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Two-generator arithmetic Fuchsian groups

BY C. MACLACHLAN† AND G. ROSENBERGER
Aberdeen University and Universität Dortmund

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1. *Introduction.* In (12), Takeuchi characterized those Fuchsian groups which are arithmetic in terms of the traces of the matrices involved in the group. Further, he used these criteria (13) to show that there are only finitely many arithmetic Fuchsian triangle groups and indeed to determine the signatures of all such groups. We note that, if two Fuchsian triangle groups have the same signature, then they are conjugate in $\mathbf{PGL}(2,\mathbb{R})$ and hence either all Fuchsian triangle groups of a given signature are arithmetic or none of them are. This will not, in general, be the case for non-triangle Fuchsian groups and in this paper, we examine Fuchsian groups with signature of one of the forms $(1; n; 0)$, $n \geqslant 2$ or $(0; 2, 2, 2, n; 0)$, $n \geqslant 3$ and odd in which case the space of conjugacy classes has dimension two. Our principal results state that for each n, there are only finitely many conjugacy classes of arithmetic Fuchsian groups of given signature $(1; n; 0)$ or $(0; 2, 2, 2, n; 0)$ and for large enough n, there are no arithmetic Fuchsian groups of that given signature (Theorems 3 and 4). Together with the results of (13) and known results for non-cocompact Fuchsian groups, these results show that there are only finitely many conjugacy classes of two-generator arithmetic Fuchsian groups (see § 7).

2. *Preliminaries on Fuchsian groups.* A Fuchsian group is a discrete subgroup of $\mathbf{PSL}(2,\mathbb{R})$. Each such group Γ operates discontinuously on the upper half-plane U. When the hyperbolic area of U/Γ is finite, Γ is said to be of the first kind. Each Fuchsian group of the first kind has a presentation of the form:

$$\left.\begin{aligned} &\text{generators:} \quad x_1, x_2, \ldots, x_r, a_1, b_1, \ldots, a_g, b_g, p_1, p_2, \ldots, p_t, \\ &\text{relations:} \quad x_i^{m_i} = 1 \ (i = 1, 2, 3, \ldots, r) \prod_{i=1}^{r} x_i \prod_{j=1}^{g} [a_j, b_j] \prod_{k=1}^{t} p_k = 1. \end{aligned}\right\} \tag{1}$$

This group has signature $(g; m_1, m_2, \ldots, m_r; t)$ and two Fuchsian groups of the first kind are isomorphic if and only if they have the same signature.

In this paper, we will be concerned with two families of signatures $(1; n; 0)$, $n \geqslant 2$

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and $(0; 2, 2, 2, n; 0)$, $n \geqslant 3$ and n odd. Note that a Fuchsian group with any of these signatures has rank 2 (see (4)). The set of conjugacy classes in $\mathbf{PGL}(2, \mathbb{R})$ of groups with signature $(1; n; 0)$ or $(0; 2, 2, 2, n; 0)$ are in one-to-one correspondence with the spaces

$$M'(n) = \left\{(x, y, z) \in \mathbb{R}^3 \middle| x^2 + y^2 + z^2 - xyz = 2 - 2\cos\frac{\pi}{n} 2 < x \leqslant y \leqslant z \leqslant \tfrac{1}{2}xy\right\},$$

and

$$M''(n) = \left\{(x, y, z) \in \mathbb{R}^3 \middle| x^2 + y^2 + z^2 - xyz = 2 - 2\cos\frac{2\pi}{n} 2 < x \leqslant y \leqslant z \leqslant \tfrac{1}{2}xy\right\},$$

respectively (see (4) and (8)).

In the first case, if the group is given, as at (1), by

$$\{x_1, a_1, b_1 | x_1^n = x_1[a_1, b_1] = 1\},$$

then the correspondence is $(x, y, z) \mapsto (\operatorname{tr} a_1, \operatorname{tr} b_1, \operatorname{tr} a_1 b_1)$. In the second, if the group is given by

$$\{x_1, x_2, x_3, x_4 | x_1^2 = x_2^2 = x_3^2 = x_4^n = x_1 x_2 x_3 x_4 = 1\},$$

then the correspondence is $(x, y, x) \mapsto (\operatorname{tr} x_1 x_2, \operatorname{tr} x_3 x_1, \operatorname{tr} x_1 x_2 x_3 x_1)$.

3. *Preliminaries on arithmetic Fuchsian groups.* Let k be a totally real algebraic number field and A a quaternion algebra over k which is unramified at the identity but ramified at all other archimedean places. Let ρ_1 be an isomorphism of A into $M(2, \mathbb{R})$. If U is the group of units of norm 1 in an order 0 of A, then $\rho_1(U) = \Gamma(A, 0)$ is a discrete subgroup of $\mathbf{SL}(2, \mathbb{R})$ whose image in $\mathbf{PSL}(2, \mathbb{R})$ is of the first kind. Let $\mathbf{P}$ denote the natural map: $\mathbf{SL}(2, \mathbb{R}) \to \mathbf{PSL}(2, \mathbb{R})$.

DEFINITION. *A Fuchsian group Γ is arithmetic if it is commensurable with some $\mathbf{P}\Gamma(A, 0)$.* ((12), (15)).

An arithmetic Fuchsian group is thus of the first kind.

In (12), Fuchsian groups are defined as discrete subgroups of $\mathbf{SL}(2, \mathbb{R})$ and the criteria for arithmeticity there are in terms of the field or ring of definition of the traces of the matrices. We find it more convenient to work in $\mathbf{PSL}(2, \mathbb{R})$, adapt the criteria accordingly and use, with slight ambiguity, the term $\operatorname{tr}\gamma$ for the trace of γ, $\gamma \in \mathbf{PSL}(2, \mathbb{R})$.

THEOREM 1 (Takeuchi). *Let Γ be a Fuchsian group of the first kind. Then Γ is an arithmetic Fuchsian group if and only if*

(i) *$k_1 = \mathbb{Q}(\operatorname{tr}\gamma : \gamma \in \Gamma)$ is an algebraic number field and the set of traces, $\operatorname{tr}\Gamma \subset 0_{k_1}$, the ring of integers in k_1 and*

(ii) *for every $\mathbb{Q}$-isomorphism $\sigma : k_1 \to \mathbb{C}$ such that*

$$\sigma|k_2 \neq Id, \quad k_2 = \mathbb{Q}((\operatorname{tr}\gamma)^2 : \gamma \in \Gamma)$$

then $\sigma(\operatorname{tr}\Gamma)$ is bounded in $\mathbb{C}$.

Throughout, we will use k_1 and k_2 to denote the fields described in this theorem, with respect to whichever Fuchsian group we are considering.

Later, parts of Takeuchi's proof will be modified and referred to so we first consider briefly some features of his work.

For any Fuchsian group Γ the vector space $A(\Gamma) = \{\Sigma a_i \gamma_i | a_i \in k_1, \gamma_i \in \Gamma\}$ is a quaternion algebra. Furthermore, two non-commuting hyperbolic elements α, β of Γ can be chosen such that $1, \alpha^2, \beta^2, \alpha^2\beta^2$ is a basis of $A(\Gamma)$ over k_1. Let $\Gamma^{(2)}$ denote the subgroup of Γ generated by squares of elements in Γ and $k_2' = \mathbb{Q}(\operatorname{tr}\gamma : \gamma \in \Gamma^{(2)})$. Let A_0

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denote the vector space spanned by $1, \alpha^2, \beta^2, \alpha^2\beta^2$ over k_2. Then it is shown that A_0 coincides with $A(\Gamma^{(2)})$ and so, in particular, $k_2 = k_2'$.

Remark. Neither discreteness nor arithmeticity is used in the above argument, the crucial feature being the existence of two hyperbolic elements which have no common fixed point. Now if Γ is a non-elementary subgroup of $\mathbf{PSL}(2, \mathbb{R})$, i.e. a subgroup containing a pair of elements of infinite order which have no common fixed point, then it necessarily contains a pair of hyperbolic elements which have no common fixed point (6). Thus the equality of A_0 and $A(\Gamma^{(2)})$ is valid when Γ is non-elementary.

Now if Γ is arithmetic and so commensurable with some $\mathbf{P}\Gamma(A, 0)$, then the quaternion algebra A is uniquely determined up to isomorphism by Γ and is the quaternion algebra $A(\Gamma^{(2)})$. Thus, in particular, it is defined over k_2 which must, thus, be totally real.

We extend Takeuchi's theorem slightly, but importantly for our purposes (see § 5) to show that k_1 must be totally real and also that $\sigma(\operatorname{tr}\Gamma)$ being bounded can be replaced by $|\sigma(\operatorname{tr}\gamma)| < 2$ for $\gamma \neq Id$. These results are essentially implicit in (12).

For simplicity, we normalize Γ so that after conjugation if necessary, $\Gamma \subset \mathbf{PSL}(2, K)$ where $[K: k_1] \leqslant 2$ (see (11)). For any $\mathbb{Q}$-isomorphism $\sigma: k_1 \to \mathbb{C}$, denote one of the extensions to K also by σ.

Let σ be such that $\sigma|k_2 \neq Id$. Let $\gamma \in \Gamma$, $\gamma \neq Id$, and suppose that $\sigma(\gamma)$ is not parabolic. Then $\sigma(\gamma)$ is conjugate in $\mathbf{PSL}(2, \mathbb{C})$ to the image of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ where $t + t^{-1} = \sigma(\operatorname{tr}\gamma)$. Let $t = re^{i\theta}$ and suppose $r \neq 1$. But then $\operatorname{tr}(\sigma(\gamma)^n) = t^n + t^{-n}$ is clearly unbounded. Thus $r = 1$ and $\sigma(\operatorname{tr}\gamma) = 2\cos\theta \in \mathbb{R}$. Hence $\sigma(\gamma)$ is either elliptic or parabolic, and in either case, $\sigma(\operatorname{tr}\gamma)$ is real, if $\sigma|k_2 \neq Id$. If $\sigma|k_2 = Id$, then

$$\sigma((tr\,\gamma)^2) = (\operatorname{tr}\gamma)^2.$$

But $\operatorname{tr}\gamma \in \mathbb{R}$ and so $\sigma(\operatorname{tr}\gamma) \in \mathbb{R}$.

Thus k_1 is totally real (cf. proposition 2 in (12)). Also note that, if $\sigma(\gamma)$ is not parabolic, then $|\sigma(\operatorname{tr}\gamma)| < 2$.

Now suppose that $\sigma(\gamma)$ is parabolic and that $\delta (\neq Id)$ is any other element of Γ. Then

$$\begin{aligned} \operatorname{tr}[\sigma(\gamma), \sigma(\delta)] &= 4 + (\operatorname{tr}\sigma(\delta))^2 + (\operatorname{tr}\sigma(\gamma\delta))^2 \pm 2\operatorname{tr}\sigma(\delta)\operatorname{tr}\sigma(\gamma\delta) - 2 \\ &= 2 + (\operatorname{tr}\sigma(\delta) \pm \operatorname{tr}\sigma(\gamma\delta))^2. \end{aligned}$$

Since $\operatorname{tr}\sigma(\delta), \operatorname{tr}\sigma(\gamma\delta)$ are real, then $\operatorname{tr}[\sigma(\gamma), \sigma(\delta)] > 2$ and so its nth power is unbounded unless $\operatorname{tr}[\sigma(\gamma), \sigma(\delta)] = 2$, for all $\delta \neq Id$. But then $\operatorname{tr}[\gamma, \delta] = 2$ and so γ, δ have a common fixed point. But not all elements of a Fuchsian group of the first kind can have a common fixed point. Thus $\sigma(\gamma)$ cannot be parabolic if $\sigma|k_2 \neq Id$. Thus if Γ contains parabolic elements and is arithmetic, then $k_2 = \mathbb{Q}$ (see Remark § 7).

Thus Takeuchi's theorem can be reformulated as

THEOREM 1′. *Let Γ be a Fuchsian group of the first kind. Then Γ is an arithmetic Fuchsian group if and only if*

(i) $k_1 = \mathbb{Q}(\operatorname{tr}\gamma: \gamma \in \Gamma)$ *is a totally real number field and* $\operatorname{tr}\Gamma \subset 0_{k_1}$ *and*

(ii) *for every $\mathbb{Q}$-isomorphism $\sigma: k_1 \to \mathbb{R}$ such that*

$$\sigma|k_2 \neq Id, \quad k_2 = \mathbb{Q}((\operatorname{tr}\gamma)^2: \gamma \in \Gamma)$$

then $|\sigma(\operatorname{tr}\gamma)| < 2$ for all $\gamma \neq Id$.

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4. *Takeuchi's Theorem for special signatures.* Theorem 1′ will now be applied to the particular cases where Γ has signature of one of the forms $(1; n; 0)$, $n \geqslant 2$ or $(0; 2, 2, 2, n; 0)$, $n \geqslant 3$, n odd. In this case Γ is generated by the images of two matrices and so the traces of all elements in the group are polynomials with integer coefficients in the traces of the generating pair and their product. From the correspondence in § 2, $k_1 = \mathbb{Q}(x, y, z)$ and if x, y, z are algebraic integers then all traces are also algebraic integers.

We now consider k_2 and adapt Takeuchi's method ((13), proposition 5) to show that $k_2 = k_0 = \mathbb{Q}(x^2, y^2, z^2, xyz)$. Take $(\alpha, \beta) = (a_1, b_1)$ or (x_1x_2, x_3x_1) as generating pair for Γ with presentation as at (1). Then $1, \alpha^2, \beta^2, \alpha^2\beta^2$ form a basis of the quaternion algebra $A(\Gamma)$ over k_1. Let A_0 be the vector space spanned by $1, \alpha^2, \beta^2, \alpha^2\beta^2$ over k_0. From the multiplication table for the basis elements of A_0, each entry can be written as a linear combination of the basis elements with coefficients in k_0. Thus A_0 is a quaternion algebra over k_0. Now $\Gamma/\Gamma^{(2)} \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ with coset representatives $1, \alpha, \beta, \alpha\beta$. A Reidemeister–Schreier calculation shows that the group $\Gamma^{(2)}$ is generated by the elements

$$\alpha^2, \beta^2, \beta\alpha\beta^{-1}\alpha^{-1}, \alpha\beta\alpha\beta^{-1}, \alpha\beta^2\alpha^{-1}.$$

But then one can deduce that $\Gamma^{(2)} \subset A_0$. It then follows that $A(\Gamma^{(2)}) = A_0$ and so $k_2 = k_0$.

Finally we show that it is sufficient to require that $|\sigma(x)| < 2$ for all σ such that $\sigma|k_2 \neq Id$. For the order of the commutator $[\alpha, \beta]$ is n and so $|\sigma \operatorname{tr}[\alpha, \beta]| < 2$. Since k_1 is totally real, $\sigma(\operatorname{tr}\gamma)$ is real for all $\gamma \in \Gamma$. Thus if $|\sigma(x)| = |\sigma(\operatorname{tr}\alpha)| < 2$, it follows that $|\sigma(\operatorname{tr}\gamma)| < 2$ for all $\gamma \neq Id$ in Γ ((2)).

Alternatively, the norm form of the quaternion algebra $A(\Gamma^{(2)})$ can be computed in terms of x, y, z and we obtain that

$$A(\Gamma^{(2)}) \cong \left(\frac{x^2(x^2-4),\ -x^2y^2(2+2\cos\pi/n)}{k_2}\right),$$

for Γ with signature $(1; n; 0)$ and with, instead, $(2+2\cos 2\pi/n)$ for Γ with signature $(0; 2, 2, 2, n; 0)$ (cf. (14)). Then, since k_1 is totally real, $A(\Gamma^{(2)})$ will be ramified at the archimedean place corresponding to σ if and only if $\sigma(x)^2 - 4$ is negative. In that case it will follow that $|\sigma(\operatorname{tr}\gamma)| < 2$ for all $\gamma \in \Gamma$, $\gamma \neq Id$.

THEOREM 2. *The triple $(x, y, z) \in M'(n)$ or $M''(n)$ represents an arithmetic Fuchsian group if and only if*

(i) *x, y, z are algebraic integers and $k_1 = \mathbb{Q}(x, y, z)$ is a totally real field and*

(ii) *for every $\mathbb{Q}$-isomorphism $\sigma: k_1 \to \mathbb{R}$ such that $\sigma|k_2 \neq Id$, $k_2 = \mathbb{Q}(x^2, y^2, z^2, xyz)$ then $|\sigma(x)| < 2$.*

Moreover, in this case, $[k_1: \mathbb{Q}(x)] \leqslant 2$ and $k_2 = \mathbb{Q}(x^2)$.

It only remains to prove the last sentence. It is clear from the fundamental equation in x, y, z that $[\mathbb{Q}(x, y, z): \mathbb{Q}(x, y)] \leqslant 2$. If it were equal to 2, then there would exist a Galois monomorphism σ such that $\sigma(z) \neq z$, but $\sigma(x) = x$ and $\sigma(y) = y$. But the fundamental equation then implies that $\sigma(z^2) \neq z^2$ and so $\sigma|k_2 \neq Id$. But $|\sigma(x)| = x > 2$ gives a contradiction. A similar argument shows that $[k_1: \mathbb{Q}(x)] \leqslant 2$.

Clearly $\mathbb{Q}(x^2) \subset k_2$. Suppose there exists $\sigma: k_2 \to \mathbb{R}$ such that $\sigma \neq Id$ but $\sigma(x^2) = x^2$. Now $|\sigma(\operatorname{tr}\gamma)| < 2$ for all $\gamma \in \Gamma$, $\gamma \neq Id$ and so $|\sigma(\operatorname{tr}\alpha^2)| = \sigma(x)^2 - 2 = x^2 - 2 > 2$. Hence $k_2 = \mathbb{Q}(x^2)$.

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5. *Arithmetic groups in $M'(n)$ and $M''(n)$.* In this section, it will be shown that, for each n, there are only finitely many conjugacy classes of arithmetic groups in $M'(n)$ or $M''(n)$.

The triples (x, y, z) satisfy the equation

$$x^2+y^2+z^2-xyz = \lambda_n \quad \left(\lambda_n = 2-2\cos\frac{\pi}{n} \text{ or } 2-2\cos\frac{2\pi}{n}\right).$$

Now λ_n and all its conjugates are bounded by $\mu_n < 4$. Since z is totally real, solving the above equation for z shows that we must have

$$(\sigma(x)^2-4)(\sigma(y)^2-4) \geqslant 4(4-\sigma(\lambda_n)).$$

Now $|\sigma(x)| < 2$ implies that $|\sigma(y)| < 2$ and so $\sigma(x^2) \leqslant \sigma(\lambda_n)$. So if $\sigma: k_1 \to \mathbb{R}$ and $\sigma|k_2 \neq Id$, then $|\sigma(x)| \leqslant w_n = \mu_n^{\frac{1}{2}} < 2$. If $\sigma|k_2 = Id$, then $\sigma(x^2) = x^2$ and so $\sigma(x) = \pm x$. Thus all conjugates of x, apart from x and $-x$, if it is a conjugate, are real numbers bounded by w_n.

THEOREM 3. *For each n, there are only finitely many conjugacy classes of arithmetic Fuchsian groups of signature $(1; n; 0)$, $n \geqslant 2$ and $(0; 2, 2, 2, n; 0)$, $n \geqslant 3$, n odd.*

Proof. Let x be the first member of a triple (x, y, z) representing an arithmetic group of one of the above signatures. We will show that the discriminant of $\mathbb{Q}(x)$ is bounded. For a fixed degree $[\mathbb{Q}(x): \mathbb{Q}]$, it is sufficient to assume that $-x$ is a conjugate of x. Thus we assume that the conjugates of x are

$$x, -x, x_1, x_2, \ldots, x_t \quad \text{with} \quad -w_n \leqslant x_1 < x_2 < \ldots\ldots < x_t \leqslant w_n.$$

The discriminant of x, $\Delta(x)$, is then given by

$$\Delta(x) = (2x)^2 \prod_{i=1}^{t} (x^2-x_i^2)^2 \prod_{i<j} (x_i-x_j)^2.$$

From the inequalities defining $M'(n)$ and $M''(n)$, it follows that $x < 3$ (see (4)). From Schur (Satz I in (10)) it follows that

$$\prod_{i<j} (x_i-x_j)^2 \leqslant w_n^{t(t-1)} M_t$$

where $M_2 = 4$ and

$$M_t = \frac{2^2.3^3.4^4.\ldots.t^t.2^2.3^3.\ldots.(t-2)^{t-2}}{3^3.5^5.\ldots.(2t-3)^{2t-3}}.$$

Thus

$$\begin{aligned}\Delta(x) &\leqslant 6^2.9^{2t}.w_n^{t(t-1)}.M_t \\ &\leqslant 6^2.9^2 \prod_{k=1}^{t-2} \left(\frac{9^2 w_n^{2(k-1)} k^{2k}}{(2k-3)^{2k-3}}\right) \cdot \frac{9^2 w_n^{2(t-2)} (t-1)^{t-1}}{(2t-5)^{2t-5}} \cdot \frac{9^2 w_n^{2(t-1)} t^t}{(2t-3)^{2t-3}}. \qquad (2)\end{aligned}$$

Let

$$a_k = \frac{9^2 w_n^{2(k-1)} k^{2k}}{(2k-3)^{2k-3}} = 9^2 \left(\frac{w_n}{2}\right)^{2k} \frac{(2k-3)^3}{w_n^2[(1-3/2k)^{2k/3}]^3}.$$

Now since $(w_n/2) < 1$ and $(1-3/2k)^{2k/3} \to e^{-1}$, there exists an integer N_0 such that for all $k > N_0$, $a_k < \frac{1}{2}$. Since the last two terms on the right-hand side of (2) are smaller than a_{t-1} and a_t, there must exist an integer N_1 such that for all $t \geqslant N_1$,

$$\Delta(x) \leqslant ((t+2)^{t+2}/(t+2)!)^2,$$

the lower bound for the discriminant of a totally real field of degree $t+2$ (see, for example, § 5·4 in (16)). Thus t is bounded and so the discriminant is bounded.

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Thus there are only a finite number of possibilities for the algebraic integer x. From the fundamental equation and the inequalities defining $M'(n)$ and $M''(n)$, it follows that

$$y^2 \leqslant \frac{x^2-\lambda_n}{x-2}.$$

Thus y and all its conjugates are bounded and so can take only finitely many values. The result is now complete.

Remark. For groups of signature $(1; 3; 0)$, $w_3 = 1$ and the above estimates show that $[\mathbb{Q}(x): \mathbb{Q}] \leqslant 10$, in that case. The totally real number x^2 is then a P.V. number (see (9)) and we note that the above method of proof shows that for a given bound L, there are only finitely many totally real P.V. numbers less than L.

6. *Arithmetic groups in $M'(n)$ and $M''(n)$ for large n.* In this section it will be shown that there are no arithmetic groups of signature $(1; n; 0)$ or $(0; 2, 2, 2, n; 0)$ for n large enough.

Since $k_2 = \mathbb{Q}(x^2, y^2, z^2, xyz)$ if Γ is arithmetic and represented by (x, y, z) then, from the fundamental equation, we have that $\mathbb{Q}(\cos \pi/n) \subset k_2$ or $\mathbb{Q}(\cos 2\pi/n) \subset k_2$.

Let $C_m = \mathbb{Q}(e^{2\pi i/m})$ and $R_m = C_m \cap \mathbb{R}$. We require an estimate on the discriminant of the field R_m. If $E \subset F$, we denote the relative discriminant by $\Delta_{F|E}$. Then

$$\Delta_{C_m|\mathbb{Q}} = \Delta^2_{R_m|\mathbb{Q}} N_{R_m|\mathbb{Q}}(\Delta_{C_m|R_m}). \tag{3}$$

Since $1, e^{2\pi i/m}$ is an integral basis for C_m over R_m and $e^{2\pi i/m}$ has minimum polynomial $f(X) = X^2 - 2\cos 2\pi/m X + 1$, then

$$N_{R_m|\mathbb{Q}}(\Delta_{C_m|R_m}) = N_{C_m|\mathbb{Q}}(f'(e^{2\pi i/m}))\,\mathbf{Z}.$$

Now $f'(e^{2\pi i/m}) = 2i \sin 2\pi/m$. So from (3) and information on $\Delta_{C_m|\mathbb{Q}}$ (see, for example § 7·5 (16)) we deduce that

$$\Delta_{R_m|\mathbb{Q}} = \begin{cases} \dfrac{m^{\phi(m)/2}}{\prod\limits_{p|m} p^{\phi(m)/2(p-1)}} & \text{if } m \neq p^\alpha, 2p^\alpha \\ (p^{\alpha p^{\alpha-1}(p-1)-p^{\alpha-1}-1})^{\frac{1}{2}} & \text{if } m = p^\alpha, 2p^\alpha, \quad p \neq 2 \\ 2^{(\alpha-1)\,2^{\alpha-2}-1} & \text{if } m = 2^\alpha, \quad \alpha \geqslant 2 \\ 1 & \text{if } m = 2. \end{cases}$$

From this one easily deduces that

$$\Delta_{R_m|\mathbb{Q}} \geqslant m^{\phi(m)/4} \text{ except for } m = 2, 3, 4, 6.$$

THEOREM 4. *There are no arithmetic Fuchsian groups of signature* $(1; n; 0)$ $n \geqslant 2$ *or* $(0; 2, 2, 2, n; 0)$, $n \geqslant 3$, *n odd for* $n \geqslant (4\pi^2)^{\frac{4}{3}}$.

Proof. Suppose that Γ is a Fuchsian group of one of these signatures which is arithmetic. Then $\Gamma^{(2)}$ has signature $(1; n, n, n, n; 0)$ and is a subgroup of finite index in some $\mathbf{P}\Gamma(A, 0)$ where the quaternion algebra $A = A(\Gamma^{(2)})$ is uniquely determined by Γ ((12), (14)). Thus the hyperbolic measure of $\mathbf{P}\Gamma(A, 0)$ satisfies the inequality

$$\mu(\mathbf{P}\Gamma(A, 0)) \leqslant \mu(\Gamma^{(2)}) = 2\pi\left(4\left(\frac{n-1}{n}\right)\right) < 8\pi.$$

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Without loss we can assume that 0 is a maximal order in A and then the volume of $\mathbf{P}\Gamma(A,0)$ depends only on the quaternion algebra A which, we recall, is defined over the field k_2. Precisely,

$$\mu(\mathbf{P}\Gamma(A,0)) = \frac{2\xi_{k_2}(2)\,\Delta_{k_2|\mathbb{Q}}^{\frac{3}{2}}}{\pi[4\pi^2]^{[k_2:\mathbb{Q}]-1}} \prod_{P|D}(NP-1)$$

(see (15), p. 109 and p. 115). Here ξ_{k_2} is the zeta-function of the field k_2 and D is the discriminant of the quaternion algebra, i.e. the product of the finite places at which it is ramified.

Now $\xi_{k_2}(2) > 1$ and $\Pi_{P|D}(NP-1) \geqslant 1$. Thus

$$\mu(\mathbf{P}\Gamma(A,0)) > 8\pi\left(\frac{\Delta_{k_2|\mathbb{Q}}^{\frac{3}{2}}}{[4\pi^2]^{[k_2:\mathbb{Q}]}}\right).$$

Now $\Delta_{k_2|\mathbb{Q}} \geqslant \Delta_{R_{2n}|\mathbb{Q}}^{[k_2:R_{2n}]}$ for signature $(1; n; 0)$ or the same with R_n for signature $(0; 2, 2, 2, n; 0)$. Thus by the given estimate on the discriminant $\Delta_{R_m|\mathbb{Q}}$ for $m \geqslant 6$

$$\mu(\mathbf{P}\Gamma(A,0)) > 8\pi\left(\frac{n^{\frac{3}{4}}}{4\pi^2}\right)^{[k_2:\mathbb{Q}]}$$

Hence for $n^{\frac{3}{4}} > 4\pi^2$ we have a contradiction.

Remark. Let Γ be a k-generator arithmetic Fuchsian group for fixed k. The above argument shows that there exists an integer $N(k)$ such that, for all $n \geqslant N(k)$, Γ contains no elements of order n.

7. *Remarks on signature* $(g; m_1, m_2, \ldots, m_r; s)$ *with* $s > 0$. We noted in § 3 that if Γ is arithmetic with signature of this form, i.e. having parabolic elements, then $k_2 = \mathbb{Q}$. The quaternion algebra is then $\mathbf{M}(2, \mathbb{Q})$ and Γ is commensurable with $\mathbf{PSL}(2, \mathbf{Z})$. This is well-known and can be interpreted alternatively in terms of isotropic ternary quadratic forms (see, for example, (3)). From this we deduce that there are only finitely many conjugacy classes of arithmetic groups for any fixed signature of the above type. For, Γ being commensurable with $\mathbf{PSL}(2, \mathbf{Z})$, means that it is conjugate to a subgroup of finite index in some $\Gamma_1(n) =$ Normalizer of $\Gamma_0(n)$ in $\mathbf{PSL}(2, \mathbb{R})$ where

$$\Gamma_0(n) = \mathbf{P}\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2, \mathbf{Z}) | b \equiv 0 \pmod{n}\right\},$$

for n square-free (1). But $\mu(\Gamma_1(n) = \pi/3\ \Pi_{p|n}(p + 1/2)$. Thus if Γ has fixed signature, then for large enough n, $\mu(\Gamma_1(n)) > \mu(\Gamma)$ and so Γ is conjugate to a subgroup of finite index in $\Gamma_1(n)$ for only finitely many values of n. But each $\Gamma_1(n)$ has only finitely many subgroups of fixed index, and hence of a given signature.

Note that this remark concludes the proof that there are only finitely many conjugacy classes of arithmetic two-generator Fuchsian groups. The results of (13) and Theorems 3 and 4 deal with the cocompact case and this section with the non-cocompact signatures.

In the case most closely allied to those treated earlier, i.e. groups Γ with signature $(1; —; 1)$, one readily calculates that $\Gamma \subset \Gamma_1(n)$ implies that $n = 1, 2, 3, 5$ or 6. From the structure of the groups $\Gamma_1(n)$ (see e.g. (3)) one finds that there are exactly four conjugacy classes of arithmetic groups of signature $(1; —; 1)$.

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Table 1

x	y	z	Signature	k_2	Discriminant
		Case A ($k_2 = \mathbb{Q}$)			
$\sqrt{5}$	$2\sqrt{3}$	$\sqrt{15}$	(1; 2; 0)	$\mathbb{Q}$	(2) (3)
$\sqrt{6}$	$2\sqrt{2}$	$2\sqrt{3}$	(1; 2; 0)	$\mathbb{Q}$	(2) (3)
$\sqrt{7}$	$\sqrt{7}$	3	(1; 2; 0)	$\mathbb{Q}$	(2) (7)
$\sqrt{5}$	4	$2\sqrt{5}$	(1; 3; 0)	$\mathbb{Q}$	(3) (5)
$\sqrt{6}$	$\sqrt{10}$	$\sqrt{15}$	(1; 3; 0)	$\mathbb{Q}$	(2) (5)
$\sqrt{7}$	$2\sqrt{2}$	$\sqrt{14}$	(1; 3; 0)	$\mathbb{Q}$	(2) (3)
$2\sqrt{2}$	$2\sqrt{2}$	3	(1; 3; 0)	$\mathbb{Q}$	(2) (3)
$\sqrt{5}$	$2\sqrt{2}$	$\sqrt{10}$	(0; 2, 2, 2, 3; 0)	$\mathbb{Q}$	(2) (5)
$\sqrt{6}$	$\sqrt{6}$	3	(0; 2, 2, 2, 3; 0)	$\mathbb{Q}$	(2) (3)
		Case B ($k_1 = k_2$ and $[k_2 : \mathbb{Q}] = 2$)			
$1+\sqrt{3}$	$1+\sqrt{3}$	$1+\sqrt{3}$	(1; 2; 0)	$\mathbb{Q}(\sqrt{3})$	P_2
$\frac{1}{2}(3+\sqrt{5})$	$\frac{1}{2}(3+\sqrt{5})$	$\frac{1}{2}(5+\sqrt{5})$	(1; 2; 0)	$\mathbb{Q}(\sqrt{5})$	P_2
$\frac{1}{2}(1+\sqrt{13})$	$\frac{1}{2}(3+\sqrt{13})$	$\frac{1}{2}(3+\sqrt{13})$	(1; 2; 0)	$\mathbb{Q}(\sqrt{13})$	P_2
$1+\sqrt{2}$	$2+\sqrt{2}$	$2+\sqrt{2}$	(1; 3; 0)	$\mathbb{Q}(\sqrt{2})$	P_3
$1+\sqrt{3}$	$1+\sqrt{3}$	$2+\sqrt{3}$	(1; 3; 0)	$\mathbb{Q}(\sqrt{3})$	P_3
$1+\sqrt{2}$	$2+\sqrt{2}$	$1+2\sqrt{2}$	(1; 4; 0)	$\mathbb{Q}(\sqrt{2})$	P_2
$\frac{1}{2}(3+\sqrt{5})$	$1+\sqrt{5}$	$1+\sqrt{5}$	(1; 5; 0)	$\mathbb{Q}(\sqrt{5})$	P_5

In case B, P_p denotes the unique (in these cases) prime ideal dividing p.

Alternatively, the space of conjugacy classes of groups of signature (1; —; 1) has the description

$$M'(\infty) = \{(x, y, z) \in \mathbb{R} \mid x^2 + y^2 + z^2 = xyz, \quad 2 < x \leqslant y \leqslant z \leqslant \tfrac{1}{2}xy\}$$

with x, y, z representing the traces of the generators and their product as before (5). Since $k_2 = \mathbb{Q}$, x^2, y^2, z^2 and xyz are all integers, the results in (7) on solutions of the above equation show again that there are exactly four points in $M'(\infty)$ corresponding to arithmetic Fuchsian groups.

8. *Examples.* We have determined all possible conjugacy classes of arithmetic Fuchsian groups of signatures of the form $(1; n; 0)$ $n \geqslant 2$, or $(0; 2, 2, 2, n; 0)$ $n \geqslant 3$, n odd, where either $k_2 = \mathbb{Q}$ or $k_1 = k_2$ and $[k_2 : \mathbb{Q}] = 2$. The solutions x, y, z in these cases are listed in Table 1. The discriminant of the corresponding quaternion algebra is also given. This is determined from the isomorphisms

$$A(\Gamma^{(2)}) \cong \left(\frac{x^2(x^2-4),\ -x^2y^2(2+2\cos\pi/n)}{k_2}\right) \quad \text{or} \quad \left(\frac{x^2(x^2-4),\ -x^2y^2(2+2\cos 2\pi/n)}{k_2}\right),$$

in the cases $(1; n; 0)$ or $(0; 2, 2, 2, n; 0)$ respectively. The arithmetic Fuchsian groups are commensurable if and only if the corresponding quaternion algebras are isomorphic (see (14)). By comparison with Takeuchi's table in (14), it can be noted that many of these examples are commensurable with triangle groups but also that several are not.

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REFERENCES

(1) HELLING, H. Bestimmung der Kommensurabilitätklasse der Hilbertschen Modulgruppe. *Math. Z.* **92** (1966), 269–280.

(2) KERN-ISBERNER, G. and ROSENBERGER, G. Einige Bemerkungen über Untergruppen der $\mathbf{PSL}(2, \mathbb{C})$. *Resultate der Math.*

(3) MACLACHLAN, C. Groups of units of zero ternary quadratic forms. *Proc. Roy. Soc. Edinburgh. Sect. A* **88** (1981), 141–157.

(4) PURZITSKY, N. and ROSENBERGER, G. Two-generator Fuchsian groups of genus one. *Math. Z.* **128** (1972), 245–251. Correction *Math. Z.* **132** (1973), 261–2.

(5) ROSENBERGER, G. Fuchssche Gruppen, die freies Product zweir zyklischer Gruppen sind, und die Gleichung $x^2+y^2+z^2 = xyz$. *Math. Ann.* **199** (1972), 213–227.

(6) ROSENBERGER, G. On discrete free subgroups of linear groups. *J. London Math. Soc.* **17** (1978), 79–85.

(7) ROSENBERGER, G. Über die Diophantische Gleichung $ax^2+by^2+cz^2 = dxyz$. *J. Reine Angew. Math.* **305** (1979), 122–125.

(8) ROSENBERGER, G. and KALIA, R. N. Automorphisms of the Fuchsian groups of type $(0; 2, 2, 2, q; 0)$. *Comm. Alg.* **6** (11), (1978) 1115–1129.

(9) SALEM, R. *Algebraic numbers and Fourier analysis* (Heath, Boston, 1963.)

(10) SCHUR, I. Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. *Math. Z.* **1** (1918), 377–402.

(11) TAKEUCHI, K. On some discrete subgroups of $\mathbf{SL}_2(\mathbb{R})$. *J. Fac. Sci. Univ. Tokyo Sect. I* **16** (1967), 97–100.

(12) TAKEUCHI, K. A characterization of arithmetic Fuchsian groups. *J. Math. Soc. Japan* **27** (1975), 600–612.

(13) TAKEUCHI, K. Arithmetic triangle groups. *J. Math. Soc. Japan* **29** (1977), 91–106.

(14) TAKEUCHI, K. Commensurability classes of arithmetic triangle groups. *J. Fac. Sci. Univ. Tokyo Sect. I A* **24** (1977), 201–222.

(15) VIGNERAS, M-F. Arithmétique des algèbres de quaternions. *Lecture Notes in Math.* no. 800 (Springer-Verlag, 1980).

(16) WEISS, E. *Algebraic number theory* (McGraw-Hill, New York, 1963.)

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