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The magnetic entropy concept

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Abstract. The magnetic entropy concept is an interpretative and predictive tool based on probability and information, defined in a suitably coarse-grained possibility space of all current density distributions under testable constraints. This paper develops the concept starting from its statistical foundations and then proceeds illustrating its potentiality in the description of the global behavior of macroscopic plasma configurations with practical interest. Finally, the paper discusses the relations of the concept to the dynamics of the underlying system of particles.

1. Introduction

In the form suitable for applications the magnetic entropy is a functional of the current density and of the vector potential pertaining to a macroscopic plasma equilibrium determined by the condition of the functional to be stationary or maximum under given constraints. From the theoretical point of view the magnetic entropy is information entropy defined in a possibility space of all suitably coarse-grained current density distributions under testable constraints.

The condition of stationary magnetic entropy (SME) determines the profile of the current density and of directly related quantities (e.g. the profile of the safety factor, the profile of temperature in the presence of Ohmic relaxation). In addition, in view of its thermodynamic meaning, SME implies stringent conditions on the profile of the thermal diffusivity (through the requirement of consistency with the power balance equation) and on the profile of the pressure (through the requirement of consistency with the Grad–Shafranov equilibrium).

The ability of SME to reproduce the profiles in tokamak machines has been tested in recent years starting from comparisons in Ohmic TCV [1,2] and proceeding with ECRH discharges in FTU [3] and ICRH + NBI discharges in JET [4]. Further work is in progress for assessing the predictive and interpretative potentiality of SME in various tokamak regimes.

The derivation and the theoretical aspects of the magnetic entropy concept are discussed in a number of papers published over a period of many years. Indeed, the present theoretical formulation is the result of a long process of development and refinement. A reader who wants to become acquainted with the theoretical aspects is forced to jump from one paper to another, trying to discern how the content of one paper is, or is not, connected with that of another. At this point, it seems worthwhile to collect the theoretical aspects of the magnetic entropy concept in one single comprehensive exposition, which can also take advantage of the new perspective offered by the recent comparisons with the observations. The purpose

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of this paper is to present this organic exposition. Since the paper is intended to be self-contained, the old papers on which the presentation is largely based are not quoted.

The paper is organized as follows. The conceptual framework at the base of the magnetic entropy, defined as information entropy or the negative of the information content of a probability distribution in a possibility space of the macroscopic magnetic configurations, is introduced and developed in Secs 3–5. In the following six sections (6–11) the theoretical aspects of the concept and their general implications for practical systems are illustrated taking, as a reference, the case of the tokamak and of the reversed field pinch. As a particular example, we show how the concept leads to a very natural thermodynamic interpretation of the tearing instability. In Secs 12–14 the relationship between the magnetic entropy and the dynamics of the underlying system of particles in the very particular case of independent particles in an external magnetic and electric field is discussed. The magnetic entropy is related, in a certain averaged sense, to the Lagrangian description of motion. In the case considered, the vanishing of its first variation is connected with the action principle. The second variation is not connected with the dynamics, which reflects the fact that the entropy as a whole is not a dynamical concept.

2. The statistics of the plasma volume elements in information space

The macroscopic magnetic configuration of a plasma under given geometrical and boundary conditions is completely determined when the distribution of the current density is known. However, the current density distribution is not completely determined by the equation for the plasma equilibrium (e.g. the Grad–Shafranov equation) in view of the arbitrary functions implied by this equation. A selection can be introduced by dividing the equilibria into stable and unstable ones but still the arbitrariness in the stable class is infinitely large. In contrast, the experimental data on the profile consistency indicate that, under certain conditions, the plasma tends to assume preferentially certain special stable equilibria. Surely, these macroscopic equilibria are ultimately the result of the complicated dynamics of the underlying system of particles including collisions and dissipation. However, limiting ourselves to a purely macroscopic point of view, the question can arise whether one can determine the more probable distributions among all possible current density distributions compatible with the given geometry and the boundary conditions.

For tackling the question one must define, as a first step, a suitable space Γ where all possible current density distributions can be represented and where the information on the system can be introduced in the form of constraints on these distributions. We adopt the Bayesian point of view of information theory [5] where the constraints have a hypothetical meaning. That is to say, one advances testable constraints that imply a determination (an assignment) of a probability distribution P by maximizing the entropy

$$S = -\int_{\Gamma} P \ln P \, d\Gamma$$

with respect to variations of P under the constraints to be tested. If the probability P and the entropy, calculated with these constraints, prove to agree with the observations, then one infers the physical consistency of the constraints and, at the same time, the internal consistency of the theory. In our specific case the existence

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of a given macroscopic current density distribution will be introduced *a priori* through 'testable' constraints whose probability in the space of all possible current density distributions is determined by the entropy principle and is to be confirmed by the experiment.

We then proceed to define the space Γ where all our information on the system will be expressed. We shall consider a plasma of volume V and a large assembly of N volume elements $\Delta V = V/N$, where ΔV is large enough to contain many particles. The N volume elements are the basic objects of our statistics. Let us introduce a six-dimensional space for the values of the current density $\mathbf{J}$ and for the position $\mathbf{x}$ in the volume V. We now take one volume element from the assembly, call it ΔV_j and place it at random in a copy of the above six-dimensional space denoted by S_j. The volume element will be at a certain position, denoted by $(\mathbf{x}_j, \mathbf{J}_j)$ (where $\mathbf{x}_j$ is for instance the center of ΔV_j). We do the same for another volume element ΔV_i, in another space S_i, taking care that ΔV_i can only occupy in S_i a position $\mathbf{x}_i \neq \mathbf{x}_j$ because the volume elements cannot overlap in V. After repeating this procedure for all N volume elements, one obtains a single point in a $6N$-dimensional space Γ which is the Cartesian product of all the S_j. This point represents a particular current density distribution of the plasma, reconstructed in the volume V with a coarse-graining ΔV.

The space Γ is called 'information space' and the current density $\mathbf{J}_j$ is the 'information variable' (other examples of information variables are the charge density in the case of the electrostatic plasma equilibrium and the mass density in the case of the gravitational equilibrium). Our purpose is to calculate the probability $P(\mathbf{J}_1 \cdots \mathbf{J}_N; \mathbf{x}_1 \cdots \mathbf{x}_N)$ for the assembly of N volume elements to occupy at random any given volume element $d\Gamma$ in information space. To see the form of $d\Gamma$ we note that the volume in three-dimensional space accessible to the volume element $j = 1$ is $V = N\Delta V$, and the volumes accessible to the volume elements $j = 2$ and 3 are $V - \Delta V = (N-1)\Delta V$ and $V - 2\Delta V = (N-2)\Delta V$, and so on. Thus the volume accessible to the assembly of N volume elements is

$$(N(N-1)(N-2)\cdots 1)(\Delta V)^N = N!\Delta V^N. \tag{2.1}$$

This must be divided by $N!$ because a permutation among the N volume elements has no physical effect and must not contribute to the total volume element in information space. Hence $d\Gamma$ is given by

$$d\Gamma = \Delta V^N d\mathbf{J}_1 \cdots d\mathbf{J}_N. \tag{2.2}$$

Further constraints are now imposed on the part of information space that concerns the current density distributions. We suppose that the total current density in each ΔV_j is formed by the coarse-grained contribution (uniform in ΔV_j) of a macroscopic (or collective) current density $\mathbf{j}(\mathbf{x}_j)$ and by a fluctuating part $\tilde{\mathbf{j}}_j$ arising from the particle structure in ΔV_j. The part $\mathbf{j}(\mathbf{x}_j)$ related to the collective equilibrium is given but not specified as it is an unknown to be determined later from the variation properties of the entropy. The total current density is then $\mathbf{J}_j = \mathbf{j}(\mathbf{x}_j) + \tilde{\mathbf{j}}_j$, where $-\infty \leqslant \tilde{\mathbf{j}}_j \leqslant +\infty$ and is a random variable in view of its fluctuating part, while $\mathbf{j}(x_j)$ can be treated as a constant. Additionally, $\mathbf{j}(x_j)$ is taken as vanishing outside a volume $\Omega \ll V$ with $\tilde{\mathbf{j}} \neq 0$ in V and with V and N that tend eventually to infinity while ΔV is fixed (thermodynamic limit: see Sec. 4). The two parts of $\mathbf{J}_j$ can be distinguished further in their physical effects by considering the part of the magnetic energy arising from the interaction of $\mathbf{J}_j$ with the vector potential $\mathbf{A}(\mathbf{x}_j)$

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created by the collective part $\mathbf{j}(x_j)$:

$$\frac{1}{2c}\int_\Gamma P(\mathbf{J}_1\cdots\mathbf{J}_N;\mathbf{x}_1\cdots\mathbf{x}_N)\sum_{j=1}^{N}(\tilde{\mathbf{j}}_j+\mathbf{j}(\mathbf{x}_j))\cdot\mathbf{A}(\mathbf{x}_j)\Delta V_j\,d\Gamma=\Phi_{\text{int}}+\Phi. \tag{2.3}$$

In view of the normalization condition

$$\int_\Gamma P(\mathbf{J}_1\cdots\mathbf{J}_N;\mathbf{x}_1\cdots\mathbf{x}_N)\,d\Gamma=1 \tag{2.4}$$

the equation (2.3) can be separated into the parts

$$\Phi_{\text{int}}=\frac{1}{2c}\int_\Gamma P(\mathbf{J}_1\cdots\mathbf{J}_N;\mathbf{x}_1\cdots\mathbf{x}_N)\sum_{j=1}^{N}\tilde{\mathbf{j}}_j\cdot\mathbf{A}(\mathbf{x}_j)\Delta V_j\,d\Gamma \tag{2.5}$$

$$\Phi=\frac{1}{2c}\sum_{j=1}^{N}\mathbf{j}(\mathbf{x}_j)\cdot\mathbf{A}(\mathbf{x}_j)\Delta V_j. \tag{2.6}$$

Here Φ_{int} is the energy of the interaction between the current fluctuations of the background and the potential of the macroscopic configuration. Equilibria described by the Vlasov equation are uncorrelated to the single particle fluctuations and this should be expressed by putting $\Phi_{\text{int}}=0$. However, Φ_{int}, when calculated explicitly in the formalism that we are developing, will describe the energy emitted or absorbed by the collective system, for instance, through the Poynting flux of electromagnetic energy (see Sec. 8) or the exchange of heat (see Sec. 10). In other words, the interaction of the collective system with the fluctuating background simulates the interaction of this system with the external world. We then consider (2.3) as a constraint to the maximization procedure of the entropy, where $\Phi_{\text{int}}+\Phi$ is fixed and is to be determined later according to the physical situation at hand.

A further constraint on P arises from the fact that $\mathbf{J}$ is a random variable, whose values are then subject to dispersion. It is convenient to characterize the dispersion by fixing the value $\overline{\Delta J^2}$ of the following average quadratic form:

$$\overline{\Delta J^2}=\frac{1}{N}\int_\Gamma P(\mathbf{J}_1\cdots\mathbf{J}_N;\mathbf{x}_1\cdots\mathbf{x}_N)\sum_{j=1}^{N}\mathbf{J}_j^2\,d\Gamma. \tag{2.7}$$

We shall see in Sec. 4 that this form is consistent in the proper limit (the thermodynamic limit) to the conventional definition $\overline{J_i^2}-\overline{J_i}^2$ of the local variance in each ΔV_i (see (4.9)). We do not need to specify the value of $\overline{\Delta J^2}$, as it will be shown to be unnecessary for the thermodynamic description of the magnetic systems. The value of $\overline{\Delta J^2}$ will be found in Sec. 13 while discussing the relation of the present model to the dynamics of the underlying system of particles.

The probability P can now be determined from the requirement that the entropy

$$S=-\int_\Gamma P\ln P\,d\Gamma$$

be stationary with respect to arbitrary variations of P under the three constraints (2.3), (2.4) and (2.7).

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3. The thermodynamic formalism

Applying the technique of the Lagrange multipliers we are led to find the extremum of the functional

$$S = -\int_\Gamma P \ln P \, d\Gamma - \alpha \int_\Gamma P \sum_{j=1}^{N} \mathbf{J}_j^2 - \frac{1}{\tau c} \int_\Gamma P \sum_{j=1}^{N} \mathbf{J}_j \cdot \mathbf{A}(\mathbf{x}_j) \Delta V_j \, d\Gamma - \gamma \int_\Gamma P \, d\Gamma, \tag{3.1}$$

where α, γ and τ are constants to be determined later in order to satisfy the constraints. The vanishing of the first variation of S with respect to arbitrary variations of P gives

$$P = P_0 \exp\left(-\alpha \sum_{j=1}^{N} \mathbf{J}_j^2 - \frac{1}{\tau c} \sum_{j=1}^{N} \mathbf{J}_j \cdot \mathbf{A}(\mathbf{x}_j) \Delta V_j\right), \tag{3.2}$$

where $P_0 = \exp(-\gamma - 1)$. Let us put

$$\alpha \mathbf{J}_j^2 + \frac{1}{\tau} \mathbf{J}_j \cdot \mathbf{A}(\mathbf{x}_j) \Delta V_j = \alpha (\mathbf{J}_j - \overline{\mathbf{J}_j})^2 + C_j, \tag{3.3}$$

where $\overline{\mathbf{J}_j}$ and C_j are independent of $\mathbf{J}_j$ and are given by

$$(\bar{J}_k)_j = -\frac{1}{2\alpha\tau c} A_k(\mathbf{x}_j) \Delta V_j, \quad C_j = -\frac{1}{4\alpha\tau^2 c^2} \sum_{k=1}^{3} A_k^2 \Delta V_j^2, \tag{3.4}$$

where k denotes the components of $\bar{\mathbf{J}}_j$ and $\mathbf{A}$. The Γ-integration then reduces to integration over Gaussian distributions. Recalling the normalization condition (2.4), we obtain

$$P = \frac{\exp\left(-\alpha\left(\sum_{j=1}^{N} (\mathbf{J}_j - \overline{\mathbf{J}_j})^2\right)\right)}{\Delta V^N (\pi/\alpha)^{3N/2}} = \frac{\exp\left(-\alpha\left(\sum_{j=1}^{N} (\tilde{\mathbf{j}}_j - \overline{\tilde{\mathbf{j}}_j})^2\right)\right)}{\Delta V^N (\pi/\alpha)^{3N/2}}, \tag{3.5}$$

$$P_0 = \frac{\exp\left(\sum_{j=1}^{N} C_j\right)}{\Delta V^N (\pi/\alpha)^{3N/2}}. \tag{3.6}$$

One can verify the relations

$$\bar{\mathbf{J}}_i = \mathbf{j}(\mathbf{x}_i) + \bar{\tilde{\mathbf{j}}}_j = \int_\Gamma P \mathbf{J}_i \, d\Gamma = -\frac{1}{2\alpha\tau c} \mathbf{A}(\mathbf{x}_i) \Delta V_i, \tag{3.7a}$$

$$\Phi_{\text{int}} = \frac{1}{2c} \sum_{j=1}^{N} \bar{\tilde{\mathbf{j}}}_j \cdot \mathbf{A}(\mathbf{x}_j) \Delta V_j = -\Phi - \frac{\Delta V^2}{4\alpha\tau c^2} \sum_{j=1}^{N} \mathbf{A}^2(\mathbf{x}_j), \tag{3.7b}$$

$$\bar{\mathbf{J}}_i^2 = \int_\Gamma P \mathbf{J}_i^2 \, d\Gamma = \frac{3}{2\alpha} + \overline{\mathbf{J}_i}^2 = \frac{3}{2\alpha} + \frac{\Delta V_i^2}{4\alpha^2\tau^2 c^2} \mathbf{A}^2(\mathbf{x}_i), \tag{3.7c}$$

$$\overline{\Delta J^2} = \frac{1}{N} \sum_{j=1}^{N} \int_\Gamma P \mathbf{J}_j^2 \, d\Gamma = \frac{3}{2\alpha} + \frac{\Delta V^2}{4\alpha^2\tau^2 c^2 N} \sum_{j=1}^{N} \mathbf{A}^2(\mathbf{x}_j). \tag{3.7d}$$

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Combining (3.7d) with (3.7b), we obtain

$$\alpha = \frac{1}{\overline{\Delta J^2}} \left(\frac{3}{2} - \frac{1}{N\tau}(\Phi + \Phi_{\rm int}) \right), \tag{3.8}$$

$$\tau = -\frac{\Delta V}{4\alpha c^2} \frac{\sum_{j=1}^{N} \mathbf{A}^2(\mathbf{x}_j)\Delta V_j}{\Phi + \Phi_{\rm int}}. \tag{3.9}$$

We can now proceed to the calculation of the entropy

$$\begin{aligned} S &= -\int_\Gamma P \ln P \, d\Gamma = N \ln \left(\Delta V \left(\frac{\pi}{\alpha} \right)^{3/2} \right) + \alpha \int_\Gamma P \sum_{j=1}^{N} (\mathbf{J}_j - \bar{\bar{\mathbf{J}}}_j)^2 \, d\Gamma \\ &= N \ln \left(\Delta V \left(\frac{\pi}{\alpha} \right)^{3/2} \right) + \alpha N \overline{\Delta J^2} - \frac{\Delta V^2}{4\alpha\tau^2 c^2} \sum_{j=1}^{N} \mathbf{A}^2(\mathbf{x}_j) \\ &= N \ln \left(\Delta V \left(\frac{\pi}{\alpha} \right)^{3/2} \right) + \frac{3N}{2} \\ &= -\frac{3N}{2} \ln \alpha + N \ln \left(\Delta V \pi^{3/2} \right) + \frac{3N}{2} \\ &= -\frac{3N}{2} \ln \left(\frac{3}{\overline{2\Delta J^2}} \left(1 - \frac{2}{3N\tau}(\Phi + \Phi_{\rm int}) \right) \right) + N \ln \left(\Delta V \pi^{3/2} \right) + \frac{3N}{2} \\ &= \frac{3N}{2} + N \ln \left(\Delta V \left(\frac{2\pi \overline{\Delta J^2}}{3} \right)^{3/2} \right) - \frac{3N}{2} \ln \left(1 - \frac{2}{3N\tau}(\Phi + \Phi_{\rm int}) \right). \end{aligned} \tag{3.10}$$

Another quantity of interest is the free energy defined by $F = \tau \ln P_0$. Combining (3.6) with (3.4), (3.7b) and (3.7d), we obtain

$$\begin{aligned} F = \tau \ln P_0 &= \tau \sum_{j=1}^{N} C_j - N\tau \ln \left(\Delta V \pi^{3/2} \right) + \frac{3N\tau}{2} \ln \alpha \\ &= \Phi + \Phi_{\rm int} - N\tau \ln \left(\Delta V \left(\frac{2\pi \overline{\Delta J^2}}{3} \right)^{3/2} \right) + \frac{3N\tau}{2} \ln \left(1 - \frac{2}{3N\tau}(\Phi + \Phi_{\rm int}) \right). \end{aligned} \tag{3.11}$$

The comparison with (3.10) gives the following relation between F and S:

$$F = -\tau S + \Phi + \Phi_{\rm int} + \frac{3N\tau}{2}. \tag{3.12}$$

4. The thermodynamic limit

We shall now consider the limit $N \to \infty, V \to \infty$, with ΔV fixed, also considering the volume Ω as fixed where the macroscopic (collective) system is localized. That is to say, the collective configuration described by $\mathbf{j}(\mathbf{x})$, localized in the volume Ω, is considered as immersed in a background (a heat bath) of freely fluctuating particles filling a volume $V \gg \Omega$ (electrically neutral in the average). In this limit the entropy and the free energy will be separated into two parts corresponding respectively to the collective system and to the background. The two systems however are not independent but they can eventually interact by exchanging energy.

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Effects related to the macroscopic configuration and to the fluctuating background are both contained in $\overline{\Delta J^2}$, which can be split as follows:

$$\begin{aligned}\overline{\Delta J^2} &= \frac{1}{N}\int_\Gamma P\sum_{j=1}^{N}\left(\mathbf{j}(\mathbf{x}_j)+\tilde{\mathbf{j}}_j\right)^2 d\Gamma \\ &= \frac{1}{N}\int_\Gamma P\sum_{j=1}^{N}\left(\mathbf{j}^2(\mathbf{x}_j)+2\mathbf{j}(\mathbf{x}_j)\cdot\tilde{\mathbf{j}}_j+\tilde{\mathbf{j}}_j^2\right) d\Gamma \\ &= \frac{1}{N}\sum_{j=1}^{N}\mathbf{j}^2(\mathbf{x}_j)+\frac{2}{N}\sum_{j=1}^{N}\mathbf{j}(\mathbf{x}_j)\cdot\overline{\tilde{\mathbf{j}}_j}+\frac{1}{N}\sum_{j=1}^{N}\overline{\tilde{\mathbf{j}}_j^2}. \end{aligned} \tag{4.1}$$

Here the first term is a purely collective part. The second term describes a spatial correlation between the collective current density and the average of the fluctuating current density of the background in interaction with the collective system, and the last term is the mean square of the fluctuations of the background.

Using (3.7a) and (3.7d) and replacing the sums $\sum_j \Delta V_j$ with integrals $\int_\Omega d\Omega$, the last two terms can be written in the form

$$\overline{\tilde{\mathbf{j}}^2} \equiv \frac{1}{N}\sum_{j=1}^{N}\overline{\tilde{\mathbf{j}}_j^2} = \frac{3}{2\alpha}+\frac{1}{V}\int_\Omega\left(\mathbf{j}(\mathbf{x})+\frac{\Delta V}{2\alpha\tau c}\mathbf{A}(\mathbf{x})\right)^2 d\Omega, \tag{4.2}$$

$$\Delta J_c \equiv \frac{2}{N}\sum_{j=1}^{N}\mathbf{j}(\mathbf{x}_j)\cdot\overline{\tilde{\mathbf{j}}_j} = -\frac{2}{V}\left(\int_\Omega \mathbf{j}^2(\mathbf{x})\,d\Omega+\frac{\Delta V}{2\alpha\tau c}\int_\Omega \mathbf{j}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})\,d\Omega\right). \tag{4.3}$$

After inserting (4.2) and (4.3) into (4.1), we obtain

$$\overline{\Delta J^2} = \overline{\tilde{\mathbf{j}}^2}\left(1-\frac{1}{V\overline{\tilde{\mathbf{j}}^2}}\left(\int_\Omega \mathbf{j}^2(\mathbf{x}_j)\,d\Omega+\frac{\Delta V}{\alpha\tau c}\int_\Omega \mathbf{j}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})\,d\Omega\right)\right). \tag{4.4}$$

In the limit $V\to\infty$, the last term in (4.4) is much smaller than unity and $\ln\overline{\Delta J^2}$ in (3.10) and (3.11) can be approximated by an expansion up to first order in Ω/V. Moreover, the limit $V\to\infty$ implies the limit $N\to\infty$ in order that ΔV remains finite. Since τ is also finite (see the explicit expression for τ given by (13.18)), it follows that $(\Phi+\Phi_{\text{int}})/N\tau$ is much less than unity, so that the last term of (3.10) and (3.11) can also be approximated by expanding the logarithm up to first order in $(\Phi+\Phi_{\text{int}})/N\tau$. Finally, recalling (4.2), one has $\alpha = 3/2\overline{\tilde{\mathbf{j}}^2}$. It follows that S can be expressed as the sum of two parts

$$S = S_{\text{b}} + S_{\text{p}}, \tag{4.5}$$

where S_{b} contains the effects of the background,

$$S_{\text{b}} = \frac{3N}{2}+N\ln\left(\Delta V\left(\frac{2\pi\overline{\tilde{\mathbf{j}}^2}}{3}\right)^{3/2}\right)+\frac{\Phi_{\text{int}}}{\tau} \tag{4.6}$$

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and $S_{\rm p}$ depends on the collective quantities of the macroscopic configuration

$$S_{\rm p} = -\frac{3}{2\Delta V\overline{\tilde{\mathbf{j}}^2}}\int_\Omega \mathbf{j}^2(\mathbf{x})\,d\Omega - \frac{\Phi}{\tau}$$
$$= -\frac{3}{2\Delta V\overline{\tilde{\mathbf{j}}^2}}\left(\int_\Omega \mathbf{j}^2(\mathbf{x})\,d\Omega + \frac{\Delta V\overline{\tilde{\mathbf{j}}^2}}{3\tau c}\int_\Omega \mathbf{j}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})\,d\Omega\right). \tag{4.7}$$

The free energy is immediately calculated from (3.12), using (4.5) and (4.6):

$$F = -N\tau\ln\left(\Delta V\left(\frac{2\pi\overline{\tilde{\mathbf{j}}^2}}{3}\right)^{3/2}\right) - \tau S_{\rm p} + \Phi. \tag{4.8}$$

The thermodynamic limits of other quantities of interest are as follows:

$$\frac{1}{N}\sum_{j=1}^{N}\left(\bar{\mathbf{J}}_j^2 - \overline{\mathbf{J}_j}^2\right) = \frac{3}{2\alpha} = \overline{\Delta J^2} = \overline{\tilde{\mathbf{j}}^2}, \tag{4.9}$$

$$\tau = -\frac{\Delta V\overline{\tilde{\mathbf{j}}^2}}{6c^2}\frac{\int_\Omega \mathbf{A}^2(\mathbf{x})}{\Phi + \Phi_{\rm int}}, \tag{4.10}$$

$$\Delta J_c = -\frac{2}{V}\left(\int_\Omega \mathbf{j}^2(\mathbf{x})\,d\Omega + \frac{\Delta V\overline{\tilde{\mathbf{j}}^2}}{3\tau}\int_\Omega \mathbf{J}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})\,d\Omega\right). \tag{4.11}$$

5. The magnetic entropy concept

Comparing the expression (4.11) with (4.7), one can write

$$S_{\rm p} = \frac{3N}{4\overline{\tilde{\mathbf{j}}^2}}\Delta J_c. \tag{5.1}$$

This relation tells us that the entropy of the macroscopic system is proportional to the correlation in space between the macroscopic current density and the fluctuating current density of the background and this correlation vanishes for $S_{\rm p} = 0$. Let us put $\Phi_{\rm int} = 0$, so that the correlation in space between the fluctuating current of the background and the collective potential also vanishes. After substituting τ (4.10) into (4.7) we obtain

$$S_{\rm p} = \frac{3}{2\Delta V\overline{\tilde{\mathbf{j}}^2}\int_\Omega \mathbf{A}^2\,d\Omega}\left(-\int_\Omega \mathbf{j}^2\,d\Omega\int_\Omega \mathbf{A}^2\,d\Omega + \left(\int_\Omega \mathbf{j}\cdot\mathbf{A}\,d\Omega\right)^2\right). \tag{5.2}$$

Applying the Schwartz inequality, we find that $S_{\rm p} \leqslant 0$, Thus, the absolute maximum $S_{\rm p} = 0$ is associated with complete absence of correlation between the macroscopic system and the background. This occurs when $\mathbf{j} = \kappa\mathbf{A}$, where κ is a constant.

By fixing $\Phi_{\rm int} = 0$ and by eliminating τ from (4.7) we have implicitly supposed that τ is varying when the macroscopic quantities are varied. A different physical situation arises when we keep τ fixed and eventually vary $\Phi_{\rm int}$ instead. Indeed, τ is a kind of generalized (negative) temperature (as can be formally seen by inspecting the formulas above) and is related to the Maxwellian temperature by a linear relation given in Sec. 13 where its relation with the underlying dynamics of the particles will be discussed. Variations with τ fixed are therefore isothermal variations. We then consider τ as a parameter of the macroscopic equilibrium and

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in view of the comparisons with practical systems we find it convenient to replace τ with a parameter μ according to the relation

$$\tau = -\frac{4\pi \Delta V \overline{\tilde{\mathbf{j}}^2}}{3\mu^2 c^2}. \tag{5.3}$$

Using this relation, S_p can be written in the form

$$S_\mathrm{p} = \frac{2\pi}{|\tau\mu^2|c^2}\left(-\int_\Omega \mathbf{j}^2\, d\Omega + \frac{\mu^2 c}{4\pi}\int_\Omega \mathbf{j}\cdot\mathbf{A}\, d\Omega\right) \tag{5.4}$$

and the interaction energy is given by the expression

$$\Phi_\mathrm{int} = -\Phi + \frac{\mu^2}{8\pi}\int_\Omega \mathbf{A}^2\, d\Omega = -\frac{1}{2c}\int_\Omega \mathbf{j}\cdot\mathbf{A}\, d\Omega + \frac{\mu^2}{8\pi}\int_\Omega \mathbf{A}^2\, d\Omega. \tag{5.5}$$

A variation of the macroscopic quantities $\mathbf{j}$ and $\mathbf{A}$ with μ (and τ) fixed, implies eventually a variation of Φ_int and, at the same time, the presence of correlation between the fluctuating and the macroscopic quantities and an energy exchange between the collective system and the background.

The variation of Φ_int will be calculated explicitly for typical situations of practical interest in Secs. 8 and 11. It will turn out that Φ_int describes the exchange of energy of the macroscopic system with the external world. Thus, the background allows for simulating the behavior of this system when it is not isolated.

6. Time evolution of the magnetic entropy

We now consider that the collective equilibrium is time dependent, but changing so slowly as to justify the application in a significant time interval of the concepts of canonical ensemble and canonical average implicit in our procedure. How long this time interval could be will be specified in Sec. 14.

In this section we shall investigate how the magnetic entropy changes in time when the plasma evolves in accordance with Maxwell equations. For this purpose it should be noted that in the derivation of Sec. 2 the localization of the plasma equilibrium in a finite volume Ω played an essential role. Thus in order to proceed consistently with our statistical model we have to express this situation in physical terms. We shall consider a situation where the plasma is carrying a volume current with density $\mathbf{j}_\mathrm{p}$ and is in contact with a surrounding conductive shell. The presence of a closed shell has a precise thermodynamic meaning because it allows us to consider the enclosed system as being isolated from the external world. However, if narrow cuts exists in the shell which allow the penetration of an electromagnetic field, then the plasma interacts with the external world and the shell merely defines the localization of the plasma equilibrium in the finite volume delimited by the shell. This model will be useful for the discussion of the physical meaning of the interaction energy Φ_int in Sec. 8.

We assume that a current $\mathbf{j}_\mathrm{s}$ exists on the shell which creates, outside the shell, a magnetic field screening the magnetic field $\mathbf{B}$ created by $\mathbf{j}_\mathrm{p}$, so that the total magnetic field vanishes outside the shell:

$$\frac{4\pi}{c}\mathbf{j}_\mathrm{p} = \nabla\times\mathbf{B}, \quad \frac{4\pi}{c}\mathbf{j}_\mathrm{s} = -\delta(\rho-\rho_\mathrm{s})\mathbf{e}_n\times\mathbf{B}. \tag{6.1}$$

The function $\delta(\rho-\rho_\mathrm{s})$, where ρ is the coordinate along the minor radius, describes the localization of $\mathbf{j}_\mathrm{s}$ on the conductive shell with effective radius ρ_s, and, whenever

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convenient, will be approximated by a δ-function; $\mathbf{e}_n$ is the unit vector orthogonal to the shell and directed outwards.

The entropy functional (5.4) is expressed in terms of the total current density $\mathbf{j} = \mathbf{j}_\mathrm{p} + \mathbf{j}_\mathrm{s}$ and $\mathbf{A}_\mathrm{p}$, where$\nabla \times \nabla \times \mathbf{A}_\mathrm{p} = (4\pi/c)\mathbf{j}_\mathrm{p}$. Thus the collective part S_p of the magnetic entropy can be subdivided into a part depending on the plasma quantities $\mathbf{j}_\mathrm{p}$ and $\mathbf{A}_\mathrm{p}$, denoted again with S_p, and a part S_s which depends only on the current density $\mathbf{j}_\mathrm{s}$ on the conductive shell:

$$S_\mathrm{p} = \frac{2\pi}{|\tau\mu^2|c^2}\left(-\int_\Omega \left(j_\mathrm{p}^2 + 2\mathbf{j}_\mathrm{p}\cdot\mathbf{j}_\mathrm{s}\right) d\Omega + \frac{\mu^2 c}{4\pi}\int_\Omega (\mathbf{j}_\mathrm{p} + \mathbf{j}_\mathrm{s})\cdot\mathbf{A}_\mathrm{p}\, d\Omega\right), \tag{6.2a}$$

$$S_\mathrm{s} = -\frac{2\pi}{|\tau\mu^2|c^2}\int_\Omega j_\mathrm{s}^2\, d\Omega = -\frac{1}{|\tau\mu^2|8\pi}\int_\Omega \delta^2(\rho-\rho_\mathrm{s})\mathbf{B}^2\, d\Omega, \tag{6.2b}$$

where $d\Omega = d\rho\, dS$ and the integration includes the shell.

We now concentrate on the part S_p pertaining to the plasma. Approximating (6.1) with the δ-function one obtains

$$\begin{aligned} S_\mathrm{p} = \frac{1}{|\tau\mu^2|8\pi}\Bigg(\Bigg(&-\int_\Omega (\nabla\times\mathbf{B})^2\, d\Omega + 2\int_{\partial\Omega} \nabla\times\mathbf{B}\cdot d\mathbf{S}\times\mathbf{B} \\ &+ \mu^2\int_\Omega \nabla\times\mathbf{B}\cdot\mathbf{A}_\mathrm{p}\, d\Omega - \mu^2\int_{\partial\Omega} d\mathbf{S}\times\mathbf{B}\cdot\mathbf{A}_\mathrm{p}\Bigg)\Bigg). \end{aligned} \tag{6.3}$$

Applying Faraday's law,

$$\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}, \quad \frac{1}{c}\frac{\partial\mathbf{A}_\mathrm{p}}{\partial t} = -\mathbf{E} \tag{6.4}$$

and after repeated application of Gauss's theorem, the time derivative $\dot{S}_\mathrm{p}$ of S_p takes the following form:

$$\begin{aligned} \dot{S}_\mathrm{p} &= \frac{1}{\tau}\Bigg(\int_\Omega \frac{\mathbf{E}}{\mu^2}(\nabla^2\mathbf{j} + \mu^2\mathbf{j})\, d\Omega - \frac{1}{\mu^2}\int_{\partial\Omega} \mathbf{E}\times\nabla\times\mathbf{j}\cdot d\mathbf{S} \\ &\quad + \frac{c}{4\pi}\int_{\partial\Omega} d\mathbf{S}\cdot\mathbf{E}\times\mathbf{B} - \frac{1}{c\mu^2}\int_{\partial\Omega} d\mathbf{S}\cdot\mathbf{B}\times\partial\mathbf{j}/\partial t\Bigg) \\ &= -\frac{1}{\tau}\left(\int_\Omega\left(\frac{1}{\mu^2}\nabla\times\mathbf{j} - \frac{c}{4\pi}\mathbf{B}\right)\cdot(\nabla\times\mathbf{E})\, d\Omega\right) - \frac{1}{\tau\mu^2 c}\int_{\partial\Omega} d\mathbf{S}\cdot\mathbf{B}\times\frac{\partial\mathbf{j}}{\partial t}, \end{aligned} \tag{6.5}$$

where $d\mathbf{S} = \mathbf{e}_n\, dS$, $\nabla\cdot\mathbf{j} = 0$ and the subscript p of $\mathbf{j}_\mathrm{p}$ has been dropped.

We distinguish two cases.

Case 1. $\mathbf{E} = \mathbf{E}_\mathrm{ext} = E_\mathrm{ext}\mathbf{e}_\phi$ is a toroidal inductive electric field that penetrates through narrow cuts in the volume enclosed by the copper shell where $\nabla\times\mathbf{E}_\mathrm{ext} = 0$, while the magnetic configuration of the plasma is time independent. This is the case of the tokamak in static equilibrium.

Case 2. The plasma is enclosed in a perfectly conductive shell without cuts so that the component of the electric field tangent to the shell vanishes and $\mathbf{E}\times d\mathbf{S} = 0$ on the shell. In general, $\nabla\times\mathbf{E} \neq 0$ and the magnetic configuration is not static due to dissipation and/or instabilities. In this case the first expression (6.5) becomes

$$\dot{S}_\mathrm{p} = \frac{1}{\tau}\int_\Omega \frac{\mathbf{E}}{\mu^2}(\nabla^2\mathbf{j} + \mu^2\mathbf{j})\, d\Omega \neq 0. \tag{6.6}$$

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The same expression for $\dot{S}_\mathrm{p}$, but with $\dot{S}_\mathrm{p} = 0$, is obtained in case 1 when one requires that the volume and surface contributions of the first expression (6.5) for $\dot{S}_\mathrm{p}$ are vanishing separately. More generally, introducing an arbitrary function F (to be determined later) the volume and the surface contributions can be separated by putting

$$\int_\Omega \frac{\mathbf{E}_\mathrm{ext}}{\mu^2}(\nabla^2\mathbf{j} + \mu^2\mathbf{j})\, d\Omega = -\int_\Omega F\, d\Omega,$$

$$\int_\Omega F\, d\Omega = -\frac{1}{\mu^2}\int_{\partial\Omega} \mathbf{E}_\mathrm{ext} \times \nabla \times \mathbf{j} \cdot d\mathbf{S} + \frac{c}{4\pi}\int_{\partial\Omega} d\mathbf{S} \cdot \mathbf{E}_\mathrm{ext} \times \mathbf{B}$$

$$-\frac{1}{c\mu^2}\int_{\partial\Omega} d\mathbf{S} \cdot \mathbf{B} \times \frac{\partial \mathbf{j}}{\partial t}, \tag{6.7}$$

where one equality is a consequence of the other.

7. The power balance of the tokamak

As is customary in statistical thermodynamics, the physical meaning of the terms contributing to the entropy change emerges after comparison with the power balance equation. Let us suppose for a moment that the plasma volume Ω is the site of a reversible process. Then in accordance with classical thermodynamics we can write

$$\dot{S}_\mathrm{p} = \frac{1}{\tau}\frac{dQ}{dt}, \tag{7.1}$$

where dQ/dt is the net heat absorbed or emitted per unit time by the plasma in Ω (in the one-fluid approximation):

$$\frac{dQ}{dt} = -\int_{\partial\Omega} \mathbf{q}_\mathrm{h} \cdot d\mathbf{S} + \int_\Omega \mathbf{E}_\mathrm{ext} \cdot \mathbf{j}_\phi\, d\Omega + \int_\Omega (p_\mathrm{A} - p_\mathrm{L})\, d\Omega. \tag{7.2}$$

Here q_h is the heat flux related to the thermal conductivity, the second term in the right-hand side describes the Ohmic contribution and the last term contains the contributions of the auxiliary power density p_A and of the non-diffusive losses p_L. This equation must be compared with (6.7) derived in case 1 for a plasma in static equilibrium:

$$\dot{S}_\mathrm{p} = \frac{1}{\tau}\int_\Omega \frac{\mathbf{E}_\mathrm{ext}}{\mu^2}(\nabla^2\mathbf{j} + \mu^2\mathbf{j})\, d\Omega + \frac{1}{\tau}\int_\Omega F\, d\Omega = 0. \tag{7.3}$$

The comparison leads to the identifications $F = p_\mathrm{A} - p_\mathrm{L}$ and

$$-\int_{\partial\Omega} \mathbf{q}_\mathrm{h} \cdot d\mathbf{S} = \frac{1}{\mu^2}\int_\Omega \mathbf{E}_\mathrm{ext} \cdot \nabla^2\mathbf{j}_\phi\, dV = \frac{1}{\mu^2}\int_{\partial\Omega} (E_\mathrm{ext}\nabla j_\phi - j_\phi \nabla E_\mathrm{ext}) \cdot d\mathbf{S}. \tag{7.4}$$

This equation is a relation between the current density profile and the heat flux. For its comparison with the observation in tokamaks see [1–4].

The comparison between the expressions (7.1) and (7.2), after a straightforward extension of the power balance to the non-stationary case, leads to the further identification

$$\frac{dS_\mathrm{p}}{dt} = \frac{1}{\tau}\frac{3}{2}\frac{d}{dt}\int_\Omega nT\, dV \tag{7.5}$$

which only holds provided that the transformation is reversible.

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It is of special interest to study the current distributions such that the magnetic entropy is stationary in any arbitrary small toroidal shell $\Delta\Omega$. As follows from (7.3) these configurations are described in the cylindrical limit, by the following equation:

$$\nabla^2 j_\phi + \mu^2 j_\phi = -\mu^2 \frac{p_\mathrm{A} - p_\mathrm{L}}{E_\mathrm{ext}} \tag{7.6}$$

whose relevance for the observations has been discussed in the references given above. For the extension of this equation to toroidal geometry, see [3].

Finally, it will be useful to note a relation between j_ϕ and the related component A_ϕ of the vector potential, which follows straightforwardly by combining (7.6) with the D'Alembert equation $\nabla^2 A_\phi = -(4\pi/c) j_\phi$. One obtains

$$j_\phi = \frac{c\mu^2}{4\pi} A_\phi + \varphi \quad \text{where } \nabla^2 \varphi = -\mu^2 \frac{p_\mathrm{A} - p_\mathrm{L}}{E_\mathrm{ext}}. \tag{7.7}$$

The term φ describes the part of the current density which is generated by the interaction of the plasma with the external world, namely with the power input and output described by p_A and p_L respectively.

8. The physical meaning of the interaction energy Φ_int

An insight into the physical aspects of the statistical model is obtained by inspecting the time derivative $\dot{\Phi}_\mathrm{int}$ of the interaction energy Φ_int. Putting $\mathbf{j} = \mathbf{j}_\mathrm{p} + \mathbf{j}_\mathrm{s}$ into (5.5) one has

$$\Phi_\mathrm{int} = -\frac{1}{2c} \int_\Omega \mathbf{j}_\mathrm{p} \cdot \mathbf{A}_\mathrm{p}\, d\Omega + \frac{1}{8\pi} \int_{\partial\Omega} d\mathbf{S} \times \mathbf{B} \cdot \mathbf{A}_\mathrm{p} + \frac{\mu^2}{8\pi} \int_\Omega A_\mathrm{p}^2\, d\Omega, \tag{8.1}$$

$$\dot{\Phi}_\mathrm{int} = -\frac{1}{4\pi} \int_\Omega \mathbf{A}_\mathrm{p} \cdot \nabla \times \frac{\partial \mathbf{B}}{\partial t}\, d\Omega + \frac{1}{2} \int_\Omega \mathbf{j}_\mathrm{p} \cdot \mathbf{E}\, d\Omega + \frac{1}{8\pi} \int_{\partial\Omega} \mathbf{A}_\mathrm{p} \cdot d\mathbf{S} \times \partial\mathbf{B}/\partial t$$

$$+ \frac{c}{8\pi} \int_{\partial\Omega} d\mathbf{S} \cdot \mathbf{E} \times \mathbf{B} - \frac{\mu^2 c}{4\pi} \int_\Omega \mathbf{A}_\mathrm{p} \cdot \mathbf{E}\, d\Omega. \tag{8.2}$$

Note that the screening field created by $\mathbf{j}_\mathrm{s}$ exists only outside Ω and the same holds for the corresponding vector potential $\mathbf{A}_\mathrm{s}$.

After application of Gauss's theorem one is led to the following expression:

$$\dot{\Phi}_\mathrm{int} = \frac{c}{4\pi} \int_{\partial\Omega} d\mathbf{S} \cdot \mathbf{E} \times \mathbf{B} + \int_\Omega \left(\mathbf{j}_\mathrm{p} - \frac{\mu^2 c}{4\pi} \mathbf{A}_\mathrm{p} \right) \cdot \mathbf{E}\, d\Omega. \tag{8.3}$$

The first term in the right-hand side describes the Poynting flux of electromagnetic energy across the boundary $\partial\Omega$. To see the meaning of the last term we specialize to the tokamak. Recalling the (7.7) this term becomes $\int_\Omega \varphi E_\mathrm{ext}\, d\Omega$, which describes the work of the electric field on that part of the current density which is generated by the auxiliary power and by the non-diffusive losses. Therefore $\dot{\Phi}_\mathrm{int}$ vanishes when the system is isolated.

In the case of a force free equilibrium which satisfies the equation

$$\nabla^2 \mathbf{j}_\mathrm{p} + \mu^2 \mathbf{j}_\mathrm{p} = 0, \tag{8.4}$$

the last term of (8.3) can be seen to vanish after a proper choice of the gauge.

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9. Adiabatic transformations

Let us consider the infinitesimal transformation $j_\phi \to j_\phi + \delta j$, where $\delta j = (\mu^2 c/4\pi)\,\delta A$ with δA satisfying the equation $\nabla^2 \delta A + \mu^2 \delta A = 0$. It is immediately seen that $j_\phi + \delta j$ satisfies the same equation (7.6) for j_ϕ. It will be shown that the magnetic entropy of the total system formed by the collective equilibrium and by the background is an invariant of the transformation above, while the entropy of the two subsystems is varying separately at first order. It follows that no heat can be emitted or absorbed by the *total* system as a consequence of a reversible transformation, which is therefore adiabatic.

We assume that the total current is conserved during the variation so that the poloidal component of $\delta\mathbf{B}$ vanishes on $\partial\Omega$, which gives a condition on the radial derivative of δA_φ. The first variation of S_p (6.2a) (where $\mathbf{j}_\mathrm{p} = j_\phi \mathbf{e}_\phi$) is then (dropping the subscripts of j_ϕ and A_ϕ)

$$\begin{aligned}\delta S_\mathrm{p} &= -\frac{2\pi}{\tau\mu^2 c^2}\left(-2\int_\Omega j\delta j\, d\Omega + \frac{\mu^2 c}{4\pi}\int_\Omega (A\delta j + j\delta A)\, d\Omega + \frac{c}{4\pi}\int_{\partial\Omega} \delta\mathbf{j}\cdot d\mathbf{S}\times\mathbf{B}\right)\\ &= \frac{2\pi}{\tau\mu^2 c^2}\int_\Omega \delta j\left(j - \frac{\mu^2 c}{4\pi}A\right) d\Omega - \frac{1}{2\tau\mu^2 c}\int_{\partial\Omega}\delta\mathbf{j}\cdot d\mathbf{S}\times\mathbf{B}. \end{aligned} \tag{9.1}$$

The first variation of Φ_int (8.1) is given by the expression

$$\begin{aligned}\delta\Phi_\mathrm{int} &= -\frac{1}{2c}\delta\int_\Omega \delta A\left(j - \frac{\mu^2 c}{4\pi}A\right) d\Omega + \frac{1}{2\mu^2 c}\int_{\partial\Omega}\delta\mathbf{j}\cdot d\mathbf{S}\times\mathbf{B}\\ &= -\frac{2\pi}{\mu^2 c^2}\int_\Omega \delta j\left(j - \frac{\mu^2 c}{4\pi}A\right) d\Omega + \frac{1}{2\mu^2 c}\int_{\partial\Omega}\delta\mathbf{j}\cdot d\mathbf{S}\times\mathbf{B}. \end{aligned} \tag{9.2}$$

Recalling that $S_\mathrm{b} = \Phi_\mathrm{int}/\tau$ is the entropy pertaining to the background we have that the first variation of the entropy of the total system vanishes:

$$\delta S = \delta S_\mathrm{p} + \delta S_\mathrm{b} = 0. \tag{9.3}$$

Let us now consider the second variation of S. One has for generic primary variations δA and $\delta j = -(c/4\pi)\nabla^2\delta A$ that

$$\delta^2 S_\mathrm{p} = \frac{2\pi}{\tau\mu^2 c^2}\left(\int_\Omega (\delta j)^2\, d\Omega - \frac{\mu^2 c}{4\pi}\int_\Omega \delta j\delta A\, d\Omega\right), \tag{9.4a}$$

$$\delta^2 S_\mathrm{b} = \frac{1}{2\tau c}\left(-\int_\Omega \delta j\delta A\, d\Omega + \frac{\mu^2 c}{4\pi}\int_\Omega (\delta A)^2\, d\Omega\right). \tag{9.4b}$$

One then obtains the inequality

$$\delta^2 S = \delta^2 S_\mathrm{p} + \delta^2 S_\mathrm{b} = \frac{1}{\tau}\int_\Omega \left(\frac{2\delta j}{\mu c} - \frac{\mu\delta A}{2\pi}\right)^2 d\Omega \leqslant 0, \tag{9.5}$$

where the equality holds only for the adiabatic variations that we are considering.

We have then seen that the entropy of the total system plasma + background is invariant under the considered variations so that no heat can be emitted or absorbed by the total system in a reversible transformation. The total system is then isolated during the transformation while to the contrary, entropy can be reversibly exchanged at first order between the collective subsystem and the background. The variation of the collective equilibrium is produced by changing the parameters that enter the solution to (7.6); for instance the safety factor,

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the boundary conditions, and so on. Then a particular state of the family of states that are solutions to (7.6) can be reached from another state performing a succession of the reversible adiabatic transformations considered above. The change of entropy between the two states is then defined through the reversible exchange of entropy with the background. The entropy change is then also defined in the realistic situations where the background does not exist and the same transformation between the two states is irreversible.

Albeit fictitious, the introduction of the background allows the definition on ideal theoretical grounds of the entropy changes of the magnetic collective equilibrium.

10. The force-free state

We now consider case 2 of Sec. 6, which applies to a plasma isolated from the external world by a perfectly conductive shell. We shall investigate the behavior of the entropy with respect to infinitesimal primary variations $\delta\mathbf{j}$, static or time dependent, around a static equilibrium, in order to determine the state of maximum entropy. The variations $\delta\mathbf{B}, \delta\mathbf{A}$ and $\delta\mathbf{j}_\mathrm{s}$ follow from $\delta\mathbf{j}$ according to the equations

$$\nabla\times\delta\mathbf{B}=\frac{4\pi}{c}\delta\mathbf{j},\quad \nabla\times\delta\mathbf{A}=\delta\mathbf{B},\quad \frac{4\pi}{c}\delta\mathbf{j}_\mathrm{s}=-\delta(\rho-\rho_\mathrm{s})\mathbf{e}_n\times\delta\mathbf{B}. \tag{10.1}$$

Putting $S_\mathrm{p}=S_{\mathrm{p}0}+\delta S_\mathrm{p}+\delta^2 S_\mathrm{p}$ one has for δS_p that

$$\delta S_\mathrm{p}=-\frac{1}{4\pi\tau\mu^2}\int_{\partial\Omega}\delta\mathbf{B}(\nabla^2\mathbf{B}+\mu^2\mathbf{B})\,d\Omega+\frac{1}{\tau\mu^2 c}\int_{\partial\Omega}d\mathbf{S}\times\delta\mathbf{j}\cdot\mathbf{B}. \tag{10.2}$$

We put $\delta\mathbf{j}=0$ at the toroidal wall $\partial\Omega$. To see the physical meaning of this condition we note that it implies $\nabla\times\delta\mathbf{B}=0$ at the wall, so that one can write $\delta\mathbf{B}=\nabla\chi$, where χ is some function that we take as a periodic function of the toroidal and poloidal coordinates on the wall. Thus, denoting with C_T any toroidal path around the wall of the torus, one has that $\oint_{C_\mathrm{T}}\delta\mathbf{B}\,dl=0$ which implies that the current through the hole of the torus is kept constant during the variation [6].

It follows from (10.2) that S_p is stationary if the magnetic configuration satisfies the equation

$$\nabla^2\mathbf{B}+\mu^2\mathbf{B}=\nabla\varphi, \tag{10.3}$$

where φ is arbitrary. A force-free state, $\nabla\times\mathbf{B}=\mu\mathbf{B}$ satisfies this equation with $\nabla\varphi=0$. The application of the curl to (10.3) leads to (8.4).

The second variation $\delta^2 S_\mathrm{p}$ has the form

$$\delta^2 S_\mathrm{p}=\frac{1}{|\tau|8\pi\mu^2}\int_\Omega\delta\mathbf{B}(\nabla^2\delta\mathbf{B}+\mu^2\delta\mathbf{B})\,d\Omega. \tag{10.4}$$

In order to discuss the sign of $\delta^2 S_\mathrm{p}$ let us consider the eigenvalues κ_m of the equation

$$\nabla\times\nabla\times\mathbf{a}_m=\kappa_m\nabla\times\mathbf{a}_m \tag{10.5}$$

with boundary condition $\mathbf{a}_m=0$ [7]. Noting that $\delta\mathbf{B}_m=\mathbf{a}_m+\nabla\psi$ is also a solution of (10.5) and determining ψ in order that $\nabla\cdot\delta\mathbf{B}_m=0$ with $(\delta\mathbf{B}_m)_n\equiv d\mathbf{S}\cdot\nabla\psi=0$ at the boundary, one has

$$\nabla^2\delta\mathbf{B}_m+\kappa_m^2\delta\mathbf{B}_m=0. \tag{10.6}$$

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Identifying the variation $\delta\mathbf{B}$ in (10.4) with a mode $\delta\mathbf{B}_m$ we obtain

$$\delta^2 S_{\mathrm{p}} = \frac{1}{|\tau| 8\pi\mu^2} \int_\Omega \delta B_m^2 \left(-\kappa_m^2 + \mu^2\right) d\Omega \tag{10.7}$$

It follows that for $\mu < \kappa_m$ one has $\delta^2 S_{\mathrm{p}} < 0$ so that S_{p} is maximum. If κ_m is the lowest eigenvalue of (10.5) the magnetic entropy is maximum with respect to all eigenmodes $\delta\mathbf{B}_m$.

One can make predictions on the time evolution of the perturbed system by inspecting the time dependence of the second variation. The first and the second variation of $\dot{S}_{\mathrm{p}}$ (6.5) are as follows:

$$\begin{aligned} \delta\dot{S}_{\mathrm{p}} = {} & \frac{1}{4\pi|\tau|\mu^2} \int_\Omega \frac{\partial\delta\mathbf{B}}{\partial t} (\nabla^2\mathbf{B} + \mu^2\mathbf{B})\, d\Omega \\ & - \frac{c}{4\pi|\tau|\mu^2} \int_\Omega (\nabla^2\delta\mathbf{B} + \mu^2\delta\mathbf{B}) \cdot \nabla \times \mathbf{E}\, d\Omega, \end{aligned} \tag{10.8}$$

$$\delta^2\dot{S}_{\mathrm{p}} = \frac{1}{4\pi|\tau|\mu^2} \int_\Omega \frac{\partial\delta\mathbf{B}}{\partial t} (\nabla^2\delta\mathbf{B} + \mu^2\delta\mathbf{B})\, d\Omega, \tag{10.9}$$

where we have put $\partial\delta j/\partial t = 0$ on $\partial\Omega$.

In order to see how the system evolves in time let us consider the case where $\delta^2 S_{\mathrm{p}} < 0$, so that the equilibrium is a maximum of S_{p}. Since the varied system, supposed to be isolated but time dependent, must evolve to states of increased entropy, one must have $\delta\dot{S}_{\mathrm{p}} > 0$. Comparing (10.9) with (10.4) one finds that a subdomain $\Delta\Omega$ of Ω exists where

$$\delta\mathbf{B} \cdot \frac{\partial\delta\mathbf{B}}{\partial t} = \frac{1}{2} \frac{\partial\delta\mathbf{B}^2}{\partial t} < 0. \tag{10.10}$$

It follows that $|\delta\mathbf{B}|$ tends to zero in $\Delta\Omega$ with time. So, after some time the contribution to the integrals (10.4) and (10.9) comes only from the domain outside $\Delta\Omega$. If one admits that $\delta^2 S_{\mathrm{p}} < 0$, $\delta^2\dot{S}_{\mathrm{p}} > 0$ in any $\Delta\Omega$, then $|\delta\mathbf{B}|$ must tend to zero everywhere. Conversely, one finds that if S_{p} is not maximum a subdomain $\Delta\Omega$ exists where $\partial|\delta\mathbf{B}|^2/\partial t > 0$ so that the equilibrium is unstable.

The formulation above of the time evolution of the perturbed force-free state is based on the same eigenvalue (10.5) that determines the stability of the reversed field pinch in the Taylor model [8]. In this model the magnetic energy is minimized under the constraint of fixed helicity. The energy is minimum and the equilibrium is stable when $\mu < \kappa_m$, where κ_m is the lowest eigenvalue of (10.5). The relevance of this mathematical formulation for the description of relaxation and magnetic reconnection in plasmas has been pointed out by Taylor [9].

Although the mathematical formulation based on (10.5) is the same as that of Taylor's model, its derivation from the entropy concept is more comprehensive. The present derivation does not assume either the helicity conservation or that the system is globally conservative. The entropy concept provides a unifying point of view in the description of the global ideal properties of the tokamak and of the reversed field pinch as well as of effects related to the local structure of the magnetic field (to be discussed in the next section). Taylor's model was the first look on a new land, ready to be explored.

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11. Thermodynamic interpretation of the tearing modes

As is known from the conventional treatment of the tearing modes, one distinguishes two physical regions in the plasma: a resistive layer localized on a rational magnetic surface inside which dissipation is taking place and a non-dissipative outer region.

In the preceding sections the properties of the magnetic entropy functional were studied with respect to changes of the magnetic configuration involving the whole plasma volume. For applying the entropy concept to the tearing modes we should see how the entropy can be defined in order to include in our thermodynamic point of view also those physical aspects that depend on the local structure of the magnetic field. A procedure convenient for this purpose consists of dividing the volume of the plasma into small toroidal shells $\Delta\Omega_n$ delimited by magnetic surfaces and in calculating the entropy S_n in each $\Delta\Omega_n$ by treating them as statistically independent systems. Of course, $\Delta\Omega_n$ should be large enough to satisfy $\Delta\Omega_n \gg V/N$ as is required by the statistical description. On the other hand, when one looks at the magnetic structure locally inside each $\Delta\Omega_n$, what really matters is not the average value of the current density or of the vector potential in $\Delta\Omega_n$, but how these quantities are changing inside $\Delta\Omega_n$, for instance with respect to the average value. In other words the information variable consistent with our present point of view to be introduced in the calculation of Sec. 2 of the entropy functional, is not $\mathbf{j}(\mathbf{x})$ but its deviation $\mathbf{j}(\mathbf{x}) - \mathbf{j}(\mathbf{x}_n)$ from its value at some point $\mathbf{x}_n$ inside $\Delta\Omega_n$. Correspondingly one should use the deviation $\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}_n)$ instead of $\mathbf{A}(\mathbf{x})$. The calculation that leads to the entropy functional (5.4) and to the interaction energy (5.5) remains unchanged when the information variable $\mathbf{j}(\mathbf{x})$ and the vector potential $\mathbf{A}(\mathbf{x})$ are replaced everywhere by $\mathbf{j}(\mathbf{x}) - \mathbf{j}(\mathbf{x}_n)$ and $\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}_n)$. Thus, in accordance with our considerations, the entropy functional takes the form

$$\begin{aligned} S_{\mathrm{p}} &= \sum_n S_{\mathrm{p}n} \\ &= \frac{3N}{2\Delta V\overline{\tilde{j}^2}} \sum_n \left(- \int_{\Delta\Omega_n} (\mathbf{j} - \mathbf{j}_n)^2 \, d\Omega_n + \frac{\mu_n^2 c}{4\pi} \int_{\Delta\Omega_n} (\mathbf{j} - \mathbf{j}_n)(\mathbf{A} - \mathbf{A}_n) \, d\Omega_n \right), \end{aligned} \tag{11.1}$$

where

$$\mu_n^2 = -\frac{4\pi\Delta V\overline{\tilde{j}^2}}{3c^2\tau_n} = \frac{4\pi}{c} \frac{\int_{\Delta\Omega_n} (\mathbf{j} - \mathbf{j}_n)(\mathbf{A} - \mathbf{A}_n) \, d\Omega_n}{\int_{\Delta\Omega_n} (\mathbf{A} - \mathbf{A}_n)^2 \, d\Omega_n}. \tag{11.2}$$

We consider a cylindrical plasma in the unperturbed equilibrium described by an axial current density $j_z(r)$ that creates a vector potential $A_z(r)$. The plasma is situated in a uniform axial magnetic field B_z so large as to be practically insensitive to small perturbations, while the other components can be slightly helically perturbed. As is well known, in view of the helical symmetry, the current density of the perturbed system is a function of space only through the helical flux $\chi = -mA_z - nrA_\theta/R$, where $A_\theta = rB_z/2$ and m, n are the poloidal and toroidal wave numbers associated with the rational magnetic surface under consideration. Let $\mathbf{x}_n \equiv (r_n, \theta_n, \varphi_n)$ be the co-ordinates of some point inside $\Delta\Omega_n$. The co-ordinates of a point varying inside the shell $\Delta\Omega_n$ with small radial thickness $\Delta r/\mathbf{x} \equiv (r_n, \theta, \varphi)$, where $r_n \in \Delta r$. Thus we can write (henceforth dropping the subscript z)

$$j - j_n = \frac{dj}{dA_n}(A - A_n) = -\frac{dj}{d\chi_n} m(A - A_n). \tag{11.3}$$

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It follows from (11.2) that

$$\mu_n^2 = -\frac{4\pi m}{c}\frac{dj}{d\chi_n}. \tag{11.4}$$

Let us calculate the first variations of S_p and of $S_\mathrm{b} = \sum_n \Phi_{n,\mathrm{int}}/\tau_n$ with respect to generic primary variations δj and δA (where $\delta j = -(c/4\pi)\nabla^2\delta A$):

$$\begin{aligned}\delta S_\mathrm{p} &= -\frac{3}{2\Delta V\overline{\tilde{j}^2}}\sum_n \int_{\Delta\Omega_n}\left(\delta j\left(2(j-j_n) - \frac{c\mu_n^2}{4\pi}(A-A_n)\right) - \frac{c\mu_n^2}{4\pi}\delta A(j-j_n)\right)d\Omega_n \\ &= -\frac{3}{2\Delta V\overline{\tilde{j}^2}}\sum_n \int_{\Delta\Omega_n}(j-j_n)\left(\delta j - \frac{c\mu_n^2}{4\pi}\delta A\right)d\Omega_n, \end{aligned}\tag{11.5}$$

$$\begin{aligned}\delta S_\mathrm{b} &= -\frac{1}{2c}\sum_n \frac{1}{\tau_n}\int_{\Delta\Omega_n}(A-A_n)\left(\delta j + m\frac{dj}{d\chi_n}\delta A\right)d\Omega_n \\ &= \frac{3}{2\Delta V\overline{\tilde{j}^2}}\sum_n \int_{\Delta\Omega_n}(j-j_n)\left(\delta j - \frac{c\mu_n^2}{4\pi}\delta A\right)d\Omega_n. \end{aligned}\tag{11.6}$$

It follows that $\delta S = \delta S_\mathrm{p} + \delta S_\mathrm{b} = 0$. For the second variations one obtains

$$\delta^2 S_\mathrm{p} = -\frac{3}{2\Delta V\overline{\tilde{j}^2}}\sum_n \int_{\Delta\Omega_n}\delta j\left(\delta j - \frac{c\mu_n^2}{4\pi}\delta A\right)d\Omega_n, \tag{11.7}$$

$$\begin{aligned}\delta^2 S_\mathrm{b} &= \frac{1}{2c}\sum_n \frac{\delta^2\Phi_{n,\mathrm{int}}}{\tau_n} = -\frac{1}{2c}\sum_n \frac{1}{\tau_n}\int_{\Delta\Omega_n}\delta A\left(\delta j - \frac{c\mu_n^2}{4\pi}\delta A\right)d\Omega_n \\ &= \frac{3c}{8\pi}\sum_n \mu_n^2\int_{\Delta\Omega_n}\delta A\left(\delta j - \frac{c\mu_n^2}{4\pi}\delta A\right)d\Omega_n \end{aligned}\tag{11.8}$$

so that

$$\delta^2 S = \delta^2 S_\mathrm{p} + \delta^2 S_\mathrm{b} = -\frac{3}{2\Delta V\overline{\tilde{j}^2}}\sum_n \int_{\Delta\Omega_n}\left(\delta j - \frac{c\mu_n^2}{4\pi}\delta A\right)^2 d\Omega_n \leqslant 0. \tag{11.9}$$

Then the total system plasma + background, considered as isolated, is thermodynamically stable because the entropy S is extremum and maximum, the adiabatic variations $\delta j = (c\mu_n^2/4\pi)\delta A$ apart. Stability is a theoretically necessary prerequisite in order that the *total* system can be identified as a not fading but present object on which conceptual experiments can be performed. It also follows that instability can arise only when the total system is not isolated but is allowed to interact with the external world by exchanging energy that we will take generically in the form of heat. We shall see presently that this is indeed the case of the tearing modes where the dissipative layer acts as a sink of magnetic energy.

Let $\Delta\Omega_\mathrm{s}$ be the site of dissipation, namely a toroidal shell centered on the rational surface $q(s) = m/n$ with infinitesimal radial thickness 2ε. Outside $\Delta\Omega_\mathrm{s}$ the variations are assumed to be adiabatic and reversible so that $\delta S_\mathrm{p} = \delta S_\mathrm{b} = 0$ and, according to (11.5) and (11.6), δA must satisfy the equation

$$\nabla^2\delta A - \frac{4\pi m}{c}\frac{dj}{d\chi_n}\delta A = 0, \tag{11.10}$$

for any n in the outer region. This is just the equation of the tearing modes in the outer region.

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We now calculate the variation of the local magnetic energy $\Phi_s = (1/2c)\int_{\Delta\Omega_s}(j - j_s)(A - A_s)\,d\Omega_s$ in $\Delta\Omega_s$. The first variation is easily seen to vanish because $j \to j_s$, $A \to A_s$ for $\varepsilon \to 0$. The second variation is given as

$$\delta^2\Phi_s = \frac{Rs}{c}\iint d\theta\,d\varphi\int_{s-\varepsilon}^{s+\varepsilon}\delta j\delta A r\,dr$$
$$= -\frac{R}{4\pi}\iint d\theta\,d\varphi\int_{s-\varepsilon}^{s+\varepsilon}\delta A\left(\frac{d}{dr}\left(r\frac{d\delta A}{dr}\right)\,dr\right). \tag{11.11}$$

Integration by parts gives in the limit $\varepsilon \to 0$ that

$$\delta^2\Phi_s = -\frac{Rs}{4\pi}\iint \Delta'(\delta A(s))^2\,d\theta\,d\varphi, \tag{11.12}$$

where

$$\Delta' = \left|\frac{d\delta A}{dr}\right|_{s-\varepsilon}^{s+\varepsilon}\Big/\delta A(s). \tag{11.13}$$

The variation $\delta^2\Phi_s$ is related to $\delta^2\Phi_{s,\mathrm{int}}$. Indeed, recalling (11.8) one has in the limit $\varepsilon \to 0$ that

$$\delta^2\Phi_{s,\mathrm{int}} = -\delta^2\Phi_s \tag{11.14}$$

because the part of the integral involving δA^2 vanishes in this limit.

On the other hand, according to our model, Φ_{int} describes the energy interaction of the system with the external world. Relation (8.3) shows that a *negative* change of Φ_{int} describes the Poynting flux of electromagnetic energy *gained* by the collective system. So, consistent with (8.3), we choose for $\delta^2\Phi_{\mathrm{int}}$ the negative sign

$$\delta^2\Phi_{\mathrm{int}} = -\delta Q, \tag{11.15}$$

where $\delta Q > 0$ denotes generically the energy *supplied* to the system.

Then one sees that in the unstable case, $\Delta' > 0$, $\delta^2\Phi_s < 0$, magnetic energy is transferred to the background as described by (11.14) and lost by the total system in form of heat, as described by (11.15). In the case $\Delta' < 0$, $\delta^2\Phi_s > 0$, the total system can be moved from equilibrium only when energy is injected from the external world because in the absence of interaction, that is to say when the system is isolated, the entropy is maximum in accordance with (11.9) and the system is stable.

In the outer region where $\delta Q = 0$, the energy conservation is expressed by $\delta^2\Phi_{\mathrm{out}} + \delta L = 0$, where $\delta^2\Phi_{\mathrm{out}}$ is the outer variation of the magnetic energy and δL is the work due to the mutual interaction of the external circuits with the varied magnetic configuration in the outer region. Since the process in the outer region is reversible, the work δL is expressed by $\delta L = -\delta^2 F$, where $\delta^2 F$ is the change of free energy. This change is given by the second variation of (4.8). Recalling that in the outer region $\delta^2 S_p = 0$ one has that $\delta^2 F = \delta^2\Phi_{\mathrm{out}}$. Therefore the free energy available to the instability is just the magnetic energy in the outer region. However, the instability can develop only when this energy is allowed to dissipate locally into heat in the resonant layer, so that it is released in form of heat to the external world. The condition for this to happen is $\delta^2\Phi_s < 0$.

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12. Lagrangian motion of the particles with macroscopic constraints

In the next three sections we shall study the relation of the magnetic entropy concept with the dynamics of the underlying system of particles. We start from the Lagrangian of a system of N_p independent particles with mass m and charge q moving with velocity $\mathbf{V}_n = d\mathbf{x}_n/dt$ in an external magnetic field described by the vector potential $\mathrm{A}(\mathbf{x}, t)$:

$$L(t) = \frac{m}{2}\sum_{n=1}^{N_\mathrm{p}} \mathbf{V}_n^2 + \frac{q}{c}\sum_{n=1}^{N_\mathrm{p}} \mathbf{V}_n \cdot \mathbf{A}(\mathbf{x}_n, t). \tag{12.1}$$

Here the effect of an external homogeneous electric field can be taken into account by putting

$$\mathbf{A}(\mathbf{x}_n, t) = \mathbf{A}_\mathrm{p}(\mathbf{x}_n) - c\mathbf{E}_\mathrm{ext}t. \tag{12.2}$$

For comparison with the point of view of the magnetic entropy concept, we must see how the coarse-graining of the magnetic configuration and the constraint (2.3) implied basically by this concept, can be taken into account in the Lagrangian formulation of the motion.

We consider the Lagrangian at a given instant of time t_0 and divide the particles into groups contained at this time into volume elements ΔV whose size is large with respect to the characteristic Larmor radius of the particles.

The average velocity of the particles present in the element ΔV_j at t_0 is then given by the expression

$$\mathbf{u}_j(t_0) = \frac{1}{N_j}\sum_{n=1}^{N_j} \mathbf{V}_n(t_0) = \frac{1}{N_j}\sum_{n=1}^{N_j} \frac{d\mathbf{x}_n}{dt}, \tag{12.3}$$

where $N_j \gg 1$ is the number of particles contained in ΔV_j at that instant.

We now look at the equation of motion of the group of N_j particles, thus identified at $t = t_0$, as is described for $t > t_0$ by the extremum of the action involving the Lagrangian (12.1). The extremum will be calculated under the condition that the motion is associated with a prescribed value of the quantity

$$\frac{q}{2c}\sum_{j=1}^{N} n_j \mathbf{u}_j \cdot \mathbf{A}(\mathbf{X}_j, t)\Delta V_j, \tag{12.4}$$

where $n_j = N_j/\Delta V_j$ is the density, $\mathbf{X}_j(t) = (1/N_j)\sum_{n=1}^{N_j} \mathbf{x}_n$ and $\mathbf{u}_j = d\mathbf{X}_j/dt$. The value of (12.4) will be determined in the next section where (12.4) will be compared with the similar constraint (2.3) introduced in Sec. 2 for the derivation of the magnetic entropy.

Introducing the Lagrangian multiplier λ, the Lagrangian of the constrained motion is expressed as

$$\begin{aligned} L(t) = {} & \frac{m}{2}\sum_{n=1}^{N_\mathrm{p}} \mathbf{V}_n^2 + \frac{q}{c}\sum_{n=1}^{N_\mathrm{p}} \mathbf{V}_n \cdot (-c\mathbf{E}_\mathrm{ext}t + \mathbf{A}_\mathrm{p}(\mathbf{x}_n)) \\ & + \lambda\frac{q}{2c}\sum_{j=1}^{N}\sum_{n=1}^{N_j} \mathbf{V}_n \cdot (-c\mathbf{E}_\mathrm{ext}t + \mathbf{A}_\mathrm{p}(\mathbf{X}_j)). \end{aligned} \tag{12.5}$$

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Let us single out the mth particle belonging to the jth group of N_j particles. The equation of motion is given by

$$\frac{d}{dt}\nabla_{\mathbf{V}_m} L - \nabla_{\mathbf{x}_m} L = 0. \tag{12.6}$$

We have

$$\begin{aligned}\frac{d}{dt}\nabla_{\mathbf{V}_m}\sum_{j}^{N}\sum_{n=1}^{N_j}\mathbf{V}_n\cdot\mathbf{A}(X_j) &= \frac{d}{dt}\mathbf{A}(\mathbf{X}_j)\\ &= \sum_{n=1}^{N_p}\mathbf{V}_n\cdot\nabla_{\mathbf{x}_n}\mathbf{A}(\mathbf{X}_j)\\ &= \sum_{n=1}^{N_j}\mathbf{V}_n\cdot\nabla_{\mathbf{x}_n}\mathbf{A}(\mathbf{X}_j)\\ &= \frac{1}{N_j}\sum_{n=1}^{N_j}\mathbf{V}_n\cdot\nabla_{\mathbf{X}_j}\mathbf{A}(\mathbf{X}_j) = \mathbf{u}_j\cdot\nabla_{\mathbf{X}_j}\mathbf{A}(\mathbf{X}_j).\end{aligned} \tag{12.7}$$

The contribution of the λ-term to the equation of motion is then

$$\begin{aligned}&\frac{\lambda q}{2c}\left(\frac{d}{dt}\nabla_{\mathbf{V}_m} - \nabla_{\mathbf{x}_m}\right)\sum_{j=1}^{N}\sum_{n=1}^{N_j}(-c\mathbf{E}_{\text{ext}}t + \mathbf{V}_n\cdot\mathbf{A}_{\text{p}}(\mathbf{X}_j))\\ &\quad= \frac{\lambda q}{2c}(-c\mathbf{E}_{\text{ext}} + \mathbf{u}_j(t)\cdot\nabla_{\mathbf{X}_j}\mathbf{A}_{\text{p}}(\mathbf{X}_j) - \nabla_{\mathbf{X}_j}\mathbf{u}_j\cdot\mathbf{A}_{\text{p}}(\mathbf{X}_j))\\ &\quad= -\frac{\lambda q}{2c}(c\mathbf{E}_{\text{ext}} + \mathbf{u}_j\times\mathbf{B}(\mathbf{X}_j))\end{aligned} \tag{12.8}$$

and the equation of motion takes the form

$$m\dot{\mathbf{V}}_m = q\left(\frac{1}{c}\mathbf{V}_m\times\mathbf{B}(\mathbf{x}_m) + \mathbf{E}_{\text{ext}}\right) + \lambda\frac{q}{2}\left(\frac{1}{c}\mathbf{u}_j\times\mathbf{B}(\mathbf{X}_j) + \mathbf{E}_{\text{ext}}\right). \tag{12.9}$$

Since the Lagrange multiplier λ is arbitrary at present, the only way to eliminate the unphysical arbitrariness of the λ-dependence is to split (12.9) into two independent equations and also independent of λ:

$$m\dot{\mathbf{V}}_m = q\left(\frac{1}{c}\mathbf{V}_m\times\mathbf{B}(\mathbf{x}_m) + \mathbf{E}_{\text{ext}}\right), \tag{12.10}$$

$$\frac{1}{c}\mathbf{u}_j\times\mathbf{B}(\mathbf{X}_j) + \mathbf{E}_{\text{ext}} = 0. \tag{12.11}$$

The first equation describes the motion at the level of the single particles while the second describes the stationary motion ($\dot{\mathbf{u}}_j = 0$) at the collective level of a group of particles. Thus the introduction of the constraint (12.4) has made possible a unified Lagrangian description of the motion at the two levels.

The macroscopic equation of motion should be completed by introducing the effect of a pressure tensor $\mathbf{P}$ and of dissipative forces $\mathbf{R}$. This can be done in a phenomenological way according to a standard procedure of classical mechanics

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[10] by defining

$$\delta \int_{t_1}^{t_2} L(t)\, dt = \sum_n \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{x}_n} - \frac{d}{dt} \frac{\partial L}{\partial \mathbf{V}_n} \right) \cdot \delta \mathbf{x}_n \, dt$$

$$= \frac{\lambda}{2} \sum_{j=1}^{N} \sum_{n=1}^{N_j} \int_{t_1}^{t_2} \mathbf{F}(\mathbf{X}_j(t)) \cdot \delta \mathbf{x}_n \, dt, \tag{12.12}$$

where

$$\mathbf{F}(X_j) = \frac{1}{n} (\nabla \cdot \mathbf{P} - \mathbf{R}) \tag{12.13}$$

so that the macroscopic equation of motion becomes

$$\frac{q}{c} (\mathbf{u}_j \times \mathbf{B}(\mathbf{X}_j) + \mathbf{E}_{\text{ext}}) - \frac{1}{n} (\nabla \cdot \mathbf{P} - \mathbf{R}) = 0. \tag{12.14}$$

13. The canonical average of the Lagrangian

We now wish to see what the Lagrangian (12.5) will become when it is averaged, at a given time t_0, according to the probability distribution P derived with our coarse-grained statistical model. Comparing the averaged quantities of the Lagrangian with the macroscopic quantities of this model we will be able to assign a dynamical meaning to the latter. For this purpose we shall express P and S in terms of the velocities rather than in terms of the current density.

To simplify the discussion, we shall take the density $n(\mathbf{X})$ to be uniform and fixed. That is, we assume that the number of particles N_j at $t = t_0$ is fixed and equal for all ΔV_j. In fact the fluctuations of $\mathbf{J}_j = qN_j\mathbf{u}_j/\Delta V_j$ arising from the Poisson fluctuations of N_j (where $N_j \gg 1$) are negligible with respect to the fluctuations arising from the random velocity $\tilde{\mathbf{v}}_n$.

We suppose that an organized or macroscopic velocity distribution $\mathbf{v}(\mathbf{X})$ is present in the system superimposed on a random velocity $\tilde{\mathbf{v}}_n$. That is to say, the particle velocity $\mathbf{V}_n$ measured at any given time t_0 is composed of a random part $\tilde{\mathbf{v}}_n$ and of a collective part $\mathbf{v}(\mathbf{X}_j)$, considered as a given function of $\mathbf{X}_j$ and supposed as essentially uniform in each ΔV_j. This part is generally much lower than $\tilde{\mathbf{v}}_n$. Its effect emerges only after averaging the velocities in ΔV_j and can be ignored when one is confined exclusively to a description at the single particle level. Thus,

$$\mathbf{V}_n = \mathbf{v}(\mathbf{X}_j) + \tilde{\mathbf{v}}_n \quad \text{for } \mathbf{x}_n(t_0), \mathbf{X}_j(t_0) \in \Delta V_j. \tag{13.1}$$

The average velocity (12.3) is then

$$\mathbf{u}_j = \mathbf{v}(\mathbf{X}_j) + \tilde{\mathbf{V}}_j, \tag{13.2}$$

where

$$\tilde{\mathbf{V}}_j = \frac{1}{N_j} \sum_{n=1}^{N_j} \tilde{\mathbf{v}}_n \quad \text{for } \mathbf{x}_n(t_0) \in \Delta V_j. \tag{13.3}$$

In accordance with the statistical model the velocity $\tilde{\mathbf{V}}_j$ is not vanishing in general but it results from the non-vanishing interaction of the fluctuating background with the collective system and determines the interaction energy Φ_{int}. Indeed, taking into account (13.2), the part of (12.4) that depends on $\mathbf{A}_{\text{p}}$ (that is, ignoring for the

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moment the electric field) becomes

$$\frac{q}{2c}\sum_{j=1}^{N} n_j(\mathbf{v}(\mathbf{X}_j)+\tilde{\mathbf{V}}_j)\cdot\mathbf{A}_{\mathrm{p}}(\mathbf{X}_j)\Delta V_j = \Phi+\tilde{\Phi}_{\mathrm{int}}, \tag{13.4}$$

where

$$\frac{q}{2c}\sum_{j=1}^{N} n_j\tilde{\mathbf{V}}_j\cdot A_{\mathrm{p}}(\mathbf{X}_j)\Delta V_j = \tilde{\Phi}_{\mathrm{int}}. \tag{13.5}$$

Performing the canonical average of $\tilde{\Phi}_{\mathrm{int}}$ with the probability distribution (3.5), which now reads

$$P=\left(\frac{\alpha}{\pi}\right)^{3N/2}\frac{1}{\Delta V^N}\exp\left(-\alpha\sum_{j=1}^{N}(\tilde{\mathbf{V}}_j-\overline{\tilde{\mathbf{V}}_j})^2\right), \tag{13.6}$$

where $\alpha=3/2\overline{\tilde{v}^2}$, one obtains

$$\int P\tilde{\Phi}_{\mathrm{int}}\,d\Gamma=\Phi_{\mathrm{int}} \tag{13.7}$$

so that (13.4) becomes the same as the constraint (2.3) of the statistical model.

Using (13.1), the Lagrangian (12.5), omitting for the moment the term involving $\mathbf{E}_{\mathrm{ext}}$, takes the form

$$\begin{aligned}L(t_0)=&\frac{m}{2}\sum_{j=2}^{N}\sum_{n=1}^{N_j}\tilde{v}_n^2+\frac{m}{2}\sum_{j=1}^{N}(\mathbf{v}^2(\mathbf{X}_j)+2\mathbf{v}(\mathbf{X}_j)\cdot\tilde{\mathbf{V}}_j)n\Delta V_j\\&+\frac{q}{c}\sum_{n=1}^{N_{\mathrm{p}}}\mathbf{V}_n\cdot\mathbf{A}_{\mathrm{p}}(\mathbf{x}_n)+\frac{\lambda}{2}\frac{q}{c}\sum_{j=1}^{N}n_j\mathbf{u}_j\cdot\mathbf{A}_{\mathrm{p}}(\mathbf{X}_j)\Delta V_j.\end{aligned} \tag{13.8}$$

The first term can be expressed with the Maxwellian temperature after performing the average with a Maxwellian distribution of the single particle velocities $\tilde{\mathbf{v}}_n$ obtaining $(1/N_j)\sum_{n=1}^{N_j}\overline{\tilde{\mathbf{v}}_n^2}=3T_j/m$, so that the term becomes equal to the plasma thermal energy

$$\frac{m}{2}\sum_{j=1}^{N}\sum_{n=1}^{N_j}\overline{\tilde{\mathbf{v}}_n^2}=\frac{3}{2}\sum_{j=1}^{N}N_jT_j=\frac{3}{2}\sum_{j=1}^{N}nT_j\Delta V_j\equiv U. \tag{13.9}$$

Let us calculate the canonical average of the third term

$$\begin{aligned}mn\int P\sum_{j=1}^{N}\mathbf{v}(\mathbf{X}_j)\cdot\tilde{\mathbf{v}}_j\Delta V_j\,d\Gamma&=\frac{2mn\Delta V}{3}\overline{\tilde{V}^2}S_{\mathrm{p}}\\&=mn\sum_{j=1}^{N}\left(-\mathbf{v}^2(\mathbf{X}_j)+\frac{\mu^2c}{4\pi nq}\mathbf{v}(\mathbf{X}_j)\cdot\mathbf{A}_{\mathrm{p}}(\mathbf{X}_j)\right),\end{aligned} \tag{13.10}$$

where relations (4.3) and (5.1) have been used.

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We now consider the fourth term

$$\sum_{n=1}^{N_\mathrm{p}} \mathbf{V}_n \cdot \mathbf{A}_\mathrm{p}(\mathbf{x}_n) = \sum_{j=1}^{N} \sum_{n=1}^{N_j} \mathbf{V}_n \cdot \mathbf{A}_\mathrm{p}(\mathbf{x}_n)$$
$$= \sum_{j=1}^{N} \left(N_j \mathbf{v}(\mathbf{X}_j) \cdot \mathbf{A}_\mathrm{p}(\mathbf{X}_j) + \sum_{n=1}^{N_j} \tilde{\mathbf{v}}_n \cdot \mathbf{A}_\mathrm{p}(\mathbf{X}_j) \right). \tag{13.11}$$

Here we take into account the fact that the positions $\mathbf{x}_n$of the particles grouped in ΔV_j at the time t_0 cannot be distinguished due to the coarse-graining of the configuration at the scale of ΔV. Thus $\mathbf{A}_\mathrm{p}(\mathbf{x}_n)$ can be identified with $\mathbf{A}_\mathrm{p}(\mathbf{X}_j)$ for every $\mathbf{x}_n \in \Delta V_j$. With this in mind and recalling (13.2) and (13.3) one has

$$\sum_{n=1}^{N_\mathrm{p}} \mathbf{V}_n \cdot \mathbf{A}_\mathrm{p}(\mathbf{x}_n) = \sum_{j=1}^{N} n \mathbf{u}_j \cdot \mathbf{A}_\mathrm{p}(\mathbf{X}_j) \Delta V_j. \tag{13.12}$$

This term combines with the last term of (13.8) to give $(\lambda + 2)(\Phi + \Phi_\mathrm{int})$ after the canonical average. We are now in position to write the canonical average of (13.8):

$$\overline{L} \equiv \int PL(t_0)\, d\Gamma = U + \frac{mn}{2} \sum_{j=1}^{N} \left(-\mathbf{v}^2(\mathbf{X}_j) + \frac{\mu^2 c}{4\pi q n} \mathbf{v}(\mathbf{X}_j) \cdot \mathbf{A}_\mathrm{p}(\mathbf{X}_j) \right) \Delta V_j$$
$$+ (\lambda + 2)(\Phi + \Phi_\mathrm{int})$$
$$= U + \frac{mn}{2} \sum_{j=1}^{N} \left(-\mathbf{v}^2(\mathbf{X}_j) + \left(\frac{\mu^2 c}{4\pi q n} + \left(1 + \frac{\lambda}{2} \right) \frac{q}{mc} \right) \mathbf{v}(\mathbf{X}_j) \cdot \mathbf{A}_\mathrm{p}(\mathbf{X}_j) \right) \Delta V_j$$
$$+ (\lambda + 2)\Phi_\mathrm{int}. \tag{13.13}$$

The relation above contains the Lagrange multiplier λ, which should be determined by the constraint (13.4) after prescribing $\Phi + \tilde{\Phi}_\mathrm{int}$. However, as is customary in statistical mechanics, one can reverse the procedure and consider instead the Lagrange multiplier λ as the primary quantity. This is the method followed in ordinary statistical mechanics for introducing the temperature and, indeed, we shall see presently that even in the present case λ is related to the generalized temperature τ.

We start by observing that if λ is chosen in such a way as to satisfy the relation

$$\frac{q}{mc} \left(\frac{\mu^2 c^2 m}{2\pi n q^2} + \lambda + 2 \right) = \frac{\mu^2 c}{4\pi n q} \quad \text{or} \quad \lambda + 2 = -\frac{\mu^2 c^2 m}{4\pi n q^2} \tag{13.14}$$

the canonical average of the Lagrangian can be expressed as a linear function of the magnetic entropy and of the plasma thermal energy *only*. Indeed, $\overline{L}$ takes the form

$$\overline{L} = U + \frac{mn\Delta V}{3} \overline{\tilde{V}^2} S_\mathrm{p} + (\lambda + 2)\Phi_\mathrm{int}. \tag{13.15}$$

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Here $\overline{\tilde{V}^2}$ is defined by (4.2) as $\overline{\tilde{V}^2} = (1/N)\sum_{j=1}^{N}\overline{\tilde{v}_j^2}$, where

$$\tilde{v}_j^2 = \frac{1}{N_j^2}\left(\sum_{n=1}^{N_j}\tilde{v}_n^2 + \sum_{\substack{n=1\\ n\neq n'}}^{N_j}\tilde{\mathbf{v}}_n\cdot\tilde{\mathbf{v}}'_n\right). \tag{13.16}$$

At the single particle level, the velocities $\tilde{\mathbf{v}}_n$ can be considered as uncorrelated, so one has

$$\overline{\tilde{V}^2} = \frac{1}{N}\sum_{j=1}^{N}\overline{\tilde{v}_j^2} = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{N_j^2}\sum_{n=1}^{N_j}\overline{\tilde{v}_n^2} = \frac{3}{mn\Delta V}T, \tag{13.17}$$

where $T = \sum_{j=1}^{N}T_j/N$. Thus, the generalized temperature τ is related, in accordance with (5.3), to the Maxwellian temperature T by the relation

$$\tau = -\frac{4\pi\Delta V n^2 q^2\overline{\tilde{V}^2}}{3\mu^2c^2} = -\frac{4\pi nq^2}{\mu^2c^2m}T. \tag{13.18}$$

It follows that

$$\lambda + 2 = -\frac{\mu^2c^2m}{4\pi nq^2} = \frac{T}{\tau} \tag{13.19}$$

and expression (13.15) for $\overline{L}$ becomes simply

$$\overline{L} = U + T(S_{\mathrm{p}} + S_{\mathrm{b}}) = U + TS. \tag{13.20}$$

Let us evaluate λ for typical tokamak machines for which $\mu s \leqslant 1$ [1–4], where s is of the order of the minor radius, $s \geqslant 20$ cm. Since the current is sustained by the electrons, m and q are the electron mass and charge. With $n \geqslant 10^{13}$ cm^{-3} one finds

$$\frac{\mu^2c^2m}{4\pi nq^2} < 6\times 10^{-5} \tag{13.21}$$

so that one can put $\lambda = -2$ for any practical purpose and (13.20) reduces to

$$\overline{L} = U + TS_{\mathrm{p}}. \tag{13.22}$$

The parameter τ, which is negative, becomes infinitely negative in the limit $\lambda \to -2$. The change of entropy of the background $\delta S_{\mathrm{b}} = \delta\Phi_{\mathrm{int}}/\tau$ vanishes in this limit but is otherwise positive (albeit infinitesimal) when $\delta\Phi_{\mathrm{int}} < 0$; for instance when heat is reversibly supplied from the background to the collective system. The parameter τ can then be interpreted formally as the temperature of the background consistently with the well-known fact that statistical systems with negative temperature are 'hotter' than systems with positive temperature, so that their entropy increases when heat is released.

14. The relation between the first variation of the magnetic entropy and Hamilton's principle

The relation found above between the magnetic entropy and the Lagrangian suggests that a relation may also exist between the variations of the two quantities. In this section we shall investigate this point.

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We start from the Lagrangian (13.8) which can be split in two parts $L(t) = \tilde{L}_{\rm p}(t) + L_{\rm X}(t)$ where, recalling (13.1),

$$\tilde{L}_{\rm p}(t) = \frac{m}{2}\sum_{j=1}^{N}\sum_{n=1}^{N_j}\tilde{v}_n^2 + \frac{q}{c}\sum_{j=1}^{N}\sum_{n=1}^{N_j}\tilde{\mathbf{v}}\cdot\mathbf{A}_{\rm p}(\mathbf{x}_n),$$

$$L_{\rm X}(t) = \frac{m}{2}\sum_{j=1}^{N}(\mathbf{v}^2(\mathbf{X}_j(t)) + 2\mathbf{v}(\mathbf{X}_j(t))\cdot\tilde{\mathbf{V}}_j)n\Delta V_j + \frac{q}{c}\sum_{j=1}^{N}\mathbf{v}(\mathbf{X}_j)\cdot\mathbf{A}_{\rm p}(X_j)n\Delta V_j$$

$$+ \frac{\lambda}{2}\frac{q}{c}\sum_{j=1}^{N}\mathbf{u}_j\cdot\mathbf{A}_{\rm p}(X_j)n\Delta V_j. \tag{14.1}$$

The Lagrangian $\tilde{L}_{\rm p}$ describes the unorganized motion at the single particle level and should be ascribed as part of the background, in accordance with our scheme. The part $L_{\rm X}$ contains the collective motion of the macroscopic system. The comparison between (13.5) and expression (5.5) for $\Phi_{\rm int}$ shows that $\tilde{\mathbf{V}}_j$ must have the form

$$\tilde{\mathbf{V}}_j = -\mathbf{v}(\mathbf{X}_j) + (c\mu^2/4\pi nq)\mathbf{A}_{\rm p} = -\mathbf{v}(\mathbf{X}_j) + \mathbf{u}_j. \tag{14.2}$$

Thus, recalling equality (13.19) for λ, $L_{\rm X}$ can be written more conveniently as

$$L_{\rm X}(t) = \frac{m}{2}\sum_{j=1}^{N}\left(-\mathbf{v}^2(\mathbf{X}_j) + \frac{\mu^2 c}{4\pi nq}\mathbf{v}(\mathbf{X}_j)\cdot\mathbf{A}_{\rm p}(\mathbf{X}_j)\right.$$

$$\left. + \frac{q}{nc}\left(\frac{\mu^2c^2}{4\pi nq^2} + \frac{2}{m}\right)\mathbf{v}(\mathbf{X}_j)\cdot\mathbf{A}_{\rm p}(\mathbf{X}_j)\right)n\Delta V_j + \frac{\lambda q}{2c}\sum_{j=1}^{N}\mathbf{u}_j\cdot\mathbf{A}_{\rm p}(\mathbf{X}_j)\Delta V_j$$

$$= \frac{m}{2}\sum_{j=1}^{N}\left(-\mathbf{v}^2(\mathbf{X}_j) + \frac{\mu^2 c}{4\pi nq}\mathbf{v}(\mathbf{X}_j)\cdot\mathbf{A}_{\rm p}(\mathbf{X}_j)\right)n\Delta V_j$$

$$+ \frac{\lambda q}{2c}\sum_{j=1}^{N}\left(-n\mathbf{v}(\mathbf{X}_j)\cdot\mathbf{A}_{\rm p}(\mathbf{X}_j) + \frac{\mu^2 c}{4\pi q}\mathbf{A}_{\rm p}^2\right)\Delta V_j. \tag{14.3}$$

We consider arbitrary primary variations $\delta\mathbf{x}_n(t)$ of the $\mathbf{x}_n(t)$, vanishing at the boundaries of a time interval $t_1 \leqslant t \leqslant t_2$. The $\delta\mathbf{x}_n$ produce a variation of the $\mathbf{X}_j$ given by

$$\delta\mathbf{X}_j(t) = \frac{1}{N_j}\sum_{n=1}^{N_j}\delta\mathbf{x}_n(t). \tag{14.4}$$

In turn the $\delta\mathbf{X}_j$ produce a variation of the $\mathbf{X}_j$-dependent quantities $\mathbf{v}(\mathbf{X}_j)$ and $\mathbf{A}_{\rm p}(\mathbf{X}_j)$ (recall that $\mathbf{A}_{\rm p}$ is treated as an external field):

$$\delta\mathbf{v}(\mathbf{X}_j, t) = \delta\mathbf{X}_j(t)\cdot\nabla\mathbf{v}(\mathbf{X}_j), \quad \delta\mathbf{A}(\mathbf{X}_j, t) = \delta\mathbf{X}_j(t)\cdot\nabla\mathbf{A}_{\rm p}(\mathbf{X}_j). \tag{14.5}$$

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One then obtains the following expression for the first variation of L_{X}:

$$\delta L_{\mathrm{X}} = \frac{mn}{2}\sum_{j=1}^{N}\Delta V_j\left(-2\mathbf{v}\delta\mathbf{v}+\frac{\mu^2 c}{4\pi nq}\delta(\mathbf{v}\cdot\mathbf{A}_{\mathrm{p}})\right) + \frac{\lambda}{2}\sum_{j=1}^{N}\Delta V_j\left(-\frac{nq}{c}\delta(\mathbf{v}\cdot\mathbf{A}_{\mathrm{p}})+\frac{\mu^2}{2\pi}\mathbf{A}_{\mathrm{p}}\cdot\delta\mathbf{A}\right). \tag{14.6}$$

For comparison with the magnetic entropy it is convenient to express the variations in terms of the current density:

$$\delta L_{\mathrm{X}}(t) = T\frac{3}{2\Delta V\bar{\tilde{\mathbf{j}}}^2}\sum_{j=1}^{N}\Delta V_j\left(-2\mathbf{j}(\mathbf{X}_j)\delta\mathbf{j}+\frac{\mu^2 c}{4\pi}(\mathbf{j}(\mathbf{X}_j)\cdot\delta\mathbf{A}+\mathbf{A}_{\mathrm{p}}(\mathbf{X}_j)\cdot\delta\mathbf{j})\right) + \lambda\sum_{j=1}^{N}\Delta V_j\left(\frac{1}{2c}(\mathbf{j}(\mathbf{X}_j)\cdot\delta\mathbf{A}_{\mathrm{p}}+\mathbf{A}_{\mathrm{p}}(\mathbf{X}_j)\cdot\delta\mathbf{j})-\frac{\mu^2}{4\pi}\mathbf{A}_{\mathrm{p}}\cdot\delta\mathbf{A}\right). \tag{14.7}$$

We now consider the time average of $\delta L_{\mathrm{X}}(t)$ in an arbitrary time interval $t_1 \leqslant t \leqslant t_2$ and in so doing we assume that the $\delta\mathbf{x}_n(t)$, and consequently the $\delta X_j(t)$, are varying in time much more rapidly than the $\mathbf{X}_j(t)$. In other words the macroscopic configuration is supposed to change in time so slowly that it can be considered as constant in a physically significant time interval (t_1, t_2), while $\mathbf{x}_n(t)$ and $\delta\mathbf{x}_n(t)$ are varying. Thus, the averaged variation of $\delta\mathbf{j}$, as well as of $\delta\mathbf{A}_{\mathrm{p}}$, can be expressed in the form $\delta\bar{\mathbf{j}} = \xi_j\cdot\nabla\mathbf{j}(\mathbf{X}_j)$, where

$$\xi_j = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} dt\frac{1}{N_j}\sum_{j=1}^{N}\delta\mathbf{x}_n(t). \tag{14.8}$$

One can assign a displacement field $\xi(\mathbf{X}_j)$ according to physical requirements (for instance, incompressibility) or one can consider the variation $\delta\bar{\vec{\mathbf{j}}}(\vec{\mathbf{X}}_j)$ as primary and determine $\xi(\mathbf{X}_j)$ by solving the inhomogeneous linear system of equations $\delta\bar{\vec{\mathbf{j}}} = \xi_j\cdot\nabla\mathbf{j}(\mathbf{X}_j)$ and then choose a set of variations $\delta\mathbf{x}_n(t)$ consistent with (14.8). It follows that the time average of the two terms in the right-hand side of (14.7) is related respectively to the first variations δS_{p} of the magnetic entropy, and $\delta\Phi_{\mathrm{int}}$ of the interaction energy, with respect to arbitrary primary variations $\delta\bar{\mathbf{j}}(\mathbf{X}_j)$ or $\xi(\mathbf{X}_j)$. The time average of $\delta L(t)$ then takes the form

$$\frac{1}{t_2-t_1}\delta\int_{t_1}^{t_2} L\,dt = T\delta S_{\mathrm{p}} + \lambda\delta\Phi_{\mathrm{int}} + \frac{1}{t_2-t_1}\delta\int_{t_1}^{t_2} L_{\mathrm{p}}\,dt \tag{14.9}$$

with

$$T\delta S_{\mathrm{p}} = \frac{m}{2nq^2}\int_{\Omega}\mathbf{j}\cdot\left(\mathbf{j}-\frac{\mu^2 c}{4\pi}\mathbf{A}_{\mathrm{p}}\right)\nabla\cdot\xi\,d\Omega,$$
$$\lambda\delta\Phi_{\mathrm{int}} = \frac{1}{c}\int_{\Omega}\mathbf{A}_{\mathrm{p}}\cdot\left(\mathbf{j}-\frac{\mu^2 c}{4\pi}\mathbf{A}_{\mathrm{p}}\right)\nabla\cdot\xi\,d\Omega, \tag{14.10}$$

where $\lambda = -2$ and $\xi\cdot d\mathbf{S} = 0$ at the boundary.

The variation of the actions involving L and $\tilde{L}_{\mathrm{p}}$ vanishes as a consequence of the equations of motion (12.10) and (12.11) described by L and the single particle motion described by $\tilde{L}_{\mathrm{p}}$. It follows that $T\delta S_{\mathrm{p}} + \lambda\delta\Phi_{\mathrm{int}} = 0$, but, in view of (14.10),

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the two terms must vanish separately; that is to say, the collective system is isolated during the variation ($\delta\Phi_{\rm int}=0$) and its entropy is stationary ($\delta S_{\rm p}=0$) provided that the vector potential satisfies the equation

$$\nabla\times\nabla\times\mathbf{A}_{\rm p}-\mu^2\mathbf{A}_{\rm p}=0. \tag{14.11}$$

At the same time one has from (14.2) that $\tilde{\mathbf{V}}=\tilde{\Phi}_{\rm int}=0$, which is consistent with the fact that the unperturbed system is isolated.

We now proceed by considering the effect of an external electric field, of the pressure and of dissipative forces. The substitution $\mathbf{A}_{\rm p}\rightarrow\mathbf{A}_{\rm p}-c\mathbf{E}_{\rm ext}t$ amounts to adding a term

$$-\left(1+\frac{\lambda}{2}\right)q\int_{t_1}^{t_2}\frac{1}{t_2-t_1}\sum_n\frac{d\delta\mathbf{x}_n}{dt}\cdot\mathbf{E}_{\rm ext}t\,dt \tag{14.12}$$

to both sides of equation (14.9). But

$$\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\sum_{n=1}^{N_{\rm p}}\frac{d\delta\mathbf{x}_n}{dt}\cdot\mathbf{E}_{\rm ext}t\,dt=-\sum_{j=1}^{N}\xi_j\cdot\mathbf{E}_{\rm ext}n\Delta V_j. \tag{14.13}$$

Denoting by $L_{\rm E}$ and $\tilde{L}_{\rm p,E}$ the Lagrangians L and $\tilde{L}_{\rm p}$ with the electric field included, one has from (14.8) that

$$\frac{1}{t_2-t_1}\delta\int_{t_1}^{t_2}L_{\rm E}\,dt=T\delta S_{\rm p}+\lambda\delta\Phi_{\rm int}+\frac{\lambda}{2}q\sum_{j=1}^{N}\xi_j\cdot\mathbf{E}_{\rm ext}n\Delta V_j+\frac{1}{t_2-t_1}\delta\int_{t_1}^{t_2}\tilde{L}_{\rm E,p}\,dt. \tag{14.14}$$

Here $\delta\int\tilde{L}_{\rm E,p}\,dt$ vanishes in view of the motion of the single particles in the electric field. On the other hand, $\delta\int L_{\rm E}\,dt$ can be expressed in terms of the pressure and dissipative forces as we did at the end of Sec. 12. Thus, recalling (12.14), we arrive at the relation

$$T\delta S_{\rm p}+\lambda\delta\Phi_{\rm int}=\frac{\lambda}{2}\sum_{j=1}^{N}(\nabla\mathbf{P}-nq\mathbf{E}_{\rm ext}-\mathbf{R})\cdot\xi_jn\Delta V_j=\frac{\lambda}{2c}\sum_{j=1}^{N}(nq\mathbf{u}_j\times\mathbf{B})\cdot\xi_j\Delta V_j. \tag{14.15}$$

Here the macroscopic velocity $\mathbf{u}_j=d\mathbf{X}_j/dt$ must be sufficiently slow so that the $\mathbf{X}_j(t)$ can be considered as constant in a physically significant time interval (t_1,t_2), consistent with our assumption. One sees from (14.15) that the condition $\delta S_{\rm p}=0$ (which entails $\delta\Phi_{\rm int}=0$ according to (14.10)) with respect to arbitrary variations ξ_j implies the local balance between the electric, the pressure and the dissipative forces and that the magnetic system is force-free (consistent with (14.11)).

If $\delta S_{\rm p}\neq 0$, the isolated collective system is not in equilibrium, the system is the site of irreversible processes and the connection with the equations of motion is lost.

While the vanishing of the first variation of the entropy is connected with the dynamics of the particles, a connection does not exist for the second variation. This is a reflection of the fact that the entropy is basically *not* a dynamical quantity, but an interpretative and predictive tool based on probability and information. The two expressions $S_{\rm p}=S_{\rm p0}+\delta S_{\rm p}$ and $S_{\rm p}=S_{\rm p0}+\delta S_{\rm p}+\delta^2S_{\rm p}$ correspond to the same magnetic configuration. But the entropy at first order $S_{\rm p}=S_0+\delta S_{\rm p}=S_0$ contains only the subset of microstates (defined by positions in *phase space*)

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determined by the initial conditions, which are conserved during the motion. Instead, the entropy at second order $S_p = S_0 + \delta^2 S_p$ should include the *enlarged* set of microstates that is not conserved but corresponds initially to the same macroscopic configuration as that of the conserved subset [5,11]. Thus one should have $\delta^2 S_p \geqslant 0$.

We have seen that the magnetic entropy concept is firmly rooted in the physical behavior of the system. This is rather surprising, considering that this concept seemed suspended in thin air by the purely formal requirements of Sec. 2.

15. Final remarks

The magnetic entropy, which is the negative of the information content in the possibility space of the current density distributions, is a tool of predictive inference, in accordance with the defining procedure of Sec. 2. For instance, in the case of the tokamak, the predictive content is given by (7.4) for the heat flux and by (7.6) for the current density profile. The comparison of these predictions with the observations in the tokamak machines provides a method for analysis and classification of the data according to general and ideal assumptions. From this point of view the magnetic entropy concept acquires a methodological value. But if the predictions are systematically verified under well defined and reproducible conditions, the magnetic entropy is no longer merely a tool of analysis, but it *becomes* a physical property of the system. In this case the conditions of stationary or maximum entropy would imply fundamental constraints on the underlying mechanisms of transport, which should operate through a kind of self-regulatory process in order to satisfy the thermodynamic requirements. Whether this is the case, and what the self-regulatory process could be, only future research can decide.

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