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GEOMETRIC-FORM BOUNDS FOR THE $GI^X/M/1$ QUEUEING SYSTEM

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The GI/M/1 queueing system was long ago studied by considering the embedded discrete-time Markov chain at arrival epochs and was proved to have remarkably simple product-form stationary distributions both at arrival epochs and in continuous time. Although this method works well also in several variants of this system, it breaks down when customers arrive in batches. The resulting $GI^X/M/1$ system has no tractable stationary distribution. In this paper we use a recent result of Miyazawa and Taylor (1997) to obtain a stochastic upper bound for the $GI^X/M/1$ system. We also introduce a class of continuous-time Markov chains which are related to the original $GI^X/M/1$ embedded Markov chain that are shown to have modified geometric stationary distributions. We use them to obtain easily computed stochastic lower bounds for the $GI^X/M/1$ system. Numerical studies demonstrate the quality of these bounds.

1. INTRODUCTION

The following service system is considered. Groups of customers arrive at a service station according to a renewal process with interarrival distribution $A(x)$. The sizes of successive arriving groups are i.i.d. random variables and independent of the arrival times. Let $(g_j : j \in \mathbf{N})$ be the group size probability distribution. There is a single server who serves one customer at a time and a waiting facility for infinitely many customers. The service times of successive customers are independent exponentially distributed random variables with parameter μ (mean $1/\mu$). Thus, customers may arrive in batches, but they are served individually and then depart singly. This model will be referred to as the $GI^X/M/1$ queueing system.

The particular case of the above system where $g_1 = 1$ and $g_j = 0, j \geq 2$, that is, the GI/M/1 system, has been studied by considering the embedded discrete-time Markov chain at arrival epochs and has become a standard example in textbooks on

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stochastic processes and queueing theory (see, e.g., Kleinrock [4], Wolff [11], and Ross [6]). This model possesses geometric stationary distribution at arrival epochs, from which many other distributions and parameters of interest have been obtained. The same method also applies in the study of several variants of this model as in the analysis of, for example, the GI/M/c systems. It breaks down, however, when the arrivals constitute a compound renewal process. In this case the form of the transition matrix of the embedded Markov chain at group-arrival epochs changes radically and the resulting chain has no tractable stationary distribution.

Over the last decade several authors, most notably among others Adan and van der Wal [1], van Dijk and Lamond [9], Hordijk and Ridder [3], van Dijk, Tsoucas, and Walrand [10], Shanthikumar and Yao [7], and Miyazawa and Taylor [5], have proposed several bounds for queueing systems whose stationary distribution is not, in principle, tractable. One basic idea is to modify slightly a nontractable system to obtain a product-form system. Then, using various approaches (Markov reward theory, coupling and sample path analysis, analytic arguments), it is proved that the product-form distributions of the modified systems may be used to obtain bounds for several quantities of interest for the original systems. These bounds are rough but computationally attractive and can be used to obtain a quick estimate of the order of magnitude for several quantities associated with the models.

In the next paragraph, we first set the notation then use a recent result of Miyazawa and Taylor [5] to obtain a stochastic upper bound for the $GI^X/M/1$ system. We then introduce a class of continuous-time Markov chains, which are obtained by appropriately modifying the continuous-time analogue of the original $GI^X/M/1$ embedded Markov chain. These chains are shown to have modified geometric stationary distributions and can be used to obtain easily computed stochastic lower bounds for the $GI^X/M/1$ system. We also present some numerical examples to demonstrate the quality of these bounds.

2. THE RESULTS

Consider the $GI^X/M/1$ system and let $Q(t)$ be the number of customers in system at an arbitrary epoch $t (t \in \mathbf{R})$. Denote by $\ldots < t_{-1} < t_0 < t_1 < t_2 < \ldots$ the successive group-arrival epochs and by $\ldots < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 < \ldots$ the successive departure (service completion) epochs in the time interval $(-\infty,\infty)$. We define the embedded processes $\{Q_n^- : n \in \mathbf{Z}\}$ and $\{Q_n^+ : n \in \mathbf{Z}\}$, by setting $Q_n^- = Q(t_n^-)$ and $Q_n^+ = Q(\tau_n^+)$, $n \in \mathbf{Z}$, that is, Q_n^- is the number of customers in system just before the nth group-arrival while Q_n^+ is the number of customers in system just after the nth departure.

Let Y_n be a random variable representing the number of departures in the time interval $T_n = \{t_{n-1}, t_n)$, given that the server is busy during this interval. Then $Y_n, n \in \mathbf{Z}$, are independent identically distributed random variables and

$$\beta_k = P[Y_n = k] = \int_0^\infty P[Y_n = k | T_n = x]\, dA(x) = \int_0^\infty e^{-\mu x} \frac{(\mu x)^k}{k!}\, dA(x), \quad k \in \mathbf{N}_0. \quad \textbf{(1)}$$

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We have that

$$Q_{n+1}^- = \begin{cases} Q_n^- + A_n - Y_{n+1} & \text{if } Y_{n+1} < Q_n^- + A_n \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where A_n is the size of the group which arrived at epoch t_n.

Let $a_k = P[A_n - Y_{n+1} = k]$, $k > 0$, $d_k = P[A_n - Y_{n+1} = -k]$, $k \geq 0$, and $c_k = \sum_{h=k}^{\infty} d_h$, $k > 0$. Note that

$$d_k = \sum_{h=1}^{\infty} g_h \beta_{h+k}, \qquad k \geq 0 \quad \text{and} \quad a_k = \sum_{h=0}^{\infty} \beta_h g_{h+k}, \qquad k > 0. \tag{3}$$

Using Eq. (2), we have

$$P[Q_{n+1}^- = j \mid Q_1^-, Q_2^-, \ldots, Q_{n-1}^-, Q_n^- = i] = P[A_n - Y_{n+1} = j - i]$$

$$= \begin{cases} d_{i-j}, \text{ if } j - i \leq 0 \\ a_{j-i}, \text{ if } j - i > 0 \end{cases} \quad \text{if } j > 0$$

$$P[Q_{n+1}^- = 0 \mid Q_1^-, Q_2^-, \ldots, Q_{n-1}^-, Q_n^- = i] = c_i, \; i \geq 0, \quad \text{if } j = 0. \tag{4}$$

By Eq. (4) we conclude that $\{Q_n^- : n \in \mathbf{Z}\}$ is a Markov chain with state space $\mathbf{N}_0$ and transition probabilities p_{ij} given by

$$\begin{aligned} p_{ii+k} &= a_k, \qquad i \geq 0, k \geq 1, \\ p_{ii-k} &= d_k, \qquad i \geq 1, 0 \leq k < i, \\ p_{i0} &= c_i, \qquad i \geq 0. \end{aligned} \tag{5}$$

Let $(r_n : n \in \mathbf{N}_0)$, $(s_n : n \in \mathbf{N}_0)$, and $(p_n : n \in \mathbf{N}_0)$ be the stationary distributions of $\{Q_n^-\}$, $\{Q_n^+\}$, and $\{Q(t)\}$, respectively. Since $\{Q_n^+\}$ and $\{Q(t)\}$ are not Markovian, we should concentrate on studying (r_n), then we can indirectly deduce properties of (s_n) and (p_n) using standard results from the theory of point processes. The distribution (r_n) satisfies the balance equations

$$\begin{aligned} r_0 &= \sum_{j=0}^{\infty} r_j c_j, \\ r_j &= \sum_{i=0}^{j-1} r_i a_{j-i} + \sum_{i=j}^{\infty} r_i d_{i-j}, \qquad j \geq 1. \end{aligned} \tag{6}$$

These equations cannot be solved explicitly, as in the case of the classic GI/M/1 model. The stochastic upper bound which we establish in this paragraph for the distribution (r_n) is based on some results of Miyazawa and Taylor [5]. Since their

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results concern a continuous-time Markov chain, we consider a continuous-time analogue of $\{Q_n^-\}$ that is a chain $\{N(t): t \in \mathbf{R}\}$ with transition rates q_{ij} given by

$$\begin{aligned} q_{ii+k} &= a_k, \qquad i \geq 0, k \geq 1, \\ q_{ii-k} &= d_k, \qquad i \geq 1, 1 \leq k < i, \\ q_{i0} &= c_i, \qquad i \geq 1. \end{aligned} \tag{7}$$

This chain has the same stationary distribution (r_n) as $\{Q_n^-\}$ since it satisfies the same balance Eqs. (6). In what follows, let $\tilde{A}(z) = \sum_{k=1}^{\infty} a_k z^k$, $\tilde{D}(z) = \sum_{k=0}^{\infty} d_k z^k$. Moreover, let

$$r = \sup\{z \geq 0 : \tilde{A}(z) < \infty\} > 1. \tag{8}$$

By the usual stability condition $\lambda \bar{g} < \mu$, we have that

$$\sum_{k=1}^{\infty} k a_k - \sum_{k=1}^{\infty} k d_k = E[A_n - Y_{n+1}] = E[A_n] - E[Y_{n+1}] = \bar{g} - \mu \frac{1}{\lambda} < 0. \tag{9}$$

Consider now a Markov chain $\{N^U(t): t \in \mathbf{R}\}$ with transition rates q_{ij}^U given by

$$\begin{aligned} q_{0k}^U &= a_k + \tilde{a}_k, \qquad k \geq 1, \\ q_{ii+k}^U &= a_k, \qquad i \geq 1, k \geq 1, \\ q_{ii-k}^U &= d_k, \qquad i \geq 1, 1 \leq k < i, \\ q_{i0}^U &= c_i, \qquad i \geq 1. \end{aligned} \tag{10}$$

where $\tilde{a}_k, k \geq 1$, are nonnegative real numbers.

This model has been studied by Miyazawa and Taylor [5], although under a slightly different notation, and our conditions (8) and (9) are exactly their regularity conditions (2.1) and (2.10). So their main results [5, Lem. 2.1, Thm. 2.1, Cor. 2.1 for $h = 0$] are valid and can be stated in the following form.

THEOREM 1: *The equation*

$$\tilde{A}(z^{-1}) = 1 - \tilde{D}(z) \tag{11}$$

has a unique solution $z_U \in (0,1)$. *If* $\tilde{a}_k, k \geq 1$, *are given by*

$$\tilde{a}_k = \sum_{n=k+1}^{\infty} a_n z_U^{-(n-k)}, \qquad k \geq 1, \tag{12}$$

then the stationary distribution $(r_n^U : n \in \mathbf{N}_0)$ *of* $\{N^U(t)\}$ *is given by*

$$r_n^U(n) = (1 - z_U) z_U^n, \qquad n \in \mathbf{N}_0. \tag{13}$$

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The distribution (r_n) *is stochastically smaller than the geometric distribution* (r_n^U); *that is*,

$$\bar{R}_n = \sum_{k=n}^{\infty} r_k \le z_U^n, \qquad n \in \mathbf{N}_0, \tag{14}$$

where the equality holds if and only if $a_k = 0$ *for* $k \ge 2$ (*equivalently*, $g_k = 0$ *for* $k \ge 2$).

To obtain stochastic lower bounds for (r_n) we consider for each $h \in \mathbf{N}$ a Markov chain $\{N^{L(h)}(t) : t \in \mathbf{R}\}$ with transition rates $q_{ij}^{L(h)}$ given by

$$\begin{aligned} q_{ii+k}^{L(h)} &= a_k, && i \ge 0, && k \ge 1, \\ q_{ii-k}^{L(h)} &= d_k, && i \ge 2, && 1 \le k < i, \\ q_{i0}^{L(h)} &= c_i, && 1 \le i \le h-1, \\ q_{i0}^{L(h)} &= c_i + \tilde{d}_i^{(h)}, && i \ge h, \end{aligned} \tag{15}$$

where $\tilde{d}_i^{(h)}, i \ge 1$, are nonnegative real numbers. We have the following theorem.

THEOREM 2: *For each* $h \in \mathbf{N}$, *the equation*

$$\sum_{k=1}^{h} a_k z^{-k} = 1 - \tilde{D}(z), \tag{16}$$

has a unique solution $z_{L(h)} \in (0,1)$. *The linear* $h \times h$ *system*

$$x_i^{(h)}(1-d_0) = \sum_{j=0}^{i-1} x_j^{(h)} a_{i-j} + \sum_{j=i+1}^{h-2} x_j^{(h)} d_{j-i} + x_{h-1}^{(h)}(1 - z_{L(h)}) \sum_{k=h-i-1}^{\infty} d_k z_{L(h)}^{k+i-h+1}, \quad 1 \le i \le h-2, \tag{17}$$

$$x_{h-1}^{(h)}(1 - z_{L(h)})(1 - \tilde{D}(z_{L(h)})) = \sum_{j=0}^{h-2} x_j^{(h)} a_{h-j-1}, \tag{18}$$

$$\sum_{i=0}^{h-1} x_i^{(h)} = 1 \tag{19}$$

has a solution $(x_i^{(h)} > 0 : 0 \le i \le h-1)$. *If* $\tilde{d}_k^{(h)}$ *are given by*

$$\tilde{d}_k^{(h)} = \frac{\sum_{n=0}^{h-2} x_n^{(h)} a_{k-n}}{x_{h-1}^{(h)}(1 - z_{L(h)}) z_{L(h)}^{k-h+1}} + \sum_{n=1}^{k-h+1} a_n z_{L(h)}^{-n} - \sum_{n=1}^{h} a_n z_{L(h)}^{-n} \ge 0, \qquad k \ge h, \tag{20}$$

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then the stationary distribution $(r_n^{L(h)} : n \in \mathbf{N}_0)$ *of* $\{N^{L(h)}(t)\}$ *is given by*

$$r_n^{L(h)} = x_n^{(h)}, \qquad 0 \le n \le h-2,$$

$$r^{L(h)}(n) = x_{h-1}^{(h)}(1 - z_{L(h)}) z_{L(h)}^{n-h+1}, \qquad n \ge h-1. \tag{21}$$

The modified geometric distribution $(r_n^{L(h)})$ *is stochastically smaller than the distribution* (r_n); *that is*,

$$\bar{R}_n^{L(h)} = \sum_{k=n}^{\infty} r_k^{L(h)} \le \bar{R}_n, \qquad n \in \mathbf{N}_0. \tag{22}$$

(*Convention*: *For* $h = 1$ *the linear system* (17)–(19) *is just* (19) *and for* $h = 2$ *it is* (18)–(19).)

Proof: Let $f(z) = 1 - \tilde{D}(z) - \sum_{k=1}^{h} a_k z^{-k}$, $z \in (0,1)$. Then,

$$f'(z) = -\tilde{D}'(z) + \sum_{k=1}^{h} k a_k z^{-k-1}, \qquad z \in (0,1),$$

$$f''(z) = -\tilde{D}''(z) - \sum_{k=1}^{h} k(k+1) a_k z^{-k-2} < 0, \qquad z \in (0,1).$$

Hence, $f'(z)$ is strictly decreasing in $(0,1)$ and $\lim_{z\to 0} f'(z) = +\infty$ while $f'(1) = \sum_{k=1}^{h} k a_k - \sum_{k=1}^{\infty} k d_k < 0$ by Eq. (9). Therefore, there exists a unique $\tau \in (0,1)$ such that $f'(\tau) = 0$, in which the maximum of f occurs. Note that $f(\tau) > f(1) = 1 - \tilde{D}(1) - \sum_{k=1}^{h} a_k \ge 0$. f is strictly increasing in $(0,\tau)$ with $(\lim_{z\to 0} f(z)) \cdot f(\tau) < 0$ and, consequently, has a unique root, while it is positive in $(\tau,1)$. Therefore, Eq. (16) has a unique solution $z_{L(h)} \in (0,1)$.

Let $\underline{y} = (y_1, y_2, \ldots, y_{h-1})$ and $\underline{F}(\underline{y}) = (F_1(\underline{y}), F_2(\underline{y}), \ldots, F_{h-1}(\underline{y}))$, where

$$F_i(\underline{y}) = \frac{1}{1-d_0}\left\{ a_i + \sum_{j=1}^{i-1} y_j a_{i-j} + \sum_{j=i+1}^{h-2} y_j d_{j-i} + y_{h-1} \sum_{k=h-i-1}^{\infty} d_k z_{L(h)}^{k+i-h+1} \right\},$$

$$1 \le i \le h-2,$$

$$F_{h-1}(\underline{y}) = \frac{1}{1 - \tilde{D}(z_{L(h)})}\left\{ a_{h-1} + \sum_{j=1}^{h-2} y_j a_{h-j-1} \right\}.$$

Then, by Eq. (16) and the relation $\tilde{A}(1) + \tilde{D}(1) = 1$, we see that $\underline{F}([0,1]^{h-1}) \subseteq [0,1]^{h-1}$ and obviously $\underline{F}$ is continuous. Therefore, by Brower's fixed point theorem, there exists $\underline{y}^*$ such that $\underline{F}(\underline{y}^*) = \underline{y}^*$. Set $x_0^{(h)} = (1 + \sum_{j=1}^{h-2} y_j + (y_{h-1}/(1 - z_{L(h)})))^{-1}$, $x_i^{(h)} = y_i x_0^{(h)}$, $1 \le i \le h-2$, and $x_{h-1}^{(h)} = x_0^{(h)}(y_{h-1}/(1 - z_{L(h)}))$. It is easy to check that this is the solution of the system (17)–(19).

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The balance equations for $(r_n^{L(h)})$ are given by

$$\begin{aligned}
r_0^{L(h)}(1-c_0) &= \sum_{j=1}^{h-1} r_j^{L(h)} c_j + \sum_{j=h}^{\infty} r_j^{L(h)}(c_j + \tilde{d}_j^{(h)}),\\
r_j^{L(h)}(1-d_0) &= \sum_{i=0}^{j-1} r_i^{L(h)} a_{j-i} + \sum_{i=j+1}^{\infty} r_i^{L(h)} d_{i-j}, \qquad 1 \le j \le h-1,\\
r_j^{L(h)}(1-d_0+\tilde{d}_j^{(h)}) &= \sum_{i=0}^{j-1} r_i^{L(h)} a_{j-i} + \sum_{i=j+1}^{\infty} r_i^{L(h)} d_{i-j}, \qquad j \ge h. \qquad \textbf{(23)}
\end{aligned}$$

By substituting $(r_n^{L(h)})$ given by Eq. (21) in (23) we obtain

$$\begin{aligned}
x_0^{(h)}(1-c_0) &= \sum_{j=1}^{h-2} x_j^{(h)} c_j + x_{h-1}^{(h)}(1-z_{L(h)})c_{h-1}\\
&\quad + x_{h-1}^{(h)}(1-z_{L(h)}) \sum_{j=h}^{\infty} z_{L(h)}^{j-h+1}(c_j + \tilde{d}_j^{(h)}),\\
x_j^{(h)}(1-d_0) &= \sum_{i=0}^{j-1} x_i^{(h)} a_{j-i} + \sum_{i=j+1}^{h-2} x_i^{(h)} d_{i-j}\\
&\quad + x_{h-1}^{(h)}(1-z_{L(h)}) \sum_{i=h-1}^{\infty} z_{L(h)}^{i-h+1} d_{i-j},\\
&\qquad\qquad 1 \le j \le h-2,\\
x_{h-1}^{(h)}(1-z_{L(h)})(1-d_0) &= \sum_{i=0}^{h-2} x_i^{(h)} a_{h-1-i} + x_{h-1}^{(h)}(1-z_{L(h)})\\
&\quad \times \sum_{i=h}^{\infty} z_{L(h)}^{i-h+1} d_{i-h+1}\\
x_{h-1}^{(h)}(1-z_{L(h)}) z_{L(h)}^{j-h+1}(1-d_0+\tilde{d}_j^{(h)}) &= \sum_{i=0}^{h-2} x_i^{(h)} a_{j-i} + x_{h-1}^{(h)}(1-z_{L(h)})\\
&\quad \times \sum_{i=h-1}^{j-1} z_{L(h)}^{i-h+1} a_{j-i} + x_{h-1}^{(h)}(1-z_{L(h)})\\
&\quad \times \sum_{i=j+1}^{\infty} z_{L(h)}^{i-h+1} d_{i-j}, \qquad j \ge h.
\end{aligned}$$

The second and third are valid because of Eqs. (17) and (18), respectively. The fourth is written equivalently as

$$\begin{aligned}
\tilde{d}_j^{(h)} &= \frac{\sum_{i=0}^{h-2} x_i^{(h)} a_{j-i}}{x_{h-1}^{(h)}(1-z_{L(h)}) z_{L(h)}^{j-h+1}} + \sum_{i=h-1}^{j-1} z_{L(h)}^{i-j} a_{j-i} + \sum_{i=j+1}^{\infty} z_{L(h)}^{i-j} d_{i-j} + d_0 - 1\\
&= \frac{\sum_{i=0}^{h-2} x_i^{(h)} a_{j-i}}{x_{h-1}^{(h)}(1-z_{L(h)}) z_{L(h)}^{j-h+1}} + \sum_{n=1}^{j-h+1} z_{L(h)}^{-n} a_n + \tilde{D}(z) - 1,
\end{aligned}$$

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which is valid because of Eqs. (16) and (20). The first follows by summing the others for $j \geq 1$. Moreover, by Eq. (19) we have that $\sum_{n=0}^{\infty} r_n^{L(h)} = 1$; therefore, $(r_n^{L(h)})$ is the stationary distribution of $\{N^{L(h)}(t)\}$. Using a standard coupling argument, or alternatively Theorem 4.2.8 of Stoyan [8], we conclude that $(r_n^{L(h)}) \leq_{st} (r_n)$. ■

The question of an intuitive, easily verifiable condition for the nonnegativity of the $\tilde{d}_k^{(h)}$'s in the general case remains open. In practical situations, for a given model and a given h we compute $\tilde{d}_k^{(h)}$ for $h \leq k \leq 2h-2$. If they are all nonnegative we can use the result of Theorem 2, because for $k \geq 2h-1$ we have $\sum_{n=1}^{k-h+1} a_n z_{L(h)}^{-n} \geq \sum_{n=1}^{h} a_n z_{L(h)}^{-n}$, therefore $\tilde{d}_k^{(h)} \geq 0$. In the following corollary we summarize two cases in which we can assure the nonnegativity of $\tilde{d}_k^{(h)}$'s in advance.

COROLLARY 1:

i. *The geometric distribution with parameter $z_{L(1)}$, as well as the distribution $(r_n^{L(2)})$ given by*

$$r_0^{L(2)} = \frac{a_1 z_{L(2)}(1 - z_{L(2)}) + a_2(1 - z_{L(2)})}{a_1 z_{L(2)} + a_2(1 - z_{L(2)})},$$

$$r_n^{L(2)} = \frac{a_1(1 - z_{L(2)}) z_{L(2)}^{n+1}}{a_1 z_{L(2)} + a_2(1 - z_{L(2)})}, \qquad n \geq 1, \tag{24}$$

are stochastically smaller than (r_n) for any parameters and underlying distributions of the $GI^X/M/1$ model.

ii. *For any $h \geq 1$ the distributions $(r_n^{L(h)})$ are stochastically smaller than (r_n) for any parameters and underlying distributions of the $GI^X/M/1$ model with geometric group-size distribution.*

PROOF:

i. For $h = 1$, it is obvious that the $\tilde{d}_k^{(h)}$'s are nonnegative. For $h = 2$, the solution of the linear system is

$$(x_0^{(2)}, x_1^{(2)}) = \left(\frac{a_1 z_{L(2)}(1 - z_{L(2)}) + a_2(1 - z_{L(2)})}{a_1 z_{L(2)} + a_2(1 - z_{L(2)})}, \frac{a_1 z_{L(2)}^2}{a_1 z_{L(2)} + a_2(1 - z_{L(2)})} \right).$$

From the remarks above we should check that $\tilde{d}_k^{(2)}$'s are nonnegative only for $k = 2$. Indeed it is easy to check that $\tilde{d}_2^{(2)} = x_0^{(2)} a_2 / x_1^{(2)}(1 - z_{L(2)}) z_{L(2)} - a_2 z_{L(2)}^{-2} > 0$.

ii. If (g_k) is geometric with parameter τ, then by Eq. (3) we have that $a_k = \sum_{h=0}^{\infty} \beta_h (1 - \tau) \tau^{h+k-1} = c\tau^k$, where $c = (1 - \tau) \sum_{h=0}^{\infty} \beta_h \tau^{h-1}$. Hence, Eq. (18) assumes the form

$$1 - \tilde{D}(z_{L(h)}) = \frac{c \sum_{n=0}^{h-2} x_n^{(h)} \tau^{h-n-1}}{x_{h-1}^{(h)}(1 - z_{L(h)})}.$$

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Therefore, by Eq. (20) for k such that $h \le k \le 2h-2$ we have

$$\tilde{d}_k^{(h)} = \frac{c\sum_{n=0}^{h-2} x_n^{(h)} \tau^{k-n}}{x_{h-1}^{(h)}(1-z_{L(h)})z_{L(h)}^{k-h+1}} - c\sum_{n=k-h+2}^{h} \tau^n z_{L(h)}^{-n}$$

$$= (\tau z^{-1})^{k-h+1}(1-\tilde{D}(z_{L(h)})) - (\tau z^{-1})^{k-h+1}\sum_{j=1}^{2h-k-1} c\tau^j z_{L(h)}^{-j}$$

$$= (\tau z_{L(h)}^{-1})^{k-h+1}\left(1-\tilde{D}(z_{L(h)}) - \sum_{j=1}^{2h-k-1} a_j z_{L(h)}^{-j}\right) \ge 0. \quad \blacksquare$$

Using the bounds of Theorems 1 and 2 we can now establish bounds for other important performance measures of the system. Let M be a random variable representing the order of a customer within his group. When groups arrive in a system according to a group size distribution (g_j), then the probability that a customer is the jth of his group is $P[M=j] = \bar{G}_j/\bar{g}$, where $\bar{g}$ is the mean group size and $\bar{G}_n = \sum_{k\ge n} g_k$ (see Burke [2] and Wolff [11, pp. 68–69]); therefore, $E[M] = E[N(N+1)]/2E[N]$, where N is a random variable having the group size distribution (g_j). Using standard results from the theory of point processes we can derive the relationship between the stationary distribution in continuous time and at arrival and departure epochs for the GI^X/M/1 queue. We have that

$$s_n = \sum_{k=0}^{n} r_k \bar{G}_{n+1-k}/\bar{g}, \qquad n \in \mathbf{N}_0,$$

$$p_0 = 1 - \frac{\lambda\bar{g}}{\mu},$$

$$p_n = \frac{\lambda\bar{g}}{\mu} s_{n-1} = \frac{\lambda}{\mu}\sum_{k=0}^{n-1} r_k \bar{G}_{n-k}, \qquad n \in \mathbf{N}. \tag{25}$$

Let $Q^-, Q_{L(h)}^-, Q_U^-$ be random variables with distributions (r_n), $(r_n^{L(h)})$, and (r_n^U), respectively. Let also S be the time in system of a customer in a stationary GI^X/M/1 queue. Then S is distributed as $\sum_{i=1}^{Q^-+M} X_i$, where $X_1, X_2, \ldots$ is a sequence of independent exponentially distributed random variables with parameter μ, independent of Q^- and M. Using the above results we can easily prove the following.

COROLLARY 2: *For a stationary GI^X/M/1 system and an $h \ge 1$ such that $\tilde{d}_k^{(h)} \ge 0$, $k \ge h$, the following are valid*:

i. $$\sum_{i=1}^{Q_{L(h)}^-+M} X_i \le_{st} S \le_{st} \sum_{i=1}^{Q_U^-+M} X_i,$$

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TABLE 1. Bounds and Exact Value for E[S]

ρ	$\bar{g}$	Interarrival Distribution	Group-size Distribution	LB1	LB2	LB5	Exact	UB
0.2	1.2	Gamma(12,2)	Geom(0.833333)	1.211345	1.21468	1.21967	1.21977	1.4143
		Exp(0.166667)	Bin(2,0.1) + 1	1.36694	1.43269	—	1.46893	1.59097
	1.5	H_2(7.5,2)	Geom(0.666667)	1.68754	1.79032	1.98345	2.02985	2.51756
		Gamma(3.75,0.5)	Poisson(0.5) + 1	1.44745	1.45374	—	1.47962	1.66951
	2	Gamma(20,2)	Geom(0.5) + 1	2.00586	2.00651	2.02043	2.0245	2.71771
		Exp(0.1)	Bin(10,0.1) + 1	1.82439	1.89253	—	2.16133	2.59884
	5	H_2(25,4)	Geom(0.2)	5.04874	5.07544	5.54105	7.79047	11.0922
		Gamma(5,0.2)	Poisson(4) + 1	3.40518	3.40583	3.42664	3.43399	4.06469
	10	Gamma(5,0.1)	Geom(0.1)	9.325	9.32534	9.35219	9.79466	13.6219
		Exp(0.02)	Bin(18,0.5) + 1	5.74198	5.74716	6.00323	8.7119	9.58279
0.5	1.2	H_2(2.4,3)	Geom(0.833333)	2.13886	2.95176	3.14495	3.40032	3.59411
		Gamma(1.2,0.5)	Poisson(0.2) + 1	1.72906	2.089	—	2.21284	2.37127
	1.5	Gamma(2,0.6666)	Geom(0.666667)	1.76802	1.98164	2.3078	2.42705	2.92705
		Exp(0.333333)	Bin(2,0.25) + 1	1.75902	2.3411	—	2.75	3.01888

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	2	$H_2(4,3)$	Geom(0.5)	2.33441	2.76235	4.57297	5.76011	6.65684
		Gamma(2,0.5)	Poisson(1) + 1	1.93903	2.13932	—	2.75219	3.25542
	5	Gamma(5,0.5)	Geom(0.2)	5.03699	5.05501	5.41047	6.97341	10.7886
		Exp(0.1)	Bin(8,0.5) + 1	3.27147	3.32972	5.27131	6.48048	7.77572
	10	$H_2(20,15)$	Geom(0.1)	9.37452	9.4039	10.1001	39.3684	51.6514
		Gamma(10,0.5)	Poisson(9) + 1	5.97356	5.98107	6.1535	6.32265	7.78588
0.8	1.2	Gamma(3.2)	Geom(0.833333)	2.14204	3.28805	3.56405	4.15132	4.35145
		Exp(0.666667)	Bin(2,0.1) + 1	2.39926	5.15334	—	5.87501	6.02754
	1.5	$H_2(1.875,2)$	Geom(0.666667)	2.2905	3.78215	8.83366	10.4424	10.955
		Gamma(1.875,1)	Poisson(0.5) + 1	1.99342	3.28879	—	5.46826	5.80718
	2	Gamma(1.25,0.5)	Geom(0.5)	2.36104	2.89056	5.91033	9.07134	10.0746
		Exp(0.4)	Bin(10,0.1) + 1	2.08251	3.00665	8.10733	8.62359	9.19233
	5	$H_2(6.25,2)$	Geom(0.2)	5.11769	5.22737	6.83304	29.1303	37.4468
		Gamma(12.5,2)	Poisson(4) + 1	3.52751	3.66036	5.36637	6.73134	7.94014
	10	Gamma(6.25,0.5)	Geom(0.1)	9.35629	9.3711	9.78394	23.1194	31.4203
		Exp(0.08)	Bin(18,0.5) + 1	5.76554	5.78749	6.75033	28.5939	30.8538

ii. $$\frac{1}{\mu}\left[\sum_{n=1}^{\infty}\bar{R}_n^{L(h)}+\frac{E[N(N+1)]}{2E[N]}\right]\le E[S]\le\frac{1}{\mu}\left[\frac{z_U}{1-z_U}+\frac{E[N(N+1)]}{2E[N]}\right],$$

iii. $$\max(0,\bar{R}_n^{L(h)}-z_U^{n+1})\le r_n\le z_U^{n+1}-\bar{R}_{n+1}^{L(h)},\quad n\in\mathbf{N}_0,$$

iv. $$\sum_{k=0}^{n}\max(0,\bar{R}_k^{L(h)}-z_U^{k+1})\bar{G}_{n+1-k}/\bar{g}\le s_n\le\sum_{k=0}^{n}(z_U^{k+1}-\bar{R}_{k+1}^{L(h)})\bar{G}_{n+1-k}/\bar{g},\quad n\in\mathbf{N}_0,$$

v. $$\frac{\lambda}{\mu}\sum_{k=0}^{n-1}\max(0,\bar{R}_k^{L(h)}-z_U^{k+1})\bar{G}_{n-k}\le p_n\le\frac{\lambda}{\mu}\sum_{k=0}^{n-1}(z_U^{k+1}-\bar{R}_{k+1}^{L(h)})\bar{G}_{n-k},\quad n\in\mathbf{N}.$$

Extensive numerical studies show that the crucial factors which influence the quality of the upper and lower bound models are the mean group-size $\bar{g}$ and the traffic intensity ρ. Both upper and lower bounds are very good for models with small $\bar{g}$. As $\bar{g}$ increases, the bounds degrade. Upper bounds are better for models with high traffic intensity while lower bounds are better for models with low traffic intensity. $L(h)$-bounds become very good for h such that $z_{L(h)}\approx z_U$. For geometric group-size distributions the $L(h)$-bounds exist for every $h\ge 1$; therefore, we can take h large enough to assure that $z_{L(h)}\approx z_U$. In this case we can approximate the original model by a lower bound model as close as we want. We present some numerical results for various parameters of the model in Table 1.

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