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# CAPITAL ALLOCATION WITH MULTIVARIATE RISK MEASURES: AN AXIOMATIC APPROACH

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Capital allocation is of central importance in portfolio management and risk-based performance measurement. Capital allocations for univariate risk measures have been extensively studied in the finance literature. In contrast to this situation, few papers dealt with capital allocations for multivariate risk measures. In this paper, we propose an axiom system for capital allocation with multivariate risk measures. We first recall the class of the positively homogeneous and subadditive multivariate risk measures, and provide the corresponding representation results. Then it is shown that for a given positively homogeneous and subadditive multivariate risk measure, there exists a capital allocation principle. Furthermore, the uniqueness of the capital allocation principe is characterized. Finally, examples are also given to derive the explicit capital allocation principles for the multivariate risk measures based on mean and standard deviation, including the multivariate mean-standard-deviation risk measures.

**Keywords:** capital allocation principle, (financial) portfolio, multivariate risk measure, positive homogeneity, subadditivity

## 1. INTRODUCTION

An important application of risk measures in portfolio management or performance measurement requires the allocation of risk capital either to subportfolios or to business units. The purpose of the present paper is to study the capital allocations with (scalar) multivariate risk measures by an axiomatic approach.

An univariate risk measure is to evaluate the risk of each financial position, by assigning a numeric value to a random variable representing the payoff of the financial position during an observed period of time. Artzner et al. [2,3] initiated the study of coherent risk measures, which satisfy four properties (or axioms): monotonicity, translation invariance, positive homogeneity and subadditivity, where both model-free and model-dependent models were studied. Further, Delbaen [6] studied coherent risk measures on general probability spaces.

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Föllmer and Schied [10] and, independently, Frittelli and Rosazza Gianin [12] introduced the broader class, named convex risk measure, by replacing the axioms of positive homogeneity and subadditivity with convexity. The convex risk measures were also studied by Deprez and Gerber [8] in the context of insurance.

In the above-mentioned framework of risk measures, the axiom of subadditivity or convexity reflects the diversification effect observed in risk measurement of financial portfolios: the sum of the *risks* of many portfolios is larger than the *risk* of the sum of the portfolios. Denault [7] suggested that this diversification advantage should be shared fairly between the portfolios, and that the risk capital of a portfolio, minus its allocated share of the diversification advantage, is effectively a firm-internal risk measure. Alternatively and more formally, assume that a univariate risk measure $\rho$ is fixed and let $X$ be a portfolio, which consists of subportfolios $X_1, \ldots, X_m$, that is, $X = X_1 + \cdots + X_m$. The capital allocation problem can be defined as to distribute the risk capital (also called the aggregate capital requirement) $k := \rho(X)$ of the portfolio $X$ to its subportfolios, that is, to compute the risk contributions $k_1, \ldots, k_m$ of $X_1, \ldots, X_m$, respectively, with $k = k_1 + \cdots + k_m$. $k_i$ is also called the risk capital allocated to $X_i, i = 1, \ldots, m$. Capital allocations with univariate risk measures have now been extensively studied in the literature. For capital allocations with coherent risk measures, see Denault [7], Fischer [9], Kalkbrener [13] Buch and Dorfleitner [4] and the references therein. For capital allocations with distortion risk measures, see Tsanakas [18,19] and the references therein. Tsanakas [20] provided the capital allocation principles for distortion-exponential risk measures, which belong to the class of convex risk measures.

Kalkbrener [13,14] suggested another axiomatic approach to capital allocation with univariate risk measures, under a basic assumption that the risk capital allocated to $X_i$ only depends on $X_i$ and $X$, but not on the decomposition of the rest

$$X - X_i = \sum_{j \neq i} X_j$$

of the portfolio.

From the perspective of practice, the investors in the markets usually use the combination of several financial positions as a trading strategy. Here and hereafter, by a *portfolio* we mean such a combination of several financial positions with its payoff being described by a random vector. Trading by portfolios allows the investors to benefit from the hedging technology. Therefore, it is reasonable and also necessary not only to evaluate the risks of the components of a portfolio separately, but also to evaluate the risk of the portfolio itself. In this case, possible dependence between the components is allowed, and should also be took into account when one evaluates the risk of the portfolio. For example, see also Definition 4.2 and Remark 4.2 below. A (scalar) multivariate risk measure is to evaluate the risk of each portfolio, by assigning a numeric value to each random vector. By proposing four axioms which are multivariate versions of the coherency axioms of Artzner et al. [3], Burgert and Rüschendorf [5] firstly studied the (scalar) coherent and convex multivariate risk measures in the name of risk measures for portfolio vectors. For more studies about (scalar) coherent and convex multivariate risk measures, see also Rüschendorf [17], Wei and Hu [21], and the references therein. On the other hand, in contrast with the extensive study about capital allocations with univariate risk measures in the literature, to our best knowledge, there are few references about capital allocations with (scalar) multivariate risk measures available in the literature, and we believe that it deserves being studied thoroughly.

The main contribution of this paper is the multivariate extension of the axiomatic approach in Kalkbrener [13,14] by incorporating both the model-free models and

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model-dependent models into a unified setup. Meanwhile, an example (i.e. the multivariate mean-standard-deviation risk measure, see Definition 4.2 below) also shows that a multivariate risk measure is more suitable and reasonable than a univariate risk measure when one evaluates the risk of a portfolio, see Remark 4.2 below, which explains why a multivariate risk measure is more suitable and reasonable compared with a univariate one. Now that a multivariate risk measure is more suitable and reasonable than a univariate one, it is natural to consider a multivariate risk measure instead of a univariate one when one concerns the capital allocation problem. In other words, it is worth studying capital allocation with multivariate risk measures.

It should also be mentioned that the main differences between the mathematical techniques used in this paper and those in Kalkbrener [13,14] are as follows. First, since all the main results and their proofs in this paper are carried out on the product space $\mathscr{X}^N$, an important issue for us to deal with is to equip this product space with a suitable norm so that it becomes a Banach space, whereas this issue can be naturally done in Kalkbrener [13,14]. Second, in order to construct meaningful multivariate risk measures, compared with the univariate ones, more delicate considerations are needed. For example, see the different multivariate risk measures $\rho_{1,c}$, $\rho_{2,c}$, and $\rho_{3,c}$ in Section 4, whereas they are the same in the univariate case.

In the present paper, we will investigate the capital allocations with (scalar) positively homogeneous and subadditive multivariate risk measures. Namely, we will first recall the class of positively homogeneous and subadditive multivariate risk measures, and provide the representation results for them. Then, we will define capital allocation principles with respect to multivariate risk measures by axioms. The characterization of the uniqueness of the capital allocation principle with respect to a positively homogeneous and subadditive multivariate risk measure is given. Finally, examples are given to illustrate the proposed capital allocation principles, where explicit capital allocation principles are given.

The reasons why we would like to work on the positively homogeneous and subadditive multivariate risk measures rather than coherent or convex multivariate risk measures are as follows. In other words, the reasons why we will drop the properties of translation invariance and monotonicity are as follows. First, if one takes into account the uncertainty of interest rate, then the property of translation invariance may be violated. In this situation, the property of translation invariance would be replaced by the one of cash subadditivity, see Karoui and Ravanelli [15]. Second, for different positions, the probabilities of losses occurring may be quite different under different generalized scenarios, and thus from the expected loss viewpoint, the property of monotonicity may not be necessary. For example, for the popular (univariate) mean-standard-deviation risk measures, the property of monotonicity may be violated. For some multivariate risk measures including the multivariate mean-standard-deviation risk measures, the property of monotonicity may be violated neither, see Propositions 4.1, 4.3, and 4.4 below. Third, it turns out that for a given positively homogeneous and subadditive multivariate risk measure, the uniqueness of the capital allocation principle can be characterized, see Theorem 3.2 below.

Although we will discuss the capital allocations only with the positively homogeneous and subadditive multivariate risk measures, we would also like to mention something about the capital allocations with the multivariate risk measures that possess three properties: translation invariance, subadditivity and positive homogeneity. Note that within the framework of the present axiomatic approach, to study capital allocations, the first step is to derive the representation results for the corresponding multivariate risk measures, because the representation results can guarantee the existence of the corresponding capital allocation principles. Unfortunately, we can not figure out in what extent the additional property of translation invariance can refine the representation result for the multivariate risk measure

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only possessing subadditivity and positive homogeneity (i.e. Theorem 2.1 below). It would be interesting to see it be worked out. Meanwhile, for the coherent multivariate risk measures that possess four properties: monotonicity, translation invariance, subadditivity and positive homogeneity, the representation results have been studied by Burgert and Rüschendorf [5], Rüschendorf [17], Wei and Hu [21].

The rest of the paper is organized as follows. In Section 2, we briefly review the definition of positively homogeneous and subadditive multivariate risk measures, and provide the representation results for them. In Section 3, we introduce the axioms of the capital allocation with multivariate risk measures. The uniqueness of the capital allocation principle with respect to a positively homogeneous and subadditive multivariate risk measures is characterized. In Section 4, examples are provided to illustrate the proposed capital allocation principles. Conclusion is summarized in Section 5.

## 2. POSITIVELY HOMOGENEOUS AND SUBADDITIVE MULTIVARIATE RISK MEASURES

In this section, we will briefly review the definition of positively homogeneous and subadditive multivariate risk measures, and provide corresponding representation results for them.

Given a measurable space $(\Omega,\ \mathscr{F})$ or a probability space $(\Omega,\ \mathscr{F},\ P)$, the payoff of a financial position is modeled by a random variable $X$ on $(\Omega,\ \mathscr{F})$ or $(\Omega,\ \mathscr{F},\ P)$. In Artzner et al. [3], $(\Omega,\ \mathscr{F})$ was called model-free model, and $(\Omega,\ \mathscr{F},\ P)$ model-dependent model. The probability $P$ was also called generalized scenarios. Throughout this paper, we denote by $L^\infty(\Omega,\ \mathscr{F})$ and $L^\infty(\Omega,\ \mathscr{F},\ P)$ the spaces of bounded random variables on $(\Omega,\ \mathscr{F})$ and essentially bounded random variables on $(\Omega,\ \mathscr{F},\ P)$, respectively. Positive values of $X$ correspond to losses, while negative values correspond to gains. We also denote by $\mathscr{G} = L^\infty(\Omega,\ \mathscr{F})$ and $\mathscr{H} = L^p(\Omega,\ \mathscr{F},\ P)$ for $1 \le p \le +\infty$, where, when $1 \le p < +\infty$, $\mathscr{H}$ denotes the space of random variables with finite $p$-order moment. For $X, Y \in \mathscr{H}$, we will identify $X$ with $Y$ if $P(X = Y) = 1$. The space $\mathscr{G}$ (or $\mathscr{H}$) represents the class of payoffs of financial positions. For $X \in \mathscr{G}$, define $\|X\|_\infty := \sup_{\omega\in\Omega} |X(\omega)|$, then $(\mathscr{G}, \|\cdot\|_\infty)$ is a Banach space. For $H \in \mathscr{H}$, define $\|H\|_p := \operatorname{esssup} |H|$, if $p = +\infty$; $[\mathbb{E}(|H|^p)]^{\frac{1}{p}}$, if $1 \le p < \infty$, where $\mathbb{E}(|H|^p)$ means the expectation of $|H|^p$ with respect to the probability $P$, then $(\mathscr{H}, \|\cdot\|_p)$ is a Banach space.

From now on, let $N \ge 1$ be a fixed positive integer. For $1 \le i \le N$, let $(\Omega_i,\ \mathscr{F}_i)$ be a fixed measurable space and $(\Omega_i,\ \mathscr{F}_i,\ P_i)$ a fixed probability space. We denote by $\mathscr{G}_i$ the space $L^\infty(\Omega_i,\ \mathscr{F}_i)$ and by $\mathscr{H}_i$ the space $L^{p_i}(\Omega_i,\ \mathscr{F}_i,\ P_i)$, $1 \le p_i \le +\infty$, $1 \le i \le N$. Let $\mathscr{X}^N$ be the product space $\mathscr{X}_1 \times \cdots \times \mathscr{X}_N$, where $\mathscr{X}_i$ is either $\mathscr{G}_i$ or $\mathscr{H}_i$ for $1 \le i \le N$. Denote by $\mathscr{H}^N$ the product space $\mathscr{H}_1 \times \cdots \times \mathscr{H}_N$. For $\mathbf{M} = (X_1, \ldots, X_N) \in \mathscr{X}^N$, define $\|\mathbf{M}\| := \|X_1\| + \cdots + \|X_N\|$, where $\|X_i\|$ equals $\|X_i\|_\infty$ if $\mathscr{X}_i$ is $\mathscr{G}_i$, and $\|X_i\|_{p_i}$ if $\mathscr{X}_i$ is $\mathscr{H}_i$ for $1 \le i \le N$. Then $(\mathscr{X}^N, \|\cdot\|)$ is a Banach space. Each element of $\mathscr{X}^N$ represents the payoff of a portfolio consisting of $N$ financial positions.

We need more notations. For $\mathbf{M_1} = (X_1, \ldots, X_N)$, $\mathbf{M_2} = (Y_1, \ldots, Y_N) \in \mathscr{X}^N$, $\mathbf{M_1} \le \mathbf{M_2}$ means $X_1 \le Y_1, \ldots, X_N \le Y_N$. $\mathbf{M_1} + \mathbf{M_2}$ stands for $(X_1 + Y_1, \ldots, X_N + Y_N)$. $a\mathbf{M}$ stands for $(aX_1, \ldots, aX_N)$ for $\mathbf{M} = (X_1, \ldots, X_N) \in \mathscr{X}^N$ and $a \in \mathbf{R}$. A sequence $\{\mathbf{M_n} = (X_1^n, \ldots, X_N^n);\ n \ge 1\}$ in $\mathscr{X}^N$ is called a bounded sequence if $\{X_1^n;\ n \ge 1\}, \ldots, \{X_N^n;\ n \ge 1\}$ are $N$ sequences of bounded random variables. That $\{\mathbf{M_n} = (X_1^n, \ldots, X_N^n);\ n \ge 1\}$ converges pointwise to $\mathbf{M} = (X_1, \ldots, X_N)$ means that $\{X_1^n;\ n \ge 1\}, \ldots, \{X_N^n;\ n \ge 1\}$ converge pointwise to $X_1, \ldots, X_N$, respectively. $\mathbf{M_n} \uparrow \mathbf{M}$ means $X_i^n \uparrow X_i$ for any $1 \le i \le N$. $\mathbf{M_n} \downarrow \mathbf{M}$

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means $X_i^n \downarrow X_i$ for any $1 \le i \le N$. We set $\mathbf{0} := (0, \dots, 0)$, $\mathbf{1} := (1, \dots, 1) \in \mathbf{R}^N$ and, for any fixed $i$ between 1 and $N$, $\mathbf{e_i} := (0, \dots, 0, 1, 0, \dots, 0)$, where 1 occupies the $i^{th}$ position. For $1 \le i \le N$, $X\mathbf{e_i}$ means $(0, \dots, 0, X, 0, \dots, 0)$, where $X$ occupies the $i^{th}$ position for any random variable $X$. Given a set $A$, $I_A$ means the indicator function of $A$.

On a general level, a (scalar) multivariate risk measure is any map from $\mathscr{X}^N$ to the real numbers $\mathbf{R}$. Given an $\mathbf{M} \in \mathscr{X}^N$, the quantity $\rho(\mathbf{M})$ represents the (minimal) amount of risk capital that the holder of the portfolio $\mathbf{M}$ has to safely invest, in order to satisfy regulators. For more details, see Burgert and Rüschendorf [5], Rüschendorf [17], Wei and Hu [21].

Next, we shall recall some axioms for the multivariate risk measures.

DEFINITION 2.1: *Let $\rho : \mathscr{X}^N \to \mathbf{R}$ be a multivariate risk measure.*

(i) *$\rho$ is called monotonous, if $\mathbf{M}_1 \le \mathbf{M}_2$ implies $\rho(\mathbf{M}_1) \le \rho(\mathbf{M}_2)$ for any $\mathbf{M}_1,\ \mathbf{M}_2 \in \mathscr{X}^N$.*

(ii) *$\rho$ is called translation invariant, if $\rho(\mathbf{M} + (a_1, \dots, a_N)) = \rho(\mathbf{M}) + \sum_{i=1}^N a_i$ for any $\mathbf{M} \in \mathscr{X}^N$ and $a_1\ \dots,\ a_N \in \mathbf{R}$.*

(iii) *$\rho$ is called positively homogeneous, if $\rho(a\mathbf{M}) = a\rho(\mathbf{M})$ for each $a \ge 0$ and $\mathbf{M} \in \mathscr{X}^N$.*

(iv) *$\rho$ is called subadditive, if $\rho(\mathbf{M_1} + \mathbf{M_2}) \le \rho(\mathbf{M_1}) + \rho(\mathbf{M_2})$ for any $\mathbf{M_1}, \mathbf{M_2} \in \mathscr{X}^N$.*

Note that if a multivariate risk measure $\rho$ satisfies axioms (i)–(iv) of Definition 2.1, then it is called coherent. For representation results for coherent multivariate risk measures, we refer to Burgert and Rüschendorf [5], Rüschendorf [17], Wei and Hu [21].

Now, we are in a position to state the representation results for positively homogeneous and subadditive multivariate risk measures, which guarantee the existence of capital allocation principles with respect to positively homogeneous and subadditive multivariate risk measures.

THEOREM 2.1: *Let $\rho : \mathscr{X}^N \to \mathbf{R}$ be a positively homogeneous and subadditive multivariate risk measure. Then*

$$\rho(\mathbf{M}) = \sup\{h(\mathbf{M}) :\ h \in H_\rho\} \tag{2.1}$$

*for all $\mathbf{M} \in \mathscr{X}^N$, where $H_\rho := \{h \in \mathscr{L}(\mathscr{X}^N) :\ h(\mathbf{M}) \le \rho(\mathbf{M})$ for all $\mathbf{M} \in \mathscr{X}^N\}$, $\mathscr{L}(\mathscr{X}^N)$ is the space of all real-valued linear functions on $\mathscr{X}^N$. The supremum over $H_\rho$ can be attained.*

PROOF: Let $\mathbf{M_0} \in \mathscr{X}^N$ and $\mathscr{X}^N_{\mathbf{M_0}}$ be the linear subspace of $\mathscr{X}^N$ spanned by $\mathbf{M_0}$. Define $f_{\mathbf{M_0}}$ on $\mathscr{X}^N_{\mathbf{M_0}}$ by $f_{\mathbf{M_0}}(a\mathbf{M_0}) := a\rho(\mathbf{M_0})$ for every real number $a$. Since $\rho(\mathbf{M_0}) + \rho(-\mathbf{M_0}) \ge \rho(\mathbf{M_0} - \mathbf{M_0}) = \rho(\mathbf{0}) = 0$, hence $\rho(-\mathbf{M_0}) \ge -\rho(\mathbf{M_0})$. Since $\rho$ is positively homogeneous,

$$f_{\mathbf{M_0}}(a\mathbf{M_0}) = \rho(a\mathbf{M_0}),$$
$$f_{\mathbf{M_0}}(-a\mathbf{M_0}) = -a\rho(\mathbf{M_0}) \le a\rho(-\mathbf{M_0}) = \rho(-a\mathbf{M_0}) \quad \text{for all} \quad a \ge 0.$$

Therefore, $f_{\mathbf{M_0}} \le \rho$ on $\mathscr{X}^N_{\mathbf{M_0}}$. It follows from the Hahn–Banach Theorem that there exists an $h_{\mathbf{M_0}} \in \mathscr{L}(\mathscr{X}^N)$ such that

$$h_{\mathbf{M_0}}(\mathbf{M}) = f_{\mathbf{M_0}}(\mathbf{M}) \quad \text{for all} \quad \mathbf{M} \in \mathscr{X}^N_{\mathbf{M_0}}$$

and

$$h_{\mathbf{M_0}}(\mathbf{M}) \le \rho(\mathbf{M}) \quad \text{for all} \quad \mathbf{M} \in \mathscr{X}^N.$$

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Consequently, $h_{\mathbf{M_0}} \in H_\rho$ with $h_{\mathbf{M_0}}(\mathbf{M_0}) = f_{\mathbf{M_0}}(\mathbf{M_0}) = \rho(\mathbf{M_0})$ and (2.1) is proved. Theorem 2.1 is proved. ■

## 3. CAPITAL ALLOCATIONS

In this section, by an axiomatic approach, we will discuss the capital allocation with positively homogenous and subadditive multivariate risk measures. The uniqueness of the capital allocation principle is also characterized.

For multivariate risk measures which are subadditive, the subadditivity reflects the diversification effect. Therefore, inspired by Denault [7], a natural and interesting issue is how about the capital allocation with these multivariate risk measures. Namely, inspired by Denault [7], this diversification advantage needs to be shared fairly between subportfolios. Precisely, given a portfolio $\mathbf{M} \in \mathscr{X}^N$ with subportfolios $\mathbf{M}_1, \ldots, \mathbf{M}_m$, that is, $\mathbf{M} = \mathbf{M}_1 + \cdots + \mathbf{M}_m$, the risk capital of a subportfolio, minus its allocated share of the diversification advantage, is effectively a firm-internal risk measure. Alternatively and formally, given a multivariate risk measure $\rho$ and a portfolio $\mathbf{M} \in \mathscr{X}^N$ with subportfolios $\mathbf{M}_1, \ldots, \mathbf{M}_m$, then the capital allocation problem can be defined as to distribute the risk capital $k := \rho(\mathbf{M})$ of the portfolio $\mathbf{M}$ to its subportfolios, that is, to compute the risk contributions $k_1, \ldots, k_m$ of $\mathbf{M}_1, \ldots, \mathbf{M}_m$, respectively, with $\rho(\mathbf{M}) = k_1 + \cdots + k_m$. $k_i$ is also called the risk capital allocated to $\mathbf{M}_i, i = 1, \ldots, m$. Note that given $m$ financial positions $X_1, \ldots, X_m$, if let $\mathbf{M} := (X_1, \ldots, X_m) \in \mathscr{X}^m$, $\mathbf{M}_i := X_i \mathbf{e_i}$, $1 \le i \le m$, and $\rho(\mathbf{M}) := \tilde{\rho}(X_1 + \cdot + X_m)$ for some fixed univariate risk measure $\tilde{\rho}$, then the above-mentioned capital allocation problem can recover that of Denault [7], as stated in the introduction part.

Inspired by Kalkbrener [13,14], we assume that the risk capital $k_i$ allocated to $\mathbf{M}_i$ depends only on $\mathbf{M}_i$ and $\mathbf{M}$, but not on the decomposition of the rest

$$\mathbf{M} - \mathbf{M}_i = \sum_{j \neq i} \mathbf{M}_j$$

of $\mathbf{M} \in \mathscr{X}^N$. More formally, let $\mathbf{M} = \mathbf{M}_1^* + \cdots + \mathbf{M}_n^*$ be another decomposition of $\mathbf{M}$ with capital allocations $l_1, \ldots, l_n$. If $\mathbf{M}_i = \mathbf{M}_j^*$ for indices $i, j$, then $k_i = l_j$.

Based on the assumption above, a capital allocation principle $\Lambda$ can be defined as a function from $\mathscr{X}^N \times \mathscr{X}^N$ to $\mathbf{R}$. $\Lambda(\mathbf{M}_i, \mathbf{M})$ represents the risk capital allocated to $\mathbf{M}_i$ if $\mathbf{M}_i$ is considered as a subportfolio of the portfolio $\mathbf{M}$. $\Lambda$ is called a capital allocation principle with respect to the risk measure $\rho$ if it satisfies $\Lambda(\mathbf{M}, \mathbf{M}) = \rho(\mathbf{M})$, that is, the risk capital allocated to $\mathbf{M}$ (considered as a stand-alone portfolio) is the risk capital $\rho(\mathbf{M})$ of $\mathbf{M}$. Hence, we introduce the following definition.

DEFINITION 3.1: *Let $\rho : \mathscr{X}^N \to \mathbf{R}$ be a multivariate risk measure. A capital allocation principle with respect to $\rho$ is a function $\Lambda$ from $\mathscr{X}^N \times \mathscr{X}^N$ to $\mathbf{R}$ such that for every $\mathbf{M} \in \mathscr{X}^N$,*

$$\Lambda(\mathbf{M}, \mathbf{M}) = \rho(\mathbf{M}).$$

Next, we will introduce some properties (or axioms) for a capital allocation principle $\Lambda$ with respect to a multivariate risk measure $\rho$. These properties are very useful in discussing the existence and uniqueness of a capital allocation principle with respect to a multivariate risk measure. On the other hand, in the one-dimensional case, the linearity and diversifying properties of the capital allocation principle $\Lambda$ alternatively formalize the capital allocation principles proposed by Denault [7]. Hence, Definition 3.2 (1) and (2) below can be

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considered as not only multidimensional extension of Kalkbrener [13], but also alternative multidimensional formalization of the capital allocation principles proposed by Denault [7].

DEFINITION 3.2: *Let $\Lambda$ be a capital allocation principle with respect to a multivariate risk measure $\rho$. Then*

(1) *$\Lambda$ is called linear, if*

$$\Lambda(a\mathbf{M}_1 + b\mathbf{M}_2, \mathbf{M}) = a\Lambda(\mathbf{M}_1, \mathbf{M}) + b\Lambda(\mathbf{M}_2, \mathbf{M})$$

*for all $a, b \in \mathbf{R}$, $\mathbf{M}$, $\mathbf{M}_1$, $\mathbf{M}_2 \in \mathscr{X}^N$.*

(2) *$\Lambda$ is called diversifying, if*

$$\Lambda(\mathbf{M}_0, \mathbf{M}) \leq \Lambda(\mathbf{M}_0, \mathbf{M}_0)$$

*for all $\mathbf{M}_0$, $\mathbf{M} \in \mathscr{X}^N$.*

(3) *$\Lambda$ is called continuous at $\mathbf{M} \in \mathscr{X}^N$, if*

$$\lim_{\varepsilon \to 0} \Lambda(\mathbf{M}_0, \mathbf{M} + \varepsilon\mathbf{M}_0) = \Lambda(\mathbf{M}_0, \mathbf{M}) \quad \textit{for all} \quad \mathbf{M}_0 \in \mathscr{X}^N.$$

The following theorem is a multidimensional extension of Theorem 3.1 of Kalkbrener [13].

THEOREM 3.1: *Let $\Lambda$ be a linear and diversifying capital allocation principle with respect to a multivariate risk measure $\rho$. If $\Lambda$ is continuous at $\mathbf{M} \in \mathscr{X}^N$, then for all $\mathbf{M}_0 \in \mathscr{X}^N$,*

$$\Lambda(\mathbf{M}_0, \mathbf{M}) = \lim_{\varepsilon \to 0} \frac{\rho(\mathbf{M} + \varepsilon\mathbf{M}_0) - \rho(\mathbf{M})}{\varepsilon}.$$

PROOF: Let $\varepsilon, \bar{\varepsilon} \in \mathbf{R}$. Since $\Lambda$ is linear and diversifying,

$$\begin{aligned}\rho(\mathbf{M} + \bar{\varepsilon}\mathbf{M}_0) &\geq \Lambda(\mathbf{M} + \bar{\varepsilon}\mathbf{M}_0, \mathbf{M} + \varepsilon\mathbf{M}_0) \\ &= \Lambda((\mathbf{M} + \varepsilon\mathbf{M}_0) + (\bar{\varepsilon} - \varepsilon)\mathbf{M}_0, \mathbf{M} + \varepsilon\mathbf{M}_0) \\ &= \rho(\mathbf{M} + \varepsilon\mathbf{M}_0) + (\bar{\varepsilon} - \varepsilon)\Lambda(\mathbf{M}_0, \mathbf{M} + \varepsilon\mathbf{M}_0)\end{aligned}$$

and

$$\begin{aligned}\rho(\mathbf{M} + \varepsilon\mathbf{M}_0) &\geq \Lambda(\mathbf{M} + \varepsilon\mathbf{M}_0, \mathbf{M} + \bar{\varepsilon}\mathbf{M}_0) \\ &= \Lambda((\mathbf{M} + \bar{\varepsilon}\mathbf{M}_0) + (\varepsilon - \bar{\varepsilon})\mathbf{M}_0, \mathbf{M} + \bar{\varepsilon}\mathbf{M}_0) \\ &= \rho(\mathbf{M} + \bar{\varepsilon}\mathbf{M}_0) + (\varepsilon - \bar{\varepsilon})\Lambda(\mathbf{M}_0, \mathbf{M} + \bar{\varepsilon}\mathbf{M}_0).\end{aligned}$$

If $\varepsilon < \bar{\varepsilon}$, then

$$\Lambda(\mathbf{M}_0, \mathbf{M} + \varepsilon\mathbf{M}_0) \leq \frac{\rho(\mathbf{M} + \bar{\varepsilon}\mathbf{M}_0) - \rho(\mathbf{M} + \varepsilon\mathbf{M}_0)}{\bar{\varepsilon} - \varepsilon} \leq \Lambda(\mathbf{M}_0, \mathbf{M} + \bar{\varepsilon}\mathbf{M}_0). \tag{3.1}$$

Since $\Lambda$ is continuous at $\mathbf{M}$, (3.1) implies

$$\Lambda(\mathbf{M}_0, \mathbf{M}) = \lim_{\varepsilon \to 0} \frac{\rho(\mathbf{M} + \varepsilon\mathbf{M}_0) - \rho(\mathbf{M})}{\varepsilon}.$$

Theorem 3.1 is proved. ■

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Theorem 3.1 says that given a linear and diversifying capital allocation principle $\Lambda$ with respect to a multivariate risk measure $\rho$, if $\Lambda$ is continuous at $\mathbf{M} \in \mathscr{X}^N$, then the risk capital $\Lambda(\mathbf{M}_0, \mathbf{M})$ allocated to $\mathbf{M}_0 \in \mathscr{X}^N$, considered as a subportfolio of $\mathbf{M} \in \mathscr{X}^N$, is uniquely determined by the derivative of $\rho$ at $\mathbf{M}$ in the direction of $\mathbf{M}_0$. Next, we will define a precise capital allocation principle with respect to any given positively homogeneous and subadditive multivariate risk measure.

DEFINITION 3.3: *Let $\rho : \mathscr{X}^N \to \mathbf{R}$ be a positively homogeneous and subadditive risk measure. For every $\mathbf{M} \in \mathscr{X}^N$, let $h_{\mathbf{M}}$ be an element of $H_\rho$ with $h_{\mathbf{M}}(\mathbf{M}) = \rho(\mathbf{M})$ and define for every $\mathbf{M}_0, \mathbf{M} \in \mathscr{X}^N$,*

$$\Lambda_\rho(\mathbf{M}_0, \mathbf{M}) := h_{\mathbf{M}}(\mathbf{M}_0).$$

Note that the existence of the element $h_{\mathbf{M}}$ in Definition 3.3 has been shown in the proof of Theorem 2.1. Moreover, $\Lambda_\rho$ is indeed a capital allocation principle with respect to $\rho$. However, $h_{\mathbf{M}}$ may not be unique in general. In other words, we may have different capital allocation principles $\Lambda_\rho$ by choosing different $h_{\mathbf{M}}$. The uniqueness of the capital allocation principle $\Lambda_\rho$ will be characterized in Theorem 3.2 below.

The following proposition states the implications between the positive homogeneity and subadditivity of a multivariate risk measure $\rho$ and the linearity and diversification of the capital allocation principle $\Lambda$ with respect to $\rho$.

PROPOSITION 3.1:

(i) *If there exists a linear and diversifying capital allocation principle $\Lambda$ with respect to a multivariate risk measure $\rho$, then $\rho$ is positively homogeneous and subadditive.*

(ii) *If a multivariate risk measure $\rho$ is positively homogeneous and subadditive, then $\Lambda_\rho$ defined in Definition 3.3 is a linear and diversifying capital allocation principle with respect to $\rho$.*

PROOF: (1) Let $\mathbf{M}_0, \mathbf{M} \in \mathscr{X}^N$. For any $a \geq 0$,

$$a\Lambda(\mathbf{M}, \mathbf{M}) = \Lambda(a\mathbf{M}, \mathbf{M}) \leq \Lambda(a\mathbf{M}, a\mathbf{M}) = a\Lambda(\mathbf{M}, a\mathbf{M}) \leq a\Lambda(\mathbf{M}, \mathbf{M}).$$

Hence,

$$a\rho(\mathbf{M}) = a\Lambda(\mathbf{M}, \mathbf{M}) = \Lambda(a\mathbf{M}, a\mathbf{M}) = \rho(a\mathbf{M}),$$

which shows that $\rho$ is positively homogeneous. Note that

$$\begin{aligned}\rho(\mathbf{M}_0 + \mathbf{M}) &= \Lambda(\mathbf{M}_0, \mathbf{M}_0 + \mathbf{M}) + \Lambda(\mathbf{M}, \mathbf{M}_0 + \mathbf{M}) \leq \Lambda(\mathbf{M}_0, \mathbf{M}_0) + \Lambda(\mathbf{M}, \mathbf{M}) \\ &= \rho(\mathbf{M}_0) + \rho(\mathbf{M}),\end{aligned}$$

which shows that $\rho$ is subadditive.

(2) From Definition 3.3 and (2.1) it follows that $\Lambda_\rho$ is a linear and diversifying capital allocation principle with respect to $\rho$. Proposition 3.1 is proved. ■

Proposition 3.1(ii) has already shown the existence of the capital allocation principle with respect to a positively homogeneous and subadditive multivariate risk measure. The following theorem characterizes the uniqueness of the capital allocation principle $\Lambda_\rho$ with respect to a positively homogeneous and subadditive multivariate risk measure $\rho$.

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THEOREM 3.2: *Let $\rho : \mathscr{X}^N \to \mathbf{R}$ be a positively homogeneous and subadditive multivariate risk measure, $\Lambda_\rho$ be defined as in Definition 3.3 and $\mathbf{M} \in \mathscr{X}^N$. Then the following three conditions are equivalent:*

(i) *$\Lambda_\rho$ is continuous at $\mathbf{M}$, that is, for all $\mathbf{M}_0 \in \mathscr{X}^N$,*

$$\lim_{\varepsilon \to 0} \Lambda_\rho(\mathbf{M}_0, \mathbf{M} + \varepsilon \mathbf{M}_0) = \Lambda_\rho(\mathbf{M}_0, \mathbf{M}).$$

(ii) *The directional derivative*

$$\lim_{\varepsilon \to 0} \frac{\rho(\mathbf{M} + \varepsilon \mathbf{M}_0) - \rho(\mathbf{M})}{\varepsilon}$$

*exists for every $\mathbf{M}_0 \in \mathscr{X}^N$.*

(iii) *There exists a unique $h \in H_\rho$ with $h(\mathbf{M}) = \rho(\mathbf{M})$.*
*Moreover, if these conditions are satisfied, then for all $\mathbf{M}_0 \in \mathscr{X}^N$,*

$$\Lambda_\rho(\mathbf{M}_0, \mathbf{M}) = \lim_{\varepsilon \to 0} \frac{\rho(\mathbf{M} + \varepsilon \mathbf{M}_0) - \rho(\mathbf{M})}{\varepsilon}. \tag{3.2}$$

PROOF: We will borrow the idea of the proof of Kalkbrener [13], Theorem 4.3 to prove the theorem. Nevertheless, more delicate arguments will be needed.

(i) $\Rightarrow$(ii). (i)$\Rightarrow$(ii) and (3.2) follow from Proposition 3.1(ii) and Theorem 3.1.

(ii) $\Rightarrow$(iii). Let $h \in H_\rho$ with $h(\mathbf{M}) = \rho(\mathbf{M})$. Then for every $\varepsilon \in \mathbf{R}$ and $\mathbf{M}_0 \in \mathscr{X}^N$,

$$\rho(\mathbf{M} + \varepsilon \mathbf{M}_0) - \rho(\mathbf{M}) \geq h(\mathbf{M} + \varepsilon \mathbf{M}_0) - h(\mathbf{M}) = \varepsilon h(\mathbf{M}_0).$$

Therefore,

$$\lim_{\varepsilon \to 0^-} \frac{\rho(\mathbf{M} + \varepsilon \mathbf{M}_0) - \rho(\mathbf{M})}{\varepsilon} \leq h(\mathbf{M}_0) \leq \lim_{\varepsilon \to 0^+} \frac{\rho(\mathbf{M} + \varepsilon \mathbf{M}_0) - \rho(\mathbf{M})}{\varepsilon}$$

and the uniqueness of $h$ follows from the equality of these limits. Moreover, by the proof of Theorem 2.1 and Definition 3.3, $h$ is exactly the $h_{\mathbf{M}}$.

(iii) $\Rightarrow$(i). Note that for every $\varepsilon \in \mathbf{R}$ and every $\mathbf{M}_0 \in \mathscr{X}^N$,

$$\begin{aligned} \rho(\mathbf{M}) - \rho(-\varepsilon \mathbf{M}_0) &= \rho(\mathbf{M} + \varepsilon \mathbf{M}_0 - \varepsilon \mathbf{M}_0) - \rho(-\varepsilon \mathbf{M}_0) \\ &\leq \rho(\mathbf{M} + \varepsilon \mathbf{M}_0) \leq \rho(\mathbf{M}) + \rho(\varepsilon \mathbf{M}_0) \end{aligned}$$

implies

$$\lim_{\varepsilon \to 0} \rho(\mathbf{M} + \varepsilon \mathbf{M}_0) = \rho(\mathbf{M}). \tag{3.3}$$

It follows from (3.1) that for a linear and diversifying capital allocation principle $\Lambda_\rho$,

$$\lim_{\varepsilon \to 0^-} \Lambda_\rho(\mathbf{M}_0, \mathbf{M} + \varepsilon \mathbf{M}_0) \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \Lambda_\rho(\mathbf{M}_0, \mathbf{M} + \varepsilon \mathbf{M}_0)$$

exist. Let $U^N$ be the linear subspace of $\mathscr{X}^N$ spanned by $\mathbf{M}_0$ and $\mathbf{M}$, and define the linear function $f : U^N \to \mathbf{R}$ by

$$f(\alpha \mathbf{M} + \beta \mathbf{M}_0) := \alpha \rho(\mathbf{M}) + \beta \lim_{\varepsilon \to 0^+} \Lambda_\rho(\mathbf{M}_0, \mathbf{M} + \varepsilon \mathbf{M}_0).$$

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It follows from (3.3) that for every $\alpha, \beta \in \mathbf{R}$

$$\begin{aligned} f(\alpha\mathbf{M}+\beta\mathbf{M}_0) &= \alpha \lim_{\varepsilon\to 0^+} \Lambda_\rho(\mathbf{M}+\varepsilon\mathbf{M}_0, \mathbf{M}+\varepsilon\mathbf{M}_0) + \beta \lim_{\varepsilon\to 0^+} \Lambda_\rho(\mathbf{M}_0, \mathbf{M}+\varepsilon\mathbf{M}_0) \\ &= \lim_{\varepsilon\to 0^+} \Lambda_\rho(\alpha\mathbf{M}+\beta\mathbf{M}_0, \mathbf{M}+\varepsilon\mathbf{M}_0) + \alpha \lim_{\varepsilon\to 0^+} \varepsilon\cdot\Lambda_\rho(\mathbf{M}_0, \mathbf{M}+\varepsilon\mathbf{M}_0) \\ &= \lim_{\varepsilon\to 0^+} \Lambda_\rho(\alpha\mathbf{M}+\beta\mathbf{M}_0, \mathbf{M}+\varepsilon\mathbf{M}_0) \\ &\le \lim_{\varepsilon\to 0^+} \Lambda_\rho(\alpha\mathbf{M}+\beta\mathbf{M}_0, \alpha\mathbf{M}+\beta\mathbf{M}_0) \\ &= \rho(\alpha\mathbf{M}+\beta\mathbf{M}_0). \end{aligned}$$

Hence, $f \le \rho$ on $U^N$. By the Hahn–Banach Theorem, there exists a linear function $h$ on $\mathscr{X}^N$ with

$$h = f \quad \text{on } U^N \quad \text{and} \quad h \le \rho \quad \text{on } \mathscr{X}^N.$$

Hence, $h$ is the unique element of $H_\rho$ with $h(\mathbf{M}) = f(\mathbf{M}) = \rho(\mathbf{M})$ and, therefore, $h(\cdot) = h_{\mathbf{M}}(\cdot) = \Lambda_\rho(\cdot, \mathbf{M})$. Consequently, from the definition of linear function $f$ it follows that

$$\Lambda_\rho(\mathbf{M}_0, \mathbf{M}) = \lim_{\varepsilon\to 0^+} \Lambda_\rho(\mathbf{M}_0, \mathbf{M}+\varepsilon\mathbf{M}_0).$$

$$\Lambda_\rho(\mathbf{M}_0, \mathbf{M}) = \lim_{\varepsilon\to 0^-} \Lambda_\rho(\mathbf{M}_0, \mathbf{M}+\varepsilon\mathbf{M}_0)$$

can be shown in the same way. Hence, $\Lambda_\rho$ is continuous at $\mathbf{M}$. Theorem 3.2 is proved. ■

## 4. EXAMPLES

In this section, we will provide examples concerning capital allocations for the multivariate risk measures based on mean and standard deviation, which are positively homogeneous and subadditive but may be lack of monotonicity.

The popular and celebrated mean-variance risk measure was introduced by Markowitz [16]. For the univariate mean-standard-deviation risk measures, we refer to Deprez and Gerber [8], Föllmer and Schied [11], Kalkbrener [13] and Ahmadi-Javid [1]. Next, we will discuss the capital allocation principles for the multivariate risk measures based on mean and standard deviation, including the multivariate mean-standard-deviation risk measures. Note that these multivariate risk measures are positively homogeneous and subadditive, but may not be monotonous, see Propositions 4.1, 4.3, and 4.4 below. In the rest of this section, we further assume that $(\Omega_1, \mathscr{F}_1, P_1) = \cdots = (\Omega_N, \mathscr{F}_N, P_N) := (\Omega, \mathscr{F}, P)$ with $2 \le p_i \le +\infty$, $1 \le i \le N$. For the reason of introducing the multivariate risk measures $\rho_{1,c}$ and $\rho_{2,c}$ below, see Remark 4.2 below.

DEFINITION 4.1: *Let $c$ be a nonnegative real number. Define the multivariate risk measures $\rho_{1,c}$ and $\rho_{2,c}$ on $\mathscr{H}^N$ and the capital allocation principles $\Lambda_{1,c}$ and $\Lambda_{2,c}$, respectively, by*

$$\rho_{1,c}(\mathbf{M}) := c\cdot\sum_{i=1}^N \sqrt{\mathrm{Var}(X_i)} + \sum_{i=1}^N \mathbb{E}(X_i),$$

$$\rho_{2,c}(\mathbf{M}) := c\cdot\sqrt{\sum_{i=1}^N \mathrm{Var}(X_i)} + \sum_{i=1}^N \mathbb{E}(X_i),$$

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$$\Lambda_{1,c}(\mathbf{M}_0,\mathbf{M}) := c\cdot\sum_{i=1}^{N}\frac{\mathrm{Cov}(Y_i,X_i)}{\sqrt{\mathrm{Var}(X_i)}}\cdot 1_{[\mathrm{Var}(X_i)>0]}+\sum_{i=1}^{N}\mathbb{E}(Y_i),$$

$$\Lambda_{2,c}(\mathbf{M}_0,\mathbf{M}) := \begin{cases} c\cdot\sum_{i=1}^{N}\dfrac{\mathrm{Cov}(Y_i,X_i)}{\sqrt{\sum_{i=1}^{N}\mathrm{Var}(X_i)}}+\sum_{i=1}^{N}\mathbb{E}(Y_i), & \text{if } \sum_{i=1}^{N}\mathrm{Var}(X_i)>0,\\ \sum_{i=1}^{N}\mathbb{E}(Y_i), & \text{otherwise,}\end{cases}$$

*for* $\mathbf{M}_0=(Y_1,\dots,Y_N),\mathbf{M}=(X_1,\dots,X_N)\in\mathscr{H}^N$, *where* $\mathbb{E}(X_i)$ *and* $\mathrm{Var}(X_i)$ *denote the expectation and the variance of* $X_i\in\mathscr{H}$, *respectively, and* $\mathrm{Cov}(Y_i,X_i)$ *means the covariance of* $Y_i$ *and* $X_i\in\mathscr{H}$. *For* $1\le i\le N$, $\mathrm{Cov}(Y_i,X_i)/\sqrt{\mathrm{Var}(X_i)}\cdot 1_{[\mathrm{Var}(X_i)>0]}$ *stands for* $\frac{\mathrm{Cov}(Y_i,X_i)}{/}\sqrt{\mathrm{Var}(X_i)}$ *if* $\mathrm{Var}(X_i)>0$ *and* $0$ *otherwise.*

*Remark 4.1*: Let $c$ be a nonnegative real number. Then

(i) $\rho_{2,c}(\mathbf{M})\le\rho_{1,c}(\mathbf{M})$ for all $\mathbf{M}\in\mathscr{H}^N$.

(ii) $\rho_{1,c}(\mathbf{M})=\sum_{i=1}^{N}\rho_{1,c}(X_i\mathbf{e}_i)$ for all $\mathbf{M}=(X_1,\dots,X_N)\in\mathscr{H}^N$. $\rho_{2,c}(\mathbf{M})\le\sum_{i=1}^{N}\rho_{2,c}(X_i\mathbf{e}_i)$ for all $\mathbf{M}=(X_1,\dots,X_N)\in\mathscr{H}^N$.

PROPOSITION 4.1: *Let* $c$ *be a nonnegative real number. Then*

(i) *The risk measures* $\rho_{1,c}$ *and* $\rho_{2,c}$ *are translation invariant, positively homogeneous and subadditive.*

(ii) *The risk measures* $\rho_{1,0}$ *and* $\rho_{2,0}$ *are monotonous.*

(iii) *If* $c>0$ *and there exists an* $A\in\mathscr{F}$ *with* $0<P(A)<c^2/(1+c^2)$, *then* $\rho_{1,c}$ *is not monotonous.*

(iv) *If* $c>0$ *and there exists an* $A\in\mathscr{F}$ *with* $0<P(A)<c^2/(N+c^2)$, *then* $\rho_{2,c}$ *is not monotonous.*

PROOF: (i) It is not hard to verify the translation invariance and the positive homogeneity of $\rho_{1,c}$ and $\rho_{2,c}$. Next, we will show the subadditivity of $\rho_{1,c}$ and $\rho_{2,c}$. Note that $\sqrt{\mathrm{Var}(X+Y)}\le\sqrt{\mathrm{Var}(X)}+\sqrt{\mathrm{Var}(Y)}$ for every $X,Y\in\mathscr{H}$, and $(a_1b_1+\cdots+a_kb_k)^2\le(a_1^2+\cdots+a_k^2)(b_1^2+\cdots+b_k^2)$ for any real numbers $a_1,\dots,a_k,b_1,\dots,b_k$. For every $\mathbf{M}=(X_1,\dots,X_N)$, $\mathbf{M}_0=(Y_1,\dots,Y_N)\in\mathscr{H}^N$, we have

$$\begin{aligned}\rho_{1,c}(\mathbf{M}+\mathbf{M}_0) &= c\cdot\sum_{i=1}^{N}\sqrt{\mathrm{Var}(X_i+Y_i)}+\sum_{i=1}^{N}\mathbb{E}(X_i+Y_i)\\ &\le c\cdot\left(\sum_{i=1}^{N}\sqrt{\mathrm{Var}(X_i)}+\sum_{i=1}^{N}\sqrt{\mathrm{Var}(Y_i)}\right)+\sum_{i=1}^{N}\mathbb{E}(X_i)+\sum_{i=1}^{N}\mathbb{E}(Y_i)\\ &= \rho_{1,c}(\mathbf{M})+\rho_{1,c}(\mathbf{M}_0)\end{aligned}$$

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and

$$\begin{aligned}
&\rho_{2,c}(\mathbf{M}+\mathbf{M}_0)\\
&\quad= c\cdot\sqrt{\sum_{i=1}^{N}\mathrm{Var}(X_i+Y_i)}+\sum_{i=1}^{N}\mathbb{E}(X_i+Y_i)\\
&\quad= c\cdot\sqrt{\sum_{i=1}^{N}\mathrm{Var}(X_i)+\sum_{i=1}^{N}\mathrm{Var}(Y_i)+2\sum_{i=1}^{N}\mathrm{Cov}(X_i,Y_i)}+\sum_{i=1}^{N}\mathbb{E}(X_i+Y_i)\\
&\quad\le c\cdot\sqrt{\sum_{i=1}^{N}\mathrm{Var}(X_i)+\sum_{i=1}^{N}\mathrm{Var}(Y_i)+2\sum_{i=1}^{N}\sqrt{\mathrm{Var}(X_i)\mathrm{Var}(Y_i)}}+\sum_{i=1}^{N}\mathbb{E}(X_i+Y_i)\\
&\quad\le c\cdot\sqrt{\sum_{i=1}^{N}\mathrm{Var}(X_i)+\sum_{i=1}^{N}\mathrm{Var}(Y_i)+2\sqrt{\left(\sum_{i=1}^{N}\mathrm{Var}(X_i)\right)\left(\sum_{i=1}^{N}\mathrm{Var}(Y_i)\right)}}\\
&\qquad+\sum_{i=1}^{N}\mathbb{E}(X_i)+\sum_{i=1}^{N}\mathbb{E}(Y_i)\\
&\quad= c\cdot\left(\sqrt{\sum_{i=1}^{N}\mathrm{Var}(X_i)}+\sqrt{\sum_{i=1}^{N}\mathrm{Var}(Y_i)}\right)+\sum_{i=1}^{N}\mathbb{E}(X_i)+\sum_{i=1}^{N}\mathbb{E}(Y_i)\\
&\quad=\rho_{2,c}(\mathbf{M})+\rho_{2,c}(\mathbf{M}_0),
\end{aligned}$$

which imply the subadditivity of $\rho_{1,c}$ and $\rho_{2,c}$, respectively.

(ii) It is obvious.

(iii) Let $c>0$ and $A\in\mathscr{F}$ with $0<p<c^2/(1+c^2)$, where $p:=P(A)$. Define $\mathbf{M}=(X_1,\dots,X_N)\in\mathscr{H}^N$ with $X_i=-1/Np$ on $A$, and 0 otherwise, for $1\le i\le N$. Then for all $1\le i\le N$,

$$\mathbb{E}(X_i)=-\frac{1}{N},\quad \mathrm{Var}(X_i)=\frac{1}{N^2}\left(\frac{1}{p}-1\right).$$

Hence,

$$\rho_{1,c}(\mathbf{M})=c\cdot\sum_{i=1}^{N}\sqrt{\mathrm{Var}(X_i)}+\sum_{i=1}^{N}\mathbb{E}(X_i)=c\cdot\sqrt{\frac{1}{p}-1}-1.$$

From $1/p>(c^2+1)/c^2=1/(c^2+1)$, we obtain $\rho_{1,c}(\mathbf{M})>0$ and thus,

$$\rho_{1,c}(\mathbf{M})>0=\rho_{1,c}(\mathbf{0}),\quad\text{but}\quad\mathbf{M}\le\mathbf{0}.$$

Therefore, $\rho_{1,c}$ is not monotonous.

(iv) The proof is similar to that of (iii). Proposition 4.1 is proved. ■

By Proposition 3.1(ii), there exist linear and diversifying capital allocation principles with respect to $\rho_{1,c}$ and $\rho_{2,c}$, respectively. The following proposition provides explicit capital allocation principles.

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PROPOSITION 4.2: *Let $c$ be a nonnegative real number. Then*

(i) *$\Lambda_{1,c}$ and $\Lambda_{2,c}$ are linear and diversifying capital allocation principles with respect to $\rho_{1,c}$ and $\rho_{2,c}$, respectively.*

(ii) *Given $\mathbf{M} = (X_1, \ldots, X_N) \in \mathscr{H}^N$, if $\mathrm{Var}(X_i) > 0$ for all $1 \le i \le N$, then $\Lambda_{1,c}$ is continuous at $\mathbf{M}$, and*

$$\Lambda_{1,c}(\mathbf{M}_0, \mathbf{M}) = \lim_{\varepsilon \to 0} \frac{\rho_{1,c}(\mathbf{M} + \varepsilon \mathbf{M}_0) - \rho_{1,c}(\mathbf{M})}{\varepsilon} \tag{4.1}$$

*for every $\mathbf{M}_0 \in \mathscr{H}^N$.*

(iii) *Given $\mathbf{M} = (X_1, \ldots, X_N) \in \mathscr{H}^N$, if $\sum_{i=1}^N \mathrm{Var}(X_i) > 0$, then $\Lambda_{2,c}$ is continuous at $\mathbf{M}$, and*

$$\Lambda_{2,c}(\mathbf{M}_0, \mathbf{M}) = \lim_{\varepsilon \to 0} \frac{\rho_{2,c}(\mathbf{M} + \varepsilon \mathbf{M}_0) - \rho_{2,c}(\mathbf{M})}{\varepsilon}$$

*for every $\mathbf{M}_0 \in \mathscr{H}^N$.*

PROOF: (i) It is not hard to verify the linearity of $\Lambda_{1,c}$ and $\Lambda_{2,c}$. Next, we will show the diversifying property of $\Lambda_{1,c}$ and $\Lambda_{2,c}$. For every $\mathbf{M} = (X_1, \ldots, X_N)$, $\mathbf{M}_0 = (Y_1, \ldots, Y_N) \in \mathscr{H}^N$,

$$\begin{aligned}
\Lambda_{1,c}(\mathbf{M}_0, \mathbf{M}) &= c \cdot \sum_{i=1}^N \frac{\mathrm{Cov}(Y_i, X_i)}{\sqrt{\mathrm{Var}(X_i)}} \cdot 1_{[\mathrm{Var}(X_i)>0]} + \sum_{i=1}^N \mathbb{E}(Y_i) \\
&\le c \cdot \sum_{i=1}^N \frac{\sqrt{\mathrm{Var}(Y_i)}\sqrt{\mathrm{Var}(X_i)}}{\sqrt{\mathrm{Var}(X_i)}} \cdot 1_{[\mathrm{Var}(X_i)>0]} + \sum_{i=1}^N \mathbb{E}(Y_i) \\
&= c \cdot \sum_{i=1}^N \sqrt{\mathrm{Var}(Y_i)} + \sum_{i=1}^N \mathbb{E}(Y_i) \\
&= \Lambda_{1,c}(\mathbf{M}_0, \mathbf{M}_0).
\end{aligned}$$

If $\sum_{i=1}^N \mathrm{Var}(X_i) > 0$, then

$$\begin{aligned}
\Lambda_{2,c}(\mathbf{M}_0, \mathbf{M}) &= c \cdot \sum_{i=1}^N \frac{\mathrm{Cov}(Y_i, X_i)}{\sqrt{\sum_{i=1}^N \mathrm{Var}(X_i)}} + \sum_{i=1}^N \mathbb{E}(Y_i) \\
&\le c \cdot \frac{\sum_{i=1}^N \sqrt{\mathrm{Var}(Y_i)}\sqrt{\mathrm{Var}(X_i)}}{\sqrt{\sum_{i=1}^N \mathrm{Var}(X_i)}} + \sum_{i=1}^N \mathbb{E}(Y_i) \\
&\le c \cdot \frac{\sqrt{\sum_{i=1}^N \mathrm{Var}(Y_i)}\sqrt{\sum_{i=1}^N \mathrm{Var}(X_i)}}{\sqrt{\sum_{i=1}^N \mathrm{Var}(X_i)}} + \sum_{i=1}^N \mathbb{E}(Y_i) \\
&= c \cdot \sqrt{\sum_{i=1}^N \mathrm{Var}(Y_i)} + \sum_{i=1}^N \mathbb{E}(Y_i) \\
&= \Lambda_{2,c}(\mathbf{M}_0, \mathbf{M}_0).
\end{aligned}$$

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Hence, $\Lambda_{1,c}$ and $\Lambda_{2,c}$ are linear and diversifying capital allocation principles with respect to $\rho_{1,c}$ and $\rho_{2,c}$, respectively.

(ii) For every $\mathbf{M}_0 = (Y_1, \ldots, Y_N), \mathbf{M} = (X_1, \ldots, X_N) \in \mathscr{H}^N$, if $\mathrm{Var}(X_i) > 0$ for all $1 \leq i \leq N$, then for all $\varepsilon$ small enough, $\mathrm{Var}(X_i + \varepsilon Y_i) > 0$ for all $1 \leq i \leq N$ since $\lim_{\varepsilon \to 0} \mathrm{Var}(X_i + \varepsilon Y_i) = \mathrm{Var}(X_i) > 0$. Thus,

$$\begin{aligned}\lim_{\varepsilon \to 0} \Lambda_{1,c}(\mathbf{M}_0, \mathbf{M} + \varepsilon \mathbf{M}_0) &= \lim_{\varepsilon \to 0} c \cdot \sum_{i=1}^{N} \frac{\mathrm{Cov}(Y_i, X_i + \varepsilon Y_i)}{\sqrt{\mathrm{Var}(X_i + \varepsilon Y_i)}} + \sum_{i=1}^{N} \mathbb{E}(Y_i) \\ &= c \cdot \sum_{i=1}^{N} \frac{\mathrm{Cov}(Y_i, X_i)}{\sqrt{\mathrm{Var}(X_i)}} + \sum_{i=1}^{N} \mathbb{E}(Y_i) \\ &= \Lambda_{1,c}(\mathbf{M}_0, \mathbf{M}).\end{aligned}$$

Hence, $\Lambda_{1,c}$ is continuous at every $\mathbf{M} \in \mathscr{H}^N$ provided with $\mathrm{Var}(X_i) > 0$ for all $1 \leq i \leq N$. Equality (4.1) follows from (i) and Theorem 3.1.

(iii) The proof is similar to that of (ii). Proposition 4.2 is proved. ■

Now, we turn to introduce the multivariate mean-standard-deviation risk measures, and discuss the capital allocation principles with respect to them.

Let $\mathbf{M} = (X_1, \ldots, X_N) \in \mathscr{H}^N$ be a portfolio. Then the total loss of $\mathbf{M}$ is $X := X_1 + \cdots + X_N$. Thus, we naturally reach at the following definition of the multivariate mean-standard-deviation risk measures.

Definition 4.2: *Let $c$ be a nonnegative real number. Define the multivariate mean-standard-deviation risk measure $\rho_{3,c}$ on $\mathscr{H}^N$ and the capital allocation principle $\Lambda_{3,c}$, respectively, by*

$$\rho_{3,c}(\mathbf{M}) := c \cdot \sqrt{\mathrm{Var}(X)} + \sum_{i=1}^{N} \mathbb{E}(X_i),$$

$$\Lambda_{3,c}(\mathbf{M}_0, \mathbf{M}) := \begin{cases} c \cdot \dfrac{\mathrm{Cov}(Y, X)}{\sqrt{\mathrm{Var}(X)}} + \displaystyle\sum_{i=1}^{N} \mathbb{E}(Y_i), & \text{if } \mathrm{Var}(X) > 0, \\ \displaystyle\sum_{i=1}^{N} \mathbb{E}(Y_i), & \text{otherwise,} \end{cases}$$

*for $\mathbf{M}_0 = (Y_1, \ldots, Y_N)$, $\mathbf{M} = (X_1, \ldots, X_N) \in \mathscr{H}^N$, where $Y := Y_1 + \cdots + Y_N$ and $X := X_1 + \cdots + X_N$.*

*Remark 4.2*: Since

$$\sqrt{\mathrm{Var}(X_1 + \cdots + X_N)} \leq \sqrt{\mathrm{Var}(X_1)} + \cdots + \sqrt{\mathrm{Var}(X_N)}$$

for every $\mathbf{M} = (X_1, \ldots, X_N) \in \mathscr{H}^N$, $\rho_{3,c}(\mathbf{M}) \leq \rho_{1,c}(\mathbf{M})$. Note that $\rho_{1,c}(\mathbf{M})$ is exactly the sum of the values of the univariate mean-standard-deviation risk measure at the components of the portfolio $\mathbf{M}$. Thus, if one would use $\rho_{1,c}$ to evaluate the risk of the portfolio $\mathbf{M}$, it would be quite possible to overestimate the risk of $\mathbf{M}$, because $\rho_{1,c}$ does not take into account the possible dependence between the components of $\mathbf{M}$. The possible dependence structure between the components of $\mathbf{M}$ reflects the hedging technology. In other words,

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when $\rho_{1,c}$ is used to evaluate the risk of $\mathbf{M}$, the advantage of hedging technology is neglected. On the contrary, if one uses $\rho_{3,c}$ to evaluate the risk of the portfolio $\mathbf{M}$, then both the risks of the components of $\mathbf{M}$ and the possible dependence structure between the components of $\mathbf{M}$ are taken into account, and hence compared with $\rho_{1,c}(\mathbf{M})$, one could reach at a more reasonable evaluation of the risk of $\mathbf{M}$. This consideration is also illustrated by the inequality $\rho_{3,c}(\mathbf{M}) \leq \rho_{1,c}(\mathbf{M})$. In summary, the multivariate risk measure $\rho_{3,c}$ is more suitable and reasonable than the basically univariate one $\rho_{1,c}$ when one evaluates the risk of the portfolio $\mathbf{M}$. Observe that if the components of the portfolio $\mathbf{M}$ are mutually independent, then $\rho_{3,c}(\mathbf{M}) = \rho_{2,c}(\mathbf{M})$, and this mainly motivates us to discuss the multivariate risk measure $\rho_{2,c}$ in Definition 4.1.

Next, we will derive explicit capital allocation principle with respect to $\rho_{3,c}$. It turns out that given a portfolio $\mathbf{M} = (X_1, \ldots, X_N) \in \mathscr{H}^N$ with subportfolios $\mathbf{M}_1, \ldots, \mathbf{M}_m$, if $\mathrm{Var}(X_1 + \cdots + X_N) > 0$, then the risk capital allocated to $\mathbf{M}_i$, $1 \leq i \leq m$, is uniquely determined by the derivative of $\rho_{3,c}$ at $\mathbf{M}$ in the direction of $\mathbf{M}_i$, and is equal to $\Lambda_{3,c}(\mathbf{M}_i, \mathbf{M})$. Note that the case of $\mathrm{Var}(X_1 + \cdots + X_N) = 0$ should not be interesting, because in this case the probability of $X_1 + \cdots + X_N$ being a constant is zero provided that $(X_1, \ldots, X_N)$ is a continuous random vector.

PROPOSITION 4.3: *Let $c$ be a nonnegative real number. Then*

(i) *The risk measure $\rho_{3,c}$ is translation invariant, positively homogeneous and subadditive.*

(ii) *The risk measure $\rho_{3,0}$ is monotonous.*

(iii) *If $c > 0$ and there exists an $A \in \mathscr{F}$ with $0 < P(A) < c^2/(1 + c^2)$, then $\rho_{3,c}$ is not monotonous.*

(iv) *$\Lambda_{3,c}$ is a linear and diversifying capital allocation principle with respect to $\rho_{3,c}$.*

(v) *Given $\mathbf{M} = (X_1, \ldots, X_N) \in \mathscr{H}^N$, if $\mathrm{Var}(X) > 0$ with $X := X_1 + \cdots + X_N$, then $\Lambda_{3,c}$ is continuous at $\mathbf{M}$, and*

$$\Lambda_{3,c}(\mathbf{M}_0, \mathbf{M}) = \lim_{\varepsilon \to 0} \frac{\rho_{3,c}(\mathbf{M} + \varepsilon \mathbf{M}_0) - \rho_{3,c}(\mathbf{M})}{\varepsilon} \tag{4.2}$$

*for every $\mathbf{M}_0 \in \mathscr{H}^N$.*

PROOF: (i) It is not hard to verify the translation invariance and the positive homogeneity of $\rho_{3,c}$. Next, we will show the subadditivity of $\rho_{3,c}$. Note that $\sqrt{\mathrm{Var}(X+Y)} \leq \sqrt{\mathrm{Var}(X)} + \sqrt{\mathrm{Var}(Y)}$ for every $X, Y \in \mathscr{H}$. For every $\mathbf{M} = (X_1, \ldots, X_N)$, $\mathbf{M}_0 = (Y_1, \ldots, Y_N) \in \mathscr{H}^N$, write $X := X_1 + \cdots + X_N$, $Y := Y_1 + \cdots + Y_N$, then

$$\begin{aligned}
\rho_{3,c}(\mathbf{M} + \mathbf{M}_0) &= c \cdot \sqrt{\mathrm{Var}(X+Y)} + \sum_{i=1}^{N} \mathbb{E}(X_i + Y_i) \\
&\leq c \cdot \left( \sqrt{\mathrm{Var}(X)} + \sqrt{\mathrm{Var}(Y)} \right) + \sum_{i=1}^{N} \mathbb{E}(X_i) + \sum_{i=1}^{N} \mathbb{E}(Y_i) \\
&= \rho_{3,c}(\mathbf{M}) + \rho_{3,c}(\mathbf{M}_0)
\end{aligned}$$

which implies the subadditivity of $\rho_{3,c}$.

(ii) It is obvious.

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(iii) Let $c > 0$ and $A \in \mathscr{F}$ with $0 < p < c^2/(1+c^2)$, where $p := P(A)$. Define $\mathbf{M} = (X_1, \dots, X_N) \in \mathscr{H}^N$ with $X_i = -1/Np$ on $A$, and 0 otherwise, for $1 \le i \le N$. Then for all $1 \le i \le N$,

$$\mathbb{E}(X_i) = -\frac{1}{N}, \quad \mathrm{Var}(X_1 + \cdots + X_N) = \frac{1}{p} - 1.$$

Hence,

$$\rho_{3,c}(\mathbf{M}) = c \cdot \sqrt{\mathrm{Var}(X)} + \sum_{i=1}^{N} \mathbb{E}(X_i) = c \cdot \sqrt{\frac{1}{p} - 1} - 1.$$

Since $1/p > (c^2+1)/c^2 = 1/(c^2+1)$, $\rho_{3,c}(\mathbf{M}) > 0$, and thus

$$\rho_{3,c}(\mathbf{M}) > 0 = \rho_{3,c}(\mathbf{0}), \quad \text{but} \quad \mathbf{M} \le \mathbf{0}.$$

Therefore, $\rho_{3,c}$ is not monotonous.

(iv) It is not hard to verify the linearity of $\Lambda_{3,c}$. Next, we will show the diversifying property of $\Lambda_{3,c}$. For every $\mathbf{M}_0 = (Y_1, \dots, Y_N)$, $\mathbf{M} = (X_1, \dots, X_N) \in \mathscr{H}^N$, write $Y := Y_1 + \cdots + Y_N$, $X := X_1 + \cdots + X_N$, if $\sqrt{\mathrm{Var}(X)} > 0$, then

$$\begin{aligned}
\Lambda_{3,c}(\mathbf{M}_0, \mathbf{M}) &= c \cdot \frac{\mathrm{Cov}(Y, X)}{\sqrt{\mathrm{Var}(X)}} + \sum_{i=1}^{N} \mathbb{E}(Y_i) \\
&\le c \cdot \frac{\sqrt{\mathrm{Var}(Y)}\sqrt{\mathrm{Var}(X)}}{\sqrt{\mathrm{Var}(X)}} + \sum_{i=1}^{N} \mathbb{E}(Y_i) \\
&= c \cdot \sqrt{\mathrm{Var}(Y)} + \sum_{i=1}^{N} \mathbb{E}(Y_i) \\
&= \Lambda_{3,c}(\mathbf{M}_0, \mathbf{M}_0).
\end{aligned}$$

Hence, $\Lambda_{3,c}$ is a linear and diversifying capital allocation principle with respect to $\rho_{3,c}$.

(v) For every $\mathbf{M}_0 = (Y_1, \dots, Y_N)$, $\mathbf{M} = (X_1, \dots, X_N) \in \mathscr{H}^N$, write $Y := Y_1 + \cdots + Y_N$, $X := X_1 + \cdots + X_N$, if $\mathrm{Var}(X) > 0$, then for all $\varepsilon$ small enough, $\mathrm{Var}(X + \varepsilon Y) > 0$ since $\lim_{\varepsilon \to 0} \mathrm{Var}(X + \varepsilon Y) = \mathrm{Var}(X) > 0$. Thus,

$$\begin{aligned}
\lim_{\varepsilon \to 0} \Lambda_{3,c}(\mathbf{M}_0, \mathbf{M} + \varepsilon \mathbf{M}_0) &= \lim_{\varepsilon \to 0} c \cdot \frac{\mathrm{Cov}(Y, X + \varepsilon Y)}{\sqrt{\mathrm{Var}(X + \varepsilon Y)}} + \sum_{i=1}^{N} \mathbb{E}(Y_i) \\
&= c \cdot \frac{\mathrm{Cov}(Y, X)}{\sqrt{\mathrm{Var}(X)}} + \sum_{i=1}^{N} \mathbb{E}(Y_i) \\
&= \Lambda_{3,c}(\mathbf{M}_0, \mathbf{M}).
\end{aligned}$$

Hence, $\Lambda_{3,c}$ is continuous at every $\mathbf{M} \in \mathscr{H}^N$ provided with $\mathrm{Var}(X) > 0$. Equality (4.2) follows from (iv) and Theorem 3.1. Proposition 4.3 is proved. ■

In the rest of this section, we will generalize the multivariate mean-standard-deviation risk measure $\rho_{3,c}$ defined in Definition 4.2 to a more general case, and discuss the capital allocation principle with respect to it.

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Let the index set $I$ be a finite subset of $\mathbf{R}^N\backslash\{\mathbf{0}\}$. For each $\alpha \in I$, $c_\alpha$ is a positive constant. Given a portfolio $\mathbf{M} = (X_1, \ldots, X_N) \in \mathscr{H}^N$ and an $\alpha := (\alpha_1, \ldots, \alpha_N) \in I$, we denote by $\mathbf{M}^\alpha$ the portfolio $(\alpha_1 X_1, \ldots, \alpha_N X_N)$, where $\alpha_i$ stands for the share of the financial position $X_i$. Note that the components of the vector $\alpha$ can be negative, and negative components are understood as short selling.

Now, we will introduce the generalized multivariate mean-standard-deviation risk measures, and discuss the capital allocation principles with respect to them.

DEFINITION 4.3: *Define the generalized multivariate mean-standard-deviation risk measure $\rho_I$ on $\mathscr{H}^N$ and the capital allocation principle $\Lambda_I$, respectively, by*

$$\rho_I(\mathbf{M}) := \sum_{\alpha\in I} c_\alpha \sqrt{\mathrm{Var}(X^\alpha)} + \sum_{i=1}^N \mathbb{E}(X_i),$$

$$\Lambda_I(\mathbf{M}_0, \mathbf{M}) := \begin{cases} \displaystyle\sum_{\alpha\in I} c_\alpha \frac{\mathrm{Cov}(Y^\alpha, X^\alpha)}{\sqrt{\mathrm{Var}(X^\alpha)}} + \sum_{i=1}^N \mathbb{E}(Y_i), & \text{if } \mathrm{Var}(X^\alpha) > 0 \quad \text{for each} \quad \alpha \in I, \\ \displaystyle\sum_{i=1}^N \mathbb{E}(Y_i), & \text{otherwise,} \end{cases}$$

*for $\mathbf{M}_0 = (Y_1, \ldots, Y_N)$, $\mathbf{M} = (X_1, \ldots, X_N) \in \mathscr{H}^N$, where $Y^\alpha := \alpha_1 Y_1 + \cdots + \alpha_N Y_N$ and $X^\alpha := \alpha_1 X_1 + \cdots + \alpha_N X_N$ for $\alpha = (\alpha_1, \ldots, \alpha_N) \in I$.*

*Remark 4.3*: Note that for each $\alpha = (\alpha_1, \ldots, \alpha_N) \in I$, the total loss of the portfolio $\mathbf{M}^\alpha = (\alpha_1 X_1, \ldots, \alpha_N X_N)$ is $X^\alpha := \alpha_1 X_1 + \cdots + \alpha_N X_N$. Thus, in the definition of $\rho_I(\mathbf{M})$, the second term reflects the expected loss of the portfolio $\mathbf{M} = (X_1, \ldots, X_N)$, while the first term reflects the combination of the volatilities of as many possible portfolios $\mathbf{M}^\alpha$ as possible. When the index set $I$ is a singleton of the unit vector $\mathbf{1} = (1, \ldots, 1) \in \mathbf{R}^N$, then $\rho_I$ and $\Lambda_I$ become $\rho_{3,c}$ and $\Lambda_{3,c}$, respectively.

Next, we will derive explicit capital allocation principle with respect to $\rho_I$. It turns out that given a portfolio $\mathbf{M} = (X_1, \ldots, X_N) \in \mathscr{H}^N$ with subportfolios $\mathbf{M}_1, \ldots, \mathbf{M}_m$, if $\mathrm{Var}(\alpha_1 X_1 + \cdots + \alpha_N X_N) > 0$ for each $\alpha = (\alpha_1, \ldots, \alpha_N) \in I$, then the risk capital allocated to $\mathbf{M}_i$, $1 \le i \le m$, is uniquely determined by the derivative of $\rho_I$ at $\mathbf{M}$ in the direction of $\mathbf{M}_i$, and is equal to $\Lambda_I(\mathbf{M}_i, \mathbf{M})$. Note that the case of $\mathrm{Var}(\alpha_1 X_1 + \cdots + \alpha_N X_N) = 0$ for some $\alpha = (\alpha_1, \ldots, \alpha_N) \in I$ should not be interesting, because in this case the probability of $\alpha_1 X_1 + \cdots + \alpha_N X_N$ being a constant is zero provided that $(\alpha_1 X_1, \ldots, \alpha_N X_N)$ is a continuous random vector.

PROPOSITION 4.4:

(i) *The risk measure $\rho_I$ is translation invariant, positively homogeneous and subadditive.*

(ii) *If $\delta := \sum_{\alpha=(\alpha_1,\ldots,\alpha_N)\in I} c_\alpha |(\alpha_1 + \cdots + \alpha_N)| > 0$ and there exists an $A \in \mathscr{F}$ with $0 < P(A) < \delta^2/(N^2 + \delta^2)$, then $\rho_I$ is not monotonous.*

(iii) *$\Lambda_I$ is a linear and diversifying capital allocation principle with respect to $\rho_I$.*

(iv) *Given $\mathbf{M} = (X_1, \ldots, X_N) \in \mathscr{H}^N$, if $\mathrm{Var}(X^\alpha) > 0$ for each $\alpha = (\alpha_1, \ldots, \alpha_N) \in I$ with $X^\alpha := \alpha_1 X_1 + \cdots + \alpha_N X_N$, then $\Lambda_I$ is continuous at $\mathbf{M}$, and*

$$\Lambda_I(\mathbf{M}_0, \mathbf{M}) = \lim_{\varepsilon\to 0} \frac{\rho_I(\mathbf{M} + \varepsilon\mathbf{M}_0) - \rho_I(\mathbf{M})}{\varepsilon}$$

*for every $\mathbf{M}_0 \in \mathscr{H}^N$.*

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PROOF: By the same arguments of the proof of Proposition 4.3, we can steadily show (i), (iii), and (iv). Next, we proceed to show (ii).

Let $\delta > 0$ and $A \in \mathscr{F}$ with $0 < p < \delta^2/(N^2 + \delta^2)$, where $p := P(A)$. Define $\mathbf{M} = (X_1, \ldots, X_N) \in \mathscr{H}^N$ with $X_i = -1/Np$ on $A$, and 0 otherwise, for $1 \le i \le N$. Then for all $1 \le i \le N$ and each $\alpha = (\alpha_1, \ldots, \alpha_N) \in I$,

$$\mathbb{E}(X_i) = -\frac{1}{N}, \quad \mathrm{Var}(\alpha_1 X_1 + \cdots + \alpha_N X_N) = \frac{(1-p)(\alpha_1 + \cdots + \alpha_N)^2}{pN^2}.$$

Hence,

$$\rho_I(\mathbf{M}) = \sum_{\alpha \in I} c_\alpha \sqrt{\mathrm{Var}(X^\alpha)} + \sum_{i=1}^{N} \mathbb{E}(X_i) = \frac{\delta}{N}\sqrt{\frac{1}{p} - 1} - 1.$$

Since $\dfrac{1}{p} > \dfrac{\delta^2 + N^2}{\delta^2} = \dfrac{N^2}{\delta^2} + 1$, $\rho_I(\mathbf{M}) > 0$, and thus

$$\rho_I(\mathbf{M}) > 0 = \rho_I(\mathbf{0}), \quad \text{but} \quad \mathbf{M} \le \mathbf{0}.$$

Therefore, $\rho_I$ is not monotonous. Proposition 4.4 is proved. ■

## 5. CONCLUSION

In the present paper, we have proposed axioms of capital allocation principle for multivariate risk measures, which generalizes those of Denault [7], Kalkbrener [13,14] from univariate risk measures to multivariate risk measures. The existence and uniqueness of the capital allocation principles for positively homogeneous and subadditive multivariate risk measures are analyzed. For some multivariate risk measures based on mean and standard deviation, including the multivariate mean-standard-deviation risk measures, the explicit formulas for the capital allocation principles are provided. These examples have illustrated the proposed capital allocation principles.

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