psi}_k(z) = \begin{bmatrix} 1 \\ c_{66}\lambda_k \end{bmatrix} \cos\left[\kappa\lambda_k(h-z)\right],$$

$$\boldsymbol{\psi}_{-k}(z) = \begin{bmatrix} 1 \\ -c_{66}\lambda_k \end{bmatrix} \cos\left[\kappa\lambda_k(h-z)\right].$$

The constants of linear combination A_{00}, A_{01}, A_k and A_{-k} ($k = 1, 2, \cdots$) in Eq. (91) are to be determined by satisfying the edge boundary conditions, Eqs. (22) ~ (23).

Imposing the edge boundary conditions on Eq. (91), we obtain

$$\mathbf{F}_a(z) = A_{00}(\boldsymbol{\psi}_{00})_{r=a} + A_{01}(\boldsymbol{\psi}_{01})_{r=a} + \sum_{k=1}^{\infty}\left[A_k(\mathbf{X}_k)_{r=a} + A_{-k}(\mathbf{X}_{-k})_{r=a}\right], \quad (93)$$

$$\mathbf{F}_b(z) = A_{00}(\boldsymbol{\psi}_{00})_{r=b} + A_{01}(\boldsymbol{\psi}_{01})_{r=b} + \sum_{k=1}^{\infty}\left[A_k(\mathbf{X}_k)_{r=b} + A_{-k}(\mathbf{X}_{-k})_{r=b}\right], \quad (94)$$

where $\mathbf{F}_a(z)$ and $\mathbf{F}_b(z)$ contain the unknown stress at $r = a$ and the unknown displacement at $r = b$ respectively, such that

$$\mathbf{F}_a(z) = [0 \quad \sigma_{r\theta}(a,z)]^T, \quad \mathbf{F}_b(z) = [u_\theta(b,z) \quad p(z)]^T.$$

with the symplectic orthogonality relations, we are able to determine the unknown constants in Eqs. (93) ~ (94) by means of eigenfunction expansion. To this end, multiplying both sides of Eq. (93) by $(\boldsymbol{\psi}_{00})_{z=0}^T\mathbf{J}$, $(\boldsymbol{\psi}_{01})_{z=0}^T\mathbf{J}$, $(\mathbf{X}_k)_{z=0}^T\mathbf{J}$ and $(\mathbf{X}_{-k})_{z=0}^T\mathbf{J}$, respectively, and integrating them over $(-h, h)$, making use of the symplectic orthogonality, we obtain

$$A_{01} = \frac{\int_{-h}^{h}(\boldsymbol{\psi}_{00})_{r=a}^T\mathbf{J}\mathbf{F}_a(z)\,dz}{\int_{-h}^{h}(\boldsymbol{\psi}_{00})_{r=a}^T\mathbf{J}(\boldsymbol{\psi}_{01})_{r=a}dz}, \quad A_{00} = \frac{\int_{-h}^{h}(\boldsymbol{\psi}_{01})_{r=a}^T\mathbf{J}\mathbf{F}_a(z)\,dz}{\int_{-h}^{h}(\boldsymbol{\psi}_{01})_{r=a}^T\mathbf{J}(\boldsymbol{\psi}_{00})_{r=a}dz}, \quad (95)$$

$$A_{-k} = \frac{\int_{-h}^{h}(\mathbf{X}_k)_{r=a}^T\mathbf{J}\mathbf{F}_a(z)\,dz}{\int_{-h}^{h}(\mathbf{X}_k)_{r=a}^T\mathbf{J}(\mathbf{X}_{-k})_{r=a}dz}, \quad A_k = \frac{\int_{-h}^{h}(\mathbf{X}_{-k})_{r=a}^T\mathbf{J}\mathbf{F}_a(z)\,dz}{\int_{-h}^{h}(\mathbf{X}_{-k})_{r=a}^T\mathbf{J}(\mathbf{X}_k)_{r=a}dz}, \quad (96)$$

where the terms containing $\int_{-h}^{h}\mathbf{X}_p^T\mathbf{J}\mathbf{X}_{\pm q}\,dz$ can be reduced by using the symplectic orthogonality, as derived in appendix A.

Likewise, there follow from Eq. (94)

$$A_{01} = \frac{\int_{-h}^{h}(\boldsymbol{\psi}_{00})_{r=b}^T\mathbf{J}\mathbf{F}_b(z)\,dz}{\int_{-h}^{h}(\boldsymbol{\psi}_{00})_{r=b}^T\mathbf{J}(\boldsymbol{\psi}_{01})_{r=b}dz}, \quad A_{00} = \frac{\int_{-h}^{h}(\boldsymbol{\psi}_{01})_{r=b}^T\mathbf{J}\mathbf{F}_b(z)\,dz}{\int_{-h}^{h}(\boldsymbol{\psi}_{01})_{r=b}^T\mathbf{J}(\boldsymbol{\psi}_{00})_{r=b}dz}, \quad (97)$$

$$A_{-k} = \frac{\int_{-h}^{h}(\mathbf{X}_k)_{r=b}^T\mathbf{J}\mathbf{F}_b(z)\,dz}{\int_{-h}^{h}(\mathbf{X}_k)_{r=b}^T\mathbf{J}(\mathbf{X}_{-k})_{r=b}dz}, \quad A_k = \frac{\int_{-h}^{h}(\mathbf{X}_{-k})_{r=b}^T\mathbf{J}\mathbf{F}_b(z)\,dz}{\int_{-h}^{h}(\mathbf{X}_{-k})_{r=b}^T\mathbf{J}(\mathbf{X}_k)_{r=b}dz}. \quad (98)$$

In Eqs. (95) ~ (98) the vectors $\mathbf{F}_0(r)$ and $\mathbf{F}_l(r)$ contain four components each, of which the traction $\sigma_{r\theta}(a,z)$ is unknown at the fixed end $r = a$, the displacement $u_\theta(b,z)$ is unknown at the free end $r = b$. Substituting Eqs. (95) ~ (96) in Eqs. (97) ~ (98) gives

$$\int_{-h}^{h}(\boldsymbol{\psi}_{00})_{r=a}^T\mathbf{J}\mathbf{F}_a(z)\,dz = \int_{-h}^{h}(\boldsymbol{\psi}_{00})_{r=b}^T\mathbf{J}\mathbf{F}_b(z)\,dz \ , \quad (99)$$

$$\int_{-h}^{h}(\boldsymbol{\psi}_{01})_{r=a}^T\mathbf{J}\mathbf{F}_a(z)\,dz = \int_{-h}^{h}(\boldsymbol{\psi}_{01})_{r=b}^T\mathbf{J}\mathbf{F}_b(z)\,dz \ , \quad (100)$$

$$c_{kb}\int_{-h}^{h}(\mathbf{X}_k)_{r=a}^T\mathbf{J}\mathbf{F}_a(z)\,dz = c_{ka}\int_{-h}^{h}(\mathbf{X}_k)_{r=b}^T\mathbf{J}\mathbf{F}_b(z)\,dz \ , \quad (101)$$

$$c_{kb}\int_{-h}^{h}(\mathbf{X}_{-k})_{r=a}^T\mathbf{J}\mathbf{F}_a(z)\,dz = c_{ka}\int_{-h}^{h}(\mathbf{X}_{-k})_{r=b}^T\mathbf{J}\mathbf{F}_b(z)\,dz \ , \quad (102)$$

in which the two constants are

$$c_{kb} = \int_{-h}^{h}(\mathbf{X}_k)_{r=b}^T\mathbf{J}(\mathbf{X}_{-k})_{r=b}\,dz \ ,$$

$$c_{ka} = \int_{-h}^{h}(\mathbf{X}_k)_{r=a}^T\mathbf{J}(\mathbf{X}_{-k})_{r=a}\,dz \ .$$

Equations (99) ~ (102) constitute a system of integral equations, solvable by the method of collocation for solving the integral equations [18]. Note that the un-

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knowns $\sigma_{r\theta}(a,z)$ in $\mathbf{F}_a(z)$ and $u_\theta(b,z)$ in $\mathbf{F}_b(z)$ are related to the constants A_{00}, A_{01}, A_k and A_{-k} by Eq. (91). Thus, substituting the relevant component equations of Eq. (91) in Eqs. (99) ~ (102), taking N terms of the series for computation, we arrive at a system of $2N + 2$ algebraic equations (2 equations from Eqs. (99)-(100); $2N$ equations from Eqs. (101) ~ (102) for $k = 1, 2, \cdots 2N$) for the $2N + 2$ unknowns (A_{00}, A_{01}, A_k and A_{-k} ($k = 1, 2, \cdots$)), which can be easily solved by using a standard method for simultaneous linear algebraic equations.

Since this problem are rather simple, its closed-form solution can be derived as in appendix C.

9. RESULTS AND DISCUSSIONS

9.1 Cylindrical Tube Fixed at $z = 0$ and Subjected to Uniformly Distributed Load at $z = l$

To examine quantitatively the stress distributions, we take $c_{44}/c_{11} = c_{66}/c_{11} = 1/3$ for transversely isotropic materials. The ratios of outer radius to inner radius of the tube are taken to be $b/a = 2$. The ratios of length to inner radius of the tube are taken to be $l/a = 10$. The tube is subjected to uniform shear stress $p(r) = 3T/2\pi(b^3 - a^3)$, where T is the torque at the free end $z = l$ and $T = 2\pi\int_a^b \sigma_{\theta z} r^2 dr$. In evaluating the stress distributions, it is necessary to take up to 20 terms to achieve satisfactory convergence.

For $T/a^3 = 1$, Fig. 3 displays the contours of stress $\sigma_{\theta z}/c_{44}$ in the r-z plane. Remarkable variation exists in the region within a distance of the outer radius b ($0.8l \le z \le l$) near the free end. The disturbance diminishes rapidly; at a distance of b from the end the solution produces essentially the same results as the Saint-Venant solution [16]. As such, the end effect is confined to a local region near the free end.

Figure 4 shows the contours of stress $\sigma_{\theta z}/c_{44}$ in the cross-sections at the free end and at $z = 0.99l$, a very short distance away from the free end. The shear stress is uniformly distributed at $z = l$ and reduces rapidly at the section $z = 0.99l$. And in the cross sections at $z < 0.8l$, the distributions are linear, as shown in Fig. 5.

The contours of stress $\sigma_{r\theta}/c_{44}$ in the r-z plane are displayed in Fig. 6. The boundary conditions $\sigma_{r\theta} = 0$ at the cylindrical surfaces at $r = a$ and $r = b$ are satisfied as required. Significant differences in the stress from the one given by the Saint-Venant solution is observed near the free end of the tube as expected since the end condition is not considered by the Saint-Venant solution. As shown in Fig. 7, the discrepancy between two solutions diminishes rapidly at a very short distance away from the free end.

9.2 Circular Plate Clamped at $r = a$ and Subjected to Parabolically Distributed Load at $r = b$

To examine quantitatively the stress distributions, we take $c_{66}/c_{11} = c_{44}/c_{11} = 1/3$ for transversely isotropic materials. The ratios of inner radius to thickness of the plates are taken to be $a/h = 5$. The ratios of outer radius to inner radius of the tube are taken to be $b/a = 4$. The plate is subjected to parabolic shear stress $p(z) = 3T/4\pi b^2 h\ z^2$, where T is the torque at the free edge $r = b$ and $T = 2\pi b^2 \int_{-h}^{h} \sigma_{r\theta}\, dz$. In evaluating the stress distributions, it is necessary to take up to 20 terms to achieve satisfactory convergence.

For $T/h^3 = 1$, Fig. 8 displays the contours of stress $\sigma_{r\theta}/c_{66}$ in the r-z plane. Remarkable variation exists in the region within a distance of the thickness $2h$ ($3.6a \le r \le 4a$) near the free edge. The disturbance diminishes rapidly; at a distance of $2h$ from the edge the stress is distributed uniformly through the thickness. As such, the end effect is confined to a local region near the free edge.

Figure 9 shows the contours of stress $\sigma_{r\theta}/c_{66}$ in the cross-sections at the top surface $z = h$ and at the middle surface $z = 0$. At the outer rim $r = b$, the shear stress is parabolically distributed but reduces rapidly away from the rim. And at $r < 3.6a$, the distribution is uniform through the thickness.

The contours of stress $\sigma_{\theta z}/c_{66}$ in the r-z plane are displayed in Fig. 10. At the bottom and top surfaces $z = -h, h$, the boundary conditions $\sigma_{\theta z} = 0$ are satisfied as required. In the stress significant differences from zero is observed near the free edge of the plate as expected since the end condition has been considered. As shown in Fig. 11, the boundary conditions $\sigma_{\theta z} = 0$ at the top surface are satisfied and the stress is non-zero only near the rim at $r = b$ on the plane at $z = 0.5h$.

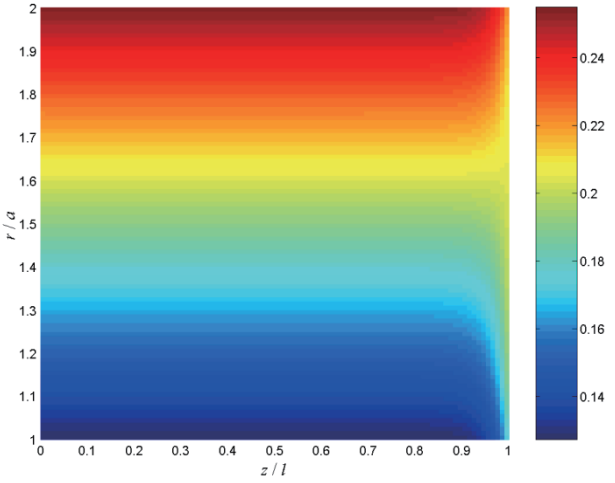


Fig. 3 Stress contours of $\sigma_{\theta z}/c_{44}$ of tube in the r-z plane.

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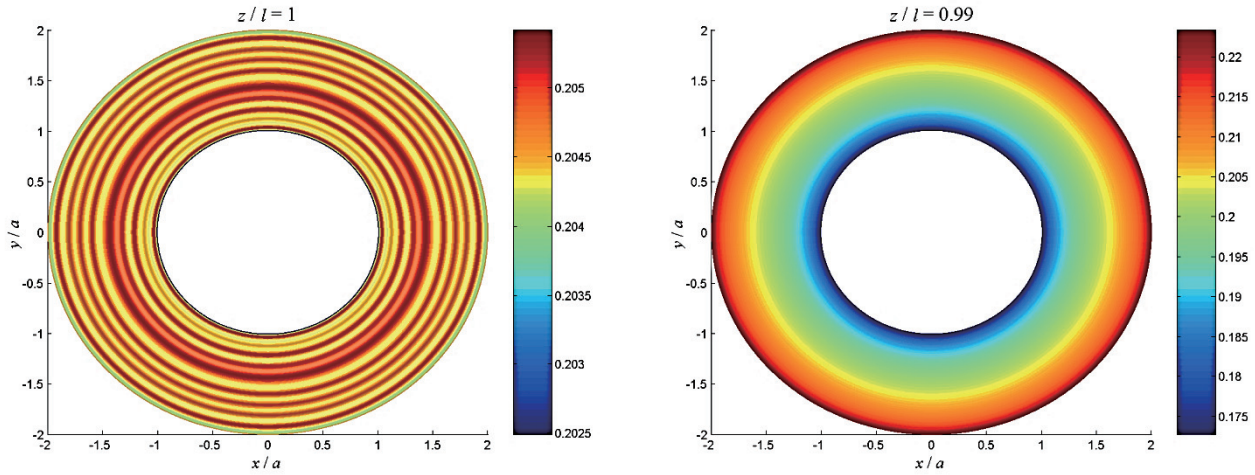


Fig. 4 Stress contours of $\sigma_{\theta z} / c_{44}$ of tube at $z/l = 1$ and at $z/l = 0.99$.

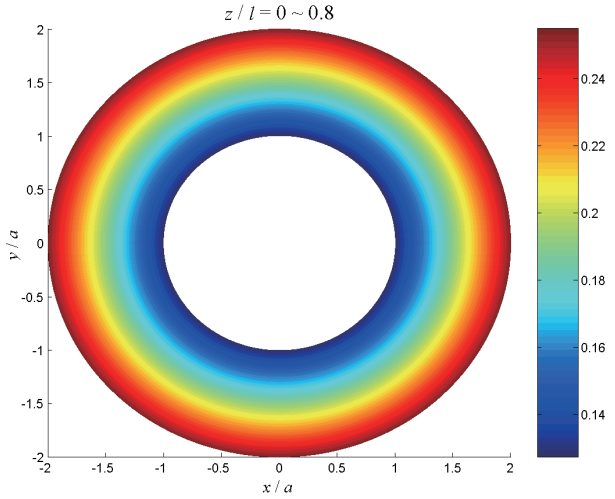


Fig. 5 Stress contours of $\sigma_{\theta z} / c_{44}$ of tube at $z/l = 0 \sim 0.8$.

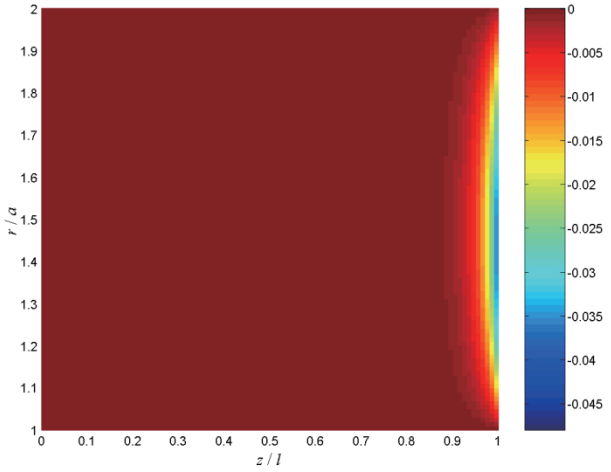


Fig. 6 Stress contours of $\sigma_{r\theta} / c_{44}$ of tube in the r-z plane.

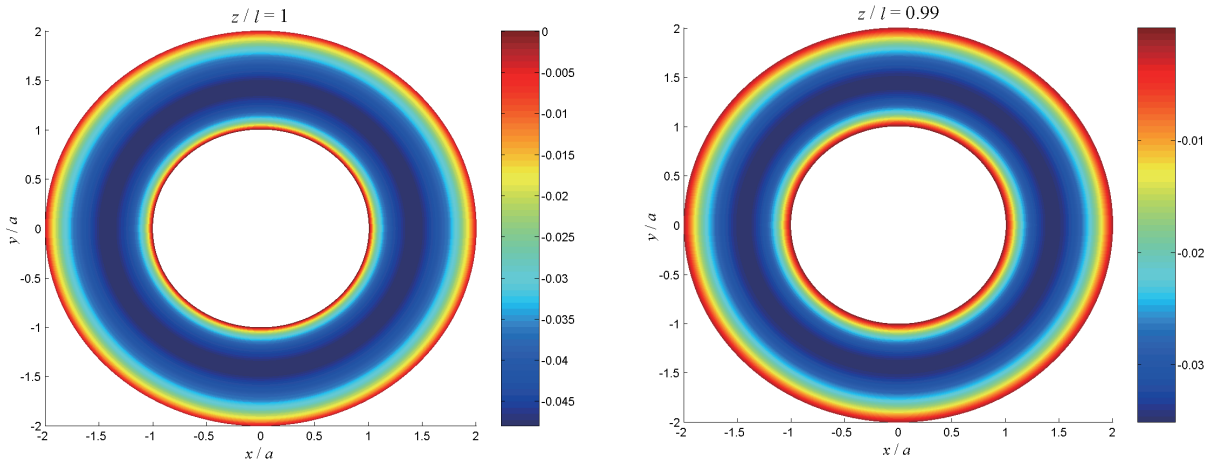


Fig. 7 Stress contours of $\sigma_{r\theta} / c_{44}$ of tube at $z/l = 1$ and at $z/l = 0.99$.

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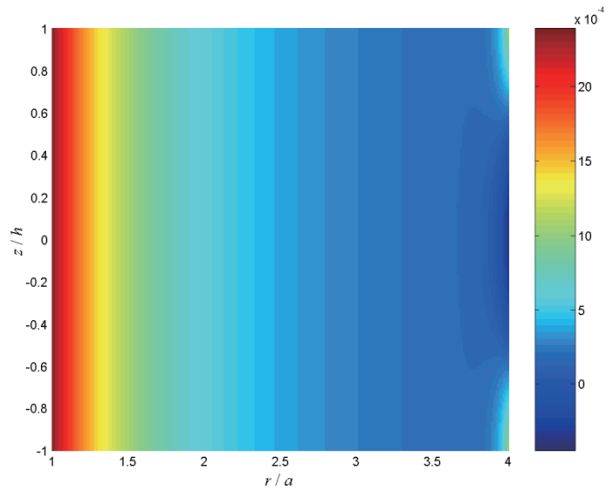


Fig. 8 Stress contours of $\sigma_{r\theta z} / c_{66}$ of plate in the *r*-*z* plane.

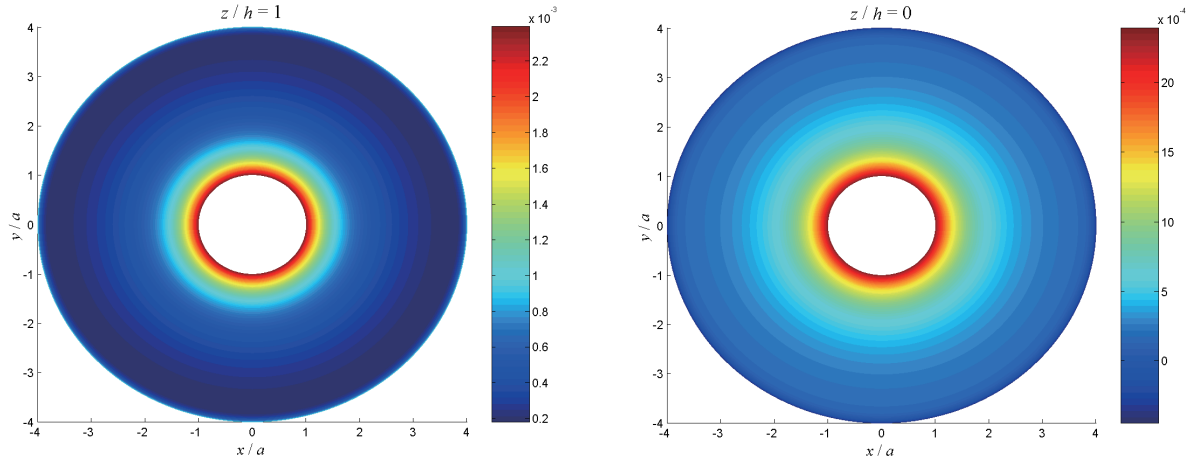


Fig. 9 Stress contours of $\sigma_{r\theta} / c_{66}$ of plate at $z/h = 1$ and at $z/h = 0$.

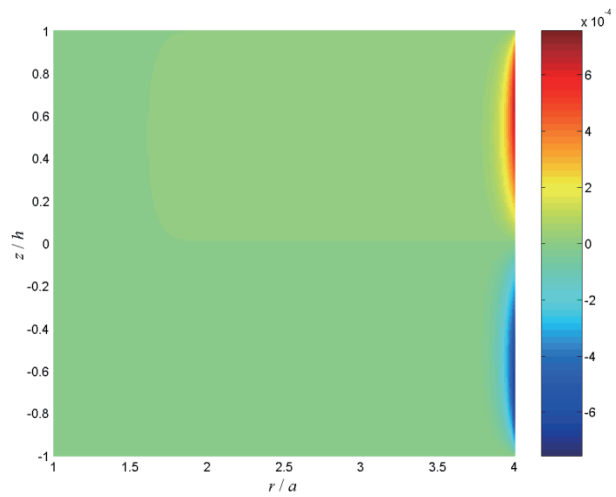


Fig. 10 Stress contours of $\sigma_{r\theta z} / c_{66}$ of plate in the *r*-*z* plane.

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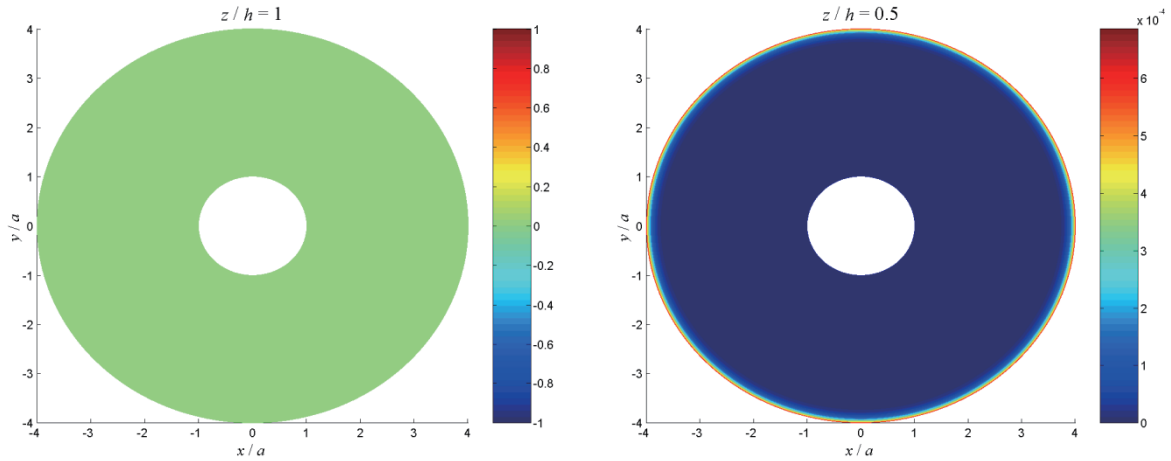


Fig. 11 Stress contours of $\sigma_{\theta z}$ / c_{66} of plate at $z/h = 1$ and at $z/h = 0.5$.

10. CONCLUSIONS

We have developed a general solution approach for analysis of transversely isotropic cylindrical bodies. The state equations and the output equations, with r and z as the distinct coordinate, respectively, are derived via the Hamiltonian variational formulation so that the systems possess the characteristic of Hamiltonian symplecticity, providing an essential basis for the systematic solution approach using eigenfunction expansion. The state equations are different in forms but equivalent in effect. Which of them is chosen for analysis depends on the solution approach for solving problem. Many mathematical approaches are applicable in the state space setting, such as methods of matrix algebra and separation of variables. With the approach, there is no need to deal with the individual components of the displacement and stress and the elastic constants. The simplicity of the approach makes treatment of a three-dimensional problem less formidable.

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APPENDIX A

Symplectic Orthogonality Relation in Equation (96):

Let $I_{nk} = I_n(\lambda_k r)$, $K_{nk} = K_n(\lambda_k r)$, thus

$$\mathbf{X}_p^T \mathbf{J}\mathbf{X}_q = (\mathbf{K}_p \boldsymbol{\psi}_p)^T \mathbf{J}(\mathbf{K}_q \boldsymbol{\psi}_q) = I_{1p}I_{2q}U_pS_q - I_{2p}I_{1q}U_qS_p\,, \tag{A1}$$

$$\begin{aligned}\mathbf{X}_p^T \mathbf{J}\mathbf{X}_{-q} &= (\mathbf{K}_p \boldsymbol{\psi}_p)^T \mathbf{J}(\mathbf{K}_{-q} \boldsymbol{\psi}_{-q}) \\ &= I_{1p}K_{2q}U_pS_{-q} - I_{2p}K_{1q}U_{-q}S_p\,.\end{aligned} \tag{A2}$$

If $p \neq q$ in Eq. (A2) then we have

$$\begin{aligned}\int_{-h}^{h} \boldsymbol{\psi}_p^T \mathbf{J}\boldsymbol{\psi}_q\, dz &= \int_{-h}^{h} (U_pS_q - U_qS_p)\, dz = 0\,, \\ \int_{-h}^{h} \boldsymbol{\psi}_p^T \mathbf{J}\boldsymbol{\psi}_{-q}\, dz &= \int_{-h}^{h} (U_pS_{-q} - U_{-q}S_p)\, dz = 0\end{aligned} \tag{A3}$$

i.e.,

$$\int_{-h}^{h} U_pS_q\, dz = \int_{-h}^{h} U_qS_p\, dz\,, \quad \int_{-h}^{h} U_pS_{-q} dz = \int_{-h}^{h} U_{-q}S_p\, dz\,. \tag{A4}$$

There are

$$\int_{-h}^{h} \mathbf{X}_p^T \mathbf{J}\mathbf{X}_q\, dz = (I_{1p}I_{2q} - I_{2p}I_{1q}) \int_{-h}^{h} U_p\, S_q\, dz\,, \tag{A5}$$

$$\int_{-h}^{h} \mathbf{X}_p^T \mathbf{J}\mathbf{X}_{-q}\, dz = (I_{1p}K_{2q} - I_{2p}K_{1q}) \int_{-h}^{h} U_pS_{-q}\, dz\,. \tag{A6}$$

Since

$$\boldsymbol{\psi}_k = \begin{bmatrix} 1 \\ c_{66}\lambda_k \end{bmatrix} \cos\left[\kappa\lambda_k(h-z)\right],$$

$$\boldsymbol{\psi}_{-k} = \begin{bmatrix} 1 \\ -c_{66}\lambda_k \end{bmatrix} \cos\left[\kappa\lambda_k(h-z)\right], \tag{A7}$$

and for eigenvalue λ_k: $\sin(2\kappa\lambda_k h) = 0$, we obtain

$$\begin{aligned}&\int_{-h}^{h} U_pS_{\pm q}\, dz \\ &= \pm c_{66}\lambda_q \int_{-h}^{h} \cos\left[\kappa\lambda_q(h-z)\right] \cos\left[\kappa\lambda_p(h-z)\right] dz = 0\,.\end{aligned} \tag{A8}$$

Then we get

$$\int_{-h}^{h} \mathbf{X}_p^T \mathbf{J}\mathbf{X}_q\, dz = 0\,, \quad \int_{-h}^{h} \mathbf{X}_p^T \mathbf{J}\mathbf{X}_{-q}\, dz = 0\,. \tag{A9}$$

If $p = q$ then we have $U_{-p} = U_p$, $S_{-p} = -S_p$, and

$$\begin{aligned}\mathbf{X}_p^T \mathbf{J}\mathbf{X}_{-p} &= I_{1p}K_{2p}U_pS_{-p} - I_{2p}K_{1p}U_{-p}S_p \\ &= -(I_{1p}K_{2p} + I_{2p}K_{1p})\, U_pS_p\,.\end{aligned} \tag{A10}$$

Since there is

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$$\int_{-h}^{h} U_p S_p \, dz = c_{66}\lambda_p \int_{-h}^{h} \cos^2\left[\kappa\lambda_p (h-z)\right] dz = c_{66}\lambda_p h \,, \tag{A11}$$

we get

$$\int_{-h}^{h} \mathbf{X}_p^T \mathbf{J} \mathbf{X}_{-p} dz = -c_{66}\lambda_p h (I_{1p}K_{2p} + I_{2p}K_{1p}) \,. \tag{A12}$$

APPENDIX B

Closed Form Solution for Twisting Problem of Cylindrical Tube:
From Eqs. (87) ~ (88):

$$\int_a^b r^2 \sigma_{\theta z}(r,\ 0)\, dr = \int_a^b r^2 p(r)\, dr \,, \quad 0 = \int_a^b \left[r^2 l p(r) - c_{44} r^2 u_\theta(r,\ l)\right] dr \tag{B1}$$

we have

$$A_{01} = \frac{4}{c_{44}(b^4 - a^4)} \int_a^b r^2 p(r)\, dr \,, \quad A_{00} = 0 \,. \tag{B2}$$

From Eqs. (89) ~ (90):

$$\int_a^b r\sigma_{\theta z}(r,\ 0) M_p(r)\, dr = \exp(\mu_p l) \int_a^b \left[rp(r) - c_{44}\mu_p r u_\theta(r,\ l)\right] M_p(r)\, dr \,, \tag{B3}$$

$$\int_a^b r\sigma_{\theta z}(r,\ 0) M_p(r)\, dr = \exp(-\mu_p l) \int_a^b \left[rp(r) + c_{44}\mu_p r u_\theta(r,\ l)\right] M_p(r)\, dr \,, \tag{B4}$$

where $M_p(r) = Y_2(\kappa\mu_p b) J_1(\kappa\mu_p r) - J_2(\kappa\mu_p b) Y_1(\kappa\mu_p r)$, we get

$$\begin{aligned} &A_{01} c_{44} \exp(-\mu_p l) \int_a^b r^2 M_p(r)\, dr + \sum_{q=1}^{\infty} c_{44}\mu_q \left[A_q \exp(-\mu_p l) - A_{-q} \exp(-\mu_p l)\right] \int_a^b r M_p(r) M_q(r)\, dr \\ &= \int_a^b rp(r) M_p(r)\, dr - A_{01} c_{44}\mu_p l \int_a^b r^2 M_p(r)\, dr + \sum_{q=1}^{\infty} c_{44}\mu_p \left[-A_q \exp(\mu_q l) - A_{-q} \exp(-\mu_q l)\right] \int_a^b r M_p(r) M_q(r)\, dr, \end{aligned} \tag{B5}$$

$$\begin{aligned} &A_{01} \exp(\mu_p l) c_{44} \int_a^b r^2 M_p(r)\, dr + \sum_{q=1}^{\infty} c_{44}\mu_q \left[A_q \exp(\mu_p l) - A_{-q} \exp(\mu_p l)\right] \int_a^b r M_p(r) M_q(r)\, dr \\ &= \int_a^b rp(r) M_p(r)\, dr + A_{01} c_{44}\mu_p l \int_a^b r^2 M_p(r)\, dr + \sum_{q=1}^{\infty} c_{44}\mu_p \left[A_q \exp(\mu_q l) + A_{-q} \exp(-\mu_q l)\right] \int_a^b r M_p(r) M_q(r)\, dr \,, \end{aligned} \tag{B6}$$

Since $p \neq q$, $\int_a^b r M_p(r) M_q(r)\, dr = 0$, the constants have been determined as

$$A_p = \frac{\int_a^b rp(r) M_p(r)\, dr - A_{01} c_{44} \left[\mu_p l + \exp(-\mu_p l)\right] \int_a^b r^2 M_p(r)\, dr}{\left[\exp(\mu_p l) + \exp(-\mu_p l)\right] c_{44}\mu_p \int_a^b r M_p^2(r)\, dr} \,, \tag{B7}$$

$$A_{-p} = -\frac{\int_a^b rp(r) M_p(r)\, dr + A_{01} c_{44} \left[\mu_p l - \exp(\mu_p l)\right] \int_a^b r^2 M_p(r)\, dr}{\left[\exp(\mu_p l) + \exp(-\mu_p l)\right] c_{44}\mu_p \int_a^b r M_p^2(r)\, dr} \,. \tag{B8}$$

APPENDIX C

Closed Form Solution for Twisting Problem of Circular Plate:
From Eqs. (99) ~ (100):

$$a\int_{-h}^{h} \sigma_{r\theta}(a,\ z)\, dz = b\int_{-h}^{h} p(z)\, dz \,, \quad \int_{-h}^{h} a^{-1} \sigma_{r\theta}(a,\ z)\, dz = \int_{-h}^{h} \left[b^{-1} p(z) + 2c_{66} b^{-2} u_\theta(b,\ z)\right] dz \,, \tag{C1}$$

we have

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$$A_{01} = -\frac{ab}{4c_{66}h}\int_{-h}^{h} p(z)\,dz\,,\quad A_{00} = \frac{ab^2(b^{-3}+a^{-3})-1}{4c_{66}h}\int_{-h}^{h} p(z)\,dz\,. \tag{C2}$$

From Eqs. (101) ~ (102):

$$c_{pb}I_{1pa}\int_{-h}^{h}\sigma_{r\theta}(a,\ z)M_p(z)\,dz = c_{pa}I_{1pb}\int_{-h}^{h} p(z)M_p(z)\,dz - c_{pa}I_{2pb}\,c_{66}\lambda_p\int_{-h}^{h}u_\theta(b,\ z)M_p(z)\,dz\,, \tag{C3}$$

$$c_{pb}K_{1pa}\int_{-h}^{h}\sigma_{r\theta}(a,\ z)M_p(z)\,dz = c_{pa}K_{1pb}\int_{-h}^{h} p(z)M_p(z)\,dz + c_{pa}K_{2pb}c_{66}\lambda_p\int_{-h}^{h}u_\theta(b,\ z)M_p(z)\,dz\,, \tag{C4}$$

where $M_p(z) = \cos[\kappa\lambda_k(h-z)]$, we get

$$\begin{aligned}
&-2A_{01}c_{66}a^{-2}\,c_{pb}I_{1pa}\int_{-h}^{h}M_p(z)\,dz + \sum_{q=1}^{\infty}c_{66}\lambda_q(A_qI_{2qa}c_{pb}I_{1pa} - A_{-q}K_{2qa}c_{pb}I_{1pa})\int_{-h}^{h}M_q(z)M_p(z)\,dz\\
&= c_{pa}I_{1pb}\int_{-h}^{h}p(z)M_p(z)\,dz - c_{66}\lambda_p(A_{00}\,bc_{pa}I_{2pb} + A_{01}b^{-1}c_{pa}I_{2pb})\int_{-h}^{h}M_p(z)\,dz\\
&\quad - \sum_{q=1}^{\infty}c_{66}\lambda_p(A_qI_{1qb}c_{pa}I_{2pb} + A_{-q}K_{1qb}c_{pa}I_{2pb})\int_{-h}^{h}M_q(z)M_p(z)\,dz\,,
\end{aligned} \tag{C5}$$

$$\begin{aligned}
&-2A_{01}c_{66}a^{-2}\,c_{pb}K_{1pa}\int_{-h}^{h}M_p(z)\,dz + \sum_{q=1}^{\infty}c_{66}\lambda_q(A_qI_{2qa}c_{pb}K_{1pa} - A_{-q}K_{2qa}c_{pb}K_{1pa})\int_{-h}^{h}M_q(z)M_p(z)\,dz\\
&= c_{pa}K_{1pb}\int_{-h}^{h}p(z)M_p(z)\,dz + c_{66}\lambda_p(A_{00}\,bc_{pa}K_{2pb} + A_{01}b^{-1}c_{pa}K_{2pb})\int_{-h}^{h}M_p(z)\,dz\\
&\quad + \sum_{q=1}^{\infty}c_{66}\lambda_p(A_qI_{1qb}c_{pa}K_{2pb} + A_{-q}K_{1qb}c_{pa}K_{2pb})\int_{-h}^{h}M_q(z)M_p(z)\,dz\,.
\end{aligned} \tag{C6}$$

Since $p \neq q$, $\int_{-h}^{h}M_p(z)M_q(z)\,dz = 0$, Eqs. (C5) ~ (C6) become

$$A_pd_{11p} + A_{-p}d_{12p} = f_{1p}\,, \tag{C7}$$

$$A_pd_{21p} + A_{-p}d_{22p} = f_{2p} \tag{C8}$$

where

$$\begin{aligned}
d_{11p} &= I_{1pa}I_{1pb}(I_{2pa}K_{2pb} + K_{2pa}I_{2pb})\\
&\quad + I_{2pa}I_{2pb}(I_{1pa}K_{1pb} + K_{1pa}I_{1pb})\,,\\
d_{12p} &= I_{2pa}I_{2pb}K_{1pa}K_{1pb} - I_{1pa}I_{1pb}K_{2pa}K_{2pb}\,,\\
d_{21p} &= I_{2pa}I_{1pb}K_{1pa}K_{2pb} - I_{1pa}I_{2pb}K_{2pa}K_{1pb}\,,
\end{aligned}$$

$$\begin{aligned}
d_{22p} &= K_{2pa}K_{2pb}(K_{1pa}I_{1pb} + I_{1pa}K_{1pb})\\
&\quad + K_{1pa}K_{1pb}(I_{2pa}K_{2pb} + K_{2pa}I_{2pb})\,,
\end{aligned}\qquad f_{1p} = \frac{-c_{pa}I_{1pb}\int_{-h}^{h}p(z)M_p(z)\,dz}{(c_{66}\lambda_ph)^2},\ f_{2p} = \frac{c_{pa}K_{1pb}\int_{-h}^{h}p(z)M_p(z)\,dz}{(c_{66}\lambda_ph)^2}.$$

Solving Eqs. (C7) ~ (C8) the constants have been determined as

$$\begin{bmatrix}A_p\\A_{-p}\end{bmatrix} = \frac{1}{d_{11p}d_{22p} - d_{21p}d_{12p}}\begin{bmatrix}d_{22p}f_{1p} - d_{12p}f_{2p}\\d_{11p}f_{2p} - d_{21p}f_{1p}\end{bmatrix}, \tag{C9}$$

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