For what values of x such an identity is valid depends on the definition one chooses to use for the inverse cotangent function. If a domain restriction of $[-\frac{\pi}{2},\ 0) \cup (0,\ \frac{\pi}{2}]$ for the cotangent function is used to define its inverse, then (2) will be valid for all $x < -1$ while if instead a domain restriction of $0 < x < \pi$ is used, (2) will be valid for all $x > -1$.

Further examples

Some further examples of integrals that can be found or evaluated using the self-similar substitution considered in this note are collected below.

(a) $\displaystyle\int \frac{dx}{(1 - x^2)^{3/2}}$ (b) $\displaystyle\int \frac{\tan^{-1} x}{(1 + x)^2}\,dx$

(c) $\displaystyle\int \ln\left(\frac{1 + x}{1 - x}\right)\frac{dx}{1 - x^2}$ (d) $\displaystyle\int \frac{dx}{\sqrt[3]{(1 + x)^2(1 - x)^4}}$

(e) $\displaystyle\int_0^1 \frac{(1 + 2x - x^2)\ln(1 + x)}{(1 + x^2)^2}\,dx$ (f) $\displaystyle\int_{-1/2}^2 \frac{\tan^{-1} x}{x^2 + x + 2}\,dx$

(g) $\displaystyle\int_{-1/2}^3 \frac{\ln(x + 2)}{(x + 1)\left[\ln(x + 3) + \ln(x + 2) - \ln(x + 1)\right]}\,dx$

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101.07 Cauchy's mean value theorem meets the logarithmic mean

The arithmetic and geometric means

Almost everybody knows that the average, or *arithmetic mean*, of two numbers a and b is

$$A(a,\ b) = \frac{a + b}{2}.$$

Fewer people know that for a and b non-negative, their *geometric mean* is

$$G(a,\ b) = \sqrt{ab}.$$

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Here are two uses for the geometric mean:

(i) (E.g., [1]). To use a balance to determine the mass of an object: Place the object on the left side of the balance and a known mass on the right side, to obtain a measurement a. Then place the object on the right side of the balance and a known mass on the left side, to obtain a measurement b. The true mass of the object is the geometric mean $\sqrt{ab}$.

(ii) ([2]) The bank in your town sells Canadian dollars at the rate \$1 = £S and buys them at the rate \$1 = £B. You and a friend want to exchange dollars and pounds between yourselves, at a rate that is fair to both. The fair exchange rate is given by the geometric mean: \$1 = £$\sqrt{SB}$.

The basic relationship between the Arithmetic and Geometric Means is as follows. We shall call this the *AGM Inequality* for short.

AGM inequality

Let a and b be non-negative numbers. Then

$$G(a, b) \leqslant A(a, b),$$

with equality if, and only if, $a = b$.

Proof: There are a zillion ways to do this. Possibly the simplest way is to observe first that $\left(\sqrt{a} - \sqrt{b}\right)^2 \geqslant 0$ with equality if, and only if, $a = b$. Now expand the left-hand side and do a little algebra.

A rather satisfying *Proof Without Words* [3] for the AGM inequality, which also suggests why $\sqrt{ab}$ is called the geometric mean, is shown in Figure 1.

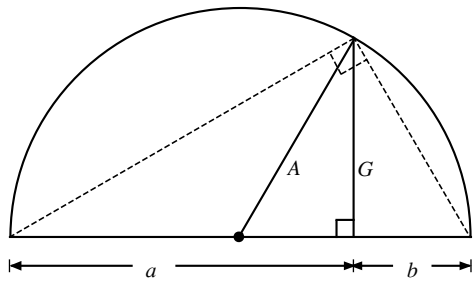


FIGURE 1: $\sqrt{ab} = G(a, b) \leqslant A(a, b) = \dfrac{a + b}{2}$

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The logarithmic mean

Only the fortunate few who have taken calculus can know that the *logarithmic mean* of two positive numbers a and b is

$$L(a, b) = \begin{cases} \dfrac{b - a}{\ln b - \ln a} & \text{if } a \neq b, \\ a & \text{if } a = b. \end{cases}$$

On its own the logarithmic mean has considerable intrinsic appeal, but it pays its rent by being useful in problems involving heat transfer and fluid mechanics, among others. It enjoys a good deal of attention in the mathematics literature.

Any function M of a and b is called a *mean* simply if it satisfies the properties

(i) $M(a, b) = M(b, a)$, and

(ii) $\min\{a, b\} \leqslant M(a, b) \leqslant \max\{a, b\}$,

for all a and b under consideration. The reader should agree that $A(a, b)$ and $G(a, b)$ are indeed means. So is the logarithmic mean.

Theorem 1: For all positive numbers a and b, $L(a, b)$ is a mean.

Proof: It is easily verified that $L(a, b) = L(b, a)$. So we may suppose that $a < b$. Applying Cauchy's mean value theorem to $f(x) = x - a$ and $g(x) = \ln(x) - \ln(a)$ on $[a, b]$, there is $c \in (a, b)$ such that

$$\frac{b - a}{\ln(b) - \ln(a)} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} = \frac{1}{1/c} = c.$$

Therefore

$$a = \min\{a, b\} \leqslant L(a, b) \leqslant \max\{a, b\} = b,$$

and so $L(a, b)$ is indeed a mean. (This justifies the choice $L(a, a) = a$, making L continuous.)

Here the astute reader may accuse us of a dodge: we didn't really use *Cauchy's* mean value theorem here, just the mean value theorem upside down! Guilty as charged, but read on.

Cauchy's mean value theorem

The following result was first obtained, using fairly complicated methods, in [4, 5]. Other methods have since been discovered (e.g. [6, 7, 8]). Here is another — as suggested in [9], the idea of using Cauchy's mean value theorem proves to be fruitful.

Theorem 2: Let a and b be positive numbers. Then

$$G(a, b) \leqslant L(a, b) \leqslant A(a, b).$$

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Proof: We may assume that $a < b$. For the right-hand inequality, we set

$$f(x) = \frac{x-a}{x+a} \qquad \text{and} \qquad g(x) = \ln(x) - \ln(a),$$

for $x \in [a, b]$. Then by Cauchy's mean value theorem, there is $c \in (a, b)$ such that

$$\frac{\frac{b-a}{b+a}}{\ln(b) - \ln(a)} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

It is easily verified that

$$\frac{f'(c)}{g'(c)} = \frac{2ac}{(c+a)^2},$$

and this is less than or equal to $\frac{1}{2}$, by the AGM inequality. Therefore $L(a, b) \leqslant A(a, b)$.

For the left-hand inequality, we set

$$f(x) = \frac{x-a}{\sqrt{ax}} \qquad \text{and} \qquad g(x) = \ln(x) - \ln(a),$$

for $x \in [a, b]$. Then by Cauchy's mean value theorem, there is $c \in (a, b)$ such that

$$\frac{\frac{b-a}{\sqrt{ab}}}{\ln(b) - \ln(a)} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Here it is easily verified that

$$\frac{f'(c)}{g'(c)} = \frac{\frac{c+a}{2}}{\sqrt{ac}},$$

and this is greater than or equal to 1, by the AGM inequality. Therefore $L(a, b) \geqslant G(a, b)$.

Remark: Setting $a = x$ and $b = x + 1$ in Theorem 2, we obtain

$$\sqrt{x(x+1)} \leqslant \frac{1}{\ln(x+1) - \ln(x)} \leqslant \frac{2x+1}{2}.$$

Then after a little manipulation, these yield the rather good estimates

$$\left(1 + \frac{1}{x}\right)^{\sqrt{x(x+1)}} \leqslant e \leqslant \left(1 + \frac{1}{x}\right)^{x+\frac{1}{2}}.$$

We also point out the agreeable fact that since $\sqrt{x(x+1)}$ is the geometric mean of $x + 1$ and x, and $x + \frac{1}{2}$ is their arithmetic mean, these estimates read:

$$\left(\frac{x+1}{x}\right)^{G(x+1,x)} \leqslant e \leqslant \left(\frac{x+1}{x}\right)^{A(x+1,x)}.$$

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Further estimates

Here we improve the inequalities of Theorem 2, viz. $G(a, b) \leqslant L(a, b) \leqslant A(a, b)$. We shall need the following extension of the AGM inequality ($t = \frac{1}{2}$ below is the AGM inequality). For the sake of completeness, and with pleasure, we supply the wonderful proof due to F. Riesz [10].

Weighted AGM inequality: Let a and b be non-negative numbers and let $t \in [0, 1]$. Then $a^t b^{(1-t)} \leqslant ta + (1-t)b$, with equality if, and only if, $a = b$.

Proof: The tangent line to $h(x) = \ln(x)$ at $x = 1$ is $y = x - 1$. Now on $(0, +\infty)$, $\ln(x)$ is concave downward ($h'' < 0$), so

$$\ln(x) \leqslant x - 1 \text{ for } x > 0.$$

Therefore, writing $M = ta + (1-t)b$, we have

$$t \ln\left(\frac{a}{M}\right) \leqslant t\left(\frac{a}{M}\right) - t \qquad \text{and} \qquad (1-t)\ln\left(\frac{b}{M}\right) \leqslant (1-t)\left(\frac{b}{M}\right) - (1-t).$$

Adding these two inequalities together gives

$$\ln\left(\left(\frac{a}{M}\right)^t \left(\frac{b}{M}\right)^{1-t}\right) \leqslant t\left(\frac{a}{M}\right) - t + (1-t)\left(\frac{b}{M}\right) - (1-t) = 0.$$

This yields the desired result, after applying the exponential function to both sides.

The following estimates, which refine Theorem 2, were obtained by very complicated methods in [11] (right-hand side) and [12] (left-hand side). But Cauchy's mean value theorem can be used here as well.

Theorem 3: Let a and b be positive numbers. Then

$$A(a, b)^{1/3} G(a, b)^{2/3} \leqslant L(a, b) \leqslant \frac{1}{3}A(a, b) + \frac{2}{3}G(a, b).$$

Proof: We may assume that $a < b$. For the right-hand inequality we set, for $x \in [a, b]$,

$$f(x) = \frac{x - a}{\frac{1}{3}\frac{a+x}{2} + \frac{2}{3}\sqrt{ax}} \qquad \text{and} \qquad g(x) = \ln(x) - \ln(a).$$

Then by Cauchy's mean value theorem, there is $c \in (a, b)$ such that

$$\frac{\frac{b-a}{\frac{1}{3}\frac{a+b}{2} + \frac{2}{3}\sqrt{ab}}}{\ln(b) - \ln(a)} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

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From here a mess ensues, but simplification reveals that

$$\frac{f'(c)}{g'(c)} = \frac{3G(a,c)[2A(a,c) + G(a,c)]}{[A(a,c) + 2G(a,c)]^2}.$$

Now the right-hand side above being less than or equal to 1 is equivalent to

$$2A(a,c)G(a,c) \leqslant A(a,c)^2 + G(a,c)^2,$$

which holds by the AGM inequality. Therefore so we have the right-hand inequality.

For the left-hand inequality we set, for $x \in [a, b]$,

$$f(x) = \frac{x - a}{\left(\frac{x+a}{2}\right)^{1/3}\left(\sqrt{xa}\right)^{2/3}} \qquad \text{and} \qquad g(x) = \ln(x) - \ln(a).$$

Again by Cauchy's mean value theorem, there is $c \in (a, b)$ such that

$$\frac{\frac{b-a}{\left(\frac{a+b}{2}\right)^{1/3}\left(\sqrt{ab}\right)^{2/3}}}{\ln(b) - \ln(a)} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Here an even bigger mess ensues, but tenacious simplification reveals that

$$\frac{f'(c)}{g'(c)} = \frac{1}{3}\frac{2A(a,c)^2 + G(a,c)^2}{G(a,c)^{2/3}A(a,c)^{4/3}}.$$

Now the right-hand side above being greater than or equal to 1 is equivalent to

$$\frac{2}{3}A(a,c)^2 + \frac{1}{3}G(a,c)^2 \geqslant A(a,c)^{4/3}G(a,c)^{2/3} = \left(A(a,c)^2\right)^{2/3}\left(G(a,c)^2\right)^{1/3},$$

which holds by the Weighted AGM inequality, with $t = \frac{2}{3}$. Therefore we have the left-hand inequality.

Okay, one last inequality

For $r \neq 0$ the *power mean* of two positive numbers a and b is

$$M_r(a, b) = \left(\frac{a^r + b^r}{2}\right)^{1/r}.$$

Obviously $M_1(a, b) = A(a, b)$, and the reader may verify that $\lim_{r \to 0} M_r(a, b) = G(a, b)$. Some other special instances of the power means are the *harmonic mean* $M_{-1}(a, b)$, the *Lorentz mean* $M_{1/3}(a, b)$, and the *root mean square* $M_2(a, b)$. (Reader: How about $M_{-\infty}(a, b)$ and $M_{+\infty}(a, b)$?)

It is a fact that for fixed a and b, the power means increase as r increases, but we do not prove it here. This follows from a simple application of what is known as *Jensen's inequality* (e.g. [9, 10]).

Clearly however, the logarithmic mean is a horse of a different colour.

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But in [13] is proved the following inequality, again using rather complicated methods. We prove it using, you guessed it, Cauchy's mean value theorem.

Theorem 4: Let a and b be positive numbers. Then

$$L(a, b) \leqslant M_{1/3}(a, b).$$

Proof: We may suppose that $a < b$. We set, for $x \in [a, b]$,

$$f(x) = \frac{x - a}{\left(\frac{x^{1/3} + a^{1/3}}{2}\right)^3} \qquad \text{and} \qquad g(x) = \ln(x) - \ln(a).$$

By Cauchy's mean value theorem, there is $c \in (a, b)$ such that

$$\frac{\frac{b-a}{\left(\frac{b^{1/3} + a^{1/3}}{2}\right)^3}}{\ln(b) - \ln(a)} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Simplification yields

$$\frac{f'(c)}{g'(c)} = \frac{A\left(a^{2/3}, c^{2/3}\right) G\left(a^{2/3}, c^{2/3}\right)}{\left(M_{1/2}\left(a^{2/3}, c^{2/3}\right)\right)^2}.$$

But it is easily verified that for any $p, q > 0$,

$$M_{1/2}(p, q) = \frac{A(p, q) + G(p, q)}{2}.$$

Therefore

$$\frac{A\left(a^{2/3}, c^{2/3}\right) G\left(a^{2/3}, c^{2/3}\right)}{\left(M_{1/2}\left(a^{2/3}, c^{2/3}\right)\right)^2} \leqslant 1$$

holds by the AGM inequality, and this immediately yields what we set out to prove.

Remarks: In case the reader is wondering (and it is natural to do so), we add finally that for fixed a and b, we have [14]:

$$M_{1/3}(a, b) \leqslant \frac{1}{3}A(a, b) + \frac{2}{3}G(a, b) \leqslant M_{1/2}(a, b).$$

The partner of the middle member here, namely $\frac{2}{3}A(a, b) + \frac{1}{3}G(a, b)$, is known as *Heron's mean*. We met Heron's mean incidentally, in the proof of Theorem 3.

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101.08 A continued fraction inspired by an identity of Euler

Introduction

In mathematics, there are many ways to express a number, either as a decimal expansion, as a sum, as a product, as a power series, in logarithmic form, in radical form, etc., each form revealing a secret pattern and symmetry. In this paper, we look at a special representation of numbers called 'continued fraction' of the form

$$[f_0, (g_0, f_1), (g_1, f_2), (g_2, f_3), \ldots] \equiv f_0 + \cfrac{g_0}{f_1 + \cfrac{g_1}{f_2 + \cfrac{g_2}{\ddots}}}.$$

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