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EXTREMAL GRAPHS FOR DEGREE SUMS AND DOMINATING CYCLES

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Abstract

A cycle C of a graph G is *dominating* if $V(C)$ is a dominating set and $V(G)\backslash V(C)$ is an independent set. Wu *et al.* ['Degree sums and dominating cycles', *Discrete Mathematics* **344** (2021), Article no. 112224] proved that every longest cycle of a k-connected graph G on $n \geq 3$ vertices with $k \geq 2$ is dominating if the degree sum is more than $(k+1)(n+1)/3$ for any $k+1$ pairwise nonadjacent vertices. They also showed that this bound is sharp. In this paper, we show that the extremal graphs G for this condition satisfy $(n-2)/3K_1 \vee (n+1)/3K_2 \subseteq G \subseteq K_{(n-2)/3} \vee (n+1)/3K_2$ or $2K_1 \vee 3K_{(n-2)/3} \subseteq G \subseteq K_2 \vee 3K_{(n-2)/3}$.

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1. Introduction

All terminology and notation not defined in this paper are the same as those in [2]. All graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph. Let H be a subgraph of G and $v \in V(G)$. We use $N_H(v)$ to denote the neighbours of v in H and $d_H(v) = |N_H(v)|$. If $H = G$, we write $N(v) = N_G(v)$ and $d(v) = |N(v)|$. If $S \subseteq V(G)$, then $N_H(S) = (\bigcup_{v\in S} N_H(v))\backslash S$, $G[S]$ denotes the subgraph induced by S and $G - S = G[V(G) - S]$. For two disjoint sets $X, Y \subseteq V(G)$, $E(X, Y)$ denotes the set of edges between X and Y. For any $x, y \in V(G)$, an (x, y)-path denotes a path starting at x and ending at y.

The *independence number* and *connectivity* of G are denoted by $\alpha(G)$ and $\kappa(G)$, respectively. Denote by K_n a complete graph of order n. Let $c(G)$ and $p(G)$ denote the order of a longest cycle and path in a graph G, respectively.

Let C be a cycle. We denote by $\overrightarrow{C}$ the cycle C with a given orientation and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse

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order, are given by $v\overleftarrow{C}u$. If $u = v$, then $u\overrightarrow{C}v = \{u\}$. We will consider both $u\overrightarrow{C}v$ and $v\overleftarrow{C}u$ as paths and vertex sets. We use u^{+i} and u^{-i} to denote the ith successor and predecessor of u, respectively. In particular, $u^{+} = u^{+1}$ and $u^{-} = u^{-1}$. If $A \subset V(C)$, we write $A^{+i} = \{a^{+i} \mid a \in A\}$ and $A^{-i} = \{a^{-i} \mid a \in A\}$. In particular, $A^{+} = A^{+1}$ and $A^{-} = A^{-1}$. Similar notation is used for paths.

Let G be a simple connected graph. For an integer $k \geq 2$, define

$$\sigma_k(G) = \min\left\{\sum_{i=1}^{k} d(x_i) \mid \{x_1, \ldots, x_k\} \text{ is an independent set of } G\right\}$$

if $\alpha(G) \geq k$ and otherwise, $\sigma_k(G) = \infty$.

A cycle C of a graph G is *dominating* if $V(C)$ is a dominating set and $V(G)\backslash V(C)$ is an independent set. The problem of existence of cycles is a classic and widely studied topic, of which the existence of dominating cycles is an interesting extension. In 1980, Bondy [1] first gave a degree sum condition on 2-connected graphs such that every longest cycle is dominating. In 2005, Lu *et al.* [4] studied the same problem for 3-connected graphs. In 2021, we extended these two results as follows.

THEOREM 1.1 (Wu *et al.* [5]). *Let G be a k-connected graph on n vertices with $k \geq 2$. If $\sigma_{k+1}(G) > (k+1)(n+1)/3$, then every longest cycle is dominating.*

In [1, 4, 5], the authors gave two examples G_1 and G_2 to show that the bounds of their results are sharp: $G_1 = kK_1 \vee (k+1)K_2$ and $G_2 = K_k \vee (k+1)K_2$. Let $G_1 \subseteq G \subseteq G_2$ and $S = V(G)\backslash V((k+1)K_2)$. Clearly, $n = |G| = 3k+2$, $k = (n-2)/3$, $d(u) = (n+1)/3$ for $u \in V(G)\backslash S$ and $d(u) \geq (2n+2)/3$ for $u \in S$. Since G contains $K_{k,2k+2}$ as its spanning subgraph and $K_{k,2k+2}$ is k-connected, G is k-connected. Any independent set of G with cardinality $k+1$ contains no vertex of S. Moreover, each component of $G - S = (k+1)K_2$ contributes only one vertex to any independent set of cardinality $k+1$. Hence, $\sigma_{k+1}(G) = (k+1)(n+1)/3$. Since S is a vertex cut of G and $G - S = (k+1)K_2$, any longest cycle uses at most k vertices of S and k components of $G - S$. Therefore, G has no dominating cycle. Moreover, if $k = 2$ and $2K_1 \vee 3K_{(n-2)/3} \subseteq G \subseteq K_2 \vee 3K_{(n-2)/3}$, then $\sigma_3(G) = n+1$, G is 2-connected and G has no dominating cycle. From this, we guess that these graphs are the extremal graphs.

In this paper, we characterise the extremal graphs when the equality of the bound of Theorem 1.1 holds. Our main result is the following.

THEOREM 1.2. *Let G be a k-connected graph on n vertices with $k \geq 2$. If $\sigma_{k+1}(G) \geq (k+1)(n+1)/3$, then every longest cycle is dominating or $(n-2)/3K_1 \vee (n+1)/3K_2 \subseteq G \subseteq K_{(n-2)/3} \vee (n+1)/3K_2$ or $2K_1 \vee 3K_{(n-2)/3} \subseteq G \subseteq K_2 \vee 3K_{(n-2)/3}$.*

2. Proof of Theorem 1.2

Before proving Theorem 1.2, we shall give some notation and lemmas which were introduced in [3, 5, 6].

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LEMMA 2.1 (Fournier and Fraisse [3]). *Let G be a k-connected graph on n vertices with $k \geq 2$. If $\sigma_{k+1}(G) \geq c$, then $c(G) \geq \min\{n, 2c/(k+1)\}$.*

Let G be a k-connected graph on n vertices with $k \geq 2$ and C a longest cycle of G with a given orientation. If C is not dominating, then we let H be a component of $G - V(C)$ with $|H| \geq 2$ and $N_C(H) = \{x_1, x_2, \ldots, x_t\}$, where the subscripts agree with the orientation of C. Let a_i be the first noninsertible vertex occurring on $x_i^+ \overrightarrow{C} x_{i+1}^-$, $A = \{a_1, a_2, \ldots, a_t\}$ and $A_h = \{a_i \mid x_i \in N(h)\}$ for $h \in V(H)$.

LEMMA 2.2 (Wu *et al.* [5]). *$A \cup \{h\}$ is an independent set for $h \in V(H)$.*

LEMMA 2.3 (Wu *et al.* [5]). *Suppose $h_1, h_2 \in V(H)$ with $h_1 \neq h_2$ and $a_i, a_j \in A_{h_1}$ with $i \neq j$. Then $d(h_2) + d(a_i) + d(a_j) \leq n + 1$.*

LEMMA 2.4 (Wu *et al.* [5]). *Suppose $h_1, h_2 \in V(H)$ with $h_1 \neq h_2$ and $a_i \in A_{h_1}, a_j \in A_{h_2}$ with $i \neq j$. Then $d(h_1) + d(a_i) + d(a_j) \leq n + 1$.*

LEMMA 2.5 (Wu *et al.* [5]). *Suppose h_1, h_2, h_3 are three distinct vertices of $V(H)$ and a_i, a_j, a_ℓ are three distinct vertices of A with $a_i \in A_{h_1}, a_j \in A_{h_2}, a_\ell \in A_{h_3}$. Then $d(a_i) + d(a_j) + d(a_\ell) \leq n$.*

LEMMA 2.6 (Wu *et al.* [5]). *Suppose $h_1, h_2 \in V(H)$ with $h_1 \neq h_2$ and a_i, a_j, a_ℓ are three distinct vertices of A with $a_i \in A_{h_1}, a_j, a_\ell \in A_{h_2}$. Then $d(a_i) + d(a_j) + d(a_\ell) \leq n + 1$.*

PROOF OF THEOREM 1.2. Let C be a longest cycle of G. Suppose C is not a dominating cycle. Then $|C| \leq n - 2$ and $G - V(C)$ has a component H with $|H| \geq 2$. Give C an orientation $\overrightarrow{C}$. Let $N_C(H) = \{x_1, x_2, \ldots, x_t\}$ where the appearance of $\{x_1, x_2, \ldots, x_t\}$ agrees with the orientation of $\overrightarrow{C}$. Since G is k-connected, we have $t \geq k \geq 2$. Define a_i, A, A_h as at the beginning of this section. By Lemma 2.2, $A \cup \{h\}$ is an independent set with at least $k + 1$ vertices for any $h \in V(H)$. Let $S = \{v \in V(G) \mid d(v) \leq (n+1)/3\}$ and $L = \{v \in V(G) \mid d(v) > (n+1)/3\}$.

Claim 1. $d(u) = (n+1)/3$ for each $u \in A \cup N_H(C)$.

PROOF. If $A \cup N_H(C) \subseteq S$, then since $\sigma_{k+1}(G) \geq (k+1)(n+1)/3$ and $A \cup \{h\}$ is an independent set with at least $k + 1$ vertices for any $h \in N_H(C)$, we can deduce that $d(u) = (n+1)/3$ for each $u \in A \cup N_H(C)$. So it suffices to show that $A \cup N_H(C) \subseteq S$.

Suppose there exists a vertex a_i of A such that $a_i \in L$ where $1 \leq i \leq t$. Assume $a_i \in A_h$ for some $h \in V(H)$. Since G is k-connected and $|H| \geq 2$, we have $|N_C(V(H)\backslash\{h\}))| \geq k - 1$. Take a subset X of $N_C(V(H)\backslash\{h\})\backslash\{x_i\}$ with $k - 2$ vertices and let $A_X = \{a_j \mid x_j \in X\}$. We take a vertex $x_\ell \in N_C(V(H) \setminus \{h\}) - X - \{x_i\}$ if possible or $x_\ell \in N_C(h) - X - \{x_i\}$ otherwise. By Lemma 2.2, $\{a_i, a_\ell, h\} \cup A_X$ is an independent set with $k + 1$ vertices. Since $X \subseteq (N_C(V(H - h))\backslash\{x_i\})$, each vertex x_j of X has a neighbour in H different from h. Therefore, by Lemma 2.4, we have $d(h) + d(a_i) + d(a_\ell) \leq n + 1$,

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and by Lemmas 2.5 and 2.6, we have $d(a_i) + d(a_p) + d(a_q) \le n + 1$ for $a_p, a_q \in A_X$. Since $a_i \in L$, we have $d(a_p) + d(a_q) < 2(n + 1)/3$ for $a_p, a_q \in A_X$. Thus,

$$\sigma_{k+1}(G) \le [d(h) + d(a_i) + d(a_\ell)] + \sum_{a_p \in A_X} d(a_p)$$
$$< (n + 1) + (k - 2)(n + 1)/3 = (k + 1)(n + 1)/3,$$

which is a contradiction. Hence, $A \subseteq S$.

Suppose there is a vertex h^* of $N_H(C)$ such that $h^* \in L$. If $t = 2$, then by Lemma 2.1, $|C| \ge 2(n + 1)/3$ and as $h^* \in L$, $n = |G| \ge |C| + |H| \ge 2(n + 1)/3 + d(h^*) - d_C(h^*) + 1 > 2(n + 1)/3 + (n + 1)/3 - 2 + 1 = n$, which is a contradiction. Thus, $t \ge 3$. Take $x_j \in N_C(h^*)$. Since $|N_C(V(H)\backslash\{h^*\}))| \ge k - 1$, we may assume that $hx_\ell \in E(G)$ where $h \in V(H)\backslash\{h^*\}$ and $\ell \ne j$. By Lemma 2.4, we have $d(h^*) + d(a_j) + d(a_\ell) \le n + 1$. Since $h^* \in L$, we have $d(a_j) + d(a_\ell) < 2(n + 1)/3$. If $t \ge k + 1$, then we take a subset B of $A\backslash\{a_j, a_\ell\}$ with $k - 1$ vertices, and hence $\sigma_{k+1}(G) \le d(a_j) + d(a_\ell) + \sum_{a_p \in B\backslash\{a_j, a_\ell\}} d(a_p) < (k + 1)(n + 1)/3$, which is a contradiction. Hence, $t \le k$. Since $t \ge k$, we have $t = k$. Therefore, $A \cup \{h^*\}$ is an independent set with $k + 1$ vertices. Since $A \subseteq S$ and $d(h^*) + d(a_j) + d(a_\ell) \le n + 1$,

$$\sigma_{k+1}(G) \le [d(h^*) + d(a_j) + d(a_\ell)] + \sum_{a_p \in A\backslash\{a_j, a_\ell\}} d(a_p) \le (k + 1)(n + 1)/3.$$

As $\sigma_{k+1}(G) \ge (k + 1)(n + 1)/3$, we have $\sigma_{k+1}(G) = (k + 1)(n + 1)/3$. Therefore, $d(h^*) + d(a_j) + d(a_\ell) = n + 1$ and $d(a_p) = (n + 1)/3$ for $a_p \in A\backslash\{a_j, a_\ell\}$. Since $h^* \in L$, we have $d(a_j) + d(a_\ell) < 2(n + 1)/3$. Hence, $\min\{d(a_j), d(a_\ell)\} < (n + 1)/3$. If $\max\{d(a_j), d(a_\ell)\} < (n + 1)/3$, then as $t \ge 3$ and $d(a_p) = (n + 1)/3$ for $a_p \in A\backslash\{a_j, a_\ell\}$, we have $d(h^*) + d(a_j) + d(a_p) > n + 1$ for $a_p \notin A_{h^*}$ or $d(h^*) + d(a_\ell) + d(a_p) > n + 1$, for otherwise, we have a contradiction to Lemma 2.4. Thus, $\max\{d(a_j), d(a_\ell)\} = (n + 1)/3$ as $A \subseteq S$. Moreover, $A\backslash\{a_\ell\} = A_{h^*}$ if $d(a_\ell) < (n + 1)/3$ and $A_{h^*} = \{a_j\}$ if $d(a_j) < (n + 1)/3$. Since $t \ge 3$, we take $a_q \in A\backslash\{a_j, a_\ell\}$. If $d(a_j) < (n + 1)/3$, then $d(a_\ell) = (n + 1)/3$ and $A_h^* = \{a_j\}$. Hence, $a_q \notin A_{h^*}$. If $a_q \in A_h$, then $d(h^*) + d(a_\ell) + d(a_q) > n + 1$, which is in contradiction to Lemma 2.3. Thus, $a_q \notin A_h$. By Lemma 2.4, $d(h) + d(a_\ell) + d(a_q) \le n + 1$. Since $d(a_p) = (n + 1)/3$ for $a_p \in A\backslash\{a_j, a_\ell\}$,

$$\sigma_{k+1}(G) \le d(h) + d(a_\ell) + d(a_q) + d(a_j) + \sum_{a_p \in A\backslash\{a_j, a_\ell, a_q\}} d(a_p) < (k + 1)(n + 1)/3,$$

which is a contradiction. Thus, $d(a_j) = (n + 1)/3$ and so $d(a_\ell) < (n + 1)/3$ and $A\backslash\{a_\ell\} = A_{h^*}$. Thus, $a_q \in A_{h^*}$ and $d(h) + d(a_q) + d(a_j) \le n + 1$ by Lemma 2.3. Since $d(a_p) = (n + 1)/3$ for $a_p \in A\backslash\{a_j, a_\ell\}$,

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$$\sigma_{k+1}(G) \le d(h) + d(a_q) + d(a_j) + d(a_\ell) + \sum_{a_p \in A\setminus\{a_j,a_\ell,a_q\}} d(a_p) < (k+1)(n+1)/3,$$

which is a contradiction. Thus, $N_H(C) \subseteq S$. □

Claim 2. $d_C(h) \ge 2$ for each $h \in V(H)$.

PROOF. By Lemma 2.2, $\{h, a_1, \ldots, a_k\}$ is an independent set with $k+1$ vertices for each $h \in V(H)$. By Claim 1, $A \subseteq S$ and hence

$$(k+1)(n+1)/3 \le \sigma_{k+1}(G) \le d(h) + \sum_{\ell=1}^{k} d(a_\ell) = d(h) + k(n+1)/3.$$

Thus, $d(h) \ge (n+1)/3$. If $d_C(h) \le 1$, then by Lemma 2.1,

$$n \ge |C| + |H| \ge |C| + d(h) - d_C(h) + 1 \ge 2(n+1)/3 + (n+1)/3 - 1 + 1 = n+1,$$

which is a contradiction. Hence, $d_C(h) \ge 2$. □

By Claim 2, $N_H(C) = V(H)$, and hence by Claim 1,

$$d(u) = (n+1)/3 \text{ for each } u \in V(H) \cup A. \tag{2.1}$$

For convenience, we assume $V(H) = \{h_1, h_2, \ldots, h_{|H|}\}$ in what follows.

Claim 3. $|H| = 2$ or $|N_C(H)| = 2$.

PROOF. If there exists a matching M with three independent edges between $V(H)$ and $V(C)$, we write $M = \{x_ih_i, x_jh_j, x_\ell h_k\}$ where $h_i, h_j, h_k \in V(H)$. By Claim 1, $d(a_i) + d(a_j) + d(a_\ell) = n+1$, which is in contradiction to Lemma 2.5. Therefore, the maximum matching number between $V(H)$ and $V(C)$ is no more than 2. If $|H| = 2$, then the result holds. So we may assume that $|H| \ge 3$. If $k \ge 3$, then as G is k-connected and $|H| \ge 3$, there exist three independent edges between $V(H)$ and $V(C)$, which is a contradiction. Hence, $k = 2$. Since $k = 2$ and $|H| \ge 3$, there exist two independent edges between $V(H)$ and $V(C)$. Suppose $x_ph_p, x_qh_q \in E(G)$ where $h_p, h_q \in V(H)$. Since the maximum matching number between $V(H)$ and $V(C)$ is no more than 2, we must have $N_C(V(H)\setminus\{h_p, h_q\}) \subseteq \{x_p, x_q\}$, and hence by Claim 2, $N_C(V(H)) = \{x_p, x_q\}$. That is, $|N_C(H)| = 2$. □

Claim 4. We have:

(1) $N_C(h_i) = N_C(H)$ for $1 \le i \le |H|$;
(2) H is complete;
(3) $|H| = 2$, $|C| = n-2$ and $t = (n-2)/3$, or $|H| = (n-2)/3$, $|C| = (2n+2)/3$ and $t = 2$;
(4) $|x_i^+\overrightarrow{C}x_{i+1}^-| = |H|$ for $1 \le i \le t$ (where $i+1$ is taken modulo t).

PROOF. First we consider the case $|H| = 2$. Clearly, $H \cong K_2$. Since $d(h_i) = (n+1)/3$ for $1 \le i \le 2$ and $H \cong K_2$, we have $d_C(h_i) = d(h_i) - d_H(h_i) = (n-2)/3$. Suppose $|N_C(h_1) \cap N_C(h_2)| = r$. Since $H \cong K_2$ and by the maximality of C, we have $|x_i^+\overrightarrow{C}x_{i+1}^-| \ge 2$ for $x_i \in N_C(h_1) \cap N_C(h_2)$ and $|x_i^+\overrightarrow{C}x_{i+1}^-| \ge 1$ otherwise. Therefore,

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$n-2 \geq |C| \geq 3r + 2[(n-2)/3 - r] + 2[(n-2)/3 - r] = (4n-8)/3 - r$ and hence $r \geq (n-2)/3$. Since $r \leq d_C(h_i) = (n-2)/3$ for $i = 1, 2$, we have $r = d_C(h_i) = (n-2)/3$ for $i = 1, 2$, and hence $N_C(H) = N_C(h_1) = N_C(h_2)$ and $t = (n-2)/3$. Finally, since $r = (n-2)/3$, we have $n-2 \geq |C| \geq (4n-8)/3 - r = n-2$ and hence $|C| = n-2$. Moreover, $|x_i^+ \overrightarrow{C} x_{i+1}^-| = 2$ for each $1 \leq i \leq t$.

Now we consider the case $|H| \geq 3$. Since $|H| \geq 3$ and by Claim 3, $|N_C(H)| = 2$, and hence by Claim 2, $N_C(h_i) = N_C(H) = \{x_1, x_2\}$ for $1 \leq i \leq |H|$. Since $d(h_i) = (n+1)/3$ and $d_C(h_i) = 2$, we have $d_H(h_i) = (n-5)/3$ for $1 \leq i \leq |H|$, and hence by Lemma 2.1, $n \geq |H| + |C| \geq 1 + d_H(h_i) + 2(n+1)/3 = n$. Therefore, $|C| = 2(n+1)/3$ and $|H| = (n-2)/3$. Since $d_H(h_i) = (n-5)/3$ for $1 \leq i \leq |H|$ and $|H| = (n-2)/3$, H is complete and so there exists a Hamiltonian path Q connecting h_1 and h_2 in H. Since $N_C(h_i) = \{x_1, x_2\}$ for $1 \leq i \leq |H|$, we have $x_1h_1, x_2h_2 \in E(G)$, and hence, $x_1h_1Qh_2x_2\overrightarrow{C}x_1$ is a cycle. By the maximality of C, $|x_1^+ \overrightarrow{C} x_2^-| \geq |Q| = |H| = (n-2)/3$. Likewise, $|x_2^+ \overrightarrow{C} x_1^-| \geq (n-2)/3$. So we have $(2n+2)/3 = |C| = |x_1^+ \overrightarrow{C} x_2^-| + |x_2^+ \overrightarrow{C} x_1^-| + |\{x_1, x_2\}| \geq (2n+2)/3$. Thus, $|x_i^+ \overrightarrow{C} x_{i+1}^-| = |H|$ for $1 \leq i \leq 2$. □

By Claim 4(1), $x_ih_1, x_{i+1}h_2 \in E(G)$ where i is taken modulo t. By Claim 4(2), H is complete. Let Q be a Hamiltonian path connecting h_1 and h_2 in H. Let $C = x_ih_1Qh_2x_{i+1}\overrightarrow{C}x_i$. Clearly, C is a cycle of G. By Claim 4(4), $|x_i^+ \overrightarrow{C} x_{i+1}^-| = |H| = |Q|$ and hence $|C| = |C|$. By the maximality of C, C is a longest cycle. Let $\mathcal{H}$ be a component of $G - V(C)$. Observing that $x_i^+ \overrightarrow{C} x_{i+1}^-$ is a path of some component of $G - C$ with $|x_i^+ \overrightarrow{C} x_{i+1}^-| = |H| = n - |C| = n - |C|$, it follows that $x_i^+ \overrightarrow{C} x_{i+1}^-$ is a Hamiltonian path of $\mathcal{H}$. Hence, $|\mathcal{H}| = |H|$. It is clear that (2.1) and Claim 4 hold for $(C, \mathcal{H})$ with respect to (C, H).

If $|H| = (n-2)/3$, then $|\mathcal{H}| = |H| = (n-2)/3$. By Claim 4(3), $|N_C(\mathcal{H})| = 2$, and since $x_i^+x_i, x_{i+1}^-x_{i+1} \in E(G)$, $x_i^+, x_{i+1}^- \in V(\mathcal{H})$ and $x_i, x_{i+1} \in V(C)$, we have $N_C(\mathcal{H}) = \{x_1, x_2\}$. By Claim 4(1), $N_C(u) = \{x_1, x_2\}$ for $u \in V(\mathcal{H})$. Additionally, $|\mathcal{H}| = (n-2)/3$ and by Claim 4(2), $\mathcal{H} \cong K_{(n-2)/3}$. Therefore, by the choice of i, $G[x_1^+ \overrightarrow{C} x_2^-] \cong K_{(n-2)/3}$, $G[x_2^+ \overrightarrow{C} x_1^-] \cong K_{(n-2)/3}$ and $N(u) = \{x_1, x_2\}$ for $u \in V(C) \backslash \{x_1, x_2\}$. By Claim 4(2) and (1), $H \cong K_{(n-2)/3}$ and $N(h) = \{x_1, x_2\}$ for any $h \in V(H)$. It follows that $G \cong 2K_1 \vee 3K_{(n-2)/3}$ or $G \cong K_2 \vee 3K_{(n-2)/3}$.

If $|H| = 2$, then $|\mathcal{H}| = |H| = 2$ and hence $H \cong K_2$. By Claim 4(3), $|N_C(\mathcal{H})| = (n-2)/3$. Since (2.1) holds for $\mathcal{H}$ and $\mathcal{H} \cong K_2$, we have $d_C(x_i^+) = (n-2)/3$. Since $x_i^+x_i \in E(G)$ where $x_i^+ \in V(\mathcal{H})$ and $x_i \in V(C)$, we have $x_i \in N_C(x_i^+)$. Since $d_C(x_i^+) = (n-2)/3$, $H \cong K_2$ and by Claim 4(1) and the maximality of C, we have $N_C(x_i^+) = \{x_1, x_2, \ldots, x_{(n-2)/3}\}$. By Claim 4(1), $N_C(u) = \{x_1, x_2, \ldots, x_{(n-2)/3}\}$ for $u \in V(\mathcal{H})$. Thus, by the choice of i, $G[x_i^+ \overrightarrow{C} x_{i+1}^-] \cong K_2$ and $N_C(u) = \{x_1, x_2, \ldots, x_{(n-2)/3}\}$ for $u \in x_i^+ \overrightarrow{C} x_{i+1}^-$ for any $1 \leq i \leq (n-2)/3$. Therefore, together with Claim 4, we deduce that $(n-2)/3K_1 \vee (n+1)/3K_2 \subseteq G \subseteq K_{(n-2)/3} \vee (n+1)/3K_2$.

This completes the proof. □

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