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Finitely summable Fredholm modules over higher rank groups and lattices

by

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Abstract

We give a classification (up to smooth homotopy) of finitely summable Fredholm representations (Fredholm modules) over higher rank groups and lattices. Our results are a direct consequence of work of Bader, Furman, Gelander and Monod on generalizations of Kazhdan's property T for locally compact groups.

Key Words: Fredholm module, Schatten ideal, bivariant K-theory, Property (T), uniformly convex Banach space, higher rank group, rigidity

Mathematics Subject Classification 2010: 19K35, 47L20, 22D12, 22E41

1. Introduction

In this note we use recent progress in rigidity theory to address the question of existence of finitely summable Fredholm modules over group C^*-algebras.

We begin by giving some motivation and recalling previous work on this problem.

According to Kasparov [8], the K-homology groups of a C^*-algebra A may be defined as groups of homotopy classes of bounded Fredholm modules over A. In [2] Connes constructed a Chern-character for Fredholm modules which satisfy a strong regularity condition, called finite summability (with respect to a dense subalgebra of A). This Chern-character is given by a completely explicit cyclic cocycle. In particular, the index pairing of the given module with K-theory can be calculated by a simple index formula. In [4] Connes exhibited a character formula which applies also in the absence of finite summability. It involves an infinite number of terms however and is very difficult to handle.

It appears therefore natural to ask whether every K-homology class can be represented by a finitely summable module.

In the case of the C^*-algebra of continuous functions on a smooth compact manifold the answer is affirmative. Every K-homology class may be represented by a module which is finitely summable over the algebra of smooth functions and Connes' character formula boils down to the classical Atiyah-Singer index formula in de Rham cohomology.

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The situation changes if one passes to noncommutative C^*-algebras.

Recall that the K-homology of C^*-algebras can be described either in terms of bounded or in terms of unbounded Fredholm modules. In [5] Connes shows that there are no finitely summable unbounded Fredholm modules over the reduced C^*-algebra of any nonamenable discrete group at all ! For bounded modules the story is different. In [2] Connes constructs bounded Fredholm modules over the reduced group C^*-algebras of free groups and lattices in $SU(1,1)$ which are finitely summable over the group ring while representing nontrivial classes in K-homology. (Note that these groups are not amenable.)

The main observation of this note is that such examples do not exist if one passes to lattices in higher rank groups. A higher rank group is defined as the group of k-rational points of a Zariski connected, almost simple algebraic group of k-rank at least two over the local field k. We have

Theorem 1.1 *i) Let G be a product of higher rank groups. Then every finitely summable Fredholm representation (see section 1.2) of G is homotopic to a virtual finite dimensional representation.*

ii) Let Γ be a lattice in G. Then every Fredholm module over $C^*_{max}(\Gamma)$, which is finitely summable over $\mathbb{C}\Gamma$, is homotopic to a virtual finite dimensional representation of Γ.

(The complete statements and the proofs can be found in section 3.) As a consequence we obtain in section 2 a complete description of the classes in $KK_G(\mathbb{C},\mathbb{C})$, $KK_*(C^*_{max}(\Gamma),\mathbb{C})$ and $KK_*(C^*_{red}(\Gamma),\mathbb{C})$ which can be realized by finitely summable Fredholm representations or Fredholm modules.

In particular, no nontrivial class in $KK_*(C^*_{red}(\Gamma),\mathbb{C})$, where Γ is a lattice in a finite product of higher rank groups, can be represented by a finitely summable module. This shows that for noncommutative C^*-algebras it is in general not possible to realize a K-homology class by a finitely summable Fredholm module.

It was actually this absence of finite summability for the natural Fredholm modules over higher rank lattices which motivated Connes to search for the index formula of [4]. The latter one applies indeed to all known Fredholm modules over higher rank groups and lattices.

The previous results are a simple consequence of recent deep work of Bader, Furman, Gelander and Monod in rigidity theory. They conjecture that every equicontinuous isometric action of a finite product of higher rank groups or a lattice in such a group on a uniformly convex Banach space admits a global fixed point. They verify this conjecture for isometric actions on classical L^p-spaces. One of their main results reads as follows:

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Theorem 1.2 (Bader, Furman, Gelander, Monod) *[1] Let G be a finite product of higher rank groups or a lattice in such a group. Let (X,μ) be a standard measure space and let ρ be an isometric linear representation of G on $L^p(X,\mu)$. Then*

$$H^1_{cont}(G,\rho) = 0$$

for $1 < p < \infty$.

Adapted to noncommutative ℓ^p-spaces (see 5.5, 5.7) their result immediately implies our main theorem. The link is given by the well known correspondence between group cohomology and the Hochschild cohomology of group algebras, as was already observed and exploited by Connes in [5].

After developing the notions of finitely summable K-homology and finitely summable Fredholm representations in section 2 we present our results in section 3. The relevant work of Bader, Furman, Gelander and Monod is recalled in section 4 and adapted to our situation in section 5. This completes the proof of the theorems stated in section 3.

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2. The setup

2.1. Smooth K-homology

The K-homology groups of a C^*-algebra A are defined, according to Kasparov [8], as the groups of homotopy classes of Fredholm modules over A.

Recall that an even **Fredholm module** $\mathcal{E} = (\mathcal{H}^{\pm}, \rho^{\pm}, F)$ over A is given by a $\mathbb{Z}/2\mathbb{Z}$-graded complex Hilbert space $\mathcal{H} = \mathcal{H}^{\pm}$, a pair $\rho = \rho^{\pm} : A \to \mathcal{L}(\mathcal{H})^{\pm}$ of representations of A, and an odd bounded linear operator $F : \mathcal{H}^{\pm} \to \mathcal{H}^{\mp}$ satisfying

$$\rho(A)(F^2 - 1) \subset \mathcal{K}(\mathcal{H}), \quad \rho(A)(F^* - F) \subset \mathcal{K}(\mathcal{H}), \quad [F, \rho(A)] \subset \mathcal{K}(\mathcal{H}). \tag{2.1}$$

Here $\mathcal{K}(\mathcal{H})$ denotes the ideal of compact operators in $\mathcal{L}(\mathcal{H})$. The notion of an odd Fredholm module is obtained by forgetting the $\mathbb{Z}/2\mathbb{Z}$-grading.

Fredholm modules should be viewed as generalized elliptic operators over A. In fact, if M is a smooth compact manifold without boundary, then every linear elliptic pseudodifferential operator over M gives rise to a Fredholm module over $A = C(M)$, the algebra of continuous complex-valued functions on M. It can be shown that such geometric modules generate the whole K-homology group $KK_*(C(M),\mathbb{C})$ of M.

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The previous example suggests a notion of smoothness (finite summability) for Fredholm modules.

Definition 2.1 Let A be a C^*-algebra and let $\mathcal{E} = (\mathcal{H}^{\pm}, \rho^{\pm}, F)$ be a Fredholm module over A. Fix $1 \leq p < \infty$. Then $\mathcal{E}$ is said to be **p-summable** over the dense (involutive) subalgebra $\mathcal{A} \subset A$ if instead of (2.1) the stronger conditions

$$\rho(\mathcal{A})(F^2-1) \subset \ell^p(\mathcal{H}), \quad \rho(\mathcal{A})(F^2-1) \subset \ell^p(\mathcal{H}), \quad [F,\rho(\mathcal{A})] \subset \ell^p(\mathcal{H}). \tag{2.2}$$

hold, where

$$\ell^p(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}), Trace(T^*T)^{\frac{p}{2}} < \infty\} \tag{2.3}$$

denotes the Schatten ideal of p-summable compact operators.

By the Weyl estimates for eigenvalues, the Fredholm module associated to an elliptic operator $\mathcal{D}$ over the smooth compact manifold M is p-summable over $\mathcal{C}^\infty(M) \subset C(M)$ for $p > \frac{dim(M)}{ord(\mathcal{D})}$.

Kasparov's K-homology groups $KK_*(A,\mathbb{C})$ are defined as the groups of equivalence classes of even (odd) Fredholm modules over A with respect to the equivalence relation generated by unitary equivalence, addition of degenerate modules, and operator homotopy.

Following Nistor [12], we introduce similar relations for smooth modules.

Definition 2.2 Let A be a C^*-algebra and let $\mathcal{A} \subset A$ be a dense (involutive) subalgebra. Denote by $\mathcal{E}_*^{(p)}((A,\mathcal{A}),\mathbb{C})$ the set of even(odd) Fredholm modules over A which are p-summable over $\mathcal{A}$.

Recall that a Fredholm module is called **degenerate** if all the terms in (2.1) are identically zero. Every degenerate module over A is thus p-summable over any dense subalgebra so that the set $\mathcal{E}_*^{(p)}((A,\mathcal{A}),\mathbb{C})$ is stable under addition of degenerate modules. There is also an obvious notion of unitary equivalence among smooth Fredholm modules. One has to be a bit careful about the appropriate notion of operator homotopy in the smooth context.

Definition 2.3 A smooth operator homotopy between $\mathcal{E}_0 = (\mathcal{H},\rho,F_0)$ and $\mathcal{E}_1 = (\mathcal{H},\rho,F_1)$, $\mathcal{E}_0,\mathcal{E}_1 \in \mathcal{E}_*^{(p)}((A,\mathcal{A}),\mathbb{C})$, is given by a family $F_t, t \in [0,1]$, of bounded operators on $\mathcal{H}$ connecting F_0 and F_1 such that

- $\mathcal{E}_t = (\mathcal{H},\rho,F_t) \in \mathcal{E}_*^{(p)}((A,\mathcal{A}),\mathbb{C}), 0 \leq t \leq 1$,
- $\{t \mapsto F_t\} \in \mathcal{C}^\infty([0,1],\mathcal{L}(\mathcal{H}))$,
- the maps $a \mapsto \rho(a)(F_t^2-1), a \mapsto \rho(a)(F_t^*-F_t)$, and $a \mapsto [F_t,\rho(a)]$ lie in $\mathcal{L}(\mathcal{A},\mathcal{C}^\infty([0,1],\ell^p(\mathcal{H}))$.

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In the last condition $\mathcal{A}$ is assumed to be equipped with a bornology turning the inclusion $\mathcal{A} \subset A$ into a bounded operator. Thus bounded linear maps from $\mathcal{A}$ to bornological (or Fréchet) spaces make sense. In our examples we will only deal with the fine bornology generated by finite subsets, so that the boundedness is automatic.

Definition 2.4 Let $1 \leq p < \infty$ and let $\mathcal{A}$ be a dense (involutive) subalgebra of the C^*-algebra A. The smooth (p-summable) K-homology groups

$$\mathcal{KK}_*^{(p)}((A,\mathcal{A}),\mathbb{C}) \quad = \quad \mathcal{E}_*^{(p)}((A,\mathcal{A}),\mathbb{C})/\sim$$

of the pair $(A,\mathcal{A})$ are defined as the groups of equivalence classes of p-summable Fredholm modules over $(A,\mathcal{A})$ with respect to the equivalence relation generated by unitary equivalence, addition of degenerate modules, and smooth operator homotopy.

There is an obvious homomorphism

$$\mathcal{KK}_*^{(p)}((A,\mathcal{A}),\mathbb{C}) \quad \longrightarrow \quad KK_*(A,\mathbb{C}) \tag{2.4}$$

One is mainly interested in the image of this map.

If a K-homology class can be represented by a smooth K-cycle, then its Chern-Connes character can be given by a simple explicit character formula. This makes index calculations for smooth K-cycles much easier than in the general case.

Proposition 2.5 *Let $\mathcal{A}$ be a dense (involutive) subalgebra of the separable C^*-algebra A. Let*

$$\check{ch}^{(p)}: \quad \mathcal{KK}_*^{(p)}((A,\mathcal{A}),\mathbb{C}) \quad \longrightarrow \quad HP^*(\mathcal{A}) \tag{2.5}$$

be Connes' Chern character on smooth K-homology with values in periodic cyclic cohomology [2] and let

$$\check{ch}: \quad KK_*(A,\mathbb{C}) \quad \longrightarrow \quad HC^*_{loc}(A) \tag{2.6}$$

be the Chern-Connes character on K-homology with values in local cyclic cohomology [13]. Then there is a natural commutative diagram

$$\begin{array}{ccccc} \mathcal{KK}_*^{(p)}((A,\mathcal{A}),\mathbb{C}) & & \longrightarrow & & KK_*(A,\mathbb{C}) \\ \check{ch}^{(p)} \downarrow & & & & \downarrow \check{ch} \\ HP^*(\mathcal{A}) & \longrightarrow & HC^*_{loc}(\mathcal{A}) & \longleftarrow & HC^*_{loc}(A) \end{array} \tag{2.7}$$

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2.2. *Representation rings of locally compact groups*

Let G be a locally compact (second countable) group. We denote by $R(G)$ the ring of unitary equivalence classes of finite dimensional unitary representations of G. Sum and product in $R(G)$ are given by the direct sum and the tensor product of (equivalence classes) of representations.

A remarkable generalization of the notion of finite dimensional unitary representation has been introduced by G. Kasparov [9]:

Definition 2.6 A **Fredholm representation** $\mathcal{E}_G = (\mathcal{H}, \rho, F)$ of G is given by

- a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $\mathcal{H}^{\pm}$
- a pair of unitary representations $\rho^{\pm} : G \longrightarrow \mathcal{U}(\mathcal{H}^{\pm})$ of G on the even(odd) part of $\mathcal{H}$.
- an odd bounded operator $F : \mathcal{H}^{\pm} \longrightarrow \mathcal{H}^{\mp}$, which almost intertwines the representations ρ^{+} and ρ^{-}, i.e.

$$\{g \mapsto \rho(g) F \rho(g)^{-1}\} \in C(G, \mathcal{L}(\mathcal{H})),$$

$$F^2 - 1 \in \mathcal{K}(\mathcal{H}), (F - F^*) \in \mathcal{K}(\mathcal{H}), \rho(g) F \rho(g)^{-1} - F \in \mathcal{K}(\mathcal{H}), \forall g \in G.$$

We propose the following notion of a smooth Fredholm representation

Definition 2.7 A Fredholm representation $\mathcal{E}_G = (\mathcal{H}, \rho, F)$ of a locally compact group G is **p-summable** if

$$F^2 - 1 \in \ell^p(\mathcal{H}), \quad F^* - F \in \ell^p(\mathcal{H}), \quad \{g \mapsto \rho(g) F \rho(g)^{-1} - F\} \in C(G, \ell^p(\mathcal{H})).$$

Denote by $Fred(G)$ ($Fred^{(p)}(G)$) the set of Fredholm representations (p-summable Fredholm representations) of G.

If $G = \Gamma$ is a discrete group, there is a tautological bijection

$$Fred^{(p)}(\Gamma) \quad \stackrel{\simeq}{\longrightarrow} \quad \mathcal{E}_0^{(p)}((C^*_{max}(\Gamma), \mathbb{C}\Gamma), \mathbb{C}) \tag{2.8}$$

between the set of p-summable Fredholm representations of Γ and the set of even Fredholm modules over the enveloping C^*-algebra $C^*_{max}(\Gamma)$ of $\ell^1(\Gamma)$, which are p-summable over $\mathbb{C}\Gamma$.

There are obvious notions of unitary equivalence, degeneracy, and operator homotopy of Fredholm representations. For finitely summable representations one puts

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Definition 2.8 A smooth operator homotopy between p-summable Fredholm representations $\mathcal{E}_0 = (\mathcal{H}, \rho, F_0)$ and $\mathcal{E}_1 = (\mathcal{H}, \rho, F_1)$ is given by a family $F_t, t \in [0,1]$ of bounded operators on $\mathcal{H}$ connecting F_0 and F_1 such that

- $\mathcal{E}_t = (\mathcal{H}, \rho, F_t) \in Fred^{(p)}(G), 0 \leq t \leq 1$,
- $\{t \mapsto F_t\} \in \mathcal{C}^\infty([0,1], \mathcal{L}(\mathcal{H}))$,
- $\{g \mapsto \rho(g) F_t \rho(g)^{-1} - F_t\} \in C(G, \mathcal{C}^\infty([0,1], \ell^p(\mathcal{H})))$.

Kasparov defines the G-equivariant K-homology of $\mathbb{C}$

$$KK_G(\mathbb{C},\mathbb{C}) \quad = \quad Fred(G)/\sim \tag{2.9}$$

as the group of equivalence classes of Fredholm representations of G with respect to the equivalence relation generated by unitary equivalence, addition of degenerate representations, and operator homotopy.

The smooth version of this is

Definition 2.9 Let G be a locally compact group and let $1 \leq p < \infty$. The p-summable, G-equivariant K-homology of $\mathbb{C}$

$$\mathcal{KK}_G^{(p)}(\mathbb{C},\mathbb{C}) \quad = \quad Fred^{(p)}(G)/\sim \tag{2.10}$$

is defined as the group of equivalence classes of p-summable Fredholm representations of G with respect to the equivalence relation generated by unitary equivalence, addition of degenerate representations, and smooth operator homotopy.

There are tautological homomorphisms

$$KK_G(\mathbb{C},\mathbb{C}) \quad \longrightarrow \quad KK(C^*_{max}(G),\mathbb{C}) \tag{2.11}$$

and

$$\mathcal{KK}_G^{(p)}(\mathbb{C},\mathbb{C}) \quad \longrightarrow \quad \mathcal{KK}^{(p)}((C^*_{max}(G),\mathcal{C}_c(G)),\mathbb{C}) \tag{2.12}$$

These are isomorphisms in the case that $G = \Gamma$ is discrete.

The Kasparov product turns the (smooth) equivariant K-groups (2.9) and (2.10) into unital, associative rings. (The tautological homomorphisms (2.11) and (2.12) then become ring homomorphisms.)

Finally there is a natural ring homomorphism

$$R(G) \quad \longrightarrow \quad KK_G(\mathbb{C},\mathbb{C}) \tag{2.13}$$

from the representation ring of G to the G-equivariant K-homology of $\mathbb{C}$, which assigns to a virtual finite dimensional unitary representation $[\rho_+] - [\rho_-]$ the Fredholm representation $\mathcal{E}_G = (\varrho_+ \oplus \varrho_-, 0)$. This homomorphism factors through the smooth equivariant K-homology rings $\mathcal{KK}_G^{(p)}(\mathbb{C},\mathbb{C})$ for all $1 \leq p < \infty$.

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3. Results

In the sequel we understand by a **higher rank group** the group of k-rational points of a Zariski-connected, almost simple algebraic group of k-rank at least two over the local field k.

Theorem 3.1 *Let G be a finite product of higher rank groups. Then*

$$\mathcal{KK}_G^{(p)}(\mathbb{C},\mathbb{C}) \simeq R(G) \tag{3.1}$$

under the canonical morphism (2.13). In particular, if G is a product of simply connected higher rank groups, then

$$\mathcal{KK}_G^{(p)}(\mathbb{C},\mathbb{C}) \simeq \mathbb{Z} \tag{3.2}$$

is generated by the trivial representation of G for $1 < p < \infty$.

Theorem 3.2 *Let Γ be a lattice (i.e. a discrete subgroup of finite covolume) in a finite product of higher rank groups. Then the tautological homomorphisms*

$$R(\Gamma) \xrightarrow{\simeq} \mathcal{KK}_\Gamma^{(p)}(\mathbb{C},\mathbb{C}) \xrightarrow{\simeq} \mathcal{KK}_0^{(p)}((C^*_{max}(\Gamma),\mathbb{C}\Gamma),\mathbb{C}) \tag{3.3}$$

are isomorphisms and

$$\mathcal{KK}_1^{(p)}((C^*_{max}(\Gamma),\mathbb{C}\Gamma),\mathbb{C}) = 0 \tag{3.4}$$

for $1 < p < \infty$.

Theorem 3.3 *Let Γ be a lattice in a finite product of higher rank groups. Then*

$$\mathcal{KK}_*^{(p)}((C^*_{red}(\Gamma),\mathbb{C}\Gamma),\mathbb{C}) = 0 \tag{3.5}$$

for $1 < p < \infty$, where $C^*_{red}(\Gamma)$ denotes the reduced C^*-algebra of Γ, i.e. the enveloping C^*-algebra of the regular representation of Γ.

Corollary 3.4 *Under the assumptions of 3.3 no nonzero class in the K-homology group $KK_*(C^*_{red}(\Gamma),\mathbb{C})$ can be represented by a Fredholm module which is finitely summable over $\mathbb{C}\Gamma$.*

Corollary 3.5 *Let G be a finite product of simple real Lie groups with finite center and suppose that each simple factor is of real rank at least two. Then the conclusion of 3.1 holds for G and the conclusions of 3.2, 3.3 and 3.4 hold for lattices in G. In particular, every finitely summable Fredholm representation of G is homotopic to a multiple of the trivial representation.*

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We will obtain these results as a consequence of a cohomological vanishing theorem which will be formulated now. Let $1 < p < \infty$. For a unitary representation ϱ of a locally compact group G on the Hilbert space $\mathcal{H}$ let ϱ_p be the isometric representation on the operator ideal $\ell^p(\mathcal{H})$ given by $\varrho_p(T) = \varrho(g)T\varrho(g^{-1})$.

Proposition 3.6 *Let G be a locally compact group. Suppose that for every unitary representation ϱ of G the continuous first cohomology group of G with coefficients in ϱ_p vanishes:*

$$H^1_{cont}(G,\varrho_p) \quad = \quad 0 \tag{3.6}$$

Then every p-summable Fredholm representation of G is smoothly operator homotopic to the direct sum of a finite dimensional virtual representation and a degenerate Fredholm representation.

Proof: Let $\mathcal{E} = (\mathcal{H},\rho,F)$ be a p-summable Fredholm representation of G. Then the map $\psi : G \to \ell^p(\mathcal{H}), g \mapsto \rho(g)F\rho(g)^{-1} - F$ defines a continuous one-cocycle $\psi \in Z^1_{cont}(G,\rho_p)$ (see (4.3)). By our assumption, this cocycle is a coboundary, i.e. $\psi(g) = \rho(g)F\rho(g)^{-1} - F = \rho(g)T\rho(g)^{-1} - T$ for some operator $T \in \ell^p(\mathcal{H})$ and all $g \in G$ (see (4.4)). Thus $\mathcal{E}_s = (\mathcal{H},\rho,F - sT)$ is a smooth operator homotopy between $\mathcal{E}$ and the Fredholm representation $\mathcal{E}' = (\mathcal{H},\rho,F'), F' = F - T$, satisfying $[F',\rho(G)] = 0$. We may suppose that F' is selfadjoint. The subspace $Ker(F') \subset \mathcal{H}$ as well as its orthogonal complement $\mathcal{H}'$ are then stable under $\rho(G)$. Therefore $\mathcal{E}'$ is the direct sum of the finite dimensional virtual unitary representation $(KerF',\rho_{KerF'},0)$ and the triple $\mathcal{E}' = (\mathcal{H}',\rho_{|\mathcal{H}'},F'' = F'_{|\mathcal{H}'})$ which is smoothly operator homotopic to the degenerate triple $\mathcal{D}' = (\mathcal{H}',\rho_{|\mathcal{H}'},\frac{F''}{|F''|})$. □

It will be shown in sections 5 (see 5.5 and 5.7) that the hypothesis (3.6) is actually verified for the class of groups considered in this paper. Assuming this fact for the moment we continue to develop the tools needed to prove the main results.

Recall that a locally compact group possesses Kazhdan's Property (T) [7], if the trivial representation is an isolated point in the unitary dual with respect to the Fell topology. In this case all finite dimensional unitary representations are isolated points in the unitary dual [16]. Every higher rank group as considered above has Kazhdan's property (T) [11]. Moreover, Property (T) is stable under taking finite products and under passage to lattices so that the higher rank lattices possess Kazhdan's property as well.

Lemma 3.7 *Let G be a locally compact group with Kazhdan's property (T). Then the composition*

$$R(G) \longrightarrow KK_G(\mathbb{C},\mathbb{C}) \longrightarrow KK(C^*_{max}(G),\mathbb{C}) \tag{3.7}$$

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is injective.

Proof: By Wang's theorem there exists for every unitary equivalence class ρ of irreducible finite dimensional representations of Γ a projector $e_\rho \in C^*_{max}(G)$ with the following property: if π is any unitary representation of G, then $\pi(e_\rho)$ is the orthogonal projection onto the ρ-isotypical component of π. Let ρ' be any unitary equivalence class of irreducible finite dimensional representations of G. Then the value of the canonical index pairing

$$\langle -,- \rangle : KK(\mathbb{C}, C^*_{max}(G)) \otimes KK(C^*_{max}(G), \mathbb{C}) \to \mathbb{C} \tag{3.8}$$

on the pair (e_ρ, ρ') is zero unless ρ equals ρ'. Thus the index pairing seperates the elements of $R(G)$ and the assertion of the lemma follows. □

Proof of Theorem 3.1: According to Theorem 5.5 a higher rank group G satisfies condition (3.6). Proposition 3.6 implies therefore that every Fredholm representation of G is equivalent to a finite dimensional virtual unitary representation. This shows that the canonical map $R(G) \to \mathcal{KK}^{(p)}_G(\mathbb{C},\mathbb{C})$ is surjective for higher rank groups. As noted above higher rank groups possess Kazhdan's property (T). Lemma 3.7 implies then that the composition $R(G) \to \mathcal{KK}^{(p)}_G(\mathbb{C},\mathbb{C}) \to KK_G(\mathbb{C},\mathbb{C})$ is injective. This proves the first assertion.

To verify the second claim we will make use of the well known Moore ergodicity theorem [1], 5.1. If $G(k)$ is the locally compact group of k-rational points of a Zariski connected, simply connected, isotropic, semisimple algebraic group G over the local field k, then the theorem asserts that all matrix coefficients of a unitary representation of $G(k)$ without fixed vectors vanish at infinity. For such a group every finite dimensional unitary representation is trivial. Otherwise, there exists a finite dimensional unitary representation ρ without fixed vectors. The function $g \mapsto det(\rho(g)), g \in G(k)$, which is a homogeneous polynomial in the matrix coefficients of ρ, has then to vanish at infinity by Moore's theorem. This contradicts the fact that ρ is unitary. □

Proof of Theorem 3.2: The proof is similar to that of 3.1. Corollary 5.7 implies that condition 3.6 holds for higher rank lattices and the arguments above apply. As higher rank lattices have Kazhdan's Property (T) one can use again lemma 3.7 to conclude. □

Proof of Theorem 1.1: The demonstrations of 3.1 and 3.2 show that every Fredholm representation of a product of higher rank groups or of a lattice in such a group is operator homotopic to the sum of a virtual finite dimensional and a degenerate

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representation. Degenerate Fredholm representations however are nullhomotopic according to [8]. □

Proof of Theorem 3.3: Using 3.6 again we see (after a smooth operator homotopy and the subtraction of a degenerate module) that every class in $\mathcal{KK}_*^{(p)}((C_{red}^*(\Gamma),\mathbb{C}\Gamma),\mathbb{C})$ may be represented by a Fredholm module $\mathcal{E} = (\mathcal{H}',\rho,0)$ over $C_{red}^*(\Gamma)$ with finite dimensional underlying Hilbert space $\mathcal{H}'$. According to [6], 18.9.5 every finite dimensional representation of the reduced group C^*-algebra of a nonamenable locally compact group is identically zero. This applies in particular to higher rank lattices which are not compact and possess property (T) and are therefore not amenable. Thus the considered Fredholm module $\mathcal{E}$ is identically zero and the theorem follows. □

Proof of Corollary 3.5: Let G be a product of real simple Lie groups with finite center and of real rank at least two. Then G fits into an extension $1 \to Z \to G \to G' \to 1$, where G' is a higher rank group in the sense of 3.1 and Z is a finite central subgroup of G ([11]). Let $\mathcal{E} = (\mathcal{H},\rho,F)$ be a p-summable Fredholm representation of G. It is canonically operator homotopic to the p-summable Fredholm representation $\mathcal{E}' = (\mathcal{H},\rho,F')$, $F' = \sum_{h\in Z} \varrho(h)F\varrho(h^{-1})$ obtained by averaging with respect to Z. The map $\psi : G' = G/Z \to \ell^p(\mathcal{H})^{\varrho_p(Z)}, g \mapsto \rho(g)F'\rho(g)^{-1} - F'$ defines then a continuous one-cocycle of G' with coefficients in $\ell^p(\mathcal{H})^{\varrho_p(Z)}$. Applying Theorem 4.2 to the spaces $B = \ell^p(\mathcal{H})^{\varrho_p(Z)}$ and $\mathcal{H} = \ell^2(\mathcal{H})^{\varrho_2(Z)}$ shows (by the arguments already used in the proof of Proposition 3.6) that $\mathcal{E}'$ is smoothly homotopic to a virtual, finite dimensional unitary representation of G. By assumption, the semisimple real Lie group G contains no compact factors so that such a representation is necessarily a multiple of the trivial one. This shows the first assertion of Corollary 3.5. The second assertion follows from a similar reasoning applied to the extension $1 \to Z' \to \Gamma \to \Gamma' \to 1$ where Γ is a lattice in G and $Z' = \Gamma \cap Z$. The group Γ' is then necessarily a lattice in the higher rank group G' and 3.2 applies to Γ'. □

4. The work of Bader, Furman, Gelander and Monod

4.1. Uniformly convex Banach spaces

Recall that a Banach space B is **strictly convex** if the midpoint of every segment joining two points of the unit sphere $S(B)$ is in the interior of the unit ball. It is **uniformly convex** if the previous condition holds uniformly in the following sense:

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for all $\delta > 0$ there exists $\epsilon > 0$ such that

$$\| x - y \| \geq \delta \quad \Rightarrow \quad \| \tfrac{x+y}{2} \| \leq 1 - \epsilon \tag{4.1}$$

for all $x, y \in S(B)$. A Banach space is uniformly smooth if its dual space is uniformly convex. Uniformly convex spaces are reflexive.

4.2. Isometric actions on uniformly convex spaces

An isometric action of a locally group G on a Banach space B is called continuous if the map $\pi : G \times B \longrightarrow B$ is continuous. Every isometry of a strictly convex space is real affine (it preserves midpoints of segments). Thus

$$\pi(g)v = \rho(g)v + \psi(g), \forall g \in G, \forall v \in B \tag{4.2}$$

where $\rho : G \longrightarrow \mathbf{O}(B)$ is a continuous orthogonal ($\mathbb{R}$-linear isometric) representation of G on B and $\psi : G \longrightarrow B$ is a continuous 1-cocycle on G with values in the G-module B:

$$\psi \in Z^1_{cont}(G,\rho) = \{\psi \in C(G,B), \psi(gh) = \psi(g) + \rho(g)\psi(h), \forall g,h \in G\}. \tag{4.3}$$

There is thus a bijection between affine isometric G-actions on B with linear part ρ and continuous 1-cocycles on G with coefficients in ρ. The topology of uniform convergence on compacta turns $Z^1_{cont}(G,\rho)$ into a Fréchet space.

An affine isometric action (with linear part ρ) possesses a fixed point if and only if the corresponding cocycle belongs to the subspace

$$B^1(G,\rho) = \{\mu \in Z^1_{cont}(G,\rho), \mu(g) = \xi - \rho(g)\xi \text{ for some } \xi \in B\}. \tag{4.4}$$

The vanishing of the cohomology group

$$H^1_{cont}(G,\rho) = Z^1_{cont}(G,\rho)/B^1(G,\rho) \tag{4.5}$$

is therefore equivalent to the assertion that every continuous isometric action of G on B with linear part ρ possesses a fixed point.

The result of Bader, Furman, Gelander and Monod that we wish to extend to noncommutative ℓ^p-spaces is the

Theorem 4.1 *[1] Let G be a product of higher rank groups or a lattice in such a group (see section 3). Let (X,μ) be a standard Borel space with σ-finite measure and let ρ be any orthogonal representation of G on the Banach space $L^p(X,\mu)$ where $1 < p < \infty$. Then $H^1_{cont}(G,\rho) = 0$.*

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It should be mentioned that Bader, Furman, Gelander and Monod conjecture that any affine isometric action of a higher rank group or lattice on a uniformly convex Banach space has a fixed point. In fact they deduce the previous theorem from the following result which has a more general flavour.

Theorem 4.2 *[1] Let G be a product of higher rank groups (see section 3) and let ρ_B be an orthogonal representation of G on a uniformly convex and uniformly smooth Banach space B. Suppose that there exists an orthogonal representation ρ of G on a Hilbert space $\mathcal{H}$ and a homeomorphism $\Phi : S(B) \longrightarrow S(\mathcal{H})$ between the unit spheres of B and $\mathcal{H}$ such that*

- *Φ intertwines the G-actions induced by ρ_B and ρ.*
- *Φ and Φ^{-1} are uniformly continuous.*

Then

$$H^1_{cont}(G, \rho_B) = 0.$$

We will show in the sequel that this theorem applies to various affine isometric actions of higher rank groups on operator ideals.

5. Isometric actions on operator ideals

5.1. Geometry of Schatten ideals

We fix from now on a real number $p, 1 < p < \infty$. The Schatten-von Neumann ideal $\ell^p(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ is the ideal of compact operators whose sequence of singular values is p-summable. It is a Banach space with respect to the norm

$$\| T \|_p^p = Trace((T^*T)^{\frac{p}{2}}) \tag{5.1}$$

The Schatten spaces are symmetrically normed operator ideals, i.e. one has

$$\| ATB \|_p \leq \| A \|_{\mathcal{L}(\mathcal{H})} \| T \|_p \| B \|_{\mathcal{L}(\mathcal{H})} \tag{5.2}$$

for $A, B \in \mathcal{L}(\mathcal{H}), T \in \ell^p(\mathcal{H})$. In particular, if $\rho : G \longrightarrow \mathcal{U}(\mathcal{H})$ is a unitary representation of a locally compact group G on $\mathcal{H}$, then

$$\rho_p : G \longrightarrow \mathcal{U}(\ell^p(\mathcal{H})), \rho_p(g)(T) = \rho(g) T \rho(g)^{-1} \tag{5.3}$$

is a strongly continuous, isometric, $\mathbb{C}$-linear representation of G on $\ell^p(\mathcal{H})$.

We will need rather precise information about the geometry of the Banach spaces $\ell^p(\mathcal{H})$ for $1 < p < \infty$. We begin with

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Theorem 5.1 (Clarkson-MacCarthy [15]) *The Banach spaces $\ell^p(\mathcal{H}), 1 < p < \infty$, are uniformly convex.*

This generalizes the correponding result for the classical Banach spaces $L^p(X,\mu)$. From [10], 1.e.9.(i) one derives further

Corollary 5.2 *Let (X,μ) be a Borel space. Then for $1 < p < \infty$ the Banach space*

$$L^p(X,\ell^p(\mathcal{H})) \tag{5.4}$$

obtained from the space $C_c(X,\ell^p(\mathcal{H}))$ of continuous compactly supported functions on X with values in $\ell^p(\mathcal{H})$ by completion with respect to the norm

$$\| f \|_p^p = \int_X Trace((f^*(t)f(t))^{\frac{p}{2}})d\mu(t) \tag{5.5}$$

is uniformly convex.

Lemma 5.3 *Let $1 < p,q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\ell^p(\mathcal{H})^* \simeq \ell^q(\mathcal{H}), \quad L^p(X,\ell^p(\mathcal{H}))^* \simeq L^q(X,\ell^q(\mathcal{H})) \tag{5.6}$$

in the notations of 5.2. The duality is given by the trace pairing.

In particular, the Schatten ideals $\ell^p(\mathcal{H}), 1 < p < \infty$, are uniformly convex and uniformly smooth Banach spaces.

The unit spheres of these spaces are uniformly homeomorphic to each other. In fact, the results of [14] imply

Proposition 5.4 *a) Let $1 < p,q < \infty$. Then the Mazur map*

$$\begin{array}{rccc} M_{p,q}: & S(\ell^p(\mathcal{H})) & \longrightarrow & S(\ell^q(\mathcal{H})) \\ & T & \mapsto & sign(T)|T|^{\frac{p}{q}} \end{array} \tag{5.7}$$

is a uniformly continuous homeomorphism.

b) Let (X,μ) be a standard Borel space with a σ-finite measure and let $1 < p,q < \infty$. Then the Mazur map

$$\begin{array}{rccc} M_{p,q}: & S(L^p(X,\ell^p(\mathcal{H}))) & \longrightarrow & S(L^q(X,\ell^q(\mathcal{H}))) \\ & f & \mapsto & sign(f)|f|^{\frac{p}{q}} \end{array} \tag{5.8}$$

is a uniformly continuous homeomorphism.

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Proof: Let $\mathcal{M}$ be a von Neumann algebra, let τ be a faithful semifinite normal trace on $\mathcal{M}$, and let $L^p(\mathcal{M},\tau), 0 < p < \infty$, be the corresponding noncommutative L^p-spaces (see [14]). Then lemma 3.2 of [14] implies that the Mazur map

$$M_{p,1} : S(L^p(\mathcal{M},\tau)) \to S(L^1(\mathcal{M},\tau)), h \mapsto sign(h)|h|^p \tag{5.9}$$

is a uniform homeomorphism. The same holds then for $M_{p,q} = M_{q,1}^{-1} \circ M_{p,1}$. Applying this in the cases $(\mathcal{M},\tau) = (\mathcal{B}(\mathcal{H}), Trace)$ and $(\mathcal{M},\tau) = (L^\infty(X,\mathcal{B}(\mathcal{H})), \mu \otimes Trace)$ yields the assertions a) and b), respectively. □

5.2. *Rigidity of actions on Schatten ideals*

We are ready to apply the results of Bader, Furman, Gelander and Monod in our framework.

Theorem 5.5 *Let G be a product of higher rank groups as in section 3 and let ρ be a unitary representation of G on the Hilbert space $\mathcal{H}$. Let ρ_p be the corresponding isometric representation on the Schatten ideals $\ell^p(\mathcal{H})$. Then*

$$H^1_{cont}(G,\rho_p) = 0 \text{ for } 1 < p < \infty. \tag{5.10}$$

Consider now a lattice Γ in a product of higher rank groups G. The homogeneous space G/Γ is then of finite volume. According to 5.2 and 5.3 the Banach spaces $L^p(G/\Gamma, \ell^p(\mathcal{H}))$ are uniformly convex and uniformly smooth for $1 < p < \infty$. Suppose that in addition a unitary representation ρ of Γ on a Hilbert space $\mathcal{H}$ is given and let ρ_p be the corresponding isometric representation on $\ell^p(\mathcal{H})$. Then $L^p(G/\Gamma, \ell^p(\mathcal{H}))$ can be identified with the completion of the space $L^{[p]}(G, \ell^p(\mathcal{H}))^\Gamma$ of Γ-equivariant maps from G to $\ell^p(\mathcal{H})$ with respect to the norm $\| f \|_p^p = \int_{\mathcal{D}} \| f(g) \|^p \, dg$ ($\mathcal{D} \subset G$ a Borel fundamental domain for Γ). The latter space carries a canonical isometric linear G-action coming from left translation.

Theorem 5.6 *Let G be a higher rank group and let Γ be a lattice in G. Let ρ be a unitary representation of Γ. Then in the previous notations*

$$H^1_{cont}(G, L^{[p]}(G, \ell^p(\mathcal{H}))^\Gamma) = 0 \text{ for } 1 < p < \infty. \tag{5.11}$$

Corollary 5.7 *Let Γ be a lattice in a product of higher rank groups as in section 3 and let ρ be a unitary representation of Γ on the Hilbert space $\mathcal{H}$. Let ρ_p be the corresponding isometric representation on the Schatten ideals $\ell^p(\mathcal{H})$. Then*

$$H^1(\Gamma,\rho_p) = 0 \text{ for } 1 < p < \infty. \tag{5.12}$$

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Proof of Theorems 5.5 and 5.6: We begin with a proof of (5.10). We want to apply theorem 4.2 to the isometric representations ρ_p on the Schatten ideals $\ell^p(\mathcal{H})$. For $1 < p < \infty$ these ideals are uniformly convex and uniformly smooth (see 5.1 and 5.3). Note that $\ell^2(\mathcal{H})$ is the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$. By construction, the Mazur map $M_{p,2} : S(\ell^p(\mathcal{H})) \to S(\ell^2(\mathcal{H}))$ (5.7) intertwines the isometric representations ρ_p and ρ_2. Moreover $M_{p,2}$ as well as its inverse $M_{2,p}$ are uniformly continuous by 5.4 a). Thus Theorem 4.2 of Bader, Furman, Gelander and Monod applies and yields (5.10). The demonstration of (5.11) is similar and uses 5.2, 5.3, and 5.4.b). □

Proof of Corollary 5.7: The well known idea is to use induction of representations and affine isometric actions from Γ to G in order to reduce the statement to (5.11). The arguments in [1], section 8.1, 8.2 apply verbatim to our situation and yield $H^1(\Gamma, \rho_p) \simeq H^1_{cont}(G, L^{[p]}(G, \ell^p(\mathcal{H}))^\Gamma)$. The claim follows then from 5.6. □

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