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Vanishing sums in function fields

BY W. D. BROWNAWELL

Department of Mathematics, Pennsylvania State University, State College, PA 16802, *U.S.A.*

AND D. W. MASSER

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, *U.S.A.*

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1. *Introduction.* Let k be a field of zero characteristic, and let F be a function field over k of genus g. We normalize each valuation v on F so that its order group consists of all rational integers, and for elements $u_1, \ldots, u_n$ of F, not all zero, we define the (projective) height as

$$H(u_1, \ldots, u_n) = -\sum_v \min(\operatorname{ord}_v u_1, \ldots, \operatorname{ord}_v u_n).$$

The sum formula on F shows that this is really a height on the projective space $\mathbb{P}_{n-1}(F)$.

We are interested in the homogeneous equation

$$u_1 + \ldots + u_n = 0 \tag{1}$$

for fixed $n \geqslant 3$, and we will say that elements $u_1, \ldots, u_n$ of F satisfying (1) give rise to a non-degenerate solution if there is no non-empty proper subset of $\{u_1, \ldots, u_n\}$ whose elements are linearly dependent over k. In this case R. C. Mason [4], generalizing his earlier work (see [3]), has given an estimate for $H(u_1, \ldots, u_n)$ in terms of g, n and the number of valuations at which $u_1, \ldots, u_n$ are not units. The purpose of this note is to sharpen Mason's inequality and give some variants. We also briefly discuss some extremal examples.

For brevity, write

$$\gamma_0 = 0, \quad \gamma_p = \tfrac{1}{2}(p-1)(p-2) \quad (p \geqslant 1).$$

Our main result is as follows.

THEOREM A. *Suppose $u_1, \ldots, u_n$ give rise to a non-degenerate solution of* (1). *For each valuation v, let $m = m(v) = m(v; u_1, \ldots, u_n)$ be the number of elements among $u_1, \ldots, u_n$ which are units at v. Then*

$$H(u_1, \ldots, u_n) \leqslant \gamma_n(2g-2) + \sum_v (\gamma_n - \gamma_m).$$

From this we deduce two corollaries. Recall that if S is a finite set of valuations, an element u of F is said to be an S-unit if it is a unit at v for all v not in S. We denote the cardinality of S by $|S|$.

COROLLARY I. *Suppose $u_1, \ldots, u_n$ give rise to a non-degenerate solution of* (1) *and that $u_1, \ldots, u_n$ are S-units for some finite set S. Then*

$$H(u_1, \ldots, u_n) \leqslant \tfrac{1}{2}(n-1)(n-2)\{|S| + 2g - 2\}.$$

In Lemma 2 of [4] this result is proved essentially with $\frac{1}{2}(n-1)(n-2)$ replaced by

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4^{n-2} (note that Mason writes (1) in non-homogeneous form, so his n is different from ours).

COROLLARY II. *Suppose $u_1, \ldots, u_n$ give rise to a non-degenerate solution of* (1) *and that u_i is an S_i-unit* $(1 \leqslant i \leqslant n)$ *for some finite sets $S_1, \ldots, S_n$. Then*

$$H(u_1, \ldots, u_n) \leqslant (n-2)(|S_1| + \ldots + |S_n|) + \tfrac{1}{2}(n-1)(n-2)(2g-2).$$

The case $g = 0$ of this result may be deduced from the 'Nevanlinna theory with bounded multiplicities' developed by Cartan [1]; see also the recent article by Hayman [2], where applications to various analogues of Waring's Problem are discussed. In fact the proof of our Theorem A, with its use of Wronskians, was suggested directly by standard arguments in Nevanlinna theory.

2. *Preliminaries.* For each valuation v on F there is a local parameter $\zeta = \zeta_v$ in F, with $\operatorname{ord}_v \zeta = 1$, such that every element of F can be expressed as a Laurent series in ζ with only finitely many negative exponents. Let $d/d\zeta$ denote the corresponding derivation on F.

LEMMA 1. *For any non-zero u in F, any valuation v, and any non-negative integer j we have*

$$\operatorname{ord}_v (u^{-1}(d/d\zeta)^j u) \geqslant -j$$

if u is not a unit at v, and

$$\operatorname{ord}_v (u^{-1}(d/d\zeta)^j u) \geqslant 0$$

if u is a unit at v.

Proof. In the first case we have an expansion $u = a\zeta^m + \ldots$ with $a \neq 0$, $m \neq 0$; thus

$$u^{-1}(d/d\zeta)^j u = m(m-1)\ldots(m-j+1)\zeta^{-j} + \ldots.$$

In the second case $m = 0$, and the calculation is even easier. This proves the lemma.

Next we fix a non-constant element z of F, and we write d/dz for the corresponding derivation. For $n \geqslant 2$ and non-zero elements $f_1, \ldots, f_{n-1}$ of F we define the logarithmic Wronskian $\lambda = \lambda(f_1, \ldots, f_{n-1})$ (with respect to z) as the determinant of order $n-1$ with $f_i^{-1}(d/dz)^{j-1} f_i$ in the ith row and jth column $(1 \leqslant i, j \leqslant n-1)$. It is the ordinary Wronskian divided by the product $f_1 \ldots f_{n-1}$, and it vanishes if and only if $f_1, \ldots, f_{n-1}$ are linearly dependent over k.

LEMMA 2. *For $n \geqslant 2$ let $f_1, \ldots, f_{n-1}$ be elements of F linearly independent over k, and for a valuation v and an integer μ with $1 \leqslant \mu \leqslant n$ suppose that exactly $\mu - 1$ among $f_1, \ldots, f_{n-1}$ are units at v. Then for $\lambda = \lambda(f_1, \ldots, f_{n-1})$ we have*

$$\operatorname{ord}_v \lambda \geqslant -\gamma_n \operatorname{ord}_v (dz/d\zeta) - (\gamma_n - \gamma_\mu).$$

Proof. By the formula for change of variable in a Wronskian (see for example [5], § 710, p. 662) we have

$$\lambda = \lambda^* (d\zeta/dz)^{\gamma_n}, \tag{2}$$

where λ^* is the logarithmic Wronskian with respect to the local variable ζ. To estimate $\operatorname{ord}_v \lambda^*$ we apply Lemma 1 to each of the columns, noting that exactly $n - \mu$ of the $f_1, \ldots, f_{n-1}$ are not units at v. We get

$$\operatorname{ord}_v \lambda^* \geqslant -\{(n-2) + (n-3) + \ldots + (\mu-1)\} = -(\gamma_n - \gamma_\mu).$$

The lemma follows from this together with (2).

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3. *Proof of Theorem A.* Let $u_1, \ldots, u_n$ be elements of F satisfying the hypotheses of the theorem. Fix any z as above.

LEMMA 3. *For $1 \leqslant i \leqslant n$ let $(-1)^{i+1}\lambda_i$ be the logarithmic Wronskian with respect to z of the functions $u_1, \ldots, u_n$ omitting u_i. Then $\lambda_1, \ldots, \lambda_n$ are all non-zero, and we have*

$$H(u_1, \ldots, u_n) = H(\lambda_1, \ldots, \lambda_n).$$

Proof. Clearly $\lambda_1, \ldots, \lambda_n$ are non-zero because of our assumption of non-degeneracy. For $1 \leqslant j \leqslant n-1$ we write down the $(j-1)$th derivative of (1) in the form

$$(u_1^{-1}(d/dz)^{j-1}u_1)u_1 + \ldots + (u_n^{-1}(d/dz)^{j-1}u_n)u_n = 0,$$

and we regard these as a system of linear equations in $u_1, \ldots, u_n$. Then Cramer's Rule shows that the points in $\mathbb{P}_{n-1}(F)$ with coordinates $u_1, \ldots, u_n$ and $\lambda_1, \ldots, \lambda_n$ are the same, and the lemma follows at once.

Now we have by definition

$$H(\lambda_1, \ldots, \lambda_n) = -\sum_v \min(\operatorname{ord}_v \lambda_1, \ldots, \operatorname{ord}_v \lambda_n).$$

Fix i with $1 \leqslant i \leqslant n$ and any valuation v. Since $m = m(v)$ of the elements $u_1, \ldots, u_n$ are units at v, it follows that at least $m-1$ of the elements $u_1, \ldots, u_n$ omitting u_i are units at v. Thus in Lemma 2 we have $\mu \geqslant m$, so $\gamma_\mu \geqslant \gamma_m$ and we conclude

$$\operatorname{ord}_v \lambda_i \geqslant -\gamma_n \operatorname{ord}_v(dz/d\zeta) - (\gamma_n - \gamma_m).$$

Therefore

$$H(\lambda_1, \ldots, \lambda_n) \leqslant \gamma_n \sum_v \operatorname{ord}_v(dz/d\zeta) + \sum_v (\gamma_n - \gamma_m).$$

But we have (see for example [4], equation (6))

$$\sum_v \operatorname{ord}_v(dz/d\zeta) = 2g-2,$$

and we end up with

$$H(\lambda_1, \ldots, \lambda_n) \leqslant \gamma_n(2g-2) + \sum_v (\gamma_n - \gamma_m).$$

The theorem follows from this and Lemma 3.

4. *Proof of Corollaries.* Suppose first that $u_1, \ldots, u_n$ are all S-units. Then $m(v) = n$ for all v not in S, and the inequality of Corollary I follows at once.

Next suppose u_i is an S_i-unit ($1 \leqslant i \leqslant n$). For each r with $1 \leqslant r \leqslant n$ let $S^{(r)}$ be the set of valuations occurring in exactly r of $S_1, \ldots, S_n$. Thus

$$|S_1| + \ldots + |S_n| = |S^{(1)}| + 2|S^{(2)}| + \ldots + n|S^{(n)}|. \tag{3}$$

For v not in any of $S^{(1)}, \ldots, S^{(n)}$ we have $m(v) = n$. And if v is in $S^{(r)}$ then clearly $m(v) \geqslant n-r$. Thus

$$\sum_v (\gamma_n - \gamma_m) \leqslant \sum_{r=1}^{n} (\gamma_n - \gamma_{n-r})|S^{(r)}|.$$

But

$$\gamma_n - \gamma_{n-r} = \tfrac{1}{2}r(2n-r-3) \leqslant r(n-2),$$

so that

$$\sum_v (\gamma_n - \gamma_m) \leqslant (n-2)\sum_{r=1}^{n} r|S^{(r)}|,$$

and now Corollary II follows from this together with (3) and Theorem A.

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5. *Extremal examples.* We start with the case $n = 3$, so that Corollary I reads

$$H = H(u_1, u_2, u_3) \leqslant |S| + 2g - 2.$$

Another proof of this was provided by Silverman [6], who also showed that in the case $g = 0$ equality is possible for every value of $|S|$. He asked whether similar examples exist for other values of g.

We can at least prove that for each $g \geqslant 0$ there is a function field F of genus g such that equality holds for infinitely many values of $|S|$. Define F by the hyperelliptic equation

$$y^2 = x^{2g+1} + 1.$$

Then for any integer $e \geqslant 1$ we have

$$y^{2e} = x^{(2g+1)e} + u_3,$$

where $$u_3 = \sum_{r=1}^{e} \binom{e}{r} x^{(2g+1)(e-r)}.$$

So we take $$u_1 = -y^{2e}, \quad u_2 = x^{(2g+1)e}.$$

We find that $$H = 2e(2g+1)$$

and $$|S| \leqslant 1 + (2g+1) + 2 + 2(e-1)(2g+1) = H - (2g-2). \tag{4}$$

Hence in fact Corollary I shows that equality holds in (4), which provides an extremal example with $|S|$ arbitrarily large (as $e \to \infty$).

This implies in particular that for any fixed $\vartheta < 1$ and $c > 0$ the inequality

$$H \leqslant \vartheta |S| + cg$$

cannot always hold. We may also show that the inequality

$$H \leqslant c|S| + 2\vartheta g \tag{5}$$

cannot always hold. For, defining F by the Fermat equation

$$x^m + y^m = 1$$

with $g = \frac{1}{2}(m-1)(m-2)$, and taking

$$u_1 = x^m, \quad u_2 = y^m, \quad u_3 = -1,$$

we find that $$H = m^2,$$

whereas $$|S| = m + m + m = 3m = H - (2g-2).$$

So this is another extremal example, and it contradicts (5) for large enough m.

Thus Corollary I may reasonably be regarded as best possible for $n = 3$. If $n > 3$ the situation is less clear. If we write any extremal example for $n = 3$ in the form $v_1 + v_2 = v_3$ then

$$v_1^{n-2} + \binom{n-2}{1} v_1^{n-3} v_2 + \ldots + \binom{n-2}{n-3} v_1 v_2^{n-3} + v_2^{n-2} = v_3^{n-2}$$

gives a solution of

$$u_1 + \ldots + u_{n-1} = -u_n,$$

which is easily seen to be non-degenerate with

$$H = (n-2)\{|S| + 2g - 2\}.$$

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Thus for each fixed $n \geqslant 3$ Corollary I would be false with $\frac{1}{2}(n-1)(n-2)$ replaced by anything less than $n-2$.

Finally, the specific example deriving from

$$v_1 = 1, \quad v_2 = x^e, \quad v_3 = 1 + x^e$$

in the case $F = k(x)$ has

$$H = e(n-2)$$

and

$$|S_1| = 0, \quad |S_2| = \ldots = |S_{n-1}| = 2, \quad |S_n| = e+1.$$

Hence, although this is not quite extremal for Corollary II, we see on making $e \to \infty$ that for each fixed $n \geqslant 3$ this corollary would be false with $n-2$ replaced by anything smaller.

6. *Weakening the hypothesis.* We next show that the inequality of Theorem A remains true, in slightly modified form, if the assumption of linear independence is replaced by a weaker hypothesis of no vanishing subsums. Write $N = \{1, 2, \ldots, n\}$ and

$$G = \max(0, 2g-2).$$

THEOREM B. *Suppose $u_1, \ldots, u_n$ satisfy* (1) *but $\sum_{i \in I} u_i \neq 0$ for every non-empty proper subset I of N. For each valuation v let $m = m(v) = m(v; u_1, \ldots, u_n)$ be the number of elements among $u_1, \ldots, u_n$ which are units at v. Then*

$$H(u_1, \ldots, u_n) \leqslant \gamma_n G + \sum_v (\gamma_n - \gamma_m).$$

Thus the inequalities of Corollaries I and II also hold in the same modified form. However, Theorem B would actually become false if G were replaced by $2g-2$; for $g = 0$ we obtain a counterexample by taking $u_1, \ldots, u_n$ as suitable constants of k.

Before we start on the proof we need some arguments from linear algebra. For the moment let $u_1, \ldots, u_n$ be elements of a vector space over k, and let $\mathscr{L}$ be the set of linear forms $L = \sum_{i=1}^n c_i x_i$ over k vanishing at $x_1 = u_1, \ldots, x_n = u_n$. Suppose I is a non-empty subset of N such that the elements u_i $(i \in I)$ are linearly dependent but for every non-empty proper subset I' of I the elements u_i $(i \in I')$ are linearly independent. Then we say that I is minimal. In this case we can find a non-zero form $L_I = \sum_{i \in I} c_i x_i$ in $\mathscr{L}$; it is unique up to non-zero multiples in k, and we shall refer to it as a minimal form.

LEMMA 4. *The minimal forms generate $\mathscr{L}$.*

Proof. We prove that every element L of $\mathscr{L}$ can be written as a linear combination of minimal forms. This will be by induction on the length l of L, i.e. the number of non-zero coefficients, the case $l = 1$ being trivial. So assume for some $l > 1$ that this holds for all elements of $\mathscr{L}$ of length strictly less than l. If L in $\mathscr{L}$ has length exactly l, we can suppose

$$L = c_1 x_1 + \ldots + c_l x_l.$$

If $I = \{1, 2, \ldots, l\}$ is minimal we are done, since then L is a multiple of L_I. Otherwise we can assume that

$$L' = c_1' x_1 + \ldots + c_k' x_k,$$

lies in $\mathscr{L}$ for some k with $1 \leqslant k < l$ and $c_1' \neq 0$. Then L' and $L'' = c_1' L - c_1 L'$ are both of length strictly less than l, and so the induction hypothesis can be applied to

$$L = (c_1/c_1') L' + (1/c_1') L''$$

giving the required decomposition. This proves the lemma.

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LEMMA 5. *Suppose* $\sum_{i=1}^{n} u_i = 0$ *but* $\sum_{i\in I} u_i \neq 0$ *for some set* $I \subseteq N$. *Then there is a minimal set meeting non-trivially both* I *and its complement* $\mathscr{C}(I)$ *in* N.

Proof. The form $L = \sum_{i=1}^{n} x_i$ is in $\mathscr{L}$, and so by Lemma 4 we have

$$L = \sum c_J L_J \tag{6}$$

for certain minimal J. If the present lemma is false, then every such J is contained either in I or $\mathscr{C}(I)$. Thus on specializing the variables x_i $(i \in \mathscr{C}(I))$, if any, to zero in (6) we obtain

$$\sum_{i\in I} x_i = \sum_{J\subseteq I} c_J L_J.$$

Hence $\sum_{i\in I} u_i = 0$, and this contradiction proves the lemma.

LEMMA 6. *Assume* $\sum_{i=1}^{n} u_i = 0$ *but that no non-empty proper subsum vanishes. Suppose however that the elements of some non-empty proper subset are linearly dependent. Then we can find an integer* $k \geqslant 2$, *a partition*

$$N = I_1 \cup \ldots \cup I_k$$

into non-empty disjoint sets $I_1, \ldots, I_k$, *and non-empty sets*

$$J_1 \subseteq I_1, J_2 \subseteq I_1 \cup I_2, \ldots, J_{k-1} \subseteq I_1 \cup \ldots \cup I_{k-1}$$

such that $I_1, I_2 \cup J_1, \ldots, I_k \cup J_{k-1}$
are minimal.

Proof. Pick any minimal I_1. By hypothesis N is not minimal, so $I_1 \neq N$. Hence $\sum_{i\in I_1} u_i \neq 0$, so by Lemma 5 we can find a minimal set I_2' meeting non-trivially both I_1 and $\mathscr{C}(I_1)$. Put $I_2 = I_2' \cap \mathscr{C}(I_1)$ and $J_1 = I_2' \cap I_1$. If $N = I_1 \cup I_2$ then it is easily verified that the lemma holds with $k = 2$. More generally suppose that for some $i \geqslant 2$ we have found non-empty disjoint $I_1, \ldots, I_i$ and non-empty

$$J_1 \subseteq I_1, \ldots, J_{i-1} \subseteq I_1 \cup \ldots \cup I_{i-1}$$

such that $I_1, \ldots, I_i \cup J_{i-1}$

are minimal. We call this situation S_i. If now $I = I_1 \cup \ldots \cup I_i$ is N then the lemma holds with $k = i$; else we proceed inductively to situation S_{i+1} as follows. By Lemma 5 there is a minimal I_{i+1}' meeting non-trivially both I and $\mathscr{C}(I)$, and we put

$$I_{i+1} = I_{i+1}' \cap \mathscr{C}(I), \quad J_i = I_{i+1}' \cap I.$$

This gives situation S_{i+1}. Since this inductive procedure must eventually terminate, the lemma is proved.

7. *Proof of Theorem B.* We give this in slightly abbreviated form. By Theorem A, we can suppose that $u_1, \ldots, u_n$ satisfy the hypotheses of Lemma 6. Denote by $n_1, \ldots, n_k$ the cardinalities of the corresponding sets $I_1, \ldots, I_k$; clearly $n_1 \geqslant 2$. Fix any z as in section 2. We write down the relation corresponding to I_1 and all its derivatives up to order $n_1 - 2$. Then we write down the relation corresponding to $I_2 \cup J_1$ and its derivatives up to order $n_2 - 1$; and so on, until we end up with the relation corresponding to $I_k \cup J_{k-1}$ and its derivatives up to order $n_k - 1$. As in the proof of Lemma 3, we regard the resulting formulae as $(n_1 - 1) + n_2 + \ldots + n_k = n - 1$ linear equations in $u_1, \ldots, u_n$.

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Let $\lambda_1, \ldots, \lambda_n$ be the determinants obtained by omitting each column in turn, with signs adjusted so that

$$u_i \lambda_j = u_j \lambda_i \quad (1 \leqslant i,j \leqslant n) \tag{7}$$

follows from Cramer's Rule.

Our first claim is that

$$\lambda_i \neq 0 \quad (1 \leqslant i \leqslant n). \tag{8}$$

For a non-empty set $I \subseteq N$ let $\lambda(I)$ denote the logarithmic Wronskian of the functions u_i $(i \in I)$ with respect to z. If i is in I_1, then the block nature of the matrix leads easily to

$$\lambda_i = c_i \lambda(I_1') \lambda(I_2) \ldots \lambda(I_k), \tag{9}$$

where $c_i \neq 0$ is in k and I_1' is the set I_1 with i removed. All the Wronskians on the right of (9) are non-zero by the minimality of the sets $I_1, I_2 \cup J_1, \ldots, I_k \cup J_{k-1}$. Hence at least (8) holds for i in I_1. But then it follows for all i using (7), since $u_1, \ldots, u_n$ are non-zero.

As in Lemma 3, we can now deduce that

$$H(u_1, \ldots, u_n) = H(\lambda_1, \ldots, \lambda_n).$$

We proceed to estimate the right-hand side at each valuation v. If $\lambda_1^*, \ldots, \lambda_n^*$ denote the determinants obtained by differentiating instead with respect to the corresponding local variable ζ, then we also have

$$u_i \lambda_j^* = u_j \lambda_i^* \quad (1 \leqslant i,j \leqslant n), \tag{10}$$

as in (7). Now for i in I_1 the formula (2) applied to (9) gives

$$\lambda_i = \lambda_i^* (d\zeta/dz)^\gamma, \tag{11}$$

with

$$\gamma = \gamma_{n_1} + \gamma_{n_2+1} + \ldots + \gamma_{n_k+1}. \tag{12}$$

So, comparing (7) and (10), we see that in fact (11) holds for all i.

Next, the same calculation as in the proofs of Lemmas 1 and 2 shows that

$$\operatorname{ord}_v \lambda_i^* \geqslant -(\gamma_n - \gamma_m) \quad (1 \leqslant i \leqslant n).$$

For any element in the rth row of λ_i^* has the form $cu^{-1}(d/d\zeta)^{j-1}u$ for some $j \leqslant r$ and c in k, and at least $m-1$ of the corresponding u's are units at v. Combining this with (11) and using the observation

$$\sum_v \operatorname{ord}_v (dz/d\zeta) = 2g - 2 \leqslant G$$

as before, we conclude that

$$H(u_1, \ldots, u_n) \leqslant \gamma G + \sum_v (\gamma_n - \gamma_m).$$

Finally, it is easily checked that

$$\gamma_a + \gamma_{b+1} \leqslant \gamma_{a+b}$$

for all $a \geqslant 1$, $b \geqslant 1$, and repeated application to (12) gives $\gamma \leqslant \gamma_n$. Since $G \geqslant 0$, this yields the theorem.

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Added in proof. Since writing this note, we have learnt that J. F. Voloch had already considered similar questions in *Bol. Soc. Brasil. Mat.* **16** (1985), 29–39. In particular it is easy to see that the proof of his Theorem 4, especially equation (17), yields Corollary I to our Theorem A.

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