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NOTES AND PROBLEMS

NONNESTED TESTING IN MODELS ESTIMATED VIA GENERALIZED METHOD OF MOMENTS

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We analyze the limiting distribution of the Rivers and Vuong (2002, *Econometrics Journal* 5, 1–39) statistic for choosing between two competing dynamic models based on a comparison of generalized method of moments minimands. It is shown that (i) if both models are misspecified then the statistic has a standard normal distribution under the null hypothesis of equal fit but the ranking could be determined by the choice of the weighting matrix; (ii) if both models are correctly specified or locally misspecified then the limiting distribution of the test statistic is nonstandard under the null.

1. INTRODUCTION

Competing economics theories often lead to econometric models that are nonnested in the sense that one model is not obtained as a special case of the other. It is therefore of interest to develop statistical procedures that discriminate between nonnested models. In these circumstances, it may be considered attractive to have some method that allows the researcher to determine which—if either—of the two models is closer to the truth in some sense. Vuong (1989) provides such a test for models estimated by quasi maximum likelihood (QML). More recently, Rivers and Vuong (2002) extend the Vuong (1989) approach to provide a very general framework for the comparison of two competing dynamic models. In this more general context, inference is based on a test statistic that compares measures of goodness of fit for the two models; one model is preferred if its goodness of fit is statistically significantly smaller than its competitor. Rivers and Vuong (2002) provide generic conditions under which the statistic has a limiting standard normal distribution under the null hypothesis that both models are "equally

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good," a concept that is defined in what follows. These generic conditions are very general, and it is argued that they cover the situation in which the competing models are estimated via generalized method of moments (GMM) and then compared using either the GMM minimands employed in the estimations or GMM type minimands that are different from those used in the estimation.[1] In spite of this seeming generality, Rivers and Vuong (2002) show that the aforementioned distributional result rests crucially on the assumption that a certain variance is nonzero.

In this paper, we investigate whether these generic conditions in fact cover GMM estimators and minimands. It turns out that the analysis depends crucially on whether the models in question are correctly specified, locally misspecified, or nonlocally misspecified. It is shown that if both models are correctly specified or locally misspecified then the Rivers and Vuong (2002) generic conditions are not satisfied and the statistic does not converge to a limiting normal distribution but to a nonstandard distribution that is a function of nuisance parameters, some of which may not be consistently estimable. However, if both models are nonlocally misspecified then the generic conditions are satisfied and the Rivers and Vuong (2002) statistic does converge to the limiting standard normal distribution. The latter result indicates that there is scope for using the Rivers and Vuong statistic to compare two misspecified models estimated via GMM although caution may need to be exercised in its use because the outcome can depend on the choice of weighting matrix. Thus our results reveal important limitations to the use of GMM minimands for model selection in this way, which in turn contribute to the wider literature on model selection via goodness of fit in econometrics; see, e.g., Rivers and Vuong (2002), Marcellino and Rossi (2008), and Kitamura (2002).

2. FRAMEWORK AND ANALYSIS

For $i = 1, 2$, let $f^{(i)} : \mathbf{V} \times \Theta^{(i)} \to \Re^{q_i}$, where $q_i < \infty$, $\Theta^{(i)} \subset \Re^{p_i}$ and let $\{v_t\}$ be a stationary and ergodic sequence of d-dimensional random vectors in $\mathbf{V}$. Suppose it is desired to compare two models denoted $\mathcal{M}_1$ and $\mathcal{M}_2$ and that each implies a population moment condition as follows:

$$\mathcal{M}_1 \Rightarrow \mathrm{E}[f^{(1)}(v_t, \bar{\theta}^{(1)})] = 0 \quad \text{for a unique } \bar{\theta}^{(1)} \in \Theta^{(1)},$$

$$\mathcal{M}_2 \Rightarrow \mathrm{E}[f^{(2)}(v_t, \bar{\theta}^{(2)})] = 0 \quad \text{for a unique } \bar{\theta}^{(2)} \in \Theta^{(2)}.$$

It is assumed that the models are nonnested; i.e., the moment conditions of one model are not a subset or subcase of another model. It is also assumed that the parameters of both models are estimated via GMM; these estimators are defined as follows:

$$\hat{\theta}_T^{(i)} = \mathrm{argmin}_{\theta^{(i)} \in \Theta^{(i)}} Q_T^{(i)}(\theta^{(i)}), \quad \text{for } i = 1, 2, \tag{1}$$

where

$$Q_T^{(i)}(\theta^{(i)}) = \left\{ T^{-1} \sum_{t=1}^{T} f^{(i)}(v_t, \theta^{(i)}) \right\}' W_T^{(i)} \left\{ T^{-1} \sum_{t=1}^{T} f^{(i)}(v_t, \theta^{(i)}) \right\} \tag{2}$$

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and $W_T^{(i)}$ is the weighting matrix. The population analogue of the GMM minimands is for $i = 1, 2$,

$$Q_0^{(i)}(\theta^{(i)}) = \mathrm{E}[f^{(i)}(v_t, \theta^{(i)})]' W^{(i)} \mathrm{E}[f^{(i)}(v_t, \theta^{(i)})], \tag{3}$$

where $W^{(i)} = \mathrm{plim}_{T\to\infty} W_T^{(i)}$.

Rivers and Vuong (2002) introduce a very general framework that includes the cases where the metric of model comparison either involves the minimands employed in the estimation or some other measure of goodness of fit. We consider the case in which the metric involves the GMM minimands and so the test statistic is

$$N_T = \frac{T^{1/2}\{Q_T^{(1)}(\hat{\theta}_T^{(1)}) - Q_T^{(2)}(\hat{\theta}_T^{(2)})\}}{\hat{\sigma}_T}, \tag{4}$$

where $\hat{\sigma}_T^2$ is a consistent estimator of σ_0^2, the limiting variance of the numerator of (4). This variance and its estimator are discussed subsequently.

To present the null and alternative hypotheses of the test, we must introduce notation for the probability limits of $\hat{\theta}_T^{(1)}$ and $\hat{\theta}_T^{(2)}$. Accordingly, we define $\mathrm{plim}_{T\to\infty}\hat{\theta}_T^{(i)} = \theta_\star^{(i)}$ for $i = 1, 2$. This convergence result can be established under certain regularity conditions that are omitted for brevity here as they are now standard in the literature. The null hypothesis of the test is that $\mathcal{M}_1$ and $\mathcal{M}_2$ are asymptotically equivalent; i.e.,

$$H_0 : Q_0^{(1)}(\theta_\star^{(1)}) = Q_0^{(2)}(\theta_\star^{(2)}). \tag{5}$$

There are two alternative hypotheses of interest: $\mathcal{M}_1$ is asymptotically better than $\mathcal{M}_2$; i.e.,

$$H_1^{(a)} : Q_0^{(1)}(\theta_\star^{(1)}) < Q_0^{(2)}(\theta_\star^{(2)}) \tag{6}$$

and $\mathcal{M}_2$ is asymptotically better than $\mathcal{M}_1$; i.e.,

$$H_1^{(b)} : Q_0^{(1)}(\theta_\star^{(1)}) > Q_0^{(2)}(\theta_\star^{(2)}). \tag{7}$$

Rivers and Vuong (2002) present regularity conditions under which N_T converges to a standard normal distribution under H_0. For the purposes of our subsequent analysis, it is useful to highlight just one of these conditions, namely, $\sigma_0^2 > 0$.

Apart from the standard assumption that the weighting matrix $W_T^{(i)}$ is positive semidefinite and converges in probability to a positive definite limit, it is assumed that $W^{(i)}$ depends on a vector of nuisance parameters $\tau_0^{(i)}$ and that $\hat{\tau}_T^{(i)}$ is an estimator of $\tau_0^{(i)}$. Therefore we have, with an obvious abuse of notation, $W^{(i)} = W^{(i)}(\tau_0^{(i)})$ and $W_T^{(i)} = W_T^{(i)}(\hat{\tau}_T^{(i)})$. It is assumed that the nuisance parameters satisfy

$$T^{1/2}(\hat{\tau}_T^{(i)} - \tau_0^{(i)}) = -A_\star^{(i)} T^{-1/2} \sum_{t=1}^{T} Y_t^{(i)} + o_p(1) \tag{8}$$

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for some symmetric matrix of constants $A_\star^{(i)}$ and vector $Y_t^{(i)}$ and that the weighting matrix satisfies

$$T^{1/2}\left(\text{vech}[W_T^{(i)}]-\text{vech}[W^{(i)}]\right) = \Delta^{(i)}T^{1/2}(\hat{\tau}_T^{(i)}-\tau_0^{(i)}) + o_p(1) \tag{9}$$

for some matrix of constants $\Delta^{(i)}$. The definitions of $A_\star^{(i)}$, $Y_t^{(i)}$, and $\Delta^{(i)}$ depend on the choice of weighting matrix.

Within our framework of GMM minimands with stationary processes, σ_0^2 has the following form:

$$\sigma_0^2 = R_\star' V_\star R_\star, \tag{10}$$

where

$$V_\star = \lim_{T\to\infty}\frac{1}{T}\text{Var}\left[\sum_{t=1}^{T}\xi_t(\theta_\star)\right] \tag{11}$$

for

$$\begin{aligned}\xi_t(\theta_\star) = \Big[&f^{(1)}(v_t,\theta_\star^{(1)})' - \text{E}[f^{(1)}(v_t,\theta_\star^{(1)})'], Y_t^{(1)'}, f^{(2)}(v_t,\theta_\star^{(2)})' \\ &-\text{E}[f^{(2)}(v_t,\theta_\star^{(2)})'], Y_t^{(2)'}\Big]'\end{aligned} \tag{12}$$

and

$$R_\star = \left[R_\star^{(1)'}, -R_\star^{(2)'}\right]',$$

$$R_\star^{(i)} = \begin{bmatrix} 2W^{(i)}\text{E}[f(v_t,\theta_\star^{(i)})] \\ -A_\star^{(i)}\Delta^{(i)\prime}B_i'\{\text{E}[f^{(i)}(v_t,\theta_\star^{(i)})]\otimes\text{E}[f^{(i)}(v_t,\theta_\star^{(i)})]\}\end{bmatrix}, \tag{13}$$

where B_i is the $q_i^2\times q_i(q_i+1)/2$ matrix such that $\text{vec}(W^{(i)}) = B_i\text{vech}(W^{(i)})$ and $A_\star^{(i)}$, $Y_t^{(i)}$, and $\Delta^{(i)}$ are defined implicitly in (8) and (9).

Some additional notation will be useful. On occasion, it is convenient to combine the parameters and moment functions from both models into one vector, and so we define $\theta = [\theta^{(1)\prime},\theta^{(2)\prime}]'$, $f(v_t,\theta) = [f^{(1)}(v_t,\theta^{(1)})', f^{(2)}(v_t,\theta^{(2)})']'$, $g_T(\theta) = [g_T^{(1)}(\theta^{(1)})', g_T^{(2)}(\theta^{(2)})']'$ for $g_T^{(i)}(\theta^{(i)}) = T^{-1}\sum_{t=1}^T f^{(i)}(v_t,\theta^{(i)})$, $G_T^{(i)}(\theta^{(i)}) = T^{-1}\sum_{t=1}^T \partial f^{(i)}(v_t,\theta^{(i)})/\partial\theta^{(i)\prime}$, $G_0^{(i)}(\theta^{(i)}) = \text{E}[\partial f^{(i)}(v_t,\theta^{(i)})/\partial\theta^{(i)\prime}]$, $G_T^{(i)}(\theta_1,\theta_2,\lambda)$ is the $(q_i\times p_i)$ matrix whose jth row is the corresponding row of $G_T^{(i)}(\bar{\theta}^{(j)})$ where $\bar{\theta}^{(j)} = \lambda_j\theta_1 + (1-\lambda_j)\theta_2$ for some $0\le\lambda_j\le 1$, and λ is the $(q\times 1)$ vector with jth element λ_j. Finally, we denote the Choleski decomposition of a matrix S by $S^{1/2}$ such that $S = S^{1/2}S^{1/2\prime}$, and we denote the inverse of $S^{1/2}$ by $S^{-1/2}\equiv[S^{1/2}]^{-1}$.

We now consider the limiting behavior of N_T under H_0. To facilitate this discussion, it is useful to relate the H_0 in (5) to the scenarios regarding the population

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moment condition employed in the analysis of model specification tests within the GMM framework. There are three such scenarios: (i) the moment condition is correctly specified, $\mathrm{E}[f^{(i)}(v_t, \theta_\star^{(i)})] = 0$; (ii) the moment condition is locally misspecified, $\mathrm{E}[f^{(i)}(v_t, \theta_\star^{(i)})] = T^{-1/2}\eta^{(i)}$, $\eta^{(i)} \neq 0$; (iii) the moment condition is misspecified, $\mathrm{E}[f^{(i)}(v_t, \theta^{(i)})] \neq 0 \; \forall \theta^{(i)} \in \Theta^{(i)}$, the so-called fixed or nonlocal alternative. Notice that H_0 in (5) can hold under any of (i)-(iii).[2]

For pedagogic convenience, we begin with the case of nonlocal misspecification, i.e., with the following assumption.

Assumption 1. $\mathcal{M}_i$ satisfies $\mathrm{E}[f(v_t, \theta)] = \mu(\theta)$ where $\mu(\theta) = [\mu^{(1)}(\theta^{(1)})', \mu^{(2)}(\theta^{(2)})']'$ and $\| \mu^{(i)}(\theta^{(i)}) \| \neq 0$ for all $\theta^{(i)} \in \Theta^{(i)}$.

To implement the estimations and test, it is necessary to choose the weighting matrices. From the results in Hall and Inoue (2003), because the models are misspecified, there is no advantage to employing a weighting matrix that converges to the inverse of the long run variance of the sample moment condition and hence to employing iterated GMM estimation. Therefore, in what follows we only study the case where the weighting matrix is a matrix of constants, such as the identity matrix.[3] To analyze the behavior of the test in this case we impose the following set of assumptions.

Assumption 2.

(a) $T^{-1/2}\sum_{t=1}^{T}\{f(v_t, \theta_\star) - \mathrm{E}[f(v_t, \theta_\star)]\} \xrightarrow{d} N(0, S(\theta_\star))$ where

$$S(\theta_\star) = \begin{bmatrix} S^{(1)}(\theta_\star^{(1)}) & S^{(1,2)}(\theta_\star) \\ S^{(1,2)}(\theta_\star)' & S^{(2)}(\theta_\star^{(2)}) \end{bmatrix}$$

is a positive definite matrix of finite constants that is partitioned conformably with $f(\cdot)$;

(b) $\mathrm{rank}\{G_0^{(i)}(\theta_\star^{(i)})\} = p_i$;

(c) $T^{1/2}(\hat{\theta}_T^{(i)} - \theta_\star^{(i)}) = O_p(1)$;

(d) $\hat{S}^{(i)}(\hat{\theta}_T^{(i)})$ is a $q_i \times q_i$ matrix satisfying $\hat{S}^{(i)}(\hat{\theta}_T^{(i)}) \xrightarrow{p} S^{(i)}(\theta_\star^{(i)})$ for $i = 1, 2$;

(e) $\hat{S}^{(1,2)}(\hat{\theta}_T)$ is a $q_1 \times q_2$ matrix satisfying $\hat{S}^{(1,2)}(\hat{\theta}_T) \xrightarrow{p} S^{(1,2)}(\theta_\star)$.

Hall and Inoue (2003, Thm. 1) provide conditions under which $T^{1/2}(\hat{\theta}_T^{(i)} - \theta_\star^{(i)})$ has a limiting normal distribution, and so Assumption 2(c) could be replaced by these lower level assumptions. The following theorem gives the limiting distribution of N_T for this choice of weighting matrix.

THEOREM 1. *Let*

(i) $\{v_t\}$, $f(.)^{(i)}$, $\theta_\star^{(i)}$, *and* $\Theta_\star^{(i)}$ *satisfy Hall (2005), where Hall's Assumptions 3.1, 3.2, 3.8–3.10, 4.2, and 4.3 hold;*

(ii) $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *satisfy Assumption 1;*

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(iii) H_0 *holds; the weighting matrix is* $W_T^{(i)} = I_{q_i}$ *and Assumption 2 holds; then* $N_T \xrightarrow{d} N(0, 1)$.

Theorem 1 confirms the results of Rivers and Vuong (2002) in that the statistic N_T has a limiting standard normal distribution under the null hypothesis if both models are misspecified in the sense of Assumption 1. This result would appear to indicate that there is scope for using this statistic to compare two misspecified models estimated via GMM. However, in our opinion, some caution needs to be exercised in its use as we now explain. The null hypothesis involves the population analogue to the minimands. These minimands depend on the weighting matrices and also the probability limits of the estimators. In general, the relative magnitudes of the minimands, $Q_0^{(i)}(\theta_\star^{(i)})$, are sensitive to the choice of weighting matrices, and so the relative ranking can be reversed by changing the weighting matrices. Whether or not this dependence on the weighting matrix is a weakness depends on the setting. In some cases, economic theory dictates an appropriate choice of weighting matrix and so only the outcome with this choice of the weighting matrix is of interest. Examples in this vein are the assessment of specification errors in asset pricing models (see e.g., Hansen and Jagannathan, 1997) or dynamic stochastic equilibrium models (see, e.g., Dridi, Guay, and Renault, 2007). However, absent these economic considerations, the choice of the weighting matrix and the relative ranking of the models can become arbitrary because there is no optimal weighting matrix in this context.

We now consider the case in which the population moment conditions are correctly specified or locally misspecified.

Assumption 3. $\mathcal{M}_i$ satisfies $S^{(i)}(\theta_\star^{(i)})^{-1/2}\mathrm{E}_T[f^{(i)}(v_t, \theta_\star^{(i)})] = T^{-1/2}\eta^{(i)}$ where $\eta^{(i)}$ is a vector of finite constants.

The operator $\mathrm{E}_T[.]$ denotes expectations with respect to the joint probability distribution of $\{v_t, t = 1, \ldots, T\}$ and $S^{(i)}(\theta_\star^{(i)}) = \lim_{T\to\infty} \mathrm{Var}[g_T^{(i)}(\theta_\star^{(i)})]$.[4] Notice that if $\eta^{(i)} = 0$ then model i is correctly specified. Rather than just consider the case of correct specification, we allow for local misspecification also because the latter casts light on the finite-sample behavior of the test statistic when the degree of misspecification is small. Given the framework in Assumption 3, we must modify the definition of the population minimands as follows: $Q_0^{(i)}(\theta^{(i)}) = \lim_{T\to\infty} \mathrm{E}_T\left[f^{(i)}(v_t, \theta^{(i)})\right]' W^{(i)} \lim_{T\to\infty} \mathrm{E}_T\left[f^{(i)}(v_t, \theta^{(i)})\right]$. Notice that Assumption 3 implies $Q_0^{(i)}(\theta_\star^{(i)}) = 0$ for both models. Therefore, although the models are not correctly specified, the local nature of this misspecification implies that the null hypothesis in (5) still holds; i.e., $H_0 : Q_0^{(1)}(\theta_\star^{(1)}) = Q_0^{(2)}(\theta_\star^{(2)})$.

Using Assumption 3 and (10)–(13), it can be seen that, for the case under consideration here, $R_\star$ is a null vector and hence $\sigma_0^2 = 0$. Therefore, if both models are either correctly specified or locally misspecified then the null distribution of N_T does not follow from the Rivers and Vuong (2002) analysis (see their

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Theorem 3). We note that Rivers and Vuong (2002, Sect. 6) provide generic conditions under which the test does not have a limiting standard normal distribution because $\sigma_0^2 = 0$. An inspection of these conditions indicates that they include the case covered here although this is not noted in their discussion of the results.

In what follows, for completeness, we present the appropriate limiting distribution theory for the test statistic in this case. To do so, it is necessary to be more specific about the construction of $\hat{\sigma}_T$ and hence about the weighting matrices employed. Because both models are assumed correctly specified or at most locally misspecified, we assume that the weighting matrices are chosen so that $W^{(i)} = \{S^{(i)}\}^{-1}$ and $W_T^{(i)}$ depends on $\hat{\tau}_T^{(i)}$, a preliminary GMM estimator of $\theta_\star^{(i)}$ using a weighting matrix, $M_T^{(i)}$, that converges to a positive definite matrix of constants, $M^{(i)}$. In this case, it follows that the matrix $A_\star^{(i)}$ and vector $Y_t^{(i)}$ in (8) are given by $A_\star^{(i)} = -\left[G_0^{(i)}(\theta_\star^{(i)})' M^{(i)} G_0^{(i)}(\theta_\star^{(i)})\right]^{-1}$ and $Y_t^{(i)} = G_0^{(i)}(\theta_\star^{(i)})' M^{(i)} f^{(i)}(v_t, \theta_\star^{(i)})$. To define $\Delta^{(i)}$, assume that $W_T^{(i)} = \{S_T^{(i)}(\hat{\tau}_T^{(i)})\}^{-1}$. It then follows that $\Delta^{(i)} = L_i[\{S^{(i)}(\theta_\star^{(i)})\}^{-1} \otimes \{S^{(i)}(\theta_\star^{(i)})\}^{-1}]\Sigma^{(i)}$ where $\Sigma^{(i)} = \mathrm{E}\left[\partial \mathrm{vec}[S_T(\theta^{(i)})]\, \partial \theta^{(i)\prime}\big|_{\theta^{(i)}=\theta_\star^{(i)}}\right]$ and L_i is a $q_i(q_i+1)/2 \times q_i^2$ selection matrix such that $\mathrm{vech}[W^{(i)}] = L_i \mathrm{vec}[W^{(i)}]$. The exact form of $\Sigma^{(i)}$ depends on the choice of covariance matrix estimator. We leave that unspecified and only impose high-level assumptions on $\Sigma^{(i)}$ in what follows. Given these definitions, it is natural to set

$$\hat{\sigma}_T^2 \;=\; \hat{R}_T' \hat{V}_T \hat{R}_T, \tag{14}$$

where $\hat{R}_T$ and $\hat{V}_T$ are consistent estimators of $R_\star$ and $V_\star$ constructed using the obvious sample analogues to the population quantities that make up these matrices.[5]

To present the limiting distribution of N_T, it is necessary to impose the following additional regularity conditions.

Assumption 4. The observed data are assumed to be a realization from a stochastic process $\{v_t; t = 1, 2, \ldots\}$ that satisfies the following conditions:

(a) $\hat{\theta}_T^{(i)} \xrightarrow{p} \theta_\star^{(i)}$;

(b) $g_T^{(i)}(\hat{\theta}_T^{(i)}) \xrightarrow{p} 0$;

(c) $G_T^{(i)}(\hat{\theta}_T^{(i)}) \xrightarrow{p} G_0^{(i)}$, $G_T^{(i)}(\hat{\theta}_T^{(i)}, \theta_\star^{(i)}, \lambda_T) \xrightarrow{p} G_0^{(i)}$;

(d) $W_T^{(i)} \xrightarrow{p} [S^{(i)}(\theta_\star^{(i)})]^{-1}$, a positive definite matrix;

(e) the limit distribution of the moment conditions satisfies

$$T^{1/2} g_T(\theta_\star) \xrightarrow{d} N\left(\begin{bmatrix} S^{(1)}(\theta_\star^{(1)})\eta^{(1)} \\ S^{(2)}(\theta_\star^{(2)})\eta^{(2)} \end{bmatrix}, S(\theta_\star)\right),$$

where $S(\theta_\star)$ is a positive definite matrix of finite constants;

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(f) equations (8) and (9) hold with the definitions given in the sentences preceding (14), and $\Sigma^{(i)}$ is a matrix of finite constants.

The limiting distribution of N_T is given in the following theorem.

THEOREM 2. *Let Assumptions 3 and 4 hold for $i = 1, 2$. Then $N_T \xrightarrow{d} \xi_1/\xi_2$ where ξ_1 and ξ_2 are two random variables defined in the Appendix whose distributions depend on q_1, q_2, certain population moments of the data, and the drift parameters $\{\eta^{(i)}\}$.*

It is evident from Theorem 2 that N_T does not have a limiting standard normal distribution in the case where it is used to compare two models via their GMM minimands and both models are either correctly specified or locally misspecified. The result for local misspecification reveals that if the models are misspecified but the degree of misspecification is small then the behavior of the statistic in moderate sized samples is more like the nonstandard distribution encountered in the correctly specified case than the standard normal distribution.

Given the sensitivity of the limiting behavior of N_T to the specification of the model or more generally the value of σ_0^2, it is desirable to implement some formal pretest of $\sigma_0^2 = 0$ against $\sigma_0^2 > 0$. Such a test is developed by Vuong (1989) in the context of QML and is suggested by Rivers and Vuong (2002) albeit in the context of their very general framework. We now consider the implementation of such a test in our context. The obvious test statistic is $T\hat{\sigma}_T^2$. From the proof of Theorem 2 in the Appendix, it follows that if the models are locally misspecified then $T\hat{\sigma}_T^2$ converges in distribution to a mixture of noncentral chi-squareds with the noncentrality parameters depending on the drift parameter, η. This means that the implementation of the test is problematic because the critical value for the asymptotically valid $100\alpha\%$ depends on η, which is itself not estimable consistently.

One solution is to adopt a decision rule based on the limiting distribution of $T\hat{\sigma}_T^2$ in the case where the models are both correctly specified ($\eta = 0$) because in this case

$$T\hat{\sigma}_T^2 \xrightarrow{d} \sum_{i=1}^{q_1+q_2} w_i z_i^2,$$

where $\{z_i\}$ are a sequence of independent and identically distributed standard normal random variables and $\{w_i\}$ are the eigenvalues of $C^{1/2'}\left\{I_{q_1+q_2} - P_0\right\}' \bar{C} \left\{I_{q_1+q_2} - P_0\right\} C^{1/2} = D$, say. The various elements composing D are defined in the proof of Theorem 2 and can be consistently estimated using sample analogues. In the simulation of the critical values, the weights are replaced by the eigenvalues of a consistent estimator of D. To consider the properties of such a strategy, define $\gamma(\alpha)$ to be the value such that $P(\zeta > \gamma(\alpha)) = \alpha$ where $\zeta \sim \sum_{i=1}^{q_1+q_2} w_i z_i^2$. Consider the decision rule

$$DR(\alpha): \text{ reject } H_0 : \sigma_0^2 \text{ in favor of } H_1 : \sigma_0^2 > 0 \quad \text{if } T\hat{\sigma}_T^2 > \gamma(\alpha).$$

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Let $P(\text{Type I}\,|\,\alpha)$ and $P(\text{Type II}\,|\,\alpha)$ denote the probabilities of Type I and Type II errors, respectively, associated with the decision rule $DR(\alpha)$.

Clearly if $DR(\alpha)$ is implemented with a fixed α then it only yields a test satisfying $\lim_{T\to\infty} P(\text{Type I}\,|\,\alpha) = \alpha$ in the case where $\mathcal{M}_i$ satisfy Assumption 3 *and* $\eta^{(i)} = 0$ for $i = 1, 2$.[6] However, if the decision rule is implemented with a value of α that tends to zero as $T \to \infty$ then this problem is mitigated in the limit as the following proposition demonstrates.

PROPOSITION 1. *Let* $\alpha = \tilde{\alpha}_T$ *with* $\tilde{\alpha}_T \to 0$ *as* $T \to \infty$ *and* $\ln[\tilde{\alpha}_T] = o(T)$.

(i) If $\mathcal{M}_i$ *satisfy Assumption 3 for* $i = 1, 2$ *and Assumption 4 holds then* $\lim_{T\to\infty} P(\text{Type I}\,|\,\alpha) = 0$.

(ii) If $\mathcal{M}_i$ *satisfy Assumption 1 for* $i = 1, 2$ *and the other assumptions of Theorem 1 hold then* $\lim_{T\to\infty} P(\text{Type II}\,|\,\alpha) = 0$.

Proposition 1 demonstrates that it is possible to develop a testing strategy that discriminates between states of the world in which $\sigma_0^2 = 0$ (correctly specified and locally misspecified models) and $\sigma_0^2 > 0$ (nonlocally misspecified models) with probability one in the limit, and thus provides a justification in the limit for the use of this test as a pretest.

Our theoretical results suggest the following decision tree. We can start by testing "$\sigma^2 = 0$" using the decision rule in Proposition 1. If this hypothesis is rejected then N_T can be used with a decision rule based on the standard normal distribution to test H_0 in (5) (subject to the previous caveats regarding the choice of weighting matrix mentioned earlier). If "$\sigma^2 = 0$" cannot be rejected then it indicates that both models are either correctly specified or locally misspecified, in which case H_0 in (5) holds by construction because $Q_0^{(i)}(\theta_\star^{(i)}) = 0$ for $i = 1, 2$. We note that N_T is consistent against the alternatives in (6) and (7) for which $\hat{\sigma}_T^2$ converges in probability to a positive constant, and this class includes the case in which one model is correctly specified or locally misspecified and the other model is nonlocally misspecified. Thus if either (6) or (7) holds then this decision tree leads to the model with the smaller population minimand with probability one in the limit.

A referee has argued that in the case where the pretest is insignificant, the statistic N_T can be compared to a critical value obtained by simulating under the null hypothesis that both models are correctly specified.[7] Such an approach offers the potential to inform with respect to the hypothesis that one model is correctly specified and the other is locally misspecified, which is different than H_0 in (5) but nevertheless of interest. However some exploratory simulations indicate that the marginal information in the test is very limited. The reason is that in the GMM context N_T is based on the difference of the overidentifying restrictions tests from each model, each of which is a test of the individual model specification. Thus, the numerator of N_T only adds to the researcher's ability to discriminate between a model that is correctly specified and one that is locally misspecified if it is significant but both overidentifying restrictions test statistics are insignificant.

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Our simulations indicate that it does not happen often, the intuition being that the power of both (the numerator of) N_T and the overidentifying restrictions test statistic are driven by the same drift, and similar magnitudes of drift make both statistics significant. We conjecture that this finding may be more generally true.

It may be desired to use a different weighting matrix in the GMM minimands used to measure the distance between the two models than the ones used in the estimation of the parameters. An inspection of the proofs of Theorems 1 and 2 indicates that the same qualitative results go through if the test is not based on the same weighting matrices as used in the estimations.

There has been growing interest in the estimation of moment models via generalized empirical likelihood (GEL) (Smith, 1997), and it is reasonable to wonder if GEL suffers deficiencies similar to those of GMM for the kind of inference problem described here. Kitamura (2000, 2002) and Shi (2010) propose extensions of the Vuong (1989) methods to GEL estimation of conditional moment restrictions models and to models defined by moment inequalities, respectively. An immediate advantage of GEL methods is that there is no weighting matrix and thus the model ranking is unambiguous. However, GEL methods do share with GMM methods the problems highlighted previously concerning the comparison of two correctly specified or locally misspecified models. Kitamura (2000) develops an analogous test for $\sigma_0^2 = 0$ within his setting. However, he concentrates on the cases in which both models are correctly specified or both are "overlapping" misspecified models with a common pseudo true measure, and he considers the test's behavior only under nonlocal alternatives for which $\sigma_0^2 \neq 0$. It is easily seen from his analysis that the same problems arise in the GEL setting when local misspecification of the population moment condition occurs. Proposition 1 can easily be extended to cover the GEL case and thus provides a justification for the pretest in the GEL setting also.[8]

NOTES

1. See Rivers and Vuong (2002, pp. 3, 13).

2. As a consequence, although the scenarios (i)–(iii) represent, respectively, the null, local alternative, and fixed alternative for the overidentifying restrictions test, these scenarios do not fulfill the same roles for the Rivers and Vuong statistics considered here. However, because (i)–(iii) are routinely applied to analyze model specification tests within the GMM framework, we believe it is of interest—and natural—to analyze the behavior of model selection tests within the GMM framework under the same scenarios.

3. In an earlier version of this paper, we consider the case in which the weighting matrix is an instrument cross-product matrix. The limiting distribution of N_T presented in Theorem 1 is the same although the construction of an appropriate $\hat{\sigma}_T$ does depend on the choice of weighting matrix; see Hall and Pelletier (2007).

4. Assumption 3 implies that the data cannot be a realization of a strictly stationary process, unless $\eta^{(i)} = 0$, because $\mathrm{E}[f(v_t, \theta)]$ changes with T. Instead the process can be viewed as a perturbation of a stationary process; see Newey (1985) and (Hall, 2005, Sect. 5.1.3).

5. See Hall and Pelletier (2007) for further details.

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6. This statement assumes the other conditions of Theorem 2 hold also.

7. If $\sigma_0^2 = 0$, then inference can be based on $T^{1/2}$ times the numerator of N_T, i.e., T times the difference between the minimands; see Vuong (1989) for a similar approach in quasi maximum likelihood estimation.

8. To match Kitamura's setting in which $\hat{\sigma}_T^2$ is calculated with a heteroskedasticity and autocorresation on consistent (HAC) estimator, we must set $\ln[\alpha_T] = o(T/b_T)$ where b_T is the bandwidth used in the HAC estimator. Kitamura (2000, p. 12), does observe that his test of σ_0^2 has power against certain local alternatives but does not relate these possibilities back to the moment conditions as in our framework or explore its implications further as done in Theorem 2.

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APPENDIX

A.1. Proof of Theorem 1. Applying the mean value theorem to $Q_T^{(i)}(\theta^{(i)})$ around $\theta^{(i)} = \theta_\star^{(i)}$, we obtain

$$Q_T^{(i)}(\hat{\theta}_T^{(i)}) = Q_T^{(i)}(\theta_\star^{(i)}) + \left\{ \frac{\partial Q_T^{(i)}(\theta^{(i)})}{\partial \theta^{(i)}}\bigg|_{\theta^{(i)}=\bar{\theta}_T^{(i)}} \right\}' \left(\hat{\theta}_T^{(i)} - \theta_\star^{(i)} \right), \tag{A.1}$$

where $\bar{\theta}_T^{(i)} = \lambda_T \theta_\star^{(i)} + (1-\lambda_T)\hat{\theta}_T^{(i)}$ for some $0 \le \lambda_T \le 1$. Now define

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$$\Phi^{(i)}(\theta_\star^{(i)}) = 2G_0^{(i)}(\theta_\star^{(i)})' W^{(i)} \mathrm{E}[f^{(i)}(v_t, \theta_\star^{(i)})]. \tag{A.2}$$

It then follows from (A.1) that under our assumptions, we have

$$Q_T^{(i)}(\hat{\theta}_T^{(i)}) = Q_T^{(i)}(\theta_\star^{(i)}) + \left\{ \frac{\partial Q_0^{(i)}(\theta_\star^{(i)})}{\partial \theta^{(i)}} \right\}' \left(\hat{\theta}_T^{(i)} - \theta_\star^{(i)} \right) + o_p(T^{-1/2}) \tag{A.3}$$

and hence

$$\begin{aligned} T^{1/2}\left[Q_T^{(1)}(\hat{\theta}_T^{(1)}) - Q_T^{(2)}(\hat{\theta}_T^{(2)}) \right] &= T^{1/2}\left[Q_T^{(1)}(\theta_\star^{(1)}) - Q_T^{(2)}(\theta_\star^{(2)}) \right] \\ &\quad + \left\{ \Phi^{(i)}(\theta_\star^{(1)}) \right\}' T^{1/2}\left(\hat{\theta}_T^{(1)} - \theta_\star^{(1)} \right) \\ &\quad - \left\{ \Phi^{(i)}(\theta_\star^{(2)}) \right\}' T^{1/2}\left(\hat{\theta}_T^{(2)} - \theta_\star^{(2)} \right) \\ &\quad + o_p(1). \end{aligned} \tag{A.4}$$

Finally, under H_0, we have $Q_0^{(1)}(\theta_\star^{(1)}) = Q_0^{(2)}(\theta_\star^{(2)})$, and so (A.4) can be written as

$$\begin{aligned} T^{1/2}\left[Q_T^{(1)}(\hat{\theta}_T^{(1)}) - Q_T^{(2)}(\hat{\theta}_T^{(2)}) \right] &= T^{1/2}\left[Q_T^{(1)}(\theta_\star^{(1)}) - Q_0^{(1)}(\theta_\star^{(1)}) \right] \\ &\quad - T^{1/2}\left[Q_T^{(2)}(\theta_\star^{(2)}) - Q_0^{(2)}(\theta_\star^{(2)}) \right] \\ &\quad + \left\{ \Phi^{(i)}(\theta_\star^{(1)}) \right\}' T^{1/2}\left(\hat{\theta}_T^{(1)} - \theta_\star^{(1)} \right) \\ &\quad - \left\{ \Phi^{(i)}(\theta_\star^{(2)}) \right\}' T^{1/2}\left(\hat{\theta}_T^{(2)} - \theta_\star^{(2)} \right) \\ &\quad + o_p(1). \end{aligned} \tag{A.5}$$

This equation simplifies further. Under our assumptions, the GMM estimator can be obtained by solving the first-order conditions associated with the minimization in (1); i.e., $G_T^{(i)}(\hat{\theta}_T^{(i)})' W_T T^{-1} \sum_{t=1}^T f^{(i)}(v_t, \hat{\theta}_T^{(i)}) = 0$. Furthermore, the probability limits must satisfy the analogous population moment condition; i.e., $\Phi^{(i)}(\theta_\star^{(i)}) = 0$. Therefore, we have

$$\begin{aligned} T^{1/2}\left[Q_T^{(1)}(\hat{\theta}_T^{(1)}) - Q_T^{(2)}(\hat{\theta}_T^{(2)}) \right] &= T^{1/2}\left[Q_T^{(1)}(\theta_\star^{(1)}) - Q_0^{(1)}(\theta_\star^{(1)}) \right] \\ &\quad - T^{1/2}\left[Q_T^{(2)}(\theta_\star^{(2)}) - Q_0^{(2)}(\theta_\star^{(2)}) \right] + o_p(1). \end{aligned} \tag{A.6}$$

Notice that $Q_T^{(i)}(.)$ and $Q_0^{(i)}(.)$ have the generic structures $\hat{h}'\hat{W}\hat{h}$ and $h'Wh$, respectively, and that

$$\hat{h}'\hat{W}\hat{h} - h'Wh = \hat{h}'\hat{W}(\hat{h} - h) + \hat{h}'(\hat{W} - W)h + (\hat{h} - h)'Wh. \tag{A.7}$$

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With the choice $W_T^{(i)} = I_{q_i}$ for the weighting matrix, it follows from (A.6) and (A.7) that

$$T^{1/2}\left[Q_T^{(1)}(\hat{\theta}_T^{(1)}) - Q_T^{(2)}(\hat{\theta}_T^{(2)})\right]$$

$$= 2\left\{\mu^{(1)}(\theta_\star^{(1)})' T^{-1/2} \sum_{t=1}^{T} [f^{(1)}(v_t, \theta_\star^{(1)}) - \mu^{(1)}(\theta_\star^{(1)})]\right.$$

$$\left. + \mu^{(2)}(\theta_\star^{(2)})' T^{-1/2} \sum_{t=1}^{T} [f^{(2)}(v_t, \theta_\star^{(2)}) - \mu^{(2)}(\theta_\star^{(2)})]\right\} + o_p(1).$$

The result then follows immediately under the stated assumptions.

A.2. Theorem 2: Definitions and Proof.

Definitions.

$$\xi_1 = d' \operatorname{diag}\left(I_{q_1} - P_0^{(1)}, -[I_{q_2} - P_0^{(2)}]\right) d, \qquad d = C^{1/2}(n_{q_1+q_2} + \bar{\eta}),$$

$$C^{1/2} = \operatorname{diag}\left(S^{(1)}(\theta_\star^{(1)})^{-1/2}, S^{(2)}(\theta_\star^{(2)})^{-1/2}\right) S(\theta_\star)^{1/2}, \qquad n_{q_1+q_2} \sim N(0, I_{q_1+q_2}),$$

$$P_0^{(i)} = F_0^{(i)}(\theta_\star^{(i)}) \left[F_0^{(i)}(\theta_\star^{(i)})' F_0^{(i)}(\theta_\star^{(i)})\right]^{-1} F_0^{(i)}(\theta_\star^{(i)})',$$

$$F_0^{(i)}(\theta_\star^{(i)}) = S^{(i)}(\theta_\star^{(i)})^{-1/2} G_0^{(i)}(\theta_\star^{(i)}),$$

$$\xi_2 = 2\sqrt{d' \left\{I_{q_1+q_2} - P_0\right\}' \bar{C} \left\{I_{q_1+q_2} - P_0\right\} d}, \qquad \bar{C} = A\, S(\theta_\star)\, A',$$

$$A = \operatorname{diag}\left(S^{(1)}(\theta_\star^{(1)})^{-1/2}, -S^{(2)}(\theta_\star^{(2)})^{-1/2}\right), \qquad P_0 = \operatorname{diag}\left(P_0^{(1)}, P_0^{(2)}\right).$$

Sketch Proof. The test statistic N_T can be written as

$$N_T = \frac{T\left\{Q_T^{(1)}\left(\hat{\theta}_T^{(1)}\right) - Q_T^{(2)}\left(\hat{\theta}_T^{(2)}\right)\right\}}{T^{1/2}\hat{\sigma}_T}. \tag{A.8}$$

Standard analysis of the overidentifying restrictions test yields (see, e.g., Hall, 2005, eqn. (3.36), p. 73)

$$T\, Q_T^{(i)}\left(\hat{\theta}_T^{(i)}\right) = \{T^{1/2} g_T^{(i)}(\theta_\star^{(i)})\}' [S^{(i)}]^{-1/2\prime} \left[I - P_0^{(i)}(\theta_\star^{(i)})\right] [S^{(i)}]^{-1/2}$$

$$\times \{T^{1/2} g_T^{(i)}(\theta_\star^{(i)})\} + o_p(1). \tag{A.9}$$

Now consider the denominator of (A.8). It is most convenient to study $T\hat{\sigma}_T^2 = T^{1/2}\hat{R}_T'\, \hat{V}_T\, T^{1/2}\hat{R}_T$. First consider $T^{1/2}\hat{R}_T$, the sample analogue of $R_\star$. Under our assumptions, it follows that

$$T^{1/2}\hat{R}_T = \left[2T^{1/2} g_T^{(1)}(\hat{\theta}_T^{(1)})' W_T^{(1)}, o_p(1), -2T^{1/2} g_T^{(2)}(\hat{\theta}_T^{(2)})' W_T^{(2)}, o_p(1)\right]'$$

$$= 2\Gamma_V T^{1/2} g_T(\hat{\theta}_T) + o_p(1), \tag{A.10}$$

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where Γ_V is defined subsequently. The result then follows by using a mean value theorem expansion for $g_T(\hat{\theta}_T)$, the standard asymptotic representation for $(\hat{\theta}_T - \theta_\star)$ (see, e.g., Hall, 2005, eqn. (3.26)), and $\hat{V}_T \xrightarrow{p} \Gamma_U S \Gamma_U'$. The matrices Γ_V and Γ_U are

$$\Gamma_V = \begin{bmatrix} W_T^{(1)} & 0 & 0 & 0 \\ 0 & 0 & -W_T^{(2)} & 0 \end{bmatrix}', \qquad \Gamma_U = \begin{bmatrix} I_{q_1} & \{G_0^{(1)\prime} M^{(1)}\}' & 0 & 0 \\ 0 & 0 & I_{q_2} & \{G_0^{(2)\prime} M^{(2)}\}' \end{bmatrix}.$$

■

A.3. Proof of Proposition 1. Because D is positive semidefinite, $w_i \geq 0$; without loss of generality, we assume the eigenvalues are ordered so that $w_i \geq w_{i+1}$. Let n_w denote the number of nonzero eigenvalues of D. It follows that

$$w_{n_w} \sum_{i=1}^{n_w} z_i^2 \leq \sum_{i=1}^{q_1+q_2} w_i z_i^2 = \sum_{i=1}^{n_w} w_i z_i^2 \leq w_1 \sum_{i=1}^{n_w} z_i^2,$$

and so $w_{n_w} c(\alpha) \leq \gamma(\alpha) \leq w_1 c(\alpha)$ where $c(\alpha)$ is the $1-\alpha$ quantile of the $\chi^2_{n_w}$ distribution. (Potscher, 1983, Thm. 5.8) establishes that for $\lim_{T\to\infty} \kappa_T = \infty$, $c(\alpha_T) = o(\kappa_T)$ if and only if $\ln[\alpha_T] = o(\kappa_T)$. The proof is completed by noting that under our assumptions if the models are correctly specified or locally misspecified then $T\hat{\sigma}_T^2 = O_p(1)$ and if the models are nonlocally misspecified then $\hat{\sigma}_T^2 \xrightarrow{p} \sigma_0^2 > 0$ and hence $T\hat{\sigma}_T^2$ diverges at rate T. ■

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