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A Spectral Erdős–Stone–Bollobás Theorem

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Let $r \geqslant 3$ and $(c/r^r)^r \log n \geqslant 1$. If G is a graph of order n and its largest eigenvalue $\mu(G)$ satisfies

$$\mu(G) \geqslant (1 - 1/(r-1) + c)n,$$

then G contains a complete r-partite subgraph with $r-1$ parts of size $\lfloor (c/r^r)^r \log n \rfloor$ and one part of size greater than $n^{1-c^{r-1}}$.

This result implies the Erdős–Stone–Bollobás theorem, the essential quantitative form of the Erdős–Stone theorem. Another easy consequence is that if $F_1, F_2, \ldots$ are r-chromatic graphs satisfying $v(F_n) = o(\log n)$, then

$$\lim_{n\to\infty} \frac{1}{n} \max\{\mu(G) : v(G) = n \text{ and } F_n \nsubseteq G\} = 1 - \frac{1}{r-1}.$$

This note is part of an ongoing project aiming to build extremal graph theory on a spectral basis. Here we give a spectral version of the Erdős–Stone theorem.

Given $r \geqslant 3$ and $c > 0$, let $g(n, r, c)$ be the maximum integer such that every graph with n vertices and $\lceil (1 - 1/(r-1) + c)n^2/2 \rceil$ edges contains a complete r-partite graph with each part of size $g(n, r, c)$. The fundamental Erdős–Stone theorem [5] states that $g(n, r, c)$ tends to infinity with n. In [3] Bollobás and Erdős found that, in fact, $g(n, r, c) = \Theta(\log n)$. Below we give a spectral version of this result.

Our notation follows [2]; thus, $K_r(s_1, \ldots, s_r)$ denotes the complete r-partite graph with parts of sizes $s_1, \ldots, s_r$, and $\mu(G)$ stands for the spectral radius of the adjacency matrix of a graph G; log denotes the logarithm base e.

The main result of this note is the following theorem.

Theorem 1. *Let $r \geqslant 3$, $(c/r^r)^r \log n \geqslant 1$, and let G be a graph with n vertices. If*

$$\mu(G) \geqslant (1 - 1/(r-1) + c)n, \tag{1}$$

then G contains a $K_r(s, \ldots, s, t)$ with $s \geqslant \lfloor (c/r^r)^r \log n \rfloor$ and $t > n^{1-c^{r-1}}$.

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We shall prove the following easy consequence of this result.

Corollary 2. *Let $r \geqslant 3$ and let $F_1, F_2, \ldots$ be r-chromatic graphs satisfying $v(F_n) = o(\log n)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \max\{\mu(G) : v(G) = n \quad \text{and} \quad F_n \nsubseteq G\} = 1 - \frac{1}{r-1} \tag{2}$$

and

$$\lim_{n\to\infty} \binom{n}{2}^{-1} \max\{e(G) : v(G) = n \quad \text{and} \quad F_n \nsubseteq G\} = 1 - \frac{1}{r-1}. \tag{3}$$

Before turning to the proofs of Theorem 1 and Corollary 2, some remarks seem in order.

1. The relation between c and n in Theorem 1 needs explanation. First, for fixed c, it shows how large n must be for the vertex classes of the required $K_r(s, \ldots, s, t)$ to be non-empty. However, c may also depend on n, *e.g.*, letting $c = 1/\log\log n$, the conclusion is meaningful for sufficiently large n.

2. Note that, in Theorem 1, if condition (1) holds for some c, the conclusion holds for every positive $c' < c$ provided n is sufficiently large, *i.e.*, as n grows, we can find a larger and more lopsided $K_r(s, \ldots, s, t)$.

3. Since $\mu(G) \geqslant 2e(G)/v(G)$, Theorem 1 implies the following form of the Erdős–Stone–Bollobás theorem:

Let $r \geqslant 3$, $(c/r^r)^r \log n \geqslant 1$, and let G be a graph with n vertices. If

$$e(G) \geqslant (1 - 1/(r-1) + c)n^2/2,$$

then G contains a $K_r(s, \ldots, s, t)$ with $s \geqslant \lfloor (c/r^r)^r \log n \rfloor$ and $t > n^{1-c^{r-1}}$.

Other lower bounds on $\mu(G)$, such as, *e.g.*,

$$\mu^2(G) \geqslant \frac{1}{n} \sum_{u \in V(G)} d^2(u),$$

imply other new versions of this theorem.

4. Suppose that c is a sufficiently small positive constant. Choosing randomly a graph G of order n with $\lceil (1 - 1/(r-1) + c)n^2/2 \rceil$ edges, we have $\mu(G) \geqslant (1 - 1/(r-1) + c)n$, but G contains no $K_2(\lfloor C \log \rfloor, \lfloor C \log n \rfloor)$ for some $C > 0$, independent of n. Hence, for constant c, Theorem 1 is best-possible up to a constant factor.

5. In the context of the project mentioned in the introduction, Corollary 2 solves asymptotically the following general extremal problem:

Given a family $\mathcal{F}$ of forbidden subgraphs with chromatic number at least 3, find the maximum spectral radius of a graph of order n containing no member of $\mathcal{F}$.

We now turn to the proofs of Theorem 1 and Corollary 2.

Write $k_r(G)$ for the number of r-cliques of a graph G. Our proof of Theorem 1 is based on the following results.

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Theorem 3 ([4], Theorem 2). *If $r \geqslant 2$ and G is a graph of order n, then*

$$k_{r+1}(G) \geqslant \left(\frac{\mu(G)}{n} - 1 + \frac{1}{r}\right)\frac{r(r-1)}{r+1}\left(\frac{n}{r}\right)^{r+1}. \qquad \square$$

Theorem 4 ([7], Theorem 1). *Let $r \geqslant 2$, $\alpha^r \log n \geqslant 1$, and let G be a graph of order n. If $k_r(G) \geqslant \alpha n^r$, then G contains a $K_r(s, \ldots, s, t)$ with $s = \lfloor \alpha^r \log n \rfloor$ and $t > n^{1-\alpha^{r-1}}$.* $\square$

Proof of Theorem 1. In view of $\mu(G) \geqslant (1 - 1/(r-1) + c)n$, Theorem 3 implies that

$$k_r(G) > c\frac{(r-1)(r-2)}{r}\left(\frac{n}{r-1}\right)^r > \frac{c}{r^r}n^r.$$

Now, letting $\alpha = c/r^r$, Theorem 4 implies that G contains a $K_r(s, \ldots, s, t)$ with

$$s \geqslant \lfloor \alpha^r \log n \rfloor = \lfloor (c/r^r)^r \log n \rfloor, \quad \text{and} \quad t > n^{1-\alpha^{r-1}} > n^{1-c^{r-1}},$$

completing the proof. $\square$

Proof of Corollary 2. Set $c_n = r^r(v(F_n)/\log n)^{1/r}$ and let G be a graph of order n not containing F_n. Then clearly G contains no $K_r(s, s, \ldots, s)$ for $s = v(F_n)$. Now

$$(c_n/r^r)^r \log n = v(F_n) \geqslant 1,$$

while, for n large enough, we have $c_n < 1/2$ and

$$n^{1-c_n^{r-1}} > (c_n/r^r)^r \log n = v(F_n).$$

Hence Theorem 1 implies that

$$\frac{\mu(G)}{n} \leqslant 1 - \frac{1}{r-1} + c_n.$$

Thus, in view of

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} r^r\left(\frac{v(F_n)}{\log n}\right)^{1/r} = r^r\left(\lim_{n\to\infty}\frac{v(F_n)}{\log n}\right)^{1/r} = 0,$$

we obtain

$$\limsup_{n\to\infty}\frac{1}{n}\max\{\mu(G) : v(G) = n \text{ and } F_n \not\subseteq G\} \leqslant 1 - \frac{1}{r-1}.$$

On the other hand, writing $T_s(n)$ for the s-partite Turán graph of order n, we see that

$$\frac{\mu(T_{r-1}(n))}{n} \geqslant \frac{\delta(T_{r-1}(n))}{n} \geqslant 1 - \frac{1}{r-1} - \frac{1}{n}.$$

Since $T_{r-1}(n)$ is $(r-1)$-partite, it contains no copy of F_n. Therefore,

$$\liminf_{n\to\infty}\frac{1}{n}\max\{\mu(G) : v(G) = n \text{ and } F_n \not\subseteq G\} \geqslant 1 - \frac{1}{r-1},$$

completing the proof of (2).

Now (3) follows since

$$\binom{n}{2}^{-1} e(G) = \frac{2e(G)}{n(n-1)} \leqslant \frac{\mu(G)}{n-1} \leqslant \frac{\mu(G)}{n} + \frac{1}{n}$$

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and

$$\binom{n}{2}^{-1} e(T_{r-1}(n)) \geqslant \binom{n}{2}^{-1} \frac{n\delta(T_{r-1}(n))}{2} = \frac{\delta(T_{r-1}(n))}{n-1} \geqslant 1 - \frac{1}{r-1}. \qquad \square$$

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After this note had been made public, it became known that Babai and Guiduli [6] had proved a weaker form of Corollary 2 using the Szemerédi Regularity Lemma; for a recent account of this matter see [1].

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