J. Plasma Physics (1997), *vol.* 58, *part* 2, *pp.* 277–285. Printed in the United Kingdom

© 1997 Cambridge University Press

Parametric decay of an upper-hybrid wave in a plasma cylinder

S. C. SHARMA

Physics Department, IIT Delhi, New Delhi 110 016, India

(Received 21 February 1996 and in revised form 15 July 1996)

A large-amplitude upper-hybrid wave in a plasma cylinder is susceptible to parametric decay into a lower-hybrid wave and an upper-hybrid wave. The perpendicular and parallel motions of electrons play equally important roles in the nonlinear interaction. The coupled-mode equations are solved using a perturbation technique. The growth rate of the instability is less sensitive to the radial mode number of the sideband. However, for higher-order radial modes it falls off as the overlap of the interacting wave functions decreases.

1. Introduction

The study of parametric instabilities of large-amplitude electrostatic (Wolf *et al.* 1980; Liu *et al.* 1982, 1984; Akimoto 1989; Stenflo 1994) and electromagnetic (Sharma *et al.* 1984; Saleem and Murtaza 1986) waves have been an active field of plasma research for the last several decades. The studies are relevant to radiofrequency current drive and heating of fusion plasmas (Tripathi *et al.* 1979; Wolf *et al.* 1980; Liu and Tripathi 1986) laser-driven fusion (Kruer 1987), space plasmas and small-scale laboratory experiments (Jain and Christiansen 1981, 1984; Praburam and Sharma 1992). Lately there has been interest in employing large-amplitude plasma waves, lower-hybrid waves and upper-hybrid waves as wigglers for the generation of short-wavelength radiation in a free-electron laser. Here again one must ensure that the wiggler wave is not susceptible to undesirable decay instability. Yu and Shukla (1977) have examined theoretically the nonlinear modulation of a upper-hybrid wave. In the present paper we examine the decay instability of an upper-hybrid electrostatic wave into lower-hybrid and upper-hybrid waves in a cylindrical plasma column. The pump wave imparts an oscillatory velocity to the electrons. When the equilibrium is perturbed by a low-frequency lower-hybrid wave $(\omega, \mathbf{k})$ and a high-frequency $(\omega_1, \mathbf{k}_1)$ upper-hybrid sideband wave, the beat of the pump wave and the sideband wave $(\omega_1, \mathbf{k}_1)$ exerts a low-frequency ponderomotive force on the electrons driving the low-frequency mode. The density perturbation at $(\omega, \mathbf{k})$ couples with the oscillatory velocity at $(\omega_0, \mathbf{k}_0)$ to produce a nonlinear current at $(\omega_1, \mathbf{k}_1)$, driving the sideband wave. In Sec. 2 we employ fluid theory to obtain the growth rate of the instability. The results are given in Sec. 3.

2. Instability analysis

Consider a cylindrical plasma column of radius a and equilibrium plasma density n_0^0, immersed in a static magnetic field $\mathbf{B}_s \parallel \hat{\mathbf{z}}$. Following Christiansen *et*

https://doi.org/10.1017/S0022377897005643 Published online by Cambridge University Press

al. (1981), the plasma is surrounded by vacuum and a metallic cylinder of radius $b > a$. An upper-hybrid pump wave propagates through it:

$$\phi_0 = \phi_0(r)\, e^{-i(\omega_0 t - k_{0z} z)}. \tag{1}$$

The equation governing $\phi_0(r)$ can be obtained from the local dispersion relation for the upper-hybrid wave,

$$k_0^2 - \frac{\omega_p^2}{\omega_0^2 - \omega_c^2} k_{0\perp}^2 - \frac{\omega_p^2}{\omega_0^2} k_{0z}^2 = 0, \tag{2}$$

by replacing $k_{0\perp}^2$ by $-\nabla_\perp^2$ and applying the resulting operator equation to ϕ_0:

$$\nabla_\perp^2 \phi_0 + p_{m0}^2 \phi_0 = 0, \tag{3}$$

where

$$p_{m0}^2 = \frac{(\omega_0^2 - \omega_p^2)(\omega_0^2 - \omega_c^2)}{(\omega_0^2 - \omega_p^2 - \omega_c^2)\,\omega_0^2} k_{0z}^2. \tag{4}$$

In the vacuum region,

$$\nabla_\perp^2 \phi_0 - k_{0z}^2 \phi_0 = 0. \tag{5}$$

For azimuthally symmetric modes ($\partial/\partial\theta = 0$), the solutions of the above equations are

$$\phi_0(r) = \begin{cases} C_0 J_0(p_{m0} r) & \text{for } r < a, \quad (6) \\ C_{01} I_0(k_{0z} r) + C_{02} K_0(k_{0z} r) & \text{for } r > a. \quad (7) \end{cases}$$

Applying the continuity of ϕ_0 and $\partial\phi_0/\partial r$ across $r = a$ and $\phi_0(b) = 0$, one obtains the constants and the dispersion relation:

$$C_{02} = -C_{01} \frac{I_0(k_{0z} b)}{k_0(k_{0z} b)}, \tag{8}$$

$$C_{01} = \frac{C_0 J_0(p_{m0} a)}{I_0(k_{0z} a) - \dfrac{I_0(k_{0z} b)}{k_0(k_{0z} b)} k_0(k_{0z} a)}, \tag{9}$$

$$\frac{p_{m0}}{k_{0z}} J_0'(p_{m0} a) = J_0(p_{m0} a) \frac{I_0'(k_{0z} a)\, K_0(k_{0z} b) - I_0(k_{0z} b)\, k_0'(k_{0z} a)}{I_0(k_{0z} a)\, k_0(k_{0z} b) - I_0(k_{0z} b)\, K_0(k_{0z} a)}. \tag{10}$$

Since k_{0z}^2 is expressible in terms of p_{m0}^2, as given above, the dispersion relation can be solved to obtain p_{m0}, which has many roots $p_0 = p_{m0}$. In terms of p_{m0} and k_{0z} the eigenfrequencies of the upper-hybrid modes are

$$\omega_0^2 = \frac{1}{2}\left\{\omega_p^2 + \omega_c^2 + \left[(\omega_p^2 + \omega_c^2)^2 - 4\omega_p^2 \omega_c^2 \frac{k_{0z}^2}{k_{0z}^2 + p_{m0}^2}\right]^{1/2}\right\}. \tag{11}$$

We consider only the m_0th mode as the pump wave. The pump wave produces an oscillatory velocity and density of electrons:

$$\mathbf{v}_{0\perp} = \frac{e}{m} \frac{\boldsymbol{\nabla}_\perp \phi_0 \times \boldsymbol{\omega}_c + i\omega_0 \boldsymbol{\nabla}_\perp \phi_0}{\omega_0^2 - \omega_c^2}, \tag{12}$$

https://doi.org/10.1017/S0022377897005643 Published online by Cambridge University Press

$$v_{0z} = \frac{-ek_{0z}\phi_0}{m\omega_0}, \tag{13}$$

$$n_0 = -n_0^0 \frac{e}{m}\left(\frac{-\nabla_\perp^2\phi_0}{\omega_0^2-\omega_c^2}+\frac{k_{0z}^2\phi_0}{\omega_0^2}\right) = -\frac{k_{0z}^2+p_{m0}^2}{4\pi e}\phi_0, \tag{14}$$

where $-e$, m, ω_0, ω_p and ω_c are the electron charge, mass, pump wave, plasma frequency and cyclotron frequency respectively.

The oscillatory electron motion couples with a high-frequency sideband wave ϕ_1 and a low-frequency wave ϕ:

$$\phi_1 = \phi_1(r)\, e^{-i(\omega_1 t - k_{1z} z)}, \tag{15}$$

$$\phi = \phi(r)\, e^{-i(\omega t - k_z z)}. \tag{16}$$

In the absence of the pump, the mode-structure equations for $\phi_1(r)$ are the same as those for ϕ_0 but with ω_0, k_{0z} and p_{m0} replaced by ω_1, k_{1z} and p_{m1}. The eigenfrequencies are given by

$$\omega_1^2 = \frac{1}{2}\left\{\omega_p^2+\omega_c^2+\left[(\omega_p^2+\omega_c^2)^2-4\omega_p^2\omega_c^2\frac{k_{1z}^2}{k_{1z}^2+p_{m1}^2}\right]^{1/2}\right\}. \tag{17}$$

The mode-structure equation for $\phi(r)$ can be deduced from the local dispersion relation for the lower-hybrid wave,

$$k^2+\frac{\omega_p^2}{\omega_c^2}k_\perp^2-\frac{\omega_p^2}{\omega^2}k_z^2-\frac{\omega_{pi}^2}{\omega^2}k^2 = 0, \tag{18}$$

by replacing $k_\perp^2$ by $-\nabla^2$ and applying the resulting operator equation to ϕ:

$$\nabla_\perp^2\phi + p_m^2\phi = 0 \quad \text{for } r < a, \tag{19}$$

where

$$p_m^2 = \frac{k_z^2(-1+\omega_p^2/\omega^2)}{(1+\omega_p^2/\omega_c^2)(1-\omega_{LH}^2/\omega^2)}, \qquad \omega_{LH}^2 = \frac{\omega_{pi}^2}{1+\omega_p^2/\omega_c^2}. \tag{20}$$

In the vacuum region,

$$\nabla_\perp^2\phi - k_z^2\phi = 0. \tag{21}$$

Following the procedure adopted for solving the mode-structure equations for ϕ_0, one obtains the same solutions as for ϕ_0 but with ω_0, k_{0z} and p_{m0} replaced by ω, k_z and p_m. The dispersion relation is also the same. However, the mode under consideration corresponds to $p = p_m$, in terms of which the eigenfrequency of the mth mode is

$$\omega^2 = \omega_{LH}^2\left(1+\frac{k_z^2\, m_i/m}{P_m^2}\right). \tag{22}$$

The phase-matching conditions for the parametric decay require that

$$\omega_1 = \omega-\omega_0, \qquad k_{1z} = k_z - k_{0z}, \tag{23}$$

i.e.

$$k_{1z} = \left[\left(\frac{\omega^2}{\omega_{LH}^2}-1\right)\frac{m}{m_i}p_m^2\right]^{1/2} - k_{0z}, \tag{24}$$

where ω_{LH}, ω_{pi} and m_i are the lower-hybrid wave frequency, the ion plasma frequency and the ions mass respectively.

https://doi.org/10.1017/S0022377897005643 Published online by Cambridge University Press

The linear response of the electrons at (ω_1, k_{1z}) can be obtained from (12)–(14) on replacing subscript '0' by '1'. The pump and sideband waves exert a low-frequency ponderomotive force

$$\mathbf{F}_p \equiv -e\mathbf{E}_p = \frac{-1}{2} m(\mathbf{v}_0 \cdot \boldsymbol{\nabla}\mathbf{v}_1 + \mathbf{v}_1 \cdot \boldsymbol{\nabla}\mathbf{v}_0)$$

on the electrons. The response of the electrons at (ω, k_z) due to $\mathbf{F}_p$ and the self-consistent field $-\boldsymbol{\nabla}\phi$ is obtained by solving the equations of motion and continuity:

$$\mathbf{v}_\perp \approx -\frac{e}{m\omega_c^2}[(i\omega(\boldsymbol{\nabla}_\perp \phi - \mathbf{E}_{p\perp}) + (\boldsymbol{\nabla}_\perp \phi - \mathbf{E}_{p\perp}) \times \boldsymbol{\omega}_c], \tag{25}$$

$$v_z = -\frac{ek_z}{m\omega}\left(\phi - \frac{E_{pz}}{ik_z}\right), \tag{26}$$

$$n_1 = n_0^0 \frac{\boldsymbol{\nabla}\cdot\mathbf{v}}{i\omega}. \tag{27}$$

The ion-density perturbation is

$$n_i = n_0^0\left(\frac{-e\nabla_\perp^2 \phi}{m_i \omega^2} + \frac{ek_z^2 \phi}{m_i \omega^2}\right). \tag{28}$$

Using (27) and (28) in Poisson's equation, we obtain

$$\begin{aligned}\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + p^2\phi = {} & \frac{\omega_p^2 e\omega_0\omega_1 C_1}{2m\omega_c^2\left(1 + \dfrac{\omega_p^2}{\omega_c^2} - \dfrac{\omega_{pi}^2}{\omega^2}\right)} \\ & \times \left\{ \frac{p_{m_1}\left[\dfrac{\partial^2}{\partial r^2}\left(\dfrac{\partial\phi_0}{\partial r} J_1(p_{m_1} r)\right) + \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(\dfrac{\partial\phi_0}{\partial r} J_1(p_{m_1} r)\right)\right]}{(\omega_0^2 - \omega_c^2)(\omega_1^2 - \omega_c^2)} \right. \\ & + \frac{k_{0z} k_{1z} J_0(p_{m_1} r)}{\omega_0^2}\left(\frac{-p_{m_1}^2 \phi_0}{\omega_1^2 - \omega_c^2} + \frac{\nabla_\perp^2 \phi_0}{\omega_0^2 - \omega_c^2}\right) \\ & - \frac{k_z \omega_c^2}{\omega^2 \omega_0 \omega_1}\left[\frac{\omega_0 k_{1z}}{\omega_1(\omega_0^2 - \omega_c^2)} - \frac{\omega_1 k_{0z}}{\omega_0(\omega_1^2 - \omega_c^2)}\right] J_1(p_{m_1} r)\, p_{m_1} \frac{\partial\phi_0}{\partial r} \\ & \left. + \frac{\phi_0}{\omega^2\omega_0^2\omega_1^2} J_0(p_{m_1} r)\, \omega_c^2 k_{0z} k_{1z} k_z^2 \right\}. \end{aligned} \tag{29}$$

The nonlinear response of the electrons at the sideband wave appears through the equation of continuity:

$$\frac{\partial n_1}{\partial t} + \boldsymbol{\nabla}\cdot(n_0^0 \mathbf{v}_1 + \tfrac{1}{2} n \mathbf{v}_0^*) = 0, \tag{30}$$

$$\begin{aligned} n_1 = {} & \frac{-n_0^0 e}{m}\left(\frac{-\nabla_\perp^2 \phi_1}{\omega_1^2 - \omega_c^2} + \frac{k_{1z}^2 \phi_1}{\omega_1^2}\right) \\ & + \frac{1}{2}\frac{n_0^0 e^2 \omega_0}{m^2\omega_1}\left(\frac{\nabla_\perp^2 \phi}{\omega_c^2} + \frac{k_z^2 \phi}{\omega^2}\right)\left(\frac{k_{0z}^2 \phi_0}{\omega_0^2} + \frac{\nabla_\perp^2 \phi_0}{\omega_0^2 - \omega_c^2}\right) - \frac{1}{2}\frac{e\omega_0}{m\omega_1}\frac{\dfrac{\partial n}{\partial r}\dfrac{\partial\phi_0}{\partial r}}{\omega_0^2 - \omega_c^2}. \end{aligned} \tag{31}$$

https://doi.org/10.1017/S0022377897005643 Published online by Cambridge University Press

Using (31) in Poisson's equation, we obtain

$$\frac{\partial^2\phi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_1}{\partial r}+p_1^2\phi_1=\frac{1}{2}\frac{\omega_p^2\,eC\omega_0}{m\omega_1\left(1-\dfrac{\omega_p^2}{\omega_1^2-\omega_c^2}\right)}\left(\frac{-p_m^2}{\omega_c^2}+\frac{k_z^2}{\omega^2}\right)$$
$$\times\left[J_0(p_m r)\left(\frac{k_{0z}^2\phi_0}{\omega_0^2}+\frac{\nabla_\perp^2\phi_0}{\omega_0^2-\omega_c^2}\right)-\frac{J_1(p_m r)\,p_m}{\omega_0^2-\omega_c^2}\frac{\partial\phi_0}{\partial r}\right]. \quad (32)$$

Equation (29) gives the linear dispersion relation $\phi(r)=C\Gamma_{0,m}J_0(p_m r)$. The right-hand side of (29) does not lead to any significant change in the eigenfunctions. However, the eigenvalues are modified. To solve (29) by a perturbation technique, we substitute $\phi=CJ_0(p_{mn}r)$ into it, multiply both sides by $rJ_0(p_{mn}r)$ and integrate over r from 0 to a. Retaining only the dominant mode $m=n$, we obtain

$$NC(p^2-p_m^2)=\frac{\omega_p^2\,e\omega_0\,\omega_1\,C_1}{m\omega_c^2\left(1+\dfrac{\omega_p^2}{\omega_c^2}-\dfrac{\omega_{pi}^2}{\omega^2}\right)a^2J_1^2(p_{mn}a)}$$
$$\times\left\{\frac{p_{m_1}I_1}{(\omega_0^2-\omega_c^2)(\omega_1^2-\omega_c^2)}+\frac{k_{0z}\,k_{1z}I_2}{\omega_0^2}\left(-\frac{p_{m_1}^2\phi_0}{\omega_1^2-\omega_c^2}+\frac{\nabla_\perp^2\phi_0}{\omega_0^2-\omega_c^2}\right)\right.$$
$$\left.-\frac{k_z\omega_c^2}{\omega^2}\left[\frac{k_{1z}}{\omega_1^2(\omega_0^2-\omega_c^2)}-\frac{k_{0z}}{\omega_0^2(\omega_1^2-\omega_c^2)}\right]p_{m_1}I_3+\frac{k_{0z}\,k_{1z}\,k_z^2}{\omega^2\omega_0^2\,\omega_1^2}\omega_c^2I_4\right\}, \quad (33)$$

where

$$I_1=\int_0^a\left\{\frac{\partial^2}{\partial r^2}\left[\frac{\partial\phi_0}{\partial r}J_1(p_{m_1}r)\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\partial\phi_0}{\partial r}J_1(p_{m_1}r)\right]\right\}rJ_0(p_{mn}r)\,dr,$$

$$I_2=\int_0^a J_0(p_{m_1}r)\,J_0(p_{mn}r)\,r\,dr,$$

$$I_3=\int_0^a J_1(p_{m_1}r)\frac{\partial\phi_0}{\partial r}J_0(p_{mn}r)\,r\,dr,$$

$$I_4=\int_0^a\phi_0\,J_0(p_{m_1}r)\,J_0(p_{mn}r)\,r\,dr,$$

$$N=\int_0^a J_0^2(p_m r)\,r\,dr+\frac{C_{01}^2}{C_0^2}\int_a^b\left[I_0(k_z r)-\frac{I_0(k_z b)\,k_0(k_z r)}{k_0(k_z b)}\right]^2 r\,dr.$$

Equation (33) can be rewritten as

$$\left(1+\frac{\omega_p^2}{\omega_c^2}\frac{p_m^2}{p_m^2+k_z^2}-\frac{\omega_p^2}{\omega^2}\frac{k_z^2}{p_m^2+k_z^2}-\frac{\omega_{pi}^2}{\omega^2}\frac{p_m^2}{p_m^2+k_z^2}\right)C$$
$$=-\frac{\omega_p^2\,e\omega_0\,\omega_1\,C_1}{m\omega_c^2\,a^2J_1^2(p_{mn}a)(p_m^2+k_z^2)}\left\{\frac{p_{m_1}I_1}{(\omega_0^2-\omega_c^2)(\omega_1^2-\omega_c^2)}+\frac{k_{0z}\,k_{1z}I_2}{\omega_0^2}\left(-\frac{p_{m_1}^2\phi_0}{\omega_1^2-\omega_c^2}+\frac{\nabla_\perp^2\phi_0}{\omega_0^2-\omega_c^2}\right)\right.$$
$$\left.-\frac{\omega_c^2}{\omega^2}\left[\frac{k_z\,k_{1z}}{\omega_1^2(\omega_0^2-\omega_c^2)}-\frac{k_z\,k_{0z}}{\omega_0^2(\omega_1^2-\omega_c^2)}\right]p_{m_1}I_3+\frac{k_{0z}\,k_{1z}\,k_z^2\,\omega_c^2}{\omega^2\omega_0^2\,\omega_1^2}I_4\right\}. \quad (34)$$

https://doi.org/10.1017/S0022377897005643 Published online by Cambridge University Press

Using a perturbation technique as was done earlier (32) gives

$$N_1(p_1^2-p_{m_1}^2)\,C_1 = \frac{\omega_p^2\, eC\omega_0}{m\omega_1\left(1-\dfrac{\omega_p^2}{\omega_1^2-\omega_c^2}\right)a^2 J_1^2(p_{m_1 n}\,a)}\left(\frac{-p_m^2}{\omega_c^2}+\frac{k_z^2}{\omega^2}\right)$$

$$\times\left[\left(\frac{k_{0z}^2\,\phi_0}{\omega_0^2}+\frac{\nabla_\perp^2\,\phi_0}{\omega_0^2-\omega_c^2}\right)I_1' - \frac{I_2'\,p_m}{(\omega_0^2-\omega_c^2)}\right], \tag{35}$$

where

$$I_1' = \int_0^a J_0(p_m\, r)\, J_0(p_{m_1 n}\, r)\, r\, dr,$$

$$I_2' = \int_0^a J_1(p_m\, r)\,\frac{\partial\phi_0}{\partial r}\, J_0(p_{m_1 n}\, r)\, r\, dr,$$

and $N_1 = N$, where $k_z = k_{1z}$ and $p_m = p_{m1}$. Equation (35) can be written as

$$\left[1-\frac{\omega_p^2}{\omega_1^2-\omega_c^2}\frac{p_{m_1}^2}{p_{m_1}^2+k_{1z}^2}-\frac{\omega_p^2}{\omega_1^2}\frac{k_{1z}^2}{(p_{m_1}^2+k_{1z}^2)}\right]C_1 = \frac{\omega_p^2\, e\omega_0\, C}{m\omega_1\, a^2 J_1^2(p_{m_1 n}\,a)\,(p_{m_1}^2+k_{1z}^2)}\left(\frac{-p_m^2}{\omega_c^2}+\frac{k_z^2}{\omega^2}\right)$$

$$\times\left[\left(\frac{k_{0z}^2\,\phi_0}{\omega_0^2}+\frac{\nabla_\perp^2\,\phi_0}{\omega_0^2-\omega_c^2}\right)I_1' - \frac{I_2'\,p_m}{\omega_0^2-\omega_c^2}\right]. \tag{36}$$

Combining (34) and (36), we obtain the nonlinear dispersion relation

$$\epsilon_1(\omega_1)\,\epsilon(\omega) = -\frac{\omega_p^4\, e^2\omega_0^2\, PQ}{m^2\omega_c^2\, a^4 J_1^2(p_{mn}\,a)\, J_1^2(p_{m_1 n}\,a)\,(p_m^2+k_z^2)\,(p_{m_1}^2+k_{1z}^2)}\left(\frac{-p_m^2}{\omega_c^2}+\frac{k_z^2}{\omega^2}\right), \tag{37}$$

where

$$\epsilon_1(\omega_1) = 1-\frac{\omega_p^2}{\omega_1^2-\omega_c^2}\frac{p_{m_1}^2}{p_{m_1}^2+k_{1z}^2}-\frac{\omega_p^2}{\omega_1^2}\frac{k_{1z}^2}{p_{m_1}^2+k_{1z}^2},$$

$$\epsilon(\omega) = 1+\frac{\omega_p^2}{\omega_c^2}\frac{p_m^2}{p_m^2+k_z^2}-\frac{\omega_p^2}{\omega^2}\frac{k_z^2}{p_m^2+k_z^2}-\frac{\omega_{pi}^2}{\omega^2}\frac{p_m^2}{p_m^2+k_z^2},$$

$$P = \frac{p_{m_1} I_1}{(\omega_0^2-\omega_c^2)\,(\omega_1^2-\omega_c^2)}-\frac{\omega_c^2}{\omega^2}\left[\frac{k_z\, k_{1z}}{\omega_1^2(\omega_0^2-\omega_c^2)}-\frac{k_z\, k_{0z}}{\omega_0^2(\omega_1^2-\omega_c^2)}\right]p_{m_1} I_3,$$

$$Q = \left(\frac{k_{0z}^2\,\phi_0}{\omega_0^2}+\frac{\nabla_\perp^2\,\phi_0}{\omega_0^2-\omega_c^2}\right)I_1' - \frac{I_2'\,p_m}{\omega_0^2-\omega_c^2}.$$

When equated to zero, $\epsilon_1(\omega_1)$ and $\epsilon(\omega)$ give the upper-hybrid wave of frequency ω_1 and the lower-hybrid wave of frequency ω as defined in (17) and (22) respectively. We write $\omega_1 = \omega_{1r} + i\gamma$ and $\omega = \omega_r + i\gamma$, and expand $\epsilon_1(\omega_1)$ and $\epsilon(\omega)$ as

$$\epsilon_1(\omega_1) = \frac{\partial\epsilon_1(\omega_1)\, i\gamma}{\partial\omega_1}, \qquad \epsilon(\omega) = \frac{\partial\epsilon(\omega)\, i\gamma}{\partial\omega}.$$

https://doi.org/10.1017/S0022377897005643 Published online by Cambridge University Press

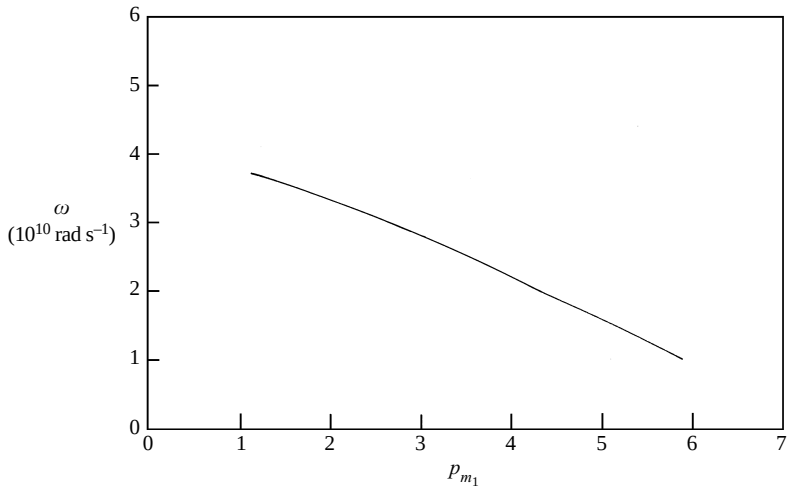


Figure 1. Frequency ω of the lower-hybrid wave as a function of the radial mode number of the sideband.

The growth rate turns out to be

$$\gamma = \frac{1}{2}\left[\frac{\omega_p^4 e^2 \omega_0^2 PQ}{m^2 \omega_c^2 a^4 J_1^2(p_{mn}a) J_1^2(p_{m_1 n}a)(p_m^2 + k_z^2)(k_{1z}^2 + p_{m_1}^2) RS}\left(\frac{-p_m^2}{\omega_c^2} + \frac{k_z^2}{\omega^2}\right)\right]^{1/2}, \quad (38)$$

where

$$R = \frac{\omega_p^2 \omega_1 p_{m_1}^2}{(\omega_1^2 - \omega_c^2)^2 (p_{m_1}^2 + k_{1z}^2)} + \frac{\omega_p^2 k_{1z}^2}{\omega_1^3 (p_{m_1}^2 + k_{1z}^2)},$$

$$S = \frac{\omega_p^2}{\omega^3}\frac{k_z^2}{p_m^2 + k_z^2} + \frac{\omega_{pi}^2}{\omega^3}\frac{p_m^2}{p_m^2 + k_z^2}.$$

In Fig. 1 we have plotted the frequency of the lower-hybrid wave ω (cf. (22) and (24)) as a function of the radial mode number of the sideband wave p_{m_1} for the following parameters: $\omega_p^2/\omega_c^2 = 0.8$, plasma density $n_0^0 = 3.1 \times 10^{12}$ cm^{-3}, static magnetic field $B_s = 6$ kG, pump-wave frequency $\omega_0 = 1.20 \times 10^{11}$ rad s^{-1}, radial mode number of the pump wave $p_{m0} = 7.4$, axial wavenumber of the pump wave $k_{0z} = 27.39$ cm^{-1}, radius of the plasma cylinder $a = 2$ cm, lower-hybrid frequency $\omega_{LH} = 8.1 \times 10^8$ rad s^{-1}, and $m_e/m_i = 1.4 \times 10^{-4}$ (helium plasma). The frequency of the lower-hybrid mode decreases linearly as the radial mode number of the sideband increases.

In Fig. 2 we have plotted the normalized growth rate of the instability as a function of the radial mode number of the sideband wave for $v_{osc} = 20c_s$, where c_s is the sound velocity. Other parameters are the same as above. The growth rate peaks at $p_{m_1} \approx 4.3$. At higher radial mode numbers it falls off monotonically as the overlap of the interacting wave functions decreases. It is also sensitive to the frequency and phase velocity of the lower-hybrid wave. By solving (38) numerically for the parameters mentioned above and $p_{m_1} = 1.2$, we find $\gamma = 3.6 \times 10^6$ s^{-1}. At $V_{\rm osc} \approx 10^{-2}c$ the growth rate is comparable to the growth rate of the parametric decay instability of the upper-hybrid wave. The power

https://doi.org/10.1017/S0022377897005643 Published online by Cambridge University Press

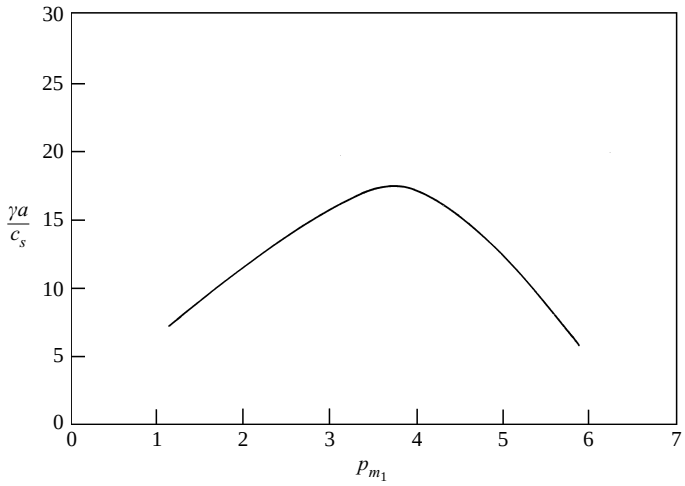


Figure 2. Normalized growth rate of the instability as a function of the radial mode number of the sideband for an oscillatory velocity $v_{\rm osc} = 20c_s$ (where c_s is the sound velocity of the ions).

density p_0 of the upper-hybrid pump wave, $\omega_0(\partial\epsilon_0/\partial\omega_0)(E_0^2/8\pi)\,v_{gz}$, necessary to achieve $v_{\rm osc} \approx 10^{-2}c$ turns out to be about 0.30 kW cm^{-2}. The growth rate scales as $k_{1\perp}$, $v_{\rm osc}$, $\omega^{1/2}$ and $\omega_p^{-1/2}$.

3. Discussion

Parametric decay of upper-hybrid waves in a low-electron-temperature plasma cylinder appears to be important at moderate powers. Boundary effects reduce the growth rate of the parametric decay instability into lower-hybrid and upper-hybrid waves, owing to incomplete overlap of the wave functions of the interacting waves. The frequency of the lower-hybrid wave falls nearly linearly with the radial mode number p_{m_1} of the sideband wave. The growth rate initially increases with p_{m_1}, reaches a maximum and then falls off monotonically. For a pump power density $P_0 \approx 0.30$ kW cm^{-2}, the growth rate of the decay instability could be a potential contender in an upper-hybrid wiggler free-electron laser. In other laboratory plasmas also, the parametric instability of upper-hybrid waves is significant. In a fusion plasma (e.g. in a tokamak), where the electron temperature is high, parametric instability is less likely to occur.

It may be mentioned here that in Lindgren *et al.* (1982) the three interacting waves are chosen to be of arbitrary frequencies; hence nonlinearity in each frequency component arises through the pondermotive force as well as the $n\mathbf{v}$ nonlinearity in the equation of motion. The coupling coefficient is thus complicated. However, in our case the low frequency $\omega \ll \omega_0$; hence the nonlinearity at $\omega, \mathbf{k}$ arises through the ponderomotive force, whereas that at ω_1, $\mathbf{k}_1$ arises through the equation of continuity. In a magnetized plasma the coupling coefficient is of the symmetric form rigorously speaking when $\mathbf{F}_p$ is derivable as the gradient of a scalar ($\mathbf{F}_p = e\boldsymbol{\nabla}\phi_p$). In this limit, we have verified that the Manley–Rowe relations are satisfied. Even when $\mathbf{F}_p$ is not derivable as $e\boldsymbol{\nabla}\phi_p$, the Manley–Rowe relations are satisfied, as shown by Lindgren *et al.* (1982) and others.

https://doi.org/10.1017/S0022377897005643 Published online by Cambridge University Press

Acknowledgements

The author is grateful to Professor V. K. Tripathi for valuable discussions. The financial support of the Department of Science and Technology, India is gratefully acknowledged.

References

Akimoto, K. 1989 *Phys. Fluids* **B1**, 1998.

Christiansen, P. J., Jain, V. K. and Stenflo, L. 1981 *Phys. Rev. Lett.* **46**, 1333.

Jain, V. K. and Christiansen, P. J. 1981 *Phys. Lett.* **82A**, 127.

Jain, V. K. and Christiansen, P. J. 1984 *Plasma Phys. Contr. Fusion* **26**, 619.

Kruer, W. L. 1987 *The Physics of Laser Plasma Interactions*. Addison-Wesley, Reading, MA.

Lindgren, T., Larsson, J. and Stenflo, L. 1982 *Plasma Phys.* **24**, 1177.

Liu, C. S. and Tripathi, V. K. 1986 *Phys. Rep.* **130**, 143.

Liu, C. S., Chan, V. S., Bhandari, D. K. and Harvey, R. W. 1982 *Phys. Fluids* **48**, 1479.

Liu, C. S., Tripathi, V. K., Chan, V. S. and Stefan, V. 1984 *Phys. Fluids* **27**, 1709.

Praburam, G. & Sharma, A. K. 1992 *J. Plasma Phys.* **48**, 3.

Saleem, H. and Murtaza, G. 1986 *J. Plasma Phys.* **36**, 295.

Sharma, R. P., Ramamurthy, K. and Yu, M. Y. 1984 *Phys. Fluids* **27**, 399.

Stenflo, L. 1994 *Physica Scripta* **T50**, 15.

Tripathi, V. K., Liu, C. S. and Grebogi, C. 1979 *Phys. Fluids* **22**, 301.

Wolf, N. S., Majeski, R., Lashinky, H., Tripathi, V. K. and Liu, C. S. 1980 *Phys. Rev. Lett.* **45**, 799.

Yu, M. Y. and Shukla, P. K. 1977 *Plasma Phys.* **19**, 889.

https://doi.org/10.1017/S0022377897005643 Published online by Cambridge University Press