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# RELIABILITY MODELING OF COHERENT SYSTEMS WITH SHARED COMPONENTS BASED ON SEQUENTIAL ORDER STATISTICS

S. ASHRAFI,* ** *University of Isfahan*
S. ZAREZADEH,*** *Shiraz University*
M. ASADI,* **** *University of Isfahan*

**Abstract**

In this paper we are concerned with the reliability properties of two coherent systems having shared components. We assume that the components of the systems are two overlapping subsets of a set of $n$ components with lifetimes $X_1, \ldots, X_n$. Further, we assume that the components of the systems fail according to the model of sequential order statistics (which is equivalent, under some mild conditions, to the failure model corresponding to a nonhomogeneous pure-birth process). The joint reliability function of the system lifetimes is expressed as a mixture of the joint reliability functions of the sequential order statistics, where the mixing probabilities are the bivariate signature matrix associated to the structures of systems. We investigate some stochastic orderings and dependency properties of the system lifetimes. We also study conditions under which the joint reliability function of systems with shared components of order $m$ can be equivalently written as the joint reliability function of systems of order $n$ ($n > m$). In order to illustrate the results, we provide several examples.

*Keywords:* Bivariate signature matrix; positive quadrant dependent; nonhomogeneous pure-birth process; stochastic ordering; equivalent systems

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## 1. Introduction

Systems with shared components have a wide range of applications in various areas of human life; for example, in network computing a server is used in common with several individual computers (see [16] [17]), and in a system of railway transport in a certain geographical area some stations are shared (see [22]). Recently, the reliability properties of systems with shared components have been addressed in the literature; see [16], [17], and [22]. Navarro *et al.* [17] considered two coherent systems having one or more shared components such that the first system has $n_1$ components with lifetimes $Y_1, \ldots, Y_{n_1}$ and the second system has $n_2$ components with lifetimes $Z_1, \ldots, Z_{n_2}$, and $n_{1,2}$ components are shared in the two systems. The authors assumed that the lifetimes $Y_1, \ldots, Y_{n_1}$ and $Z_1, \ldots, Z_{n_2}$ are subsets of a set of random lifetimes $X_1, \ldots, X_n$, where $n = n_1 + n_2 - n_{1,2}$. Under these conditions, they presented the joint reliability function of the lifetimes of the two systems as a mixture based on the notion of the

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* Postal address: Department of Statistics, University of Isfahan, Isfahan, 81744, Iran.
** Email address: s.ashrafi@sci.ui.ac.ir
*** Postal address: Department of Statistics, Shiraz University, Shiraz, 71454, Iran.
Email address: s.zarezadeh@shirazu.ac.ir
**** Email address: m.asadi@sci.ui.ac.ir



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bivariate signature matrix (BSM). The BSM is, in fact, a probability matrix defined as follows. Let $T_1$ and $T_2$ denote the lifetimes of the two systems and the components of the systems have independent and identically distributed (i.i.d.) lifetimes $X_1, \ldots, X_n$ with a common absolutely continuous cumulative distribution function (CDF) $F$. Assume that $X_{1:n}, \ldots, X_{n:n}$ denote the order statistics corresponding to the $X_i$ and define the discrete random variables (RVs) $I$ and $J$ such that

$$\{I = i, J = j\} \equiv \{T_1 = X_{i:n}, T_2 = X_{j:n}\}. \tag{1}$$

The signature matrix $\mathcal{S} = \{s_{i,j}\}_{n\times n}$ is a probability matrix with elements $s_{i,j}$, $i, j = 1, \ldots, n$, defined as

$$s_{i,j} := \mathbb{P}(I = i, J = j) = \mathbb{P}(T_1 = X_{i:n}, T_2 = X_{j:n}), \qquad i, j = 1, \ldots, n.$$

The element $s_{i,j}$ of the matrix indicates, in fact, the probability that the $i$th component failure causes the failure of the system with lifetime $T_1$ and the $j$th component failure causes the failure of the system with lifetime $T_2$. We see that $s_{i,j} = n_{i,j}/n!$, where $n_{i,j}$ is the number of permutations of the order of component failures in which $\{T_1 = X_{i:n}, T_2 = X_{j:n}\}$; see [17]. The joint reliability function of $(T_1, T_2)$ can be expressed in terms of the BSM and the joint reliability of $X_{i:n}$ and $X_{j:n}$, i.e.

$$\mathbb{P}(T_1 > t_1, T_2 > t_2) = \sum_{i=1}^{n}\sum_{j=1}^{n} s_{i,j}\mathbb{P}(X_{i:n} > t_1, X_{j:n} > t_2), \qquad t_1 > 0,\ t_2 > 0. \tag{2}$$

For results on the properties of the joint reliability (2) and examples on systems with shared components; see [16] and [17]. Equation (2) is proved under the assumption that the components of the systems have i.i.d. lifetimes. This means that the failure of any component in the system does not affect the lifetimes of the remaining components. However, in real-life situations, it is more reasonable to assume that the components of both systems are dependent. In particular, a situation arises when a component fails in the system and the load on the remaining components increases (decreases), and, hence, the failure of one component affects the lifetimes of the other remaining components. In order to describe this situation, the model of sequential order statistics (SOS), as an extension of ordinary order statistics, can be employed for modeling the ordered lifetimes of the system components. In an SOS model, it is assumed that at the starting time of the system operation, the lifetimes of the components are i.i.d. with a common CDF $F_1$. After the failure of the first component at a time instant, say $t_1$, the CDF of the residual lives of the $n-1$ remaining components changes to that of residual lives of i.i.d. components with age $t_1$ and a CDF $F_2$ (possibly different from $F_1$). If the $i$th component fails at time instant $t_i$, the CDF of the residual lives of the $n-i$ remaining components changes to that of residual lives of i.i.d. component with CDF $F_{i+1}$ truncated at $t_i$, $i = 1, \ldots, n-1$. If we proceed this way, we obtain a set of ordered failure times for the components, denoted by $X^*_{1:n}, \ldots, X^*_{n:n}$, which are called the SOS based on the CDFs $F_1, \ldots, F_n$; see [7], [8] and [12]. Throughout the paper, we assume that the CDFs $F_1, F_2, \ldots, F_n$ are absolutely continuous with a common support on $(0, +\infty)$. The joint probability density function (PDF) of $(X^*_{1:n}, \ldots, X^*_{n:n})$ is

$$f^*_{1,\ldots,n:n}(\boldsymbol{x}) = n!\left[\prod_{i=1}^{n-1}\left(\frac{\bar{F}_i(x_i)}{\bar{F}_{i+1}(x_i)}\right)^{n-i} f_i(x_i)\right] f_n(x_n), \qquad x_1 < x_2 < \cdots < x_n.$$

The SOS model forms a Markov chain with transition probabilities

$$\mathbb{P}(X^*_{i:n} > t \mid X^*_{i-1:n} = s) = \left(\frac{\bar{F}_i(t)}{\bar{F}_i(s)}\right)^{n-i+1}, \qquad t \geq s,\ i = 2, \ldots, n.$$

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We refer the reader to, for example, [4]–[6], [15], and [20] for more details on stochastic properties and applications of the SOS model in reliability.

The model of component failures based on SOS is closely related to the model of component failures based on nonhomogeneous pure-birth (NHPB) process. The NHPB process, which arises naturally in many applications, is defined as follows. A counting process $\{N(t),\ t \geq 0\}$ is said to be an NHPB process with intensity functions $\lambda_i(t) \geq 0,\ i = 1, 2, \ldots$, if

- $\{N(t),\ t \geq 0\}$ has the Markov property;
- $\mathbb{P}(N(t + \Delta t) = i + 1 \mid N(t) = i) = \lambda_i(t)\Delta t + o(\Delta t),\ i \geq 1$;
- $\mathbb{P}(N(t + \Delta t) > i + 1 \mid N(t) = i) = o(\Delta t),\ i \geq 1$,

where each $\lambda_i(t),\ i = 1, 2, \ldots$, satisfies $\int_t^\infty \lambda_i(u)\,\mathrm{d}u = \infty$ for all $t \geq 0$. When the $\lambda_i(t)$ are identical, an NHPB process is called a nonhomogeneous Poisson process (NHPP). In an NHPP, $\Lambda(t) = \int_0^t \lambda(u)\,\mathrm{d}u$ is called the mean value function; see [2]. Assuming that $Z_1, Z_2, \ldots, Z_n$ are the first $n$ epoch times of the process $\{N(t),\ t \geq 0\}$, Zhuang and Hu [23] showed that under the condition $\lambda_i(u) = (n - i + 1)r_i(u),\ i = 1, \ldots, n$, where $r_i(u)$ denotes the hazard rate of $F_i$, the joint reliability of the ordered component failures based on the SOS model and the joint reliability of component failures based on the NHPB process are identical. That is,

$$(X^*_{1:n}, \ldots, X^*_{n:n}) \stackrel{\mathrm{D}}{=} (Z_1, Z_2, \ldots, Z_n),$$

where '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution.

In this paper we study the reliability properties of two coherent systems with shared components. We consider two coherent systems with respective components $n_1$ and $n_2$ among which $n_{1,2}$ components are shared, and we assume that the components of the two systems fail according to the SOS model $X^*_{1:n}, \ldots, X^*_{n:n}$. In Section 2, the signature-based joint reliability function of system lifetimes is expressed in terms of the joint reliability function of the SOS model $(X^*_{i:n}, X^*_{j:n})$. Stochastic ordering results are obtained in order to compare the lifetimes of two pairs of systems with shared components based on stochastic orderings between the elements of the BSMs associated to the systems. We also study the signature-based dependency between the lifetimes of the two systems. In order to illustrate the results, we also provide examples of systems with shared components. In particular, we provide a detailed example regarding a three-state network with 14 components (as a special case of the systems with shared components) where the components are assumed to fail according to the NHPP model. In Section 3 we explore conditions under which the joint reliability function of two systems of order $m$, with shared components, can be expressed equivalently as the joint reliability function of two systems with shared components of order $n$ $(n > m)$. Section 4 is devoted to some concluding remarks.

## 2. The joint reliability of the systems under the SOS model

Consider two systems having some shared components. Let the first system have $n_1$ components and the second system $n_2$ components of which $n_{1,2}$ components are shared. Hence, the total number of components of the two systems is $n = n_1 + n_2 - n_{1,2}$. Suppose that RVs $X_1, \ldots, X_n$ denote the lifetimes of the components of the two systems and denote $T_1$ and $T_2$ as the lifetimes of the systems, respectively. In what follows, we assume that the failure of the components occurs according to the SOS $X^*_{1:n}, \ldots, X^*_{n:n}$. That is, $X^*_{i:n}$ represents the $i$th component failure time in the system as described in Section 1 for $i = 1, 2, \ldots, n$.

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Furthermore, we assume that the reliability functions $\bar{F}_1, \ldots, \bar{F}_n$ characterize the SOS model. Under these assumptions, the joint reliability function of $T_1$ and $T_2$ can be represented in terms of the reliability functions of the $(X^*_{i:n}, X^*_{j:n})$ as

$$\mathbb{P}(T_1 > t_1, T_2 > t_2) = \sum_{i=1}^{n}\sum_{j=1}^{n} s_{i,j}\mathbb{P}(X^*_{i:n} > t_1, X^*_{j:n} > t_2), \qquad t_1 > 0,\ t_2 > 0, \tag{3}$$

where the validity of this representation follows from the fact that (2) is valid for the systems with exchangeable components (see [17]), and using [15, Lemma 2.1], which states that there exists an exchangeable random vector $\boldsymbol{X}^* = (X^*_1, \ldots, X^*_n)$ such that the corresponding vector of its usual order statistics is identically distributed as the vector of the SOS $(X^*_{1:n}, \ldots, X^*_{n:n})$.

Among the special cases of the general form of the SOS model, we refer to two important cases which are of interest in many applications. A special case, which is important in different applications, is when the reliability functions $\bar{F}_i$ have proportional hazard rates. Recall that two reliability functions $\bar{F}$ and $\bar{G}$ are said to have proportional hazard rates if there exists $\alpha > 0$ such that $\bar{F}(x) = (\bar{G}(x))^{\alpha}$. Now assuming that for parameters $\alpha_i > 0$, $i = 1, \ldots, n$, we have, in the SOS model, $\bar{F}_i = \bar{F}^{\alpha_i}$. The joint PDF of $X^*_{i:n}$ and $X^*_{j:n}$, is

$$f^{*\boldsymbol{\beta}}_{i,j:n}(x_i, x_j) = L_{j-1}\sum_{k=i+1}^{j} b^{(i)}_k(j)\left(\frac{\bar{F}(x_j)}{\bar{F}(x_i)}\right)^{\beta_k}\sum_{k=1}^{i} b_k(i)(\bar{F}(x_i))^{\beta_k}\frac{f(x_i)f(x_j)}{\bar{F}(x_i)\bar{F}(x_j)}, \tag{4}$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, $\beta_i = (n - i + 1)\alpha_i$, provided that

$$\beta_i \neq \beta_j, \qquad i \neq j \text{ for all } i, j \in \{1, \ldots, n\},$$

$$L_{j-1} = \prod_{k=1}^{j}\beta_k, \qquad b^{(i)}_k(j) = \prod_{l=i+1,\, l\neq k}^{j}\frac{1}{\beta_l - \beta_k}, \qquad i + 1 \leq k \leq j,$$

$$b_k(i) = \prod_{l=1,\, l\neq k}^{i}\frac{1}{\beta_l - \beta_k}, \qquad 1 \leq k \leq i \leq n.$$

The empty products are defined to be 1; see [13]. In particular, the PDF of $X^*_{i:n}$ is

$$f^{*\boldsymbol{\beta}}_{i:n}(x) = L_{i-1}\sum_{k=1}^{i} b_k(i)(\bar{F}(x))^{\beta_k - 1}f(x). \tag{5}$$

As mentioned in Section 1, the SOS model is related to the model of component failures based on the NHPB process. In particular, when the intensity functions in the NHPB process are identical, it reduces to the NHPP. In this case, if the failures of components occur according to an NHPP and if all orders of component failures have equal probabilities, then the joint reliability function in (2) can be expressed analogously as

$$\mathbb{P}(T_1 > t_1, T_2 > t_2) = \sum_{i=1}^{n}\sum_{j=1}^{n} s_{i,j}\mathbb{P}(\vartheta_i > t_1, \vartheta_j > t_2), \qquad t_1 > 0,\ t_2 > 0,$$

where $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$ are the first $n$ occurrence times of the NHPP.

We now turn our attention to the following examples.

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**Example 1.** Let $X_1, X_2, X_3, X_4$ be nonnegative RVs denoting the lifetimes of four components. Consider two coherent systems built from the $X_i$ with lifetimes

$$T_1 = \min\{X_1, \max\{\min\{X_2, X_3\}, X_4\}\}, \qquad T_2 = \min\{\max\{X_1, X_2\}, \max\{X_3, X_4\}\}.$$

For these systems, it is straightforward to verify that the positive elements of the BSM are

$$s_{1,2} = \tfrac{1}{12}, \qquad s_{2,2} = \tfrac{1}{4}, \qquad s_{1,3} = \tfrac{1}{6}, \qquad s_{2,3} = \tfrac{1}{3}, \qquad s_{3,3} = \tfrac{1}{6}.$$

Assume that the components fail according to the SOS model based on the CDFs $F_1, \ldots, F_4$, where

$$\bar{F}_i(t) = \mathrm{e}^{-\alpha_i t}, \qquad t > 0,\ \alpha_i > 0,\ i = 1, \ldots, 4.$$

With $\beta_i = (n - i + 1)\alpha_i$, $\beta_i \neq \beta_j$, $i \neq j$, using (4) and (5), we see that

$$\mathbb{P}(X^*_{1:4} > x, X^*_{2:4} > y) = \frac{1}{\beta_{12}}(\beta_1 \mathrm{e}^{-\beta_{12}x - \beta_2 y} - \beta_2 \mathrm{e}^{-\beta_1 y}),$$

$$\mathbb{P}(X^*_{1:4} > x, X^*_{3:4} > y) = \frac{\beta_1\beta_2\beta_3}{\beta_{23}}\left(\frac{\mathrm{e}^{-\beta_{13}x - \beta_3 y}}{\beta_3\beta_{13}} - \frac{\mathrm{e}^{-\beta_{12}x - \beta_2 y}}{\beta_2\beta_{12}} + \frac{\beta_{23}\mathrm{e}^{-\beta_1 y}}{\beta_1\beta_{12}\beta_{13}}\right),$$

$$\mathbb{P}(X^*_{2:4} > x, X^*_{3:4} > y) = \frac{\beta_1\beta_2\beta_3}{\beta_{12}}\left(\frac{\mathrm{e}^{-\beta_{23}x - \beta_3 y}}{\beta_3\beta_{23}} - \frac{\mathrm{e}^{-\beta_{13}x - \beta_3 y}}{\beta_3\beta_{13}} + \frac{\mathrm{e}^{-\beta_1 y}}{\beta_1\beta_{13}} - \frac{\mathrm{e}^{-\beta_2 y}}{\beta_2\beta_{23}}\right),$$

and

$$\mathbb{P}(X^*_{2:4} > y) = \frac{1}{\beta_{21}}(\beta_2 \mathrm{e}^{-\beta_1 y} - \beta_1 \mathrm{e}^{-\beta_2 y}),$$

$$\mathbb{P}(X^*_{3:4} > y) = \frac{1}{\beta_{21}\beta_{32}\beta_{31}}\left(\beta_2\beta_3\beta_{32}\mathrm{e}^{-\beta_1 y} - \beta_1\beta_3\beta_{31}\mathrm{e}^{-\beta_2 y} + \beta_1\beta_2\beta_{21}\mathrm{e}^{-\beta_3 y}\right),$$

where $\beta_{ij} = \beta_i - \beta_j$, $i \neq j$. Therefore, the joint reliability function of $(T_1, T_2)$ can be written as

$$\begin{aligned}\mathbb{P}(T_1 > t_1, T_2 > t_2) = {} & \tfrac{1}{12}\mathbb{P}(X^*_{1:4} > t_1, X^*_{2:4} > t_2) + \tfrac{1}{6}\mathbb{P}(X^*_{1:4} > t_1, X^*_{3:4} > t_2) \\ & + \tfrac{1}{3}\mathbb{P}(X^*_{2:4} > t_1, X^*_{3:4} > t_2) + \tfrac{1}{4}\mathbb{P}(X^*_{2:4} > t_2) + \tfrac{1}{6}\mathbb{P}(X^*_{3:4} > t_2).\end{aligned}$$

In Figure 1 we present plots of the joint and marginal reliability functions of $T_1$ and $T_2$, where $\alpha_1 = \frac{1}{2} < \alpha_2 = 1 < \alpha_3 = 2$ and, hence, $\beta_1 = 2 < \beta_2 = 3 < \beta_3 = 4$.

The networks arise in different real-life situations; for example, communications, computing, and the internet. A network consists of a set of nodes and a set of links where some particular nodes are considered as terminals. Let a network have three states (up, partial performance, and down). Suppose that the links are subject to failure and all nodes are absolutely reliable. Denote by $T_1$ and $T_2$ the entrance times of the network into partial performance and down state, respectively. If $X_{1:n}, \ldots, X_{n:n}$ denote the ordered lifetimes of the network links then the joint reliability function of $T_1$ and $T_2$ can be expressed as (2) where in this case the bivariate signature is a matrix with elements $s_{i,j} = \mathbb{P}(T_1 = X_{i:n}, T_2 = X_{j:n}) = 0$ for $i \geq j$; see [10] and [11]. We see that the three-state network (system) described here can be considered as two networks with lifetimes $T_1$ and $T_2$ ($T_1 < T_2$ with probability 1) in which all components are shared.

Next we provide an example of a three-state network with 14 links (components) where the links fail according to an NHPP.

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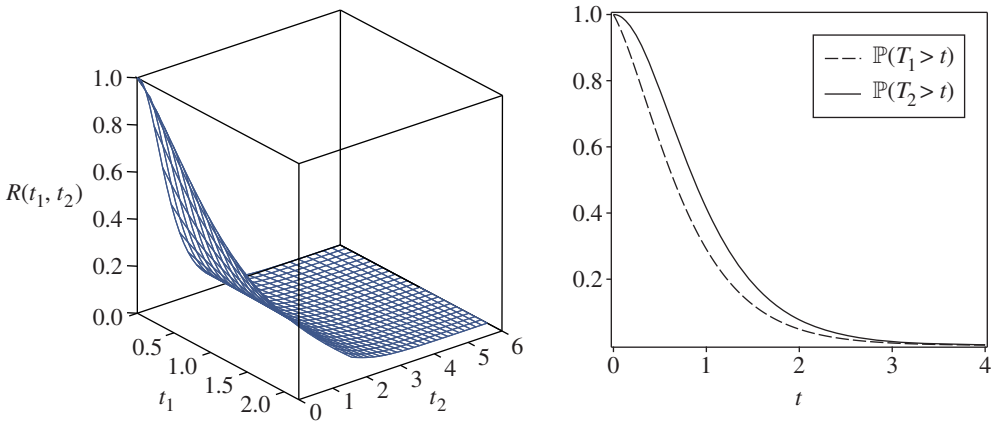



FIGURE 1: (a) The joint reliability function of $(T_1, T_2)$, and (b) the marginal reliability functions of $T_1$, and $T_2$ from Example 1.

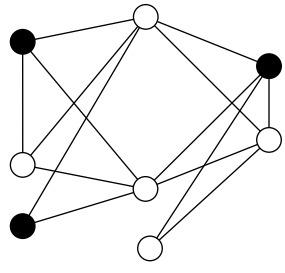


FIGURE 2: The network with eight nodes and 14 links. The terminals are shown as solid circles.

**Example 2.** Consider a network with the graph as depicted in Figure 2. The network has 14 links and eight nodes. We assume that the links are subject to failure and the nodes are absolutely reliable. The states of the network are defined according to the connection between the terminals of the network, where the terminals are shown as solid circles. We further assume that the network is in the up state if all terminals are connected, in the down state if all terminals are disconnected, and otherwise in the partial performance state.

Let the network links fail according to an NHPP with mean value function $\Lambda(t) = -\ln \bar{G}(t)$, where $\bar{G}(t) = \mathrm{e}^{-t^a}$, $t > 0$, $a > 0$. Then the joint reliability function of $T_1$ and $T_2$ is

$$\begin{aligned}\mathbb{P}(T_1 > t_1, T_2 > t_2) &= \sum_{i=1}^{14}\sum_{j=i+1}^{14} s_{i,j} \sum_{k=0}^{i-1}\sum_{l=k}^{j-1} \frac{(-\ln \bar{G}(t_1))^k}{k!} \frac{(-\ln(\bar{G}(t_2)/\bar{G}(t_1)))^{l-k}}{(l-k)!} \bar{G}(t_2) \\ &= \sum_{i=1}^{14}\sum_{j=i+1}^{14} s_{i,j} \sum_{k=0}^{i-1}\sum_{l=k}^{j-1} \frac{t_1^{ak}}{k!} \frac{(t_2^a - t_1^a)^{l-k}}{(l-k)!} \mathrm{e}^{-t_2^a}.\end{aligned}$$

The elements $s_{i,j}$ of the BSM are calculated using an algorithm from [21] for different values of $i$ and $j$ (the calculated matrix is available on request from the authors). Using the values of

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TABLE 1: Some descriptive quantities of $T_1$ and $T_2$ for different values of $a$.

| $a$ | $\mathbb{E}(T_1)$ | $\mathbb{E}(T_2)$ | $V(T_1)$ | $V(T_2)$ | $\text{cov}(T_1, T_2)$ | $\text{corr}(T_1, T_2)$ |
|---|---|---|---|---|---|---|
| $\frac{1}{2}$ | 70.3674 | 125.2500 | 3822.1035 | 7210.1823 | 3951.3880 | 0.7527 |
| $\frac{2}{3}$ | 22.7661 | 36.0061 | 224.0946 | 327.3205 | 202.2583 | 0.7468 |
| 1 | 7.6556 | 10.6114 | 11.7599 | 12.6485 | 8.9355 | 0.7326 |
| 2 | 2.6925 | 3.2114 | 0.4062 | 0.2982 | 0.2454 | 0.7051 |

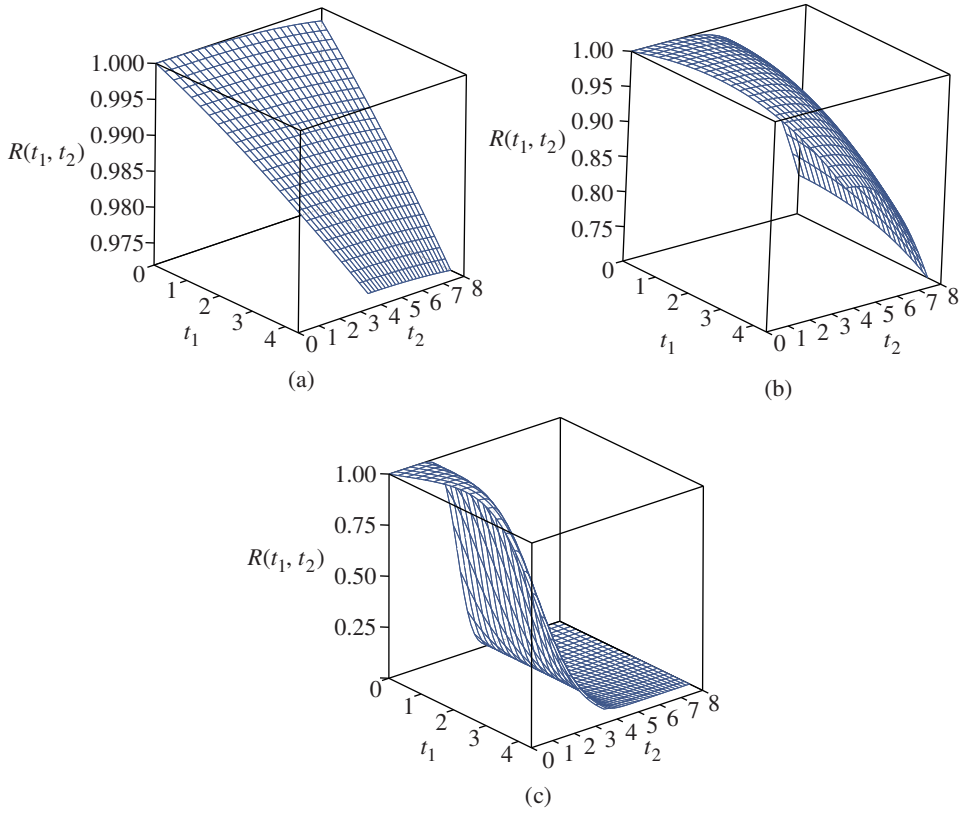



FIGURE 3: The joint reliability function of $(T_1, T_2)$ for (a) $a = 0.5$, (b) $a = 1$, and (c) $a = 2$.

the bivariate signature, we plot the joint reliability function of $(T_1, T_2)$ for different values of shape parameter $a$; see Figure 3.

In Table 1 we present the mean $(\mathbb{E}(T_i))$ and variance $(V(T_i))$ of $T_i$, $i = 1, 2$, together with the covariance and correlation between $T_1$ and $T_2$.

Before proceeding to our next results, we recall the following definitions of some partial orderings; see [19] for more details on the different concepts of partial orderings.

**Definition 1.** Let $\boldsymbol{X}_1 = (X_{11}, \ldots, X_{1n})$ and $\boldsymbol{X}_2 = (X_{21}, \ldots, X_{2n})$ be two random vectors with survival functions $\bar{F}_i(t_1, \ldots, t_n) = \mathbb{P}(X_{i1} > t_1, \ldots, X_{in} > t_n)$, $i = 1, 2$, respectively.

We say that $\boldsymbol{X}_1$ is smaller than $\boldsymbol{X}_2$ in the usual stochastic order, denoted by $\boldsymbol{X}_1 \leq_{\text{st}} \boldsymbol{X}_2$ if $\mathbb{E}(\phi(\boldsymbol{X}_1)) \leq \mathbb{E}(\phi(\boldsymbol{X}_2))$ for every increasing function $\phi$ provided that the expectations exist.

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**Definition 2.** Let $f$ denote the joint PDF of the random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$.

(a) We say that $f$ is multivariate totally positive of order 2 ($\mathrm{MTP}_2$) if, for every $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{x}' = (x_1', \ldots, x_n')$,

$$f(\boldsymbol{x}) f(\boldsymbol{x}') \leq f(\boldsymbol{x} \vee \boldsymbol{x}') f(\boldsymbol{x} \wedge \boldsymbol{x}'),$$

where $\boldsymbol{x} \vee \boldsymbol{x}' = (x_1 \vee x_1', \ldots, x_n \vee x_n')$ and $\boldsymbol{x} \wedge \boldsymbol{x}' = (x_1 \wedge x_1', \ldots, x_n \wedge x_n')$ such that $x_i \vee x_i' = \max\{x_i, x_i'\}$ and $x_i \wedge x_i' = \min\{x_i, x_i'\}$, $i = 1, \ldots, n$.

(b) The RVs $X_1$ and $X_2$ are said to be positively quadrant dependent (PQD) if

$$\mathbb{P}(X_1 > x_1, X_2 > x_2) \geq \mathbb{P}(X_1 > x_1)\mathbb{P}(X_2 > x_2) \quad \text{for all } x_1 > 0,\ x_2 > 0,$$

or, equivalently, $\mathrm{cov}(h_1(X_1), h_2(X_2)) \geq 0$ for every pair of increasing functions $h_1(\cdot)$ and $h_2(\cdot)$.

(c) The RVs $X_1$ and $X_2$ are said to be associated if $\mathrm{cov}(h_1(X_1, X_2), h_2(X_1, X_2)) \geq 0$ for every pair of increasing functions $h_1(x_1, x_2)$ and $h_2(x_1, x_2)$.

For $n = 2$, $\mathrm{MTP}_2$ is also known as totally positive of order 2 ($\mathrm{TP}_2$). Put simply, for the random vector $(X_1, X_2)$ with joint PDF $f(x_1, x_2)$, $f$ is $\mathrm{TP}_2$ if, for all $x_1 < x_1'$ and $x_2 < x_2'$,

$$f(x_1, x_2) f(x_1', x_2') \geq f(x_1, x_2') f(x_1', x_2).$$

It is remarkable that the $\mathrm{TP}_2$ property implies associated and associated implies PQD.

Consider two systems with respective lifetimes $T_1^{(1)}$ and $T_2^{(1)}$ having some shared components, and let $\mathcal{S}^{(1)}$ be the corresponding BSM. Assume that the components of the two systems fail according to the SOS $X_{1:n}^*, \ldots, X_{n:n}^*$ based on the CDFs $F_1, \ldots, F_n$. Also, consider two other systems with respective lifetimes $T_1^{(2)}$ and $T_2^{(2)}$ having shared components. Let $\mathcal{S}^{(2)}$ be the corresponding BSM where we assume that the components of these two systems fail according to the SOS $Y_{1:n}^*, \ldots, Y_{n:n}^*$ based on the CDFs $G_1, \ldots, G_n$. If $F_1 \leq_{\mathrm{st}} G_1$ and $F_i \leq_{\mathrm{hr}} G_i$, $i = 2, \ldots, n$ then, from [23, Theorem 5.1], $(X_{1:n}^*, \ldots, X_{n:n}^*) \leq_{\mathrm{st}} (Y_{1:n}^*, \ldots, Y_{n:n}^*)$ which, in turn, implies that $(X_{i:n}^*, X_{j:n}^*) \leq_{\mathrm{st}} (Y_{i:n}^*, Y_{j:n}^*)$. Therefore, if we assume that $\mathcal{S}^{(1)} \leq_{\mathrm{st}} \mathcal{S}^{(2)}$ then, from [3, Theorem 3.3], we obtain $(T_1^{(1)}, T_2^{(1)}) \leq_{\mathrm{st}} (T_1^{(2)}, T_2^{(2)})$.

As an application of this result we present the following example.

**Example 3.** Consider two pairs of systems having shared components with the following structures.

(I) The first pair is made of four components $X_1, \ldots, X_4$ with lifetimes

$$T_1^{(1)} = \min\{X_1, \max\{X_2, X_3\}\},$$
$$T_2^{(1)} = \min\{\max\{X_2, X_3\}, \max\{X_2, X_4\}, \max\{X_3, X_4\}\}.$$

Let the components of these systems fail according to an SOS model based on the CDFs $F_1, \ldots, F_4$. The positive elements of the BSM $\mathcal{S}^{(1)}$ are

$$s_{1,3}^{(1)} = \tfrac{1}{4}, \qquad s_{2,2}^{(1)} = \tfrac{1}{6}, \qquad s_{2,3}^{(1)} = \tfrac{1}{4}, \qquad s_{3,2}^{(1)} = \tfrac{1}{3}.$$

(II) Let $Y_1, \ldots, Y_4$ be the lifetimes of four components that are used to build the second pair of the systems where we assume that the lifetimes of the systems are

$$T_1^{(2)} = \max\{Y_1, \min\{Y_2, Y_3\}\}, \qquad T_2^{(2)} = \max\{Y_2, Y_3, Y_4\}.$$

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Let also the components of the two systems fail according to an SOS model based on the CDFs $G_1, \ldots, G_4$. For these systems, we can easily see that the positive elements of the BSM $\mathcal{S}^{(2)}$ are

$$s^{(2)}_{2,4} = \tfrac{1}{3}, \qquad s^{(2)}_{3,4} = \tfrac{5}{12}, \qquad s^{(2)}_{4,3} = \tfrac{1}{4}.$$

It is straightforward to see that for every increasing function $\phi(i, j)$,

$$\sum_{i=1}^{4}\sum_{j=1}^{4} s^{(1)}_{i,j}\phi(i, j) \le \sum_{i=1}^{4}\sum_{j=1}^{4} s^{(2)}_{i,j}\phi(i, j),$$

which, in turn, implies $\mathcal{S}^{(1)} \le_{\mathrm{st}} \mathcal{S}^{(2)}$. If $F_1 \le_{\mathrm{st}} G_1$ and $F_i \le_{\mathrm{hr}} G_i$, $i = 2, \ldots, n$, then $(T^{(1)}_1, T^{(1)}_2) \le_{\mathrm{st}} (T^{(2)}_1, T^{(2)}_2)$.

In the next theorem we explore the dependency between the lifetimes of systems with shared components.

**Theorem 1.** *Consider a pair of systems having some shared components. Let the components fail according to the SOS $X^*_{1:n}, \ldots, X^*_{n:n}$ based on the CDFs $F_1, \ldots, F_n$. Further, assume that $T_1$ and $T_2$ are the lifetimes of the systems, respectively, and suppose that $I$ and $J$ are defined as in (1).*

(a) *If $I$ and $J$ are PQD then $T_1$ and $T_2$ are PQD.*

(b) *If $I$ and $J$ are associated then $T_1$ and $T_2$ are associated.*

*Proof.* Since $(X^*_{1:n}, \ldots, X^*_{n:n})$ forms a Markov chain with $\mathrm{TP}_2$ transition probability densities then $X^*_{1:n}, \ldots, X^*_{n:n}$ are $\mathrm{MTP}_2$, which implies that $X^*_{i:n}$ and $X^*_{j:n}$ are $\mathrm{TP}_2$; see [14]. Hence, $X^*_{i:n}$ and $X^*_{j:n}$ are associated and PQD. By considering (3), the proof follows using the same steps as in the proof of [1, Theorem 5]. □

As an application of this theorem we present the following example.

**Example 4.** Consider again Example 1. Let $\bar{S}_{i,j} = \sum_{k=i+1}^{n}\sum_{l=j+1}^{n} s_{k,l}$. We see that

$$\bar{S}_{1,0} = \tfrac{3}{4}, \qquad \bar{S}_{2,0} = \tfrac{1}{6}, \qquad \bar{S}_{0,2} = \tfrac{2}{3}, \qquad \bar{S}_{1,2} = \tfrac{1}{2}, \qquad \bar{S}_{2,2} = \tfrac{1}{6}.$$

Thus, we have $\bar{S}_{i,j} \ge \bar{S}_{i,0}\bar{S}_{0,j}$ for $i, j = 1, 2, 3, 4$, implying that $I$ and $J$ are PQD. Under the assumption that the components fail according to an SOS model, from Theorem 1 we conclude that $T_1$ and $T_2$ are also PQD.

## 3. Equivalent systems with shared components

Two different coherent systems with exchangeable component lifetimes are said to be equivalent if the lifetimes of the systems have identical distributions. Navarro *et al.* [18] introduced this concept and showed that if $X_1, X_2, \ldots, X_n$ are exchangeable RVs and $T$ is the lifetime of a mixed (or coherent) system with component lifetimes $X_1, X_2, \ldots, X_k$ ($k \le n$), then the reliability of $T$ can be equivalently represented as the reliability of a mixed system based on $X_1, X_2, \ldots, X_n$; see [18, Theorem 3.4]. The notion of equivalent systems is particularly important for comparing the reliability and ageing properties of the systems with different numbers of components. Ashrafi and Asadi [1] extended the notion of equivalent systems to three-state systems and made stochastic comparisons of the lifetimes of three-state systems with different sizes according to some notion of stochastic orderings.

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In this section we aim to investigate the conditions under which systems with shared components are equivalent. Before stating the definition of equivalent systems, we recall that in a system the components may fail according to different random failure mechanisms. For example, the components may fail according to ageing over time, the components may be subjected to failure according to a counting process, or the components may fail as a result of fatal shocks arriving based on some stochastic process. In this section we consider two types of random mechanism for the component failures. In Subsection 3.1 we assume that the components fail according to ageing and the lifetimes of the components are exchangeable, and in Subsection 3.2 we assume that the components fail according to the SOS model.

**Definition 3.** Let $(T_1^{(1)}, T_2^{(1)})$ be the lifetimes of a pair of systems with shared components, and $(T_1^{(2)}, T_2^{(2)})$ be the lifetimes of another pair of systems with shared components. Assuming that failure of components in the two pairs occurs according to the same random mechanisms, we say that these pairs of systems are equivalent if $(T_1^{(1)}, T_2^{(1)}) \overset{\mathrm{D}}{=} (T_1^{(2)}, T_2^{(2)})$.

### 3.1. The components with exchangeable lifetimes

In this subsection we consider two pairs of systems of different orders such that each pair consists of components with exchangeable lifetimes and some components are shared between the two systems. Under these circumstances, we investigate the conditions under which the two pairs are equivalent. Suppose that $\mathcal{C}_{i,j:n}$ denotes a matrix of order $n$ whose $(i, j)$th element is 1 and the other elements are 0. Further, for $n > m$, let

$$d_{k+i,\,l+j} = \frac{\binom{i-1+k}{k}\binom{j-i-1+l-k}{l-k}\binom{n-j-l}{n-m-l}}{\binom{n}{m}}, \qquad k = 0, \ldots, n-m,\ l = k, \ldots, n-m, \tag{6}$$

$$d^*_{k+i} = \frac{\binom{i-1+k}{k}\binom{n-i-k}{n-m-k}}{\binom{n}{m}}, \qquad k = 0, \ldots, n-m. \tag{7}$$

In the next theorem we explore the conditions under which the two pairs of systems, with shared components, are equivalent. In order to prove the result, we impose the condition that the corresponding BSM is symmetric (i.e. it is equal to its transpose). We see that when the lifetimes of components are exchangeable absolutely continuous RVs, a sufficient condition for the BSM to be symmetric is that the two systems have the same structure and the positions of the shared components are the same in both systems. In this case, we see that

$$s_{i,j} = \mathbb{P}(T_1 = X_{i:n}, T_2 = X_{j:n}) = \mathbb{P}(T_1 = X_{j:n}, T_2 = X_{i:n}) = s_{j,i},$$

which, in turn, implies that the associated BSM is symmetric. As an application of this, see Example 5.

**Theorem 2.** *Assume that $X_1, \ldots, X_n$ are exchangeable RVs from the CDF $F$. Let $T_1^{(1)}$ and $T_2^{(1)}$ denote the lifetimes of two systems with shared components based on $X_1, \ldots, X_m$, $m < n$, where we assume that the corresponding BSM is symmetric. If $T_1^{(2)}$ and $T_2^{(2)}$ denote the lifetimes of two other systems with shared components based on $X_1, \ldots, X_n$, then*

$$(T_1^{(1)}, T_2^{(1)}) \overset{\mathrm{D}}{=} (T_1^{(2)}, T_2^{(2)}),$$

*where $T_1^{(2)}$ and $T_2^{(2)}$ have the joint reliability function*

$$\mathbb{P}(T_1^{(2)} > t_1, T_2^{(2)} > t_2) = \sum_{k=1}^{n}\sum_{l=k}^{n} p_{k,l}\mathbb{P}(X_{k:n} > \min\{t_1, t_2\}, X_{l:n} > \max\{t_1, t_2\}),$$

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*in which $p_{k,l}$ is the $(k,l)$th element of the matrix*

$$\mathcal{P} = \sum_{i=1}^{m}\sum_{j=i+1}^{m}\sum_{k=0}^{n-m}\sum_{l=k}^{n-m} s_{i,j}\, d_{k+i,\,l+j}\, \mathcal{C}_{k+i,\,l+j:n} + \sum_{j=1}^{m}\sum_{i=j}^{m}\sum_{k=0}^{n-m} s_{i,j}\, d^*_{k+j}\, \mathcal{C}_{k+j,\,k+j:n}.$$

*Proof.* We prove only =the $t_1 \le t_2$ case. David and Joshi [9] showed that the following recurrence relationships hold for the reliability functions of order statistics corresponding to the exchangeable RVs $X_1, \ldots, X_n$:

$$\begin{aligned}&\mathbb{P}(X_{i:m} > t_1, X_{j:m} > t_2)\\ &\quad = \sum_{k=0}^{n-m}\sum_{l=k}^{n-m} d_{k+i,\,l+j}\,\mathbb{P}(X_{k+i:n} > t_1, X_{l+j:n} > t_2), \qquad t_1 \le t_2,\ m < n,\end{aligned}$$

and

$$\mathbb{P}(X_{j:m} > t) = \sum_{k=0}^{n-m} d^*_{k+j}\,\mathbb{P}(X_{k+j:n} > t), \qquad m < n.$$

Hence, from (2) for $t_1 \le t_2$, we can write

$$\mathbb{P}(T_1^{(1)} > t_1, T_2^{(1)} > t_2) = \sum_{i=1}^{m}\sum_{j=i+1}^{m} s_{i,j}\,\mathbb{P}(X_{i:m} > t_1, X_{j:m} > t_2) + \sum_{i=1}^{m}\sum_{j=1}^{i} s_{i,j}\,\mathbb{P}(X_{j:m} > t_2),$$

where

$$\begin{aligned}&\sum_{i=1}^{m}\sum_{j=i+1}^{m} s_{i,j}\,\mathbb{P}(X_{i:m} > t_1, X_{j:m} > t_2)\\ &\quad = \sum_{i=1}^{m}\sum_{j=i+1}^{m}\sum_{k=0}^{n-m}\sum_{l=k}^{n-m} s_{i,j}\, d_{k+i,\,l+j}\,\mathbb{P}(X_{k+i:n} > t_1, X_{l+j:n} > t_2)\\ &\quad = \sum_{i=1}^{m}\sum_{j=i+1}^{m}\sum_{k=i}^{n-m+i}\sum_{l=k-i+j}^{n-m+j} s_{i,j}\, d_{k,l}\,\mathbb{P}(X_{k:n} > t_1, X_{l:n} > t_2)\\ &\quad = \sum_{k=1}^{n}\sum_{l=k+1}^{n}\sum_{i=\max\{k-n+m,1\}}^{k}\sum_{j=\max\{i+1,l-n+m\}}^{l-k+i} s_{i,j}\, d_{k,l}\,\mathbb{P}(X_{k:n} > t_1, X_{l:n} > t_2)\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{i} s_{i,j}\,\mathbb{P}(X_{j:m} > t_2) &= \sum_{j=1}^{m}\sum_{i=j}^{m}\sum_{k=0}^{n-m} s_{i,j}\, d^*_{k+j}\,\mathbb{P}(X_{k+j:n} > t_2)\\ &= \sum_{j=1}^{m}\sum_{i=j}^{m}\sum_{k=j}^{n-m+j} s_{i,j}\, d^*_{k}\,\mathbb{P}(X_{k:n} > t_2)\\ &= \sum_{k=1}^{n}\sum_{j=\max\{k-(n-m),1\}}^{k}\sum_{i=j}^{m} s_{i,j}\, d^*_{k}\,\mathbb{P}(X_{k:n} > t_2).\end{aligned}$$

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Therefore, the joint reliability function of $(T_1^{(1)}, T_2^{(1)})$ can be written as

$$\begin{aligned}\mathbb{P}(T_1^{(1)} > t_1, T_2^{(1)} > t_2) &= \sum_{k=1}^{n}\sum_{l=k}^{n} p_{k,l}\mathbb{P}(X_{k:n} > t_1, X_{l:n} > t_2),\\ &= \mathbb{P}(T_1^{(2)} > t_1, T_2^{(2)} > t_2), \qquad t_1 \le t_2, \end{aligned} \tag{8}$$

where $p_{k,l}$ is the $(k,l)$th element of the BSM

$$\begin{aligned}\mathcal{P} &= \sum_{k=1}^{n}\sum_{l=k+1}^{n}\sum_{i=\max\{k-(n-m),1\}}^{k}\sum_{j=\max\{i+1,l-(n-m)\}}^{l-k+i} s_{i,j}d_{k,l}\mathcal{C}_{k,l:n}\\ &\quad+\sum_{k=1}^{n}\sum_{j=\max\{k-(n-m),1\}}^{k}\sum_{i=j}^{m} s_{i,j}d_k^*\mathcal{C}_{k,k:n}\\ &= \sum_{i=1}^{m}\sum_{j=i+1}^{m}\sum_{k=0}^{n-m}\sum_{l=k}^{n-m} s_{i,j}d_{k+i,l+j}\mathcal{C}_{k+i,l+j:n} + \sum_{j=1}^{m}\sum_{i=j}^{m}\sum_{k=0}^{n-m} s_{i,j}d_{k+j}^*\mathcal{C}_{k+j,k+j:n},\end{aligned}$$

and $d_{k+i,l+j}$ and $d_{k+j}^*$ are defined in (6) and (7), respectively. Thus, the proof for the $t_1 \le t_2$ case is complete. Similarly, using the assumption that the BSM of the two systems is symmetric, it is straightforward to see that

$$\begin{aligned}\mathbb{P}(T_1^{(1)} > t_1, T_2^{(1)} > t_2) &= \mathbb{P}(T_1^{(1)} > t_2, T_2^{(1)} > t_1)\\ &= \sum_{k=1}^{n}\sum_{l=k}^{n} p_{k,l}\mathbb{P}(X_{k:n} > t_2, X_{l:n} > t_1), \qquad t_1 \ge t_2. \end{aligned} \tag{9}$$

Therefore, from (8) and (9), we finally conclude that for every $t_1, t_2 > 0$,

$$\begin{aligned}\mathbb{P}(T_1^{(1)} > t_1, T_2^{(1)} > t_2) &= \sum_{k=1}^{n}\sum_{l=k}^{n} p_{k,l}\mathbb{P}(X_{k:n} > \min\{t_1,t_2\}, X_{l:n} > \max\{t_1,t_2\})\\ &= \mathbb{P}(T_1^{(2)} > t_1, T_2^{(2)} > t_2).\end{aligned}$$

□

**Example 5.** Let $X_1, \ldots, X_6$ denote the lifetimes of the six components that are exchangeable. Consider two systems made up of four components $X_1, X_2, X_3, X_4$ with lifetimes $T_1 = \min\{X_1, \max\{X_2, X_3\}\}$ and $T_2 = \min\{\max\{X_2, X_3\}, X_4\}$, where the components $X_2$ and $X_3$ are shared. The corresponding BSM is

$$\mathcal{S} = \begin{pmatrix} 0 & \frac{1}{12} & \frac{1}{6} & 0\\ \frac{1}{12} & \frac{1}{6} & \frac{1}{6} & 0\\ \frac{1}{6} & \frac{1}{6} & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Theorem 2, one can verify that the BSM of equivalent systems, with shared compo-

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nents, of order 6 is

$$\mathcal{P} = \begin{pmatrix} 0.1667 & 0.0333 & 0.0500 & 0.0500 & 0.0333 & 0 \\ 0 & 0.2000 & 0.0500 & 0.0500 & 0.0333 & 0 \\ 0 & 0 & 0.1500 & 0.0500 & 0.0333 & 0 \\ 0 & 0 & 0 & 0.0667 & 0.0333 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

### 3.2. The components fail based on an SOS model

We now investigate the conditions under which two pairs of systems with shared components are equivalent in the case that the system components fail based on an SOS model. Kamps and Cramer [13] obtained a recurrence relation for the PDF of the $i$th SOS according to survival functions $\bar{F}^{\alpha_1}, \ldots, \bar{F}^{\alpha_{n+1}}$, i.e.

$$\beta_1 f_{i:n}^{*\tilde{\boldsymbol{\beta}}}(x) = \beta_{i+1} f_{i:n+1}^{*\boldsymbol{\beta}}(x) + (\beta_1 - \beta_{i+1}) f_{i+1:n+1}^{*\boldsymbol{\beta}}(x), \tag{10}$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{n+1})$, $\tilde{\boldsymbol{\beta}} = (\beta_2, \ldots, \beta_{n+1})$, $\beta_i = (n-i+2)\alpha_i$, and $\beta_i \neq \beta_j$, $i \neq j$ for all $i, j \in \{1, \ldots, n+1\}$. In the next lemma, we present a recurrence relation for the joint PDF of the SOS.

**Lemma 1.** *Let $f_{i,j:n+1}^{*\boldsymbol{\beta}}(x, y)$ denote the joint PDF of $(X_{i:n+1}^*, X_{j:n+1}^*)$ defined in (4) based on parameters $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{n+1})$. Assuming that $\beta_i \neq \beta_j$, $i \neq j$ and the $\beta_i$ are decreasing in $i$, we have*

$$\begin{aligned} \beta_1 f_{i,j:n}^{*\tilde{\boldsymbol{\beta}}}(x, y) &= \beta_{j+1} f_{i,j:n+1}^{*\boldsymbol{\beta}}(x, y) + (\beta_{i+1} - \beta_{j+1}) f_{i,j+1:n+1}^{*\boldsymbol{\beta}}(x, y) \\ &\quad + (\beta_1 - \beta_{i+1}) f_{i+1,\,j+1:n+1}^{*\boldsymbol{\beta}}(x, y), \qquad 1 \le i < j \le n. \end{aligned} \tag{11}$$

*Proof.* Let $U_{1:n+1}^*, \ldots, U_{n+1:n+1}^*$ be the SOS based on survival functions $\bar{H}^{\alpha_1}, \ldots, \bar{H}^{\alpha_{n+1}}$, where $\bar{H}(u) = 1 - u$, $0 < u < 1$. Assume that $h_{i,j:n+1}^{*\boldsymbol{\beta}}(u, v)$ denotes the joint PDF of $(U_{i:n+1}^*, U_{j:n+1}^*)$ in which $\beta_i = (n-i+2)\alpha_i$, $i = 1, \ldots, n+1$. From (4), we have

$$\begin{aligned} &\beta_{j+1} h_{i,j:n+1}^{*\boldsymbol{\beta}}(u, v) \\ &\quad = L_j \sum_{k=i+1}^{j+1} b_k^{(i)} (j+1)(\beta_{j+1} - \beta_k) \left(\frac{1-v}{1-u}\right)^{\beta_k} \sum_{k=1}^{i} b_k(i) \frac{(1-u)^{\beta_k - 1}}{(1-v)} \\ &\quad = (\beta_{j+1} - \beta_{i+1}) h_{i,j+1:n+1}^{*\boldsymbol{\beta}}(u, v) \\ &\qquad + L_j \sum_{k=i+1}^{j+1} b_k^{(i)} (j+1)(\beta_{i+1} - \beta_k) \left(\frac{1-v}{1-u}\right)^{\beta_k} \sum_{k=1}^{i} b_k(i) \frac{(1-u)^{\beta_k - 1}}{(1-v)} \\ &\quad = (\beta_{j+1} - \beta_{i+1}) h_{i,j+1:n+1}^{*\boldsymbol{\beta}}(u, v) \\ &\qquad + L_j \sum_{k=i+2}^{j+1} b_k^{(i+1)} (j+1) \left(\frac{1-v}{1-u}\right)^{\beta_k} \sum_{k=1}^{i+1} b_k(i+1)(\beta_{i+1} - \beta_k) \frac{(1-u)^{\beta_k - 1}}{(1-v)} \\ &\quad = (\beta_{j+1} - \beta_{i+1}) h_{i,j+1:n+1}^{*\boldsymbol{\beta}}(u, v) + (\beta_{i+1} - \beta_1) h_{i+1,\,j+1:n+1}^{*\boldsymbol{\beta}}(u, v) + \beta_1 h_{i,j:n}^{*\tilde{\boldsymbol{\beta}}}(u, v). \end{aligned}$$

Hence, the proof is complete. □

As a result of Lemma 1, we have the following theorem.

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**Theorem 3.** *Let $T_1^{(1)}$ and $T_2^{(1)}$ denote the lifetimes of two systems with shared components where we assume that the corresponding BSM is symmetric. Suppose that the two systems have n components that fail according to the SOS $X_{1:n}^*, \ldots, X_{n:n}^*$ based on survival functions $\bar{F}^{\alpha_2}, \ldots, \bar{F}^{\alpha_{n+1}}$. Assume that $T_1^{(2)}$ and $T_2^{(2)}$ denote the lifetimes of two other systems having shared components whose components fail according to the SOS $X_{1:n+1}^*, \ldots, X_{n+1:n+1}^*$ based on survival functions $\bar{F}^{\alpha_1}, \ldots, \bar{F}^{\alpha_{n+1}}$. Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{n+1})$ in which $\beta_i = (n-i+2)\alpha_i$, the $\beta_i$ are decreasing, and $\beta_i \neq \beta_j$, $i \neq j$. If $\bar{F}_{i,j:n+1}^{*\boldsymbol{\beta}}(t_1, t_2)$ denotes the joint reliability function of $(X_{i:n+1}^*, X_{j:n+1}^*)$ then*

$$(T_1^{(1)}, T_2^{(1)}) \overset{\mathrm{D}}{=} (T_1^{(2)}, T_2^{(2)}),$$

*where $(T_1^{(2)}, T_2^{(2)})$ has the joint reliability function*

$$\mathbb{P}(T_1^{(2)} > t_1, T_2^{(2)} > t_2) = \sum_{k=1}^{n+1} \sum_{l=k}^{n+1} a_{k,l} \bar{F}_{k,l:n+1}^{*\boldsymbol{\beta}}(\min\{t_1, t_2\}, \max\{t_1, t_2\})$$

*in which $a_{k,l}$ is the $(k,l)$th element of the matrix*

$$\begin{aligned}\mathcal{A} = &\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=0}^{1} \sum_{l=k}^{1} s_{i,j} \eta_{j+1}^{(1-k)(1-l)} (\eta_{i+1} - \eta_{j+1})^{(1-k)l} (1-\eta_{i+1})^{kl} \mathcal{C}_{k+i,\, l+j:n+1} \\ &+ \sum_{j=1}^{n} \sum_{i=j}^{n} \sum_{k=0}^{1} s_{i,j} \eta_{j+1}^{(1-k)} (1-\eta_{j+1})^{k} \mathcal{C}_{k+j,\, k+j:n+1}, \end{aligned} \tag{12}$$

*where $\eta_i = \beta_i / \beta_1$.*

*Proof.* We prove the $t_1 \leq t_2$ case. From (3), for $t_1 \leq t_2$ we can write

$$\mathbb{P}(T_1^{(1)} > t_1, T_2^{(1)} > t_2) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} s_{i,j} \bar{F}_{i,j:n}^{*\tilde{\boldsymbol{\beta}}}(t_1, t_2) + \sum_{i=1}^{n} \sum_{j=1}^{i} s_{i,j} \bar{F}_{j:n}^{*\tilde{\boldsymbol{\beta}}}(t_2),$$

where $\bar{F}_{i,j:n}^{*\tilde{\boldsymbol{\beta}}}(t_1, t_2)$ is the joint reliability function of $(X_{i:n}^*, X_{j:n}^*)$ and $\bar{F}_{j:n}^{*\tilde{\boldsymbol{\beta}}}(t_2)$ is the reliability function of $X_{j:n}^*$ based on parameters $\tilde{\boldsymbol{\beta}} = (\beta_2, \ldots, \beta_{n+1})$. From (11), we have

$$\begin{aligned}&\sum_{i=1}^{n} \sum_{j=i+1}^{n} s_{i,j} \bar{F}_{i,j:\,n}^{*\tilde{\boldsymbol{\beta}}}(t_1, t_2) \\ &\quad = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=0}^{1} \sum_{l=k}^{1} s_{i,j} \eta_{j+1}^{(1-k)(1-l)} (\eta_{i+1} - \eta_{j+1})^{(1-k)l} (1-\eta_{i+1})^{kl} \bar{F}_{i+k,\, j+1:\, n+1}^{*\boldsymbol{\beta}}(t_1, t_2) \\ &\quad = \sum_{k=1}^{n+1} \sum_{l=k+1}^{n+1} \sum_{i=\max\{k-1,1\}}^{k} \sum_{j=\max\{i+1,l-1\}}^{l-k+i} s_{i,j} \eta_{j+1}^{(1-k+i)(1-l+j)} (\eta_{i+1} - \eta_{j+1})^{(1-k+i)(l-j)} \\ &\qquad\qquad \times (1-\eta_{i+1})^{(k-i)(l-j)} \bar{F}_{k,l:\, n+1}^{*\boldsymbol{\beta}}(t_1, t_2), \end{aligned}$$

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and using (10), we can write

$$\sum_{i=1}^{n}\sum_{j=1}^{i} s_{i,j}\bar{F}^{*\tilde{\boldsymbol{\beta}}}_{j:n}(t_2) = \sum_{j=1}^{n}\sum_{i=j}^{n}\sum_{k=0}^{1} s_{i,j}\eta_{j+1}^{(1-k)}(1-\eta_{j+1})^k \bar{F}^{*\boldsymbol{\beta}}_{j+k:n+1}(t_2)$$
$$= \sum_{k=1}^{n+1}\sum_{j=\max\{k-1,1\}}^{k}\sum_{i=j}^{n} s_{i,j}\eta_{j+1}^{(1-k+j)}(1-\eta_{j+1})^{k-j}\bar{F}^{*\boldsymbol{\beta}}_{k:n+1}(t_2).$$

Therefore, the joint reliability function of $(T_1^{(1)}, T_2^{(1)})$ is

$$\mathbb{P}(T_1^{(1)} > t_1, T_2^{(1)} > t_2) = \sum_{k=1}^{n+1}\sum_{l=k}^{n+1} a_{k,l}\bar{F}^{*\boldsymbol{\beta}}_{k,l:n+1}(t_1,t_2), \qquad t_1 \le t_2, \tag{13}$$

where $a_{k,l}$ is the $(k,l)$th element of

$$\mathcal{A} = \sum_{k=1}^{n+1}\sum_{l=k+1}^{n+1}\sum_{i=\max\{k-1,1\}}^{k}\sum_{j=\max\{i+1,l-1\}}^{l-k+i} s_{i,j}\eta_{j+1}^{(1-k+i)(1-l+j)}(\eta_{i+1}-\eta_{j+1})^{(1-k+i)(l-j)}$$
$$\times (1-\eta_{i+1})^{(k-i)(l-j)}\mathcal{C}_{k,l:n+1}$$
$$+\sum_{k=1}^{n+1}\sum_{j=\max\{k-1,1\}}^{k}\sum_{i=j}^{n} s_{i,j}\eta_{j+1}^{(1-k+j)}(1-\eta_{j+1})^{k-j}\mathcal{C}_{k,k:n+1}$$
$$=\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=0}^{1}\sum_{l=k}^{1} s_{i,j}\eta_{j+1}^{(1-k)(1-l)}(\eta_{i+1}-\eta_{j+1})^{(1-k)l}(1-\eta_{i+1})^{kl}\mathcal{C}_{k+i,l+j:n+1}$$
$$+\sum_{j=1}^{n}\sum_{i=j}^{n}\sum_{k=0}^{1} s_{i,j}\eta_{j+1}^{(1-k)}(1-\eta_{j+1})^k\mathcal{C}_{k+j,k+j:n+1}.$$

Thus, the proof for the $t_1 \le t_2$ case is complete. Similarly, using the assumption that the BSM of two systems is symmetric, we see that

$$\mathbb{P}(T_1^{(1)} > t_1, T_2^{(1)} > t_2) = \mathbb{P}(T_1^{(1)} > t_2, T_2^{(1)} > t_1)$$
$$= \sum_{k=1}^{n+1}\sum_{l=k}^{n+1} a_{k,l}\bar{F}^{*\boldsymbol{\beta}}_{k,l:n+1}(t_2,t_1), \qquad t_1 \ge t_2. \tag{14}$$

Therefore, from (13) and (14), we finally conclude that for every $t_1, t_2 > 0$,

$$\mathbb{P}(T_1^{(1)} > t_1, T_2^{(1)} > t_2) = \sum_{k=1}^{n+1}\sum_{l=k}^{n+1} a_{k,l}\bar{F}^{*\boldsymbol{\beta}}_{k,l:n+1}(\min\{t_1,t_2\},\max\{t_1,t_2\})$$
$$= \mathbb{P}(T_1^{(2)} > t_1, T_2^{(2)} > t_2). \qquad \square$$

**Example 6.** Consider again the systems presented in Example 5. Let the components fail according to the SOS model based on $\bar{F}^{\alpha_2}, \ldots, \bar{F}^{\alpha_5}$, where $\alpha_1 = 2$, $\alpha_2 = 2.3$, $\alpha_3 = 2.5$,

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$\alpha_4 = 3$, and $\alpha_5 = 4$. We see that $\beta_1 = 10$, $\beta_2 = 9.2$, $\beta_3 = 7.5$, $\beta_4 = 6$, and $\beta_5 = 4$. Using Theorem 3, for the equivalent systems with shared components of order 5, we have

$$\mathcal{A} = \begin{pmatrix} 0.2300 & 0.0625 & 0.1142 & 0.0533 & 0 \\ 0 & 0.2700 & 0.1067 & 0.0383 & 0 \\ 0 & 0 & 0.0833 & 0.0417 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

**Remark 1.** Note that, although the elements of matrix $\mathcal{A}$ in (12) depend on parameters $\alpha_i$, $i = 1, \ldots, n+1$, these elements do not depend on the baseline distribution function $F$.

**Theorem 4.** *Consider two systems with respective lifetimes $T_1^{(1)}$ and $T_2^{(1)}$ having some shared components and let $\mathcal{S}^{(1)}$ be the corresponding BSM. Assume that the components of the two systems fail according to the SOS $X^*_{1:n+1}, \ldots, X^*_{n+1:n+1}$ based on survival functions $\bar{F}^{\alpha_1}, \ldots, \bar{F}^{\alpha_{n+1}}$. Also, consider two other systems with respective lifetimes $T_1^{(2)}$ and $T_2^{(2)}$ having shared components. Let $\mathcal{S}^{(2)}$ be the corresponding symmetric BSM. Assume that the components of these two systems fail according to the SOS $X^*_{1:n}, \ldots, X^*_{n:n}$ based on survival functions $\bar{F}^{\alpha_2}, \ldots, \bar{F}^{\alpha_{n+1}}$. With $\mathcal{A}^{(2)}$ defined in (12) corresponding to $\mathcal{S}^{(2)}$, if $\mathcal{S}^{(1)}(\leq_{\rm st} [\geq_{\rm st}])\mathcal{A}^{(2)}$ then $(T_1^{(1)}, T_2^{(1)})(\leq_{\rm st} [\geq_{\rm st}])(T_1^{(2)}, T_2^{(2)})$.*

*Proof.* For every increasing function $\varphi(t_1, t_2)$, we have

$$\mathbb{E}(\varphi(T_1^{(1)}, T_2^{(1)})) = \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} s_{i,j}^{(1)} \mathbb{E}(\varphi(X^*_{i:n+1}, X^*_{j:n+1}))$$

and from Theorem 3, we have

$$\mathbb{E}(\varphi(T_1^{(2)}, T_2^{(2)})) = \sum_{i=1}^{n+1}\sum_{j=i}^{n+1} a_{i,j}^{(2)} \mathbb{E}(\varphi(X^*_{i:n+1}, X^*_{j:n+1})).$$

Thus, the required result follows from the assumption that $\mathcal{S}^{(1)}(\leq_{\rm st} [\geq_{\rm st}])\mathcal{A}^{(2)}$. □

## 4. Conclusions

In this paper we studied the reliability properties of two coherent systems with shared components in the case where the components of the systems fail according to the SOS model. This is a general model which includes the ordinary model of component failures based on i.i.d. random lifetimes. The model also includes, under some mild conditions, the model of component failures according to an NHPB process and, in particular, the model of failures based on an NHPP. We considered two systems with $n_1$ and $n_2$ components, where $n_{1,2}$ of them are shared between the two systems, $n = n_1 + n_2 - n_{1,2}$. We obtained the joint reliability function of the system lifetimes in terms of the BSM and the joint reliability function of the vector $(X^*_{i:n}, X^*_{j:n})$, where $X^*_{i:n}$ and $X^*_{j:n}$ are the $i$th and the $j$th SOS associated to the component lifetimes. We provided a detailed example concerning the reliability properties of the lifetime states of a three-state network (as a special case of systems with shared components) for which the components fail under an NHPP. We explored the dependency between the lifetimes of two systems based on the dependency between the elements of the BSM associated to the systems. We showed that when the elements of the BSM are PQD (associated), the lifetimes of the systems are also PQD (associated). We also obtained some stochastic ordering results

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in order to compare two pairs of systems with shared components. We defined the concept of equivalent systems with shared components (as an extension of the concept defined in [18]) and gave conditions under which the joint reliability of two systems of order $m$ can be expressed as the joint reliability of two systems with shared components of order $n$ ($n > m$). We provided several examples throughout the paper to illustrate our results.

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