Math. Proc. Camb. Phil. Soc. (1977), **82**, 249
Printed in Great Britain

Exact sequences of conjugacy classes and rationalization

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(*Received 5 April* 1977)

1. *Introduction.* The purpose of this paper is to exhibit two exact sequences involving the sets of conjugacy classes in various groups. One of them is applied to prove the following result.

THEOREM. *The rationalization map of a finitely generated nilpotent group induces a finite-to-one map of conjugacy classes.*

The principal reference for this paper is Hilton, Mislin and Roitberg(1), where in particular the rationalization of a nilpotent group is described.

The work in this paper is a piece of pure algebra, but it is motivated by topological questions, which I shall now describe. The reader can ignore them if he wishes.

Let X be a connected nilpotent CW complex and let W be a finite connected complex. In (1) the authors prove various results about the sets of based homotopy classes of maps from W to X and its localizations. In (2) they ask if these results extend to the sets of free homotopy classes, and in connexion with this they ask if the theorem stated above is true. Now the theorems in (1) are mostly proved by obstruction theory using the skeleta of W. However, one can use the refined Postnikov systems of X and its localizations instead; see the appendix of (3) for an example. This has two advantages. The first is that one does not need W to be connected; the second is that one needs algebraic results only for abelian groups, not for nilpotent ones. Indeed one can obtain results for nilpotent groups as special cases of results for nilpotent spaces by considering Eilenberg–MacLane spaces.

Since W need no longer be connected one can answer the questions of (2) by saying that free homotopy classes behave in the same way as based ones. For free homotopy classes of maps from W may be equated with based homotopy classes of maps from W with a disjoint base point. This could be written out in detail, but it would be a lengthy duplication of (1). The present paper is intended to illustrate the other point: how these topological methods can be used to prove results in group theory.

2. *The exact sequences.* In this section we give the two exact sequences. They require only the most elementary group theory. We shall use the following notation. A star $*$ denotes a one-element set or group. If G is a group and $y \in G$, then $C_G(y)$ is the centralizer of y in G, $[G]$ is the set of conjugacy classes in G, and $[y] \in [G]$ is the conjugacy class containing y.

EXACT SEQUENCE 1. *Let*

$$* \to K \xrightarrow{\alpha} G \xrightarrow{\beta} H \to *$$

https://doi.org/10.1017/S0305004100053858 Published online by Cambridge University Press

be a short exact sequence of groups and let $x \in K$. Then there is a natural exact sequence of groups and pointed sets

$$* \to C_K(x) \xrightarrow{\alpha} C_G(\alpha x) \xrightarrow{\beta} H \xrightarrow{\delta_x} ([K],[x]) \xrightarrow{\alpha} ([G],[\alpha x]) \xrightarrow{\beta} ([H],[1]) \to *.$$

Explanation. The maps marked α and β are the obvious ones. The map marked δ_x is defined as follows. If $h \in H$, then $h = \beta g$ for some $g \in G$. Since $\beta(g(\alpha x)g^{-1}) = 1$, $g(\alpha x)g^{-1} = \alpha k$ for a unique $k \in K$. Set $\delta_x h = [k]$; one can verify that this does not depend on the choice of g. The assertion that the sequence is exact means the following.

(i) The portion $* \to C_K(x) \to C_G(\alpha x) \to H$ is an exact sequence of groups and group homomorphisms in the usual sense.

(ii) $\delta_x^{-1}[x] = \beta(C_G(\alpha x))$, $\alpha^{-1}[\alpha x] = \delta_x(H)$, $\beta^{-1}[1] = \alpha([K])$, $\beta\colon [G] \to [H]$ is surjective.

(iii) If $h_1, h_2 \in H$, then $\delta_x h_1 = \delta_x h_2$ if and only if $h_1^{-1}h_2 \in \beta(C_G(\alpha x))$.

(iv) There is a well-defined left action of H on $[K]$ given by

$$(\beta g).[k] = [\alpha^{-1}(g(\alpha k)g^{-1})] \quad \text{for} \quad k \in K,\ g \in G;$$

if $k_1, k_2 \in K$, then $\alpha[k_1] = \alpha[k_2]$ if and only if $[k_1]$ and $[k_2]$ lie in the same orbit under this action.

Perhaps the most striking consequence is that $\alpha^{-1}[\alpha x]$ has a natural group structure if $\beta(C_G(\alpha x))$ is normal in H (see (ii) and (iii)).

Proof. It is possible to verify all this directly. Alternatively one can use topological methods. For any space X, let $\Omega_+ X$ be the space of free maps of the circle into X. If A is a group, then let $K(A,1)$ be the corresponding Eilenberg–MacLane space and recall that $\pi_0 \Omega_+ K(A,1)$ is naturally equivalent to $[A]$. One can compute that for $a \in A$,

$$\pi_i(\Omega_+ K(A,1),[a]) = \begin{cases} C_A(a) & \text{for } i = 1, \\ * & \text{for } i \geqslant 2; \end{cases}$$

here $[a]$ is used by abuse of notation for a point in the corresponding path-component. Consider now the fibration $(\Omega_+ K(K,1),[x]) \to (\Omega_+ K(G,1),[\alpha x]) \to (\Omega_+ K(H,1),\ [1])$. Its sequence of homotopy groups and sets gives the required exact sequence, apart from the one-element set at the right-hand end; and one can add this because $\beta\colon [G] \to [H]$ is surjective.

EXACT SEQUENCE 2. *Let*

$$* \to K \xrightarrow{\alpha} G \xrightarrow{\beta} H \to *$$

be a central extension of groups (a short exact sequence with $\alpha(K)$ central in G) and let $y \in G$. Then there is a natural exact sequence of groups and pointed sets

$$* \to K \xrightarrow{\alpha} C_G(y) \xrightarrow{\beta} C_H(\beta y) \xrightarrow{\delta_y} K \xrightarrow{\alpha_y} ([G],[y]) \xrightarrow{\beta} ([H],[\beta y]) \to *.$$

Explanation. The maps marked α and β are the obvious ones; α_y is defined by $\alpha_y k = [(\alpha k)y]$ for $k \in K$; δ_y is well defined by $\delta_y(\beta g) = \alpha^{-1}(gyg^{-1}y^{-1})$ for $g \in G$ and $\beta g \in C_H(\beta y)$. The assertion that the sequence is exact means the following.

(i) The portion $* \to K \to C_G(y) \to C_H(\beta y) \to K$ is an exact sequence of groups and group homomorphisms in the usual sense.

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(ii) $\alpha_y^{-1}[y] = \delta_y(C_H(\beta y))$, $\beta^{-1}[\beta y] = \alpha_y(K)$, $\beta: [G] \to [H]$ is surjective.

(iii) For $k_1, k_2 \in K$, $\alpha_y k_1 = \alpha_y k_2$ if and only if $k_1^{-1} k_2 \in \delta_y(C_H(\beta y))$.

(iv) There is a well-defined left action of K on $[G]$ given by

$$k.[g] = [(\alpha k)\, g] \quad \text{for} \quad k \in K, \quad g \in G;$$

if $g_1, g_2 \in G$, then $\beta[g_1] = \beta[g_2]$ if and only if $[g_1]$ and $[g_2]$ lie in the same orbit under this action.

Here $\beta^{-1}[\beta y] = \alpha_y(K)$ always has an abelian group structure because K is abelian. This will be used in the next section.

Proof. Much as for Exact sequence 1. If one wishes to use topological methods rather than direct verification, then one should consider the fibration

$$(\Omega_+ K(G,1), [y]) \to (\Omega_+ K(H,1), [\beta y]) \to (\Omega_+ K(K,2), c);$$

here c is a member of the path-component containing the constant loops. One computes that

$$\pi_i(\Omega_+ K(K,2), c) = K \quad \text{for} \quad i = 1 \quad \text{or} \quad 2,$$
$$* \quad \text{for} \quad i = 0 \quad \text{or} \quad i \geqslant 3.$$

3. *Rationalizations of nilpotent groups.* In this section we prove the theorem stated in the introduction. We shall need the following properties of rationalization, taken from (1). Rationalization is an exact functor $G \mapsto G_0$ from nilpotent groups to nilpotent groups. It preserves lower central series. There is a natural homomorphism $e: G \to G_0$ called a rationalization map, or just rationalization. Its kernel is the torsion subgroup of G, so is finite if G is finitely generated. Finally, there is a 'rationalization five-lemma': if

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_3} & & \downarrow{\scriptstyle e_4} & & \downarrow{\scriptstyle e_5} \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

is a commutative diagram of nilpotent groups with exact rows and e_1, e_2, e_4 and e_5 are rationalization maps, then so also is e_3.

Now let G be a finitely generated nilpotent group with lower central series

$$G = \Gamma^1 G \supset \Gamma^2 G \supset \ldots \supset \Gamma^c G \supset \Gamma^{c+1} G = *.$$

Let $e: G \to G_0$ be rationalization and let $\beta: G \to G/\Gamma^c G$ be the quotient map. Before proving the theorem we give a lemma stating information derived from Exact sequence 2.

LEMMA. *Let $y \in G$. Then*

(i) $e|C_G(y): C_G(y) \to C_{G_0}(ey)$ *is a rationalization map;*

(ii) $\beta^{-1}[\beta y] \subset [G]$ *is a finitely generated abelian group,* $\beta_0^{-1}[\beta_0 ey] \subset [G_0]$ *is an abelian group, and* $e|\beta^{-1}[\beta y]: \beta^{-1}[\beta y] \to \beta_0^{-1}[\beta_0 ey]$ *is a rationalization map.*

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Proof. (i) We use induction on c. Write $K = \Gamma^c G$ and $H = G/\Gamma^c G$. Because rationalization preserves lower central series, we have a commutative diagram of groups

$$\begin{array}{ccccccccc} * & \to & K & \to & G & \xrightarrow{\beta} & H & \to & * \\ & & \downarrow & & \downarrow & & \downarrow & & \\ * & \to & K_0 & \to & G_0 & \xrightarrow{\beta_0} & H_0 & \to & *, \end{array}$$

whose rows are central extensions and whose columns are rationalizations. Exact sequence 2 therefore gives a commutative diagram of groups

$$\begin{array}{ccccccccc} * & \to & K & \to & C_G(y) & \to & C_H(\beta y) & \to & K \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ * & \to & K_0 & \to & C_{G_0}(ey) & \to & C_{H_0}(\beta_0 ey) & \to & K_0 \end{array}$$

with exact rows. The first and fourth columns we know to be rationalizations and the third column is a rationalization by the inductive hypothesis. By the rationalization five-lemma the second column is a rationalization, as required.

(ii) From Exact sequence 2 we have a commutative diagram

$$\begin{array}{ccccccc} C_H(\beta y) & \to & K & \to & \beta^{-1}[\beta y] & \to & * \\ \downarrow & & \downarrow & & \downarrow & & \\ C_{H_0}(\beta_0 ey) & \to & K_0 & \to & \beta_0^{-1}[\beta_0 ey] & \to & * \end{array}$$

with exact rows. Because K and K_0 are abelian, $\beta^{-1}[\beta y]$ and $\beta_0^{-1}[\beta_0 ey]$ can be given abelian group structures such that this diagram is a diagram of groups. Also $K = \Gamma^c G$ is finitely generated, so $\beta^{-1}[\beta y]$ is finitely generated. Finally the first column in this diagram is rationalization by (i) and the second column is known to be rationalization, so the third column is rationalization by the rationalization five-lemma. This completes the proof of (ii).

We can now prove the theorem. Let z be a member of G_0; we have to show that $e^{-1}[z] \subset [G]$ is finite. We continue to use induction on c. Consider the commutative square

$$\begin{array}{ccc} [G] & \xrightarrow{\beta} & [H] \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e'} \\ [G_0] & \xrightarrow{\beta_0} & [H_0], \end{array}$$

where H is $G/\Gamma^c G$ as before. The inductive hypothesis tells us that e' is finite-to-one. It follows that $e^{-1}[z]$ is contained in the union of finitely many sets of the form $\beta^{-1}[\beta y]$ with $y \in G$. But by part (ii) of the lemma, $e|\beta^{-1}[\beta y]$ is finite-to-one for each such y. Therefore $e^{-1}[z]$ is finite. This concludes the proof.

Remark. Let S and T be sets of primes with $S \supset T$ and let G be a nilpotent group such that the factors of its lower central series are finitely generated modules over the integers localized at S. The same argument then shows that the localization of G at T (see (1)) induces a finite-to-one map of conjugacy classes.

I am grateful to J. F. Adams, P. J. Hilton and W. Meier for encouragement and Sidney Sussex College for financial support. The material of this paper is taken from a thesis to be submitted for a doctorate to the University of Cambridge.

https://doi.org/10.1017/S0305004100053858 Published online by Cambridge University Press

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