NOTE ON THE EVALUATION OF COMPLEX DETERMINANTS

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A square matrix A, of order n, having complex coefficients can be inverted without the aid of any operations involving complex numbers. This can be done if the coefficients $a_{rs}+i\alpha_{rs}$ are replaced by their matrix equivalents* $\begin{pmatrix} a_{rs} & \alpha_{rs} \\ -\alpha_{rs} & a_{rs} \end{pmatrix}$ and the resulting $2n \times 2n$ real matrix A_1 is inverted. The inverse will be a $2n \times 2n$ matrix of similar form in which the subsidiary 2×2 matrices can be replaced by the equivalent complex numbers, thus yielding the inverse of A. It is the purpose of this note to show that a similar technique can be employed to evaluate $\det A$.

The direct evaluation of $\det A$ is most readily performed† by expressing A as the product BC of a unit lower triangular matrix B and an upper triangular matrix C. Thus

$$\begin{pmatrix} a_{11}+i\alpha_{11} & a_{12}+i\alpha_{12} & a_{13}+i\alpha_{13} & \dots \\ a_{21}+i\alpha_{21} & a_{22}+i\alpha_{22} & a_{23}+i\alpha_{23} & \dots \\ a_{31}+i\alpha_{31} & a_{32}+i\alpha_{32} & a_{33}+i\alpha_{33} & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ b_{21}+i\beta_{21} & 1 & 0 & 0 & \dots \\ b_{31}+i\beta_{31} & b_{32}+i\beta_{32} & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} c_{11}+i\gamma_{11} & c_{12}+i\gamma_{12} & c_{13}+i\gamma_{13} & \dots \\ 0 & c_{22}+i\gamma_{22} & c_{23}+i\gamma_{23} & \dots \\ 0 & 0 & c_{33}+i\gamma_{33} & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{1}$$

It follows immediately that

$$\det A = \prod_{r=1}^{n} (c_{rr}+i\gamma_{rr}). \tag{2}$$

If the matrices A, B, C are rewritten in their equivalent real forms equation (1) is replaced by $A_1 = B_1 C_1$ or

$$\begin{pmatrix} a_{11} & \alpha_{11} & a_{12} & \alpha_{12} & a_{13} & \alpha_{13} & \dots \\ -\alpha_{11} & a_{11} & -\alpha_{12} & a_{12} & -\alpha_{13} & a_{13} & \dots \\ a_{21} & \alpha_{21} & a_{22} & \alpha_{22} & a_{23} & \alpha_{23} & \dots \\ -\alpha_{21} & a_{21} & -\alpha_{22} & a_{22} & -\alpha_{23} & a_{23} & \dots \\ a_{31} & \alpha_{31} & a_{32} & \alpha_{32} & a_{33} & \alpha_{33} & \dots \\ -\alpha_{31} & a_{31} & -\alpha_{32} & a_{32} & -\alpha_{33} & a_{33} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

* See, for example, Frazer, Duncan and Collar, *Elementary matrices* (Cambridge, 1938), p. 35.

† For a detailed computational layout of the process of triangular resolution see L. Fox, 'Practical methods for the solution of linear equations and the inversion of matrices', *J.R. Statist. Soc.* (in the Press).

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$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ b_{21} & \beta_{21} & 1 & 0 & 0 & 0 & \dots \\ \beta_{21} & -b_{21} & 0 & 1 & 0 & 0 & \dots \\ b_{31} & \beta_{31} & b_{32} & \beta_{32} & 1 & 0 & \dots \\ -\beta_{31} & b_{31} & -\beta_{32} & b_{32} & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} c_{11} & \gamma_{11} & c_{12} & \gamma_{12} & c_{13} & \gamma_{13} & \dots \\ -\gamma_{11} & c_{11} & -\gamma_{12} & c_{12} & -\gamma_{13} & c_{13} & \dots \\ 0 & 0 & c_{22} & \gamma_{22} & c_{23} & \gamma_{23} & \dots \\ 0 & 0 & -\gamma_{22} & c_{22} & -\gamma_{23} & c_{23} & \dots \\ 0 & 0 & 0 & 0 & c_{33} & \gamma_{33} & \dots \\ 0 & 0 & 0 & 0 & -\gamma_{33} & c_{33} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{3}$$

The right-hand side of equation (3) is not quite in the form obtained by triangular resolution of the matrix on the left, since there are some non-zero terms below the diagonal of C_1. These are, however, readily removed, and the product $B_1 C_1$ unaltered, if to the $2r$th row of C_1 is added γ_{rr}/c_{rr} times the $(2r-1)$th row, while from the $(2r-1)$th column of B_1 is subtracted γ_{rr}/c_{rr} times the $2r$th column. The triangular resolution of A_1 therefore expresses it in the form $B_2 C_2$, where

$$B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\gamma_{11}/c_{11} & 1 & 0 & 0 & 0 & 0 & \dots \\ b'_{21} & \beta_{21} & 1 & 0 & 0 & 0 & \dots \\ -\beta'_{21} & b_{21} & -\gamma_{22}/c_{22} & 1 & 0 & 0 & \dots \\ b'_{31} & b_{31} & b'_{32} & \beta_{32} & 1 & 0 & \dots \\ -\beta'_{31} & b_{31} & -\beta'_{32} & b_{32} & -\gamma_{33}/c_{33} & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{4}$$

$$C_2 = \begin{pmatrix} c_{11} & \gamma_{11} & c_{12} & \gamma_{12} & c_{13} & \gamma_{13} & \dots \\ 0 & c'_{11} & -\gamma'_{12} & c'_{12} & -\gamma'_{13} & c'_{13} & \dots \\ 0 & 0 & c_{22} & \gamma_{22} & c_{23} & \gamma_{23} & \dots \\ 0 & 0 & 0 & c'_{22} & -\gamma'_{23} & c'_{23} & \dots \\ 0 & 0 & 0 & 0 & c_{33} & \gamma_{33} & \dots \\ 0 & 0 & 0 & 0 & 0 & c'_{33} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{5}$$

The dashed elements in B_2, C_2 are of no further interest in this context, except for the elements c'_{rr} which are given by the equation

$$c'_{rr} = c_{rr} + \frac{\gamma_{rr}^2}{c_{rr}}. \tag{6}$$

It is apparent that when this triangular resolution, which does not involve any complex numbers, is completed the individual factors of the expression (2) for $\det A$ can be obtained by inspection of the diagonal and adjacent terms of the odd rows of C_2. Multiplication of these complex factors is inconvenient, however, and can be avoided in the following way: $|c_{rr} + i\gamma_{rr}|^2 = c_{rr}^2 + \gamma_{rr}^2 = c_{rr} c'_{rr}$.

Thus $$|\det A|^2 = \prod_{r=1}^{n} c_{rr} c'_{rr}$$

$$= \text{product of diagonal terms of } C_2. \tag{7}$$

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$$\arg(c_{rr}+i\gamma_{rr}) = \tan^{-1}(\gamma_{rr}/c_{rr})$$
$$= -\tan^{-1}[\text{element in } 2r\text{th row and } (2r-1)\text{th column of } B_2],$$

and so
$$\phi = \arg\det A$$
$$= -\sum_{r=1}^{n} \tan^{-1}[(2r, 2r-1)\text{th element of } B_2]. \quad (8)$$

The ambiguity in the inverse tangent is eliminated by the knowledge that its cosine has the sign of c_{rr}. The real and imaginary parts of $\det A$ are then $|\det A|\cos\phi$ and $|\det A|\sin\phi$ respectively.

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