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Rigidity for partially hyperbolic diffeomorphisms

RÉGIS VARÃO

IMECC-UNICAMP, Rua Sérgio Buarque de Holanda, 651, Campinas, SP, CEP 13083-859, Brazil
(e-mail: regisvarao@ime.unicamp.br)

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Abstract. In this work we completely classify C^∞ conjugacy for smooth conservative (pointwise) partially hyperbolic diffeomorphisms homotopic to a linear Anosov automorphism on the 3-torus by its center foliation behavior. We prove that the uniform version of absolute continuity for the center foliation is the natural hypothesis to obtain C^∞ conjugacy to its linear Anosov automorphism. Avila, Viana and Wilkinson [Absolute continuity, Lyapunov exponents and rigidity I: Geodesic flows. *J. Eur. Math. Soc. (JEMS)* **17**(6) (2015), 1435–1462] proved that for a perturbation in the volume preserving case of the time-one map of an Anosov flow absolute continuity of the center foliation implies smooth rigidity. The absolute version of absolute continuity is the appropriate scenario for our context since it is not possible to obtain a result analogous to that of Avila, Viana and Wilkinson for our class of maps, for absolute continuity alone fails miserably to imply smooth rigidity for our class of maps. Our theorem is a global rigidity result as we do not assume the diffeomorphism to be at some distance from the linear Anosov automorphism. We also do not assume ergodicity. In particular, a metric condition on the center foliation implies ergodicity and C^∞ center foliation.

1. *Introduction*

A fundamental problem in dynamical systems is the description of the orbits of elements under a given law of motion; in our case this law will be given by diffeomorphisms. One of the most effective ways to do so is through classification, that is, understanding a given dynamics by its relation with some other better-known model. A typical and important class of examples of dynamical systems are the hyperbolic diffeomorphisms, also known as Anosov diffeomorphisms. Anosov [**1**] proved that these maps (here we always assume at least $C^{1+\alpha}$) are ergodic with respect to the volume measure. One of the key steps of his proof relies on the understanding of the behavior of the two associate invariant foliations,

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the stable and the unstable, which are uniformly contracting and uniformly expanding, respectively.

These hyperbolic maps exhibit enough structure that one can analyze them using geometric and metric tools. A diffeomorphism $f : M \to M$ on a smooth manifold M is said to be transitive if there exists $x \in M$ such that $\{f^n(x)\}_n$ is dense in M. We say that f is robustly transitive if there exists a C^1 neighborhood of f such that all diffeomorphisms in this neighborhood are transitive. Among the first examples of robustly transitive systems were the Anosov diffeomorphisms (see [**3**] and references therein). It was then thought that hyperbolic diffeomorphisms were the only robustly transitive systems; in fact Mañé [**12**] proved that C^1 robustly transitive diffeomorphisms on a surface are the hyperbolic ones. But Shub [**20**] and Mañé [**11**] gave examples of robustly transitive systems which were not hyperbolic. Their systems fall on a larger class known as partially hyperbolic diffeomorphisms.

The partially hyperbolic diffeomorphisms are similar to Anosov diffeomorphisms as they have expanding and contracting directions and the respectively associated stable and unstable foliations. The difference is that there is a center direction for a partially hyperbolic diffeomorphism which is dominated by the stable and unstable directions (see Definition 2.1). A major difficulty in understanding these systems arises when predicting the behavior of the center direction since it may not exhibit any kind of hyperbolicity. As one might expect, this brings new difficulties, but most importantly it gives rise to new dynamical behaviors.

The stable and unstable foliations of a partially hyperbolic diffeomorphism are absolutely continuous; this means that for a set of full volume almost every leaf intersects this set in a full leaf-volume measure. That is, one should see absolute continuity of a foliation as a Fubini-like behavior, which may also be seen as some sort of regularity condition of these foliations. In contrast, the center foliation may exhibit a new type of behavior, atomic disintegration: there is a set of full volume which intersects each center leaf in a finite number of points. At first this might sound like a pathological behavior, but it turns out that this is in fact a common behavior for the center foliation [**2**, **15**, **19**].

If one imposes some regularity on a center foliation, for instance absolute continuity as it happens for the stable and unstable foliations, one may get a very rigid structure. For instance, assuming absolute continuity of the center foliation for volume preserving perturbations of a time-one map of the geodesic flow on a constant negatively curved surface, Avila *et al* [**2**] obtain that in fact this perturbation is a time-one map of a C^∞ Anosov flow.

So far the main results on the rigidity problem concerning the center foliation are those presented in [**2**], as described above, and in [**22**]. In [**22**] the authors study the rigidity problem for the class derived from Anosov systems, which are partially hyperbolic diffeomorphisms homotopic to a linear Anosov. These systems have a hyperbolic memory; more precisely, they are semi-conjugate to a linear Anosov, also called the linearization of this diffeomorphism (see §2 for details).

It is assumed in [**22**] that the center foliation is C^1 and it has transversely absolutely continuous foliations with bounded Jacobian, and the implication is C^1 conjugacy with a linear Anosov. Our main result largely extends this last result since it does not assume any

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smoothness from the center foliation; in fact it assumes an almost everywhere condition, and then it implies a much stronger regularity, C^∞.

The low-regularity assumption in our result is simply a uniform version of absolute continuity of a foliation. More precisely, the disintegration of volume on any foliated box has conditional measures with a uniformly bounded density with respect to the leaf-volume (see Definition 2.5). Micena and Tahzibi [**13**] called this the UBD property. Our main result also answers their conjecture that the UBD property would imply C^1 conjugacy; in fact we obtain C^∞.

Transversely absolutely continuous foliations with bounded Jacobian means that the holonomies of transversal foliations have uniformly bounded Jacobian, One can easily check that this implies the UBD property (the proof follows from appropriate modifications of [**5**, Proposition 6.2.2]).

In the context of Avila *et al* [**2**] absolute continuity implies smoothness; in our context this is far from possible and one cannot drop the UBD property as, by [**22**, Theorem 1.3], the center foliation may be C^1 and f not even C^1 smoothly conjugate to its linearization. We point out that being a C^1 foliation implies having locally uniform bounded densities. Hence, our hypothesis (the uniform version of absolute continuity) is a natural one and, from the foregoing, can be seen as a sharp condition. Our main result, Theorem 1.1, in contrast to other results, is a global rigidity result, that is, one does not assume any proximity of the derived from Anosov diffeomorphism (see Definition 2.2) to its linearization. We also point out that we do not assume ergodicity.

We now state our main result.

THEOREM 1.1. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a C^∞ conservative partially hyperbolic diffeomorphism homotopic to a linear Anosov. The center foliation has the uniform bounded density property if and only if f is C^∞ conjugate to its linearization.*

Moreover, if f is only $C^{1+\alpha}$, then the center foliation has the uniform bounded density property if and only if f is C^1 conjugate to its linearization.

The next section presents preliminaries, among them the formal definitions and results mentioned in this section and elsewhere. After the preliminaries we prove our main result. Remark 3.1, where we propose a general program to tackle rigidity problems for the center foliations of partially hyperbolic diffeomorphisms in a more general setting, comes after the proof of Theorem 1.1.

2. *Preliminaries*

We briefly present the main definitions and results we will be using throughout this work.

Definition 2.1. Let M be a smooth compact Riemannian manifold. A diffeomorphism $f : M \to M$ is called partially hyperbolic if the tangent bundle of the ambient manifold admits an invariant decomposition $TM = E^s \oplus E^c \oplus E^u$ and if (for some Riemannian metric on M) for all unit vectors v^s, v^c, v^u respectively on E^s_x, E^c_x, E^u_x,

$$\|D_x f v^s\| < \|D_x f v^c\| < \|D_x f v^u\|$$

and $\|D_x f v^s\| < 1 < \|D_x f v^u\|$.

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One may also find in the literature the above definition referred to as a *pointwise partially hyperbolic* rather than *absolutely partially hyperbolic diffeomorphism* where the inequalities above hold uniformly, that is, $\|D_x f v^s\| < \|D_y f v^c\| < \|D_z f v^u\|$ and $\|D_x f v^s\| < 1 < \|D_z f v^u\|$ for all $x, y, z \in M$.

It is well known that for partially hyperbolic diffeomorphisms, there are foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ tangent to the sub-bundles E^s and E^u called *stable* and *unstable foliations*, respectively (for more details see, for example, [**14**]). On the other hand, *a priori* there is no guarantee of the existence of a center foliation for a partially hyperbolic diffeomorphism. But that is not our concern since in the context in which we will be working (partially hyperbolic diffeomorphisms on $\mathbb{T}^3$) all partially hyperbolic diffeomorphisms (as defined above) on $\mathbb{T}^3$ admit a central foliation tangent to E^c by [**4**, **16**]. We say that f is an Anosov diffeomorphism if it satisfies Definition 2.1 but there is no center direction. Anosov diffeomorphisms also have stable and unstable invariant foliations as described above.

Definition 2.2. We say that $f : \mathbb{T}^n \to \mathbb{T}^n$ is a derived from Anosov diffeomorphism or just a DA diffeomorphism if it is a partially hyperbolic diffeomorphism homotopic to a linear Anosov automorphism $A : \mathbb{T}^n \to \mathbb{T}^n$.

If f is a DA diffeomorphism we call the linear Anosov A the linearization of f; this also means that f has hyperbolic memory. More precisely, by Franks [**7**] there is a semi-conjugacy $h : \mathbb{T}^3 \to \mathbb{T}^3$ between f and its linearization A, that is,

$$A \circ h = h \circ f. \tag{1}$$

Moreover, this semi-conjugacy has the property that there exists a constant $\Omega \in \mathbb{R}$ such that if $H, F : \mathbb{R}^3 \to \mathbb{R}^3$ denote the lift of h and f to $\mathbb{R}^3$ respectively, we have $\|H(x) - x\| \leq \Omega$ for all $x \in \mathbb{R}^3$, and, given two points $a, b \in \mathbb{R}^3$,

$$H(a) = H(b) \Leftrightarrow \|F^n(a) - F^n(b)\| < \Omega \quad \text{for all } n \in \mathbb{Z}. \tag{2}$$

Let $(M, \mu, \mathcal{B})$ be a probability space, where M is a compact metric space, μ a probability measure and $\mathcal{B}$ a Borel σ-algebra. Given a partition $\mathcal{P}$ of M by measurable sets, we construct a probability space $(\mathcal{P}, \widetilde{\mu}, \widetilde{\mathcal{B}})$ in the following way. Let $\pi : M \to \mathcal{P}$ be the canonical projection, that is, π associates to a point x of M the partition element of $\mathcal{P}$ that contains it. Then we define $\widetilde{\mu} := \pi_* \mu$ and $\widetilde{\mathcal{B}} := \pi_* \mathcal{B}$.

Definition 2.3. Given a partition $\mathcal{P}$. A family $\{\mu_P\}_{P \in \mathcal{P}}$ is a *system of conditional measures* for μ (with respect to $\mathcal{P}$) if

(i) given $\phi \in C^0(M)$, then $P \mapsto \int \phi \, d\mu_P$ is measurable;

(ii) $\mu_P(P) = 1$ $\widetilde{\mu}$-almost everywhere;

(iii) if $\phi \in C^0(M)$, then $\displaystyle\int_M \phi \, d\mu = \int_{\mathcal{P}} \left(\int_P \phi \, d\mu_P \right) d\widetilde{\mu}$.

When it is clear which partition we are referring to, we say that the family $\{\mu_P\}$ *disintegrates* the measure μ.

PROPOSITION 2.1. [**6**, **18**] *Given a partition $\mathcal{P}$, if $\{\mu_P\}$ and $\{\nu_P\}$ are conditional measures that disintegrate μ on $\mathcal{P}$, then $\mu_P = \nu_P$ $\widetilde{\mu}$-almost everywhere.*

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An easy consequence of this proposition, which will be useful to us, is the following corollary.

COROLLARY 2.2. *If $T : M \to M$ preserves a probability μ and the partition $\mathcal{P}$, then $T_*\mu_P = \mu_{T(P)}$, $\widetilde{\mu}$-almost everywhere.*

Proof. This follows from the fact that $\{T_*\mu_P\}_{P\in\mathcal{P}}$ is also a disintegration of μ. □

Definition 2.4. We say that a partition $\mathcal{P}$ is a measurable partition (or countably generated) with respect to μ if there exist a measurable family $\{A_i\}_{i\in\mathbb{N}}$ and a measurable set Y of full measure such that if $B \in \mathcal{P}$, then there exists a sequence $\{B_i\}$ where $B_i \in \{A_i, A_i^c\}$ such that $B \cap Y = \bigcap_i B_i \cap Y$.

The next theorem guarantees the existence of conditional measures with respect to a measurable partition.

THEOREM 2.3. (Rokhlin's disintegration [**18**]) *Let $\mathcal{P}$ be a measurable partition of a compact metric space M and μ a probability. Then there exists a disintegration by conditional measures for μ.*

Recall that a foliation is absolutely continuous if the conditional measures on the leaves of the foliation are equivalent to the Lebesgue measure. In other words, a set of full volume intersects almost every leaf in a set of full leaf-volume. The UBD property is the uniform version of a foliation being absolutely continuous.

Definition 2.5. We say that a foliation $\mathcal{F}$ has the uniform bounded density property (UBD for short) if there exists a uniform constant K such that for any foliated box $\mathcal{B}$,

$$K^{-1} \leq \frac{dm_x^{\mathcal{B}}}{d\widehat{\mathrm{Leb}}_x} \leq K,$$

where $m_x^{\mathcal{B}}$ is the conditional measure of volume on the segments of $\mathcal{F}$ on $\mathcal{B}$ and $\widehat{\mathrm{Leb}_x}$ means the normalized Lebesgue measure on the connected component of $\mathcal{F}(x) \cap \mathcal{B}$ which contains x.

3. *The proof of Theorem 1.1*

Proof. We work on the universal cover of $\mathbb{T}^3$, $\mathbb{R}^3$. Let $\pi : \mathbb{R}^3 \to \mathbb{T}^3$ be the canonical projection. We denote by F and H the lifted function of f and the semi-conjugacy h, respectively. And for the other lifts we use the symbol $\sim$, for instance $\widetilde{\mathcal{F}}^c$ is the lift of the center foliation $\mathcal{F}^c$. We shall use $\widetilde{\mathcal{F}}^c$ to refer to the center foliation of F, and $\widetilde{\mathcal{F}}^c_x$, $\widetilde{\mathcal{F}}^c(x)$ for the lifted center foliation through x. Suppose that the splitting for A is of the form $E^{ss} \oplus E^{wu} \oplus E^{uu}$, otherwise work with F^{-1}. Throughout the proof we shall denote by λ the eigenvalue of the center direction of A, which is greater than one since we suppose A to have an expanding center direction. We also see A as a partially hyperbolic with the splitting of the tangent bundle as $E^s_A \oplus E^c_A \oplus E^u_A$.

At the universal cover let us define a 'base space'

$$B := \bigcup_{x\in\widetilde{\mathcal{F}}^s(0)} \widetilde{\mathcal{F}}^u(x).$$

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By [**10**] all the center leaves of F intersect B at a unique point. We may also assume that 0 is a fixed point for F, otherwise take a periodic point p of period n for f and work with f^n instead of f. The lifted function F is such that $\pi(0) = p$. This implies that B is F invariant, $F(B) = B$. On $\mathbb{R}^3$ the center foliation is an oriented foliation. We also assume that F preserves the center foliation orientation, otherwise work with f^2 instead of f (and the respective lift). Note that these changes do not affect the result since h is also a semi-conjugacy between f^n and A^n for all $n \in \mathbb{N}$.

Let us define the set

$$\mathcal{B}_0 = \{y \in \mathbb{R}^3 \mid d^c(P(y), y) \leq \Omega \text{ and } y \in \widetilde{\mathcal{F}}^{c,+}\}$$

where $d^c(\cdot, \cdot)$ is the distance inside the center leaf, $\mathcal{F}^{c,+}$ stands for the positive half side of the orientable foliation $\widetilde{\mathcal{F}}^c$, and Ω is as defined in §2.

We consider the iterations $\mathcal{B}_k := F^k(\mathcal{B}_0)$ for all $k \in \mathbb{N}$. Now let $m_x^{\mathcal{B}_k}$ be the conditional measure defined on $\widetilde{\mathcal{F}}_x^c \cap \mathcal{B}_k$ which is the disintegration of volume restricted to $\mathcal{B}_k$ and the partition given by $\{\widetilde{\mathcal{F}}_\xi^c \cap \mathcal{B}_k\}_{\xi \in B}$. In particular, $m_x^{\mathcal{B}_k}$ is a probability measure. The existence of $m_x^{\mathcal{B}_k}$ is guaranteed by Theorem 2.3, but there are two remarks concerning the application of Rokhlin's theorem: first, that Theorem 2.3 is for a probability measure; and second, that it should be checked whether $\widetilde{\mathcal{F}}_x^c \cap \mathcal{B}_k$ is a measurable partition. Let us see how to deal with these two issues. Note that it is not difficult to see that B is a metrizable space with the topology of a subspace of $\mathbb{R}^3$. Hence, let $Z \subset B$ be a ball in B and consider $Z_k := (P|\mathcal{B}_k)^{-1}(Z)$, let S_Z be a countable dense set in Z, and for all disks $D(s, r)$ inside B of center $s \in S_Z$ and radius $r \in \mathbb{Q}$ let $A_{s,r} := (P|\mathcal{B}_k)^{-1}(D(s, r))$; then these $A_{s,r}$ (there are a countable number of them) play the rule of A_i in Definition 2.4 (and in Definition 2.4 one may assume Y to be $\mathbb{R}^3$). Hence, we may apply Theorem 2.3 to disintegrate $\mathrm{Vol}|Z_k$ when normalized. Then the conditional measures $m_x^{\mathcal{B}_k}$ come from this disintegration. But the normalization does not affect the disintegration since the only change one gets is for the projected measure (which in the notation of Definition 2.3 would be $\widetilde{\mu}$).

Let us denote $\mu_{\mathcal{B}_k} := P_* \mathrm{Vol}|\mathcal{B}_k$. That is, $\mu_{\mathcal{B}_k}$ is the projection on B of the volume measure restricted to $\mathcal{B}_k$. We are able to relate two different conditional measures; by [**2**, Lemma 3.2] we have that

$$m_x^{\mathcal{B}_k} \frac{d\mu_{\mathcal{B}_k}}{d\mu_{\mathcal{B}_0}}(x) = m_x^{\mathcal{B}_0}, \tag{3}$$

where this equality is understood to be defined only on $\widetilde{\mathcal{F}}_x^c \cap \mathcal{B}_0$.

Note that these measures differ up to multiplication by the constant $d\mu_{\mathcal{B}_k}/d\mu_{\mathcal{B}_0}(x)$. Observe as well that

$$m_x^{\mathcal{B}_k}(\cdot) = \rho_k(x, \cdot)\widehat{\lambda}_{\mathcal{B}_k}(x, \cdot) \quad \text{for all } k \in \mathbb{N},$$

where $\lambda_{\mathcal{B}_k}(x, \mathrm{cot})$ is the Lebesgue (length) measure on the leaf segment $\widetilde{\mathcal{F}}_x^c \cap \mathcal{B}_k$, $\widehat{\lambda}_{\mathcal{B}_k}(x, \mathrm{cot})$ is the normalized Lebesgue measure on the leaf segment $\widetilde{\mathcal{F}}_x^c \cap \mathcal{B}_k$, Leb_x is the Lebesgue measure on the entire leaf $\widetilde{\mathcal{F}}_x^c$, and $\rho_k(x, \mathrm{cot}) : \widetilde{\mathcal{F}}_x^c \cap \mathcal{B}_k \to \mathbb{R}$ is the density function. By the UBD property of the center foliation, $\rho_k \in [K^{-1}, K]$. Then

$$\begin{aligned} m_x^{\mathcal{B}_k}(I) &= \int_I \rho_k(x, \zeta)\, d\widehat{\lambda}_{\mathcal{B}_k}(x, \mathrm{cot})(\zeta) = \int_I \rho_k(x, \zeta)\, d\frac{\mathrm{Leb}_x}{\mathrm{Leb}_x(\widetilde{\mathcal{F}}_x^c \cap \mathcal{B}_k)}(\zeta) \\ &\leq \frac{K}{\mathrm{Leb}_x(\widetilde{\mathcal{F}}_x^c \cap \mathcal{B}_k)} \int_I d\,\mathrm{Leb}_x(\zeta) \leq \frac{K\mathrm{Leb}_x(I)}{\mathrm{Leb}_x(\widetilde{\mathcal{F}}_x^c \cap \mathcal{B}_k)} \end{aligned}$$

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for all $k \in \mathbb{N}$ and $I \subset \widetilde{\mathcal{F}}^c_x$ a measurable set. Analogously,

$$m^{\mathcal{B}_k}_x(I) \geq \frac{K^{-1}\,\mathrm{Leb}_x(I)}{\mathrm{Leb}_x(\widetilde{\mathcal{F}}^c_x \cap \mathcal{B}_k)}.$$

From the above inequalities we have

$$\frac{d\mu_{\mathcal{B}_k}}{d\mu_{\mathcal{B}_0}}(x) = \frac{m^{\mathcal{B}_0}_x(\mathrm{cot})}{m^{\mathcal{B}_k}_x(\mathrm{cot})} \leq K^2 \frac{\mathrm{Leb}_x(\mathcal{F}^c_x \cap \mathcal{B}_k)}{\mathrm{Leb}_x(\mathcal{F}^c_x \cap \mathcal{B}_0)}.$$

Similarly, we get

$$\frac{d\mu_{\mathcal{B}_k}}{d\mu_{\mathcal{B}_0}}(x) \geq K^{-2} \frac{\mathrm{Leb}_x(\mathcal{F}^c_x \cap \mathcal{B}_k)}{\mathrm{Leb}_x(\mathcal{F}^c_x \cap \mathcal{B}_0)}.$$

So far we have taken for granted that if the conditional measure exists for $\mathcal{B}_0$ it also exists for $\mathcal{B}_k$. This is shown in the next lemma.

LEMMA 3.1. *There is a set $D \subset \mathbb{R}^3$ which is F-invariant, has full volume measure and is $\widetilde{\mathcal{F}}^c$ foliated and for which $m^{\mathcal{B}_k}_x$ is well defined for all $x \in D$ and $k \in \mathbb{N}$.*

Proof. Let $S \subset B$ be a countable dense set of B, let $S' := \pi(S)$ and $B' := \pi(B)$ (recall that $\pi : \mathbb{R}^3 \to \mathbb{T}^3$ is the canonical projection). Then S' is a dense set of B'. For a given $s \in S'$ and $r \in \mathbb{Q}$, define $C'(s, r)$ to be a foliated box such that all segments of the center foliation have length r and $\mathrm{Vol}(\partial C'(s, r)) = 0$ and satisfying also that $\mathrm{Vol}(\bigcup_{s\in S'} C'(s, r)) = 1$.

We disintegrate volume on the foliated box $C'(s, r)$, hence its conditional measures are defined on a set $M(s, r)$ which has $\mathrm{Vol}|C'(s, r)$ full measure. Then $\mathrm{Vol}(M(s, r) \cup C'(s, r)^c) = 1$. Now define

$$D_0 := \bigcap_{r\in\mathbb{Q}} \left(\bigcup_{s\in S'} C'(s, r) \right) \quad \text{and} \quad D_1 := D_0 \Big\backslash \bigcup_{(s,r)\in S'\times\mathbb{Q}} \partial(C'(s, r)).$$

Notice that D_0 and D_1 have full volume and that if $x \in D_1$ then there are conditional measures defined on the center leaf of x with arbitrary size. Let $D_2 := \bigcap_{i\in\mathbb{Z}} f^i(D_2)$, which is an f invariant full volume set. Then the lemma is proved by taking $D := \pi^{-1}(D_2)$. □

Let $\eta_{x,k}$ be a measure (not a probability) defined on $\widetilde{\mathcal{F}}^c_x \cap \mathcal{B}_k$ as

$$\eta_{x,k} := \lambda^k m^{\mathcal{B}_k}_x.$$

Recall that λ is the eigenvalue of the center direction of A. Now, using equation (3), we have

$$\eta_{x,k} = \lambda^k m^{\mathcal{B}_k}_x = \left(\frac{d\mu_{\mathcal{B}_k}}{d\mu_{\mathcal{B}_0}}(x) \right)^{-1} \lambda^k m^{\mathcal{B}_0}_x \quad \text{on } \widetilde{\mathcal{F}}^c_x \cap \mathcal{B}_0,$$

and from the inequalities above,

$$\frac{d\mu_{B_k}}{d\mu_{\mathcal{B}_0}}(x) = \alpha_{x,k} \frac{\mathrm{Leb}_x(\mathcal{F}^c_x \cap \mathcal{B}_k)}{\mathrm{Leb}_x(\mathcal{F}^c_x \cap \mathcal{B}_0)},$$

where $\alpha_{x,k} \in [K^{-2}, K^2]$, for all $x \in A \subset \mathbb{R}^3$ and $k \in \mathbb{N}$.

Combining the above, we may rewrite

$$\eta_{x,k} = \left(\alpha_{x,k} \frac{\mathrm{Leb}_x(\mathcal{F}^c_x \cap \mathcal{B}_k)}{\mathrm{Leb}_x(\mathcal{F}^c_x \cap \mathcal{B}_0)} \right)^{-1} \lambda^k m^{\mathcal{B}_0}_x \quad \text{on } \widetilde{\mathcal{F}}^c_x \cap \mathcal{B}_0.$$

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LEMMA 3.2. *There is* $\Delta \in (0, \infty)$ *such that* $\lambda^k/\mathrm{Leb}_x(\widetilde{\mathcal{F}}^c_x \cap \mathcal{B}_k) \in [1/\Delta, \Delta]$ *for all* $x \in \mathbb{R}^3$. *Recall that* Leb_x *is the Lebesgue (length) measure of the leaf* $\widetilde{\mathcal{F}}^c_x$.

Proof. Let $x, y \in \mathbb{R}^3$ be the extreme points of the segment $\widetilde{\mathcal{F}}^c_x \cap \mathcal{B}_0$. Then $F^n(x)$ and $F^n(y)$ are the extreme end points of $\widetilde{\mathcal{F}}^c(F^n(x)) \cap \mathcal{B}_k$. Hence we want to measure the growth of these extreme points inside the center foliation. By [**10**] the center foliation $\widetilde{\mathcal{F}}^c$ is quasi-isometric. That means that there exists a constant Q such that, for all $z \in \widetilde{\mathcal{F}}^c(w)$, we have $Q^{-1}\|z - w\| \le d^c(z, w) \le Q\|z - w\|$, where d^c is the distance inside the center foliation. Hence, due the quasi-isometry of the center foliation, we only need to analyze the growth of $\|F^n(x) - F^n(y)\|$. We will consider only the superior estimative as the inferior one can be found by an analogous argument.

$$\begin{aligned}\|F^n(x) - F^n(y)\| \le{}& \|F^n(x) - H \circ F^n(x)\| \\ &+ \|H \circ F^n(x) - H \circ F^n(y)\| + \|H \circ F^n(y) - F^n(y)\|.\end{aligned}$$

Because H is uniformly close to the identity, as exposed in §2, the first and third terms of the right-hand side are uniformly bounded. We have to see that the term

$$\|H \circ F^n(x) - H \circ F^n(y)\| = \|A^n(H(x)) - A^n(H(y))\|$$

grows at rate λ. On the last identity we used $H \circ F = A \circ H$.

By [**21**] the semi-conjugacy sends center leaf into center leaf (i.e. $H(\widetilde{\mathcal{F}}^c_F) = \widetilde{\mathcal{F}}^c_A$), therefore $\|A^n(H(x)) - A^n(H(y))\|$ does grow at rate λ since $H(x) \in \widetilde{\mathcal{F}}^c_A(H(y))$. □

LEMMA 3.3. *For all* $x \in D$, *there exists a measure* η_x *such that* $F_*\eta_x = \lambda^{-1}\eta_{F(x)}$ *and* $\eta_x = \rho_x \mathrm{Leb}_x$, *where the density* $\rho_x(\mathrm{cot})$ *is uniformly bounded (independent of* $x \in A$*).*

Proof. Since the disintegration is unique and F is volume preserving, the family $\{F_*\eta_{x,k}\}_{x\in B}$ when normalized is a disintegration of volume restricted to $\mathcal{B}_{k+1}$ and the family $\{\eta_{x,k+1}\}_{x\in B}$ when normalized is a disintegration of volume restricted to $\mathcal{B}_{k+1}$, hence these normalized measures are the same by Proposition 2.1. As a consequence $F_*\eta_{x,k}$ and $\eta_{x,k+1}$ are the same up to multiplication by a constant. To find this constant one could simply evaluate these measures on the same set. Using the F invariance of $\widetilde{\mathcal{F}}^c$ and the definition of $\eta_{x,k}$, we obtain

$$\begin{aligned}F_*\eta_{x,k}(F(\mathcal{B}_k \cap \widetilde{\mathcal{F}}^c_x)) &= \eta_{x,k}(F^{-1}(F(\mathcal{B}_k \cap \widetilde{\mathcal{F}}^c_x))) = \eta_{x,k}(\mathcal{B}_k \cap \widetilde{\mathcal{F}}^c_x) \\ &= \lambda^k = \lambda^{-1}\lambda^{k+1} = \lambda^{-1}\eta_{F(x),k+1}(\mathcal{B}_{k+1} \cap \widetilde{\mathcal{F}}^c_{F(x)}) \\ &= \lambda^{-1}\eta_{F(x),k+1}(F(\mathcal{B}_k \cap \widetilde{\mathcal{F}}^c_x)),\end{aligned}$$

therefore $F_*\eta_{x,k} = \lambda^{-1}\eta_{F(x),k+1}$.

Hence, if there is a subsequence $k_{i(x)}$ such that $\eta_{x,k_{i(x)}}$ converges weakly to η_x and $\eta_{F(x),k_{i(x)}+1}$ converges weakly to $\eta_{F(x)}$, then $F_*\eta_x = \lambda^{-1}\eta_{F(x)}$.

We now show how to define these subsequences. By the axiom of choice we can choose a set $W \subset A$ such that if $x, y \in C$ and $x \neq y$ then $\{F^n(x)\}_{n\in\mathbb{Z}} \cap \{F^n(y)\}_{n\in\mathbb{Z}} = \emptyset$. For $x \in W$, let $\alpha_{x,k_{i(x)}}$ be a subsequence such that $\{\alpha_{x,k_{i(x)}}\}_{i(x)}$ is a convergent subsequence. Now consider the subsequence $k_{i(x)} + 1$ and take $i(F(x))$ a subsequence of $i(x)$ such that $\{\alpha_{F(x),k_{i(F(x))+1}}\}_{i(F(x))}$ is a convergent subsequence. Repeat the argument for the

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positive iterates $F^n(x)$. By cantor diagonal process there exists a sequence $j(x)$ such that $\alpha_{F^n(x),(k_{j(x)}+n)_{j(x)}}$ is a convergent subsequence for all $n \in \mathbb{N}$. Repeating the argument for the backward iteration yields the desired subsequence in the orbit of x. □

Some comments on the arbitrariness of the set W used in the lemma above may be found in Remark 3.1, after the proof of our main theorem.

One of our goals is to calculate the center Lyapunov exponent for all points of $\mathbb{R}^3$. For that we will need the dynamically defined measures η_x, but they are defined only almost everywhere. We wish we could define these measures for all center leaves. *A priori* this is not possible. What we shall do is to construct some sort of fake dynamically defined conditional measures, good enough for us to compute the Lyapunov exponents for all points.

Given $\xi \in \mathbb{R}^3$, take any sequence $\{\xi_n\} \subset \mathbb{R}^3$ for which η_{ξ_n} is defined for all n and $\lim_{n\to\infty} \xi_n = \xi$. Let $\mathcal{I}_\xi$ be the set of all connected intervals on $\mathcal{F}^c_\xi$. Given $I \in \mathcal{I}_\xi$. define

$$\mathfrak{m}_\xi(I) := \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \eta_{\xi_i}(I_{\xi_i}) \quad \text{for all } I \in \mathcal{I}_\xi,$$

$$\mathfrak{M}_\xi(I) := \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \eta_{\xi_i}(I_{\xi_i}) \quad \text{for all } I \in \mathcal{I}_\xi,$$

where $I_{\xi_i} \in \mathcal{I}_{\xi_i}$ is defined as

$$I_{\xi_i} := \text{bounded interval of } \widetilde{\mathcal{F}}^c_{\xi_i} \setminus \left(\left(\bigcup_{z \in \widetilde{\mathcal{F}}^s(a_I)} \widetilde{\mathcal{F}}^u(z) \right) \cup \left(\bigcup_{z \in \widetilde{\mathcal{F}}^s(b_I)} \widetilde{\mathcal{F}}^u(z) \right) \right)$$

in which a_I and b_I are respectively the bottom and top extreme points of the interval I.

LEMMA 3.4. *There exists $\gamma > 1$ independent of the choice of ξ and $\{\xi_n\}_n$ such that*

$$\gamma^{-1} \operatorname{Leb}_\xi(I) \le \mathfrak{m}_\xi(I) \le \mathfrak{M}_\xi(I) \le \gamma \operatorname{Leb}_\xi(I),$$

for all $I \in \mathcal{I}_\xi$ small enough.

Proof. By [**17**, Theorem B] the unstable holonomy inside the center-unstable leaf and the stable holonomy inside the center-stable leaf are uniformly C^1 as long as I is small enough (i.e. having uniformly bounded length). The uniformity is defined for interval I with a uniform bounded length and the holonomies are uniformly C^1 if we consider them to be close, of some uniformly bounded distance. And from Lemma 3.3 we know that the η_{ξ_i} have uniformly bounded densities, hence the lemma follows. □

Since $F(\mathcal{I}_\xi) = \mathcal{I}_{F(\xi)}$ it makes sense to state the following result.

LEMMA 3.5. *For all $I \in \mathcal{I}_{F^n(\xi)}$ small enough*

$$F^n_* \mathfrak{m}_\xi(I) = \lambda^{-n} \mathfrak{m}_{F^n(\xi)}(I),$$
$$F^n_* \mathfrak{M}_\xi(I) = \lambda^{-n} \mathfrak{M}_{F^n(\xi)}(I).$$

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Proof. We prove the result for $\mathfrak{m}_\xi$; the case $\mathfrak{M}_\xi$ is analogous. Let $I \in \mathcal{I}_{F^n(\xi)}$. Since the stable and unstable foliations are F-invariant and F preserves intervals on center leaves, we have $(F^{-n}(I))_{\xi_i} = F^{-n}(I_{F^n(\xi_i)})$. Then

$$\begin{aligned} F_*^n \mathfrak{m}_\xi(I) = \mathfrak{m}_\xi(F^{-n}(I)) &= \liminf_{J\to\infty} \frac{1}{J} \sum_{i=1}^{J} \eta_{\xi_i}((F^{-n}(I))_{\xi_i}) \\ &= \liminf_{J\to\infty} \frac{1}{J} \sum_{i=1}^{J} \eta_{\xi_i}(F^{-n}(I_{F^n(\xi_i)})) \\ &= \liminf_{J\to\infty} \frac{1}{J} \sum_{i=1}^{J} \lambda^{-n} \eta_{\xi_i}(I_{F^n(\xi_i)}) \\ &= \lambda^{-n} \mathfrak{m}_{F^n(\xi)}(I). \end{aligned}$$

□

Now let us show that the center Lyapunov exponent is defined everywhere and equals $\log(\lambda)$. Using the above two lemmas, we have that, for a given n,

$$\begin{aligned} \gamma^{-1} F_*^n \operatorname{Leb}_\xi(I) = F_*^n(\gamma^{-1} \operatorname{Leb}_\xi)(I) &\le F_*^n(\mathfrak{m}_\xi)(I) \\ &\le \lambda^{-n} \mathfrak{m}_{F^n(\xi)}(I) \le \lambda^{-n} \mathfrak{M}_{F^n(\xi)}(I) \le \gamma F_*^n \operatorname{Leb}_\xi(I). \end{aligned}$$

Dividing through by $F_*^n \operatorname{Leb}_\xi(I)$ and applying $(1/n)\log$ yields

$$\frac{1}{n}\log(\gamma^{-1}) \le \frac{1}{n}\log\left(\lambda^{-n}\frac{\mathfrak{m}_{F^n(\xi)}(I)}{F_*^n \operatorname{Leb}_\xi(I)}\right) \le \frac{1}{n}\log\left(\lambda^{-n}\frac{\mathfrak{M}_{F^n(\xi)}(I)}{F_*^n \operatorname{Leb}_\xi(I)}\right) \le \frac{1}{n}\log(\gamma).$$

Now let us shrink the interval I. Let $x \in I$ and define I_ϵ as the interval inside the center foliation of radius ϵ centered on x. Therefore,

$$\begin{aligned} \lim_{\epsilon\to 0} \lambda^{-n} \frac{\mathfrak{m}_{F^n(\xi)}(I_\epsilon)}{F_*^n \operatorname{Leb}_\xi(I_\epsilon)} &= \lim_{\epsilon\to 0} \lambda^{-n} \frac{\mathfrak{m}_{F^n(\xi)}(I_\epsilon)}{\operatorname{Leb}_\xi(F^{-n}(I_\epsilon))} \\ &= \lim_{\epsilon\to 0} \lambda^{-n} \frac{\int_{I_\epsilon} \rho_{F^n(\xi)}\, d\operatorname{Leb}_{F^n(\xi)}}{\operatorname{Leb}_\xi(F^{-n}(I_\epsilon))}. \end{aligned} \tag{4}$$

But we know that $\mathfrak{m}_{F^n(\xi)} (= \rho_{F^n(\xi)} \operatorname{Leb}_{F^n(\xi)})$ has bounded density with respect to the Lebesgue measure, so for some universal constant β,

$$\beta^{-1} \int_{I_\epsilon} d\operatorname{Leb}_{F^n(\xi)} \le \int_{I_\epsilon} \rho_{F^n(\xi)}\, d\operatorname{Leb}_{F^n(\xi)} \le \beta \int_{I_\epsilon} d\operatorname{Leb}_{F^n(\xi)}.$$

The two estimates above imply that

$$\begin{aligned} \beta^{-1}\lambda^{-n}\|DF^{-n}|\widetilde{E}^c(x)\|^{-1} &\le \lim_{\epsilon\to 0} \lambda^{-n} \frac{\int_{I_\epsilon} \rho_{F^n(\xi)}\, d\operatorname{Leb}_{F^n(\xi)}}{\operatorname{Leb}_\xi(F^{-n}(I_\epsilon))} \\ &\le \beta\lambda^{-n}\|DF^{-n}|\widetilde{E}^c(x)\|^{-1}. \end{aligned}$$

Hence, applying $(1/n)\log$ and passing to the limit as n goes to infinity, we get

$$\begin{aligned} -\log\lambda - \lim_{n\to\infty}\frac{1}{n}\|DF^{-n}|\widetilde{E}^c(x)\| &\le \lim_{n\to\infty}\lim_{\epsilon\to 0} \lambda^{-n} \frac{\int_{I_\epsilon} \rho_{F^n(\xi)}\, d\operatorname{Leb}_{F^n(\xi)}}{\operatorname{Leb}_\xi(F^{-n}(I_\epsilon))} \\ &\le -\log\lambda - \lim_{n\to\infty}\frac{1}{n}\|DF^{-n}|\widetilde{E}^c(x)\|. \end{aligned}$$

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From equation (4) take I to be I_ϵ as defined above and, passing to the limit as n goes to infinity, we get

$$0 \leq -\log\lambda - \lim_{n\to\infty} \frac{1}{n}\|DF^{-n}|\widetilde{E^c}(x)\| \leq 0.$$

This implies that

$$\lim_{n\to\infty} \frac{1}{-n}\|DF^{-n}|\widetilde{E^c}(x)\| = \log\lambda \quad \text{for all } x \in \mathbb{R}^3. \tag{5}$$

The above means that the center Lyapunov exponent of F^{-1} equals $\log(\lambda^{-1})$ for every point. In particular, because $\pi : \mathbb{R}^3 \to \mathbb{T}^3$ is a local isometry and $\pi \circ F^{-1} = f^{-1} \circ \pi$ the center Lyapunov exponent of $f^{-1} : \mathbb{T}^3 \to \mathbb{T}^3$ is defined for every point in $\mathbb{T}^3$ and equals $\log(\lambda^{-1})$.

LEMMA 3.6. *The diffeomorphism f is in fact an Anosov diffeomorphism.*

Proof. We know that $\lim_{n\to\infty} \|Df^n|E^c_f\| = \log(\lambda)$, for all $x \in \mathbb{T}^3$. Consider $\varepsilon > 0$ satisfying $\lambda_\varepsilon := \lambda - \varepsilon > 0$. Since the center exponent exists for every x it follows that, given $x \in \mathbb{T}^3$, there are $n_x \in \mathbb{N}$ and a neighborhood $\mathcal{U}_x$ of x such that for all $x \in \mathcal{U}_x$, $|Df^{n_x}|E^c| \geq e^{n_x\lambda_\varepsilon}$. Since $\mathbb{T}^3$ is a compact manifold, take a finite cover $\mathcal{U}_{x_1} \ldots \mathcal{U}_{x_l}$. Let $C_i < 1$ be small enough so that for $x \in \mathcal{U}_{x_i}$. Then $|Df^n(x)|E^c| \geq C_{x_i} e^{n\lambda_\varepsilon}$ for all $n \in \{0, 1, \ldots, n_{x_i}\}$. Let $C := \min_i\ C_{x_i}$. Then $|Df^n(x)|E^c| \geq Ce^{n\lambda_\varepsilon}$ for all $x \in \mathbb{T}^3$ and $n \in \mathbb{N}$. Hence the center foliation is expanding, therefore f is in fact an Anosov diffeomorphism. □

LEMMA 3.7. *f is C^∞ conjugate to its linearization.*

Proof. Let us first prove that, for all $x \in \mathrm{Per}(f)$,

$$\lambda^s_f(x) = \lambda^s_A, \quad \lambda^c_f(x) = \lambda^c_A, \quad \lambda^u_f(x) = \lambda^u_A,$$

where λ^*_g is the Lyapunov exponent in the direction $* \in \{s, c, u\}$ of the map $g \in \{f, A\}$. From equation (5) we know that $\lambda^c_f(x) = \lambda^c_A$ on any periodic point $x \in \mathbb{T}^3$ for f. Since the center foliation is absolutely continuous and f is an Anosov diffeomorphism, we get that $\lambda^u_f(x) = \lambda^u_A$ by [**8**]. Since f and A are volume preserving, $\lambda^s_f(x) + \lambda^c_f(x) + \lambda^u_f(x) = 0$ and $\lambda^s_A(x) + \lambda^c_A(x) + \lambda^u_A(x) = 0$, which implies that $\lambda^s_f(x) = \lambda^s_A$ for all periodic points $x \in \mathbb{T}^3$ of f. Hence, if the Lyapunov exponents are constant on periodic points for any direction, then f is C^1 conjugate to its linearization by [**22**, Proposition 1.1].

Recall that $f = h^{-1} \circ A \circ h$. Take $h_0 : \mathbb{T}^3 \to \mathbb{T}^3$ to be a C^∞ diffeomorphism C^1 close to h. Note that

$$h_0 \circ f \circ h_0^{-1} = h_0 \circ h^{-1} \circ A \circ h \circ h_0^{-1} = (h \circ h_0^{-1})^{-1} \circ A \circ (h \circ h_0^{-1}).$$

Then $h_0 \circ f \circ h_0^{-1}$ is C^1 close to A since $(h \circ h_0^{-1})^{-1}$ and $(h \circ h_0^{-1})$ are close to the identity. Observe that $h \circ h_0$ is a C^1 conjugacy between $h_0 \circ f \circ h_0^{-1}$ and A; then it naturally satisfies the smooth conjugacy hypothesis from [**9**], therefore we obtain that $h_0 \circ f \circ h_0^{-1}$ is C^∞ conjugate to A. Now because h_0 and $h \circ h_0^{-1}$ are C^∞, h is also C^∞, as we wanted to show. □

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We have proven above that UBD property implies C^∞ conjugacy. It is easy to see that the converse is true. The theorem is now proven. □

Remark 3.1. It is worth noting that the construction of the measures η_x (Lemma 3.3) on the universal cover are used simply as a tool to obtain information on the Lyapunov exponent. Although the measures η_x are intimately connected with the Rohklin disintegration of volume on the center foliations, they lack a very important property: they do not *a priori* vary measurably. In our case we did not need them to vary in such a way. Besides that, these measures really live on the universal cover, that is, we cannot project them coherently on $\mathbb{T}^3$: two measures η_x and η_y in $\widetilde{\mathcal{F}}^c_x$ and $\widetilde{\mathcal{F}}^c_y$ respectively and such that $\pi(x) = \pi(y)$ then most likely satisfy $\pi_*\eta_x \neq \pi_*\eta_y$. But because we know that in the end f is smoothly conjugate to its linearization we can induce the length measure from the linearization to the dynamics of f and obtain such dynamically defined measures varying measurably. Hence, an interesting program of study to tackle rigidity problems related to invariant foliations would be to better understand partially hyperbolic dynamics which admits measurably dynamically defined measures (e.g. $f_*\mu_x = \lambda\mu_{f(x)}$). With this generality one should not expect to obtain smooth rigidity results *a priori*, but one should start by obtaining rigidity results from the measurability point of view. That would most likely lead to implications for the Lyapunov exponent, entropy, metric isomorphisms and so on.

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