IMPROVED HARMONY SEARCH METHODS TO REPLACE VARIATIONAL PRINCIPLE IN GEOTECHNICAL PROBLEMS

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ABSTRACT

Variational principle is an important principle in engineering discipline. This principle is suitable for simple problems where an analytical expression can be determined, but there are many practical problems where the classical variational principle is practically impossible to be applied. In this paper, the authors will try to demonstrate that the variational principle can be replaced by the use of modern artificial intelligence based optimization method (harmony search method) which can be applied to much more complicated problems. Two different improved harmony search algorithms are proposed in this paper. The new algorithms differ from the original algorithm in that: (1) The harmonies are rearranged into several pairs and the better pairs are used to develop several new harmonies; (2) Different probabilities are assigned to different harmonies. The robustness of the proposed methods is demonstrated by using three difficult examples, and the sensitivities of the related optimization parameters are investigated through statistical orthogonal analysis.

Keywords : Variational principle, Slope stability analysis, Harmony search algorithm, Minimum factor of safety, Random number.

1. INTRODUCTION

A variational principle in physics is a method for determining the state or dynamics of a physical system, by identifying it as an extremum (minimum, maximum or saddle point) of a function or functional. Variational principle is expressed in terms of the calculus of variations. According to Lanczos [1], any physical law which can be expressed as a variational principle describes an expression which is self-adjoint. These expressions are also called Hermitian and are invariant under a Hermitian transformation.

The formulation of the principle of least action is commonly attributed to Maupertuis [2] who stated that "Nature is thrifty in all its actions" and "The laws of movement and of rest deduced from this principle being precisely the same as those observed in nature, we can admire the application of it to all phenomena". Euler [3] gave a formulation of the principle in 1744. The full importance of the principle to mechanics was stated by Lagrange [4], and the variational principle was used to derive the equations of motion by Hamilton [5,6] which is now called the Lagrangian equations of motion. Jacobi [7] tackled the problem of whether the variational principle found minima or other extrema (*e.g.* a saddle point); and most of his work focused on geodesics on two-dimensional surfaces.

In geotechnical engineering, variational principle has provided the solutions to many problems. For slope stability problem, Baker and Garber [8], Baker [9] and Revilla and Castillo [10] have applied the principle to determine the factor of safety of a slope which does not require any assumption on the internal force distribution. Through the use of the variational principle, the critical failure surface can be determined automatically. The variational formulation by Baker [9] was criticized by De Jong [11,12] as the stationary value may have an indefinite character rather than a minimum. The global minimum is not necessarily given by the gradient of the function being zero if the global minimum lies along the boundary of the solution domain. This conclusion was also supported by Castilo and Luenco [13,14] which was based on a series of counter-examples. Baker [15] later incorporated some addi-

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tional physical restrictions into the basic limiting equilibrium framework so as to guarantee that the slope stability problem has a well-defined minimum solution. Although the variational principle requires very few assumptions with no convergence problems during the solution, it is difficult to be adopted when the geometry or the ground/loading conditions are complicated. Variational principle is hence not adopted in any commercial program or used in any practical problem.

Besides the variational principle, optimization search for the critical failure surface (large number of control variables) has also been considered by various methods (mainly gradient type method) by Baker and Gaber [8], Nguyen [16], Celestino and Duncan [17], Arai and Tagyo [18], Baker [9], Yamagami and Jiang [19], Chen and Shao [20]. These classical methods are all limited by the presence of local minimum as the local minimum close to the initial trial will be obtained in the analysis. In view of the limitations of the classical optimization methods, the current approach is the adoption of the stochastic global optimization methods for the present problem. These algorithms usually find a solution close to the best one with good efficiency. Greco [21] and Malkawi *et al*. [22] adopted the Monte Carlo technique for searching the critical slip surface with success for some cases, but there is no precision control on the accuracy of the global minimum. Zolfaghari *et al*. [23] adopted the genetic algorithm while Bolton *et al*. [24] used the leap-frog optimization technique to evaluate the minimum factor of safety. Among the modern stochastic global optimization methods that have evolved in recent years, there are only limited applications in geotechnical engineering. The simulated annealing method, PSO, HM, ant-colony and Tabu search are first adopted by Cheng [25-28], while the genetic algorithm have been adopted by Zolfaghari *et al*. [23] and Cheng *et al*. [27] and the leap-frog algorithm adopted by Bolton *et al*. [24]. Cheng *et al*. [27] have carried out a detailed comparisons between six major types of stochastic global optimization methods, and the sensitivity of these methods under different optimization parameters are investigated.

Classically, static bounds to the control variables are used where the solution domain for each control variable is fixed and pre-determined by engineering experience. Cheng [25,28] has developed the first procedure which transformed the various constraints and requirement of a kinematically acceptable failure mechanism to the evaluation of the upper and lower bounds of the control variables. The control variables are defined with dynamic domains where the bounds are controlled by the requirement of kinematically acceptable failure mechanism and are changing during the solution. Through such approach, there is no need to define the static solution domain to each control variable by engineering experience. The approach by Cheng [25] is actually equivalent to the enforcement of convex slip surface by limiting the upper and lower bounds of the control variables sequentially, and this approach should be applicable to other disciplines which require a convex geometry (but not necessarily a convex function).

In this paper, the variational principle will first be demonstrated to be equivalent to the global optimization method. If a robust global optimization method is developed, the analysis can avoid the problem as mentioned by De Josseline [11,12]. Towards this issue, the authors will adopt the efficient harmony search algorithm and propose two improved versions of this method. It will be demonstrated that the new algorithms are more stable over wider conditions, and are recommended to be used for larger scale as compared with the original harmony search method.

2. EQUIVALENCE BETWEEN VARIATIONAL PRINCIPLE AND GLOBAL OPTIMIZATION SEARCH

Since the variational principle is difficult to be used for general problems, the authors propose to use the modern global optimization as a replacement of the variational principle. In the present paper, the authors will use slope stability problem for illustration. Wu and Tsai [29] have introduced the variational principle for a slope in cohesive soil which is a highly simplified case. In this paper, a more general formulation is considered and the applicability of the modern global optimization method is demonstrated. For simplicity, the Janbu's method [30] which assumes the interslice shear force to be zero is adopted in the illustration. Consider a simple slope as shown in Fig. 1 where the ground profile is given by Eq. (1) as:

$$f(x)=\begin{cases} 0 & x\le 0 \\ x\tan\beta & 0\le x\le d_1 \\ h & d_1\le x \end{cases} \tag{1}$$

in which, β is the slope angle and d_1 is the horizontal extent of the slope from toe to the crest. According to the Mohr-Coulomb failure criteria, ultimate shear strength τ_f is expressed as:

$$\tau_f = c+\sigma\tan\phi \tag{2}$$

Vertical force equilibrium gives:

$$W - T\sin\alpha - P\cos\alpha = 0 \tag{3}$$

Where c and ϕ are the shear strength parameters of soil; σ and ds are the base normal force and base length at the base of a slice in Fig. 1(b); $P=\sigma_n\, ds$ is the base normal force; τ is the mobilized shear stress at base of slip surface; α is the inclination of the base of the slip surface; W is the weight of the slice and is given by $W=\gamma(f-y)\,dx$; T is the shear force at the bottom of the slice and $T=\tau ds$. Using Eqs. (1) and (2) and let dx approach the infinity, the mobilized shear stress τ could be expressed as:

$$\tau=\frac{c+\gamma(f-y)\tan\phi}{F+y'\tan\phi} \tag{4}$$

where F is the factor of safety, y is the failure surface, γ,

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c, and ϕ are the bulk unit weight, cohesion and friction angle respectively. Using the Janbu's method [30] in the present study where the interslice shear force is assumed to be zero:

$$F = \frac{\sum (c + W \tan\phi)/\eta_\alpha}{W \tan\alpha} \qquad \eta_\alpha = \left(1 + \frac{\tan\alpha \tan\phi}{F}\right) \cos^2\alpha \tag{5}$$

The total energy of the system comprises of three components, gravity, bottom shear force and interslice vertical force and horizontal force. If there is no external horizontal force acting on the slope, based on the Janbu's simplified method [30] with no vertical interslice force, the total potential energy S is expressed by Wu and Tsai [29] as Eq. (6)

$$\int_a^b \left(\tau(1+y'^2) - \gamma(f-y)y'\right) dx \tag{6}$$

Substituting Eq. (4) into Eq. (6) yields

$$H = \frac{c + \gamma(f-y)\tan\phi}{F + y'\tan\phi}(1+y'^2) - \gamma(f-y)\,y' \tag{7}$$

The Euler's equation and the transeverality conditions of Eq. (6) could be expressed respectively as follows:

$$\frac{dH_{y'}}{dx} - H_y = 0 \tag{8}$$

$$H + (f' - y')\,H_{y'} = 0,\ x = a,\ b \tag{9}$$

where a, b are the x abscissa of the points where the failure surface intersects with the surface of the slope. From the boundary condition, for two ends of the possible failure surface, we have:

$$f(x) = y(x)\ , \quad x = a,\ b \tag{10}$$

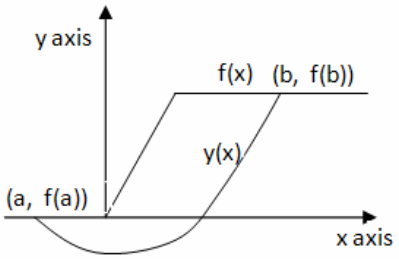
Solve Eq. (9) using boundary conditions Eq. (10) gives:

$$y'(x) = \frac{-Ff'(x) - \tan\varphi \pm \sqrt{F^2 f'(x)^2 + \tan^2\varphi + F^2 + \tan^2\varphi f'(x)^2}}{\tan\varphi\, f'(x)^2 - F} \qquad x = a,\ b \tag{11}$$

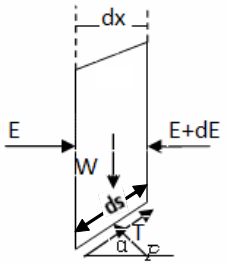
From Fig. 1, it's easy to find that $y'(a) < y'(b)$, then

$$y'(a) = \frac{-Ff'(a) - \tan\varphi + \sqrt{F^2 f'(a)^2 + \tan^2\varphi + F^2 + \tan^2\varphi f'(a)^2}}{\tan\varphi\, f'(a)^2 - F} \tag{12}$$

$$y'(b) = \frac{-Ff'(b) - \tan\varphi - \sqrt{F^2 f'(b)^2 + \tan^2\varphi + F^2 + \tan^2\varphi f'(b)^2}}{\tan\varphi\, f'(b)^2 - F} \tag{13}$$



(a) Coordinates system and a slip surface



(b) Internal forces in a slice from Janbu's simplified method (interslice shear force is assumed to be zero)

Fig. 1 A simple slope to illustrate the variational principle and harmony search method

Solving Eq. (8), a second order differential equation governing the slip surface will be obtained:

$$y''(x) = \frac{B}{A} \tag{14}$$

where

$$B = \gamma(F + y'\tan\varphi)(f'F^2 + Fy'^2\tan\varphi - F\tan\varphi - 2y'\tan^2\varphi + f'\tan^2\varphi) \tag{15}$$

$$A = 2(\tan^2\varphi + F^2)\left[c + \gamma(f-y)\tan\varphi\right] \tag{16}$$

There are totally five unknowns: a, b, F, and two factors for the second order differential equation. There are also five equations: Two boundary conditions (Eq. 8), two transeverality conditions (Eq. 9) and one governing equation (Eq. 12). Analytical solution to this system of equations arising from the use of variational principle is not easy to formulate, so a finite difference method is adopted for the solution in the present study:

(1) Set increment Δx.
(2) Assume point A and F, then $f(a)$, $y(a)$, $f'(a)$ are all determined from Eq. (10), $y'(a)$ is calculated from Eq. (12) and $y''(a)$ is calculated from Eq. (14).
(3) Use Runge-Kutta method to approximate the solution of this initial value problem.
(4) Trace where the failure surface intersect the slope profile, *i.e.* $f(x) = y(x)$ and the number of steps required in the calculation of the factor of safety N is recorded.
(5) Calculate the Factor of safety F using Janbu's simplified method [30].

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(6) Check if the F obtained is close enough to previously one obtained in tracing the failure surface.

(7) When (6) is achieved, check whether $f'(b)$ satisfies the transeverality condition (Eq. 11), if not, A will be altered, then go to (3). If it does, the critical failure surface and the corresponding FOS are determined.

For the present problem, the critical solution is determined by the variational principle as well as the harmony search method. For illustration, a simple homogeneous slope without water is considered. The height of the slope is 5m, slope angle = 45°, c' = 10kPa and ϕ' = 5°, 10°, 15° and 20°. The comparisons between the critical factors of safety from the variational principle and harmony search method are shown in Table 1. It is interesting to note that the results by the two formulations are virtually the same, which has indirectly demonstrated that the two methods are actually equivalent in analysis. The critical failure surfaces obtained from the two methods are also virtually the same. Since the use of global optimization method is much more convenient for complicated problem, it will be adopted in the later study to replace the use of variational principle in upper bound analysis.

Table1 Minimum factor of safety for different ϕ by variational principle and global optimization method

ϕ	5	10	20	30	40
Eq. (12)	0.705	0.872	1.175	1.490	1.856
Modified harmony search	0.710	0.878	1.186	1.494	1.857

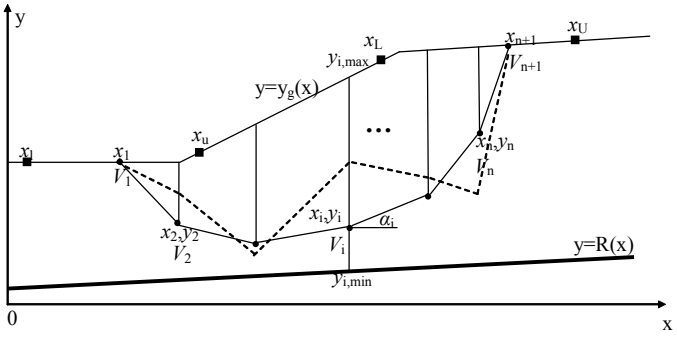


Fig. 2 A typical slip surface obtained by conventional procedure

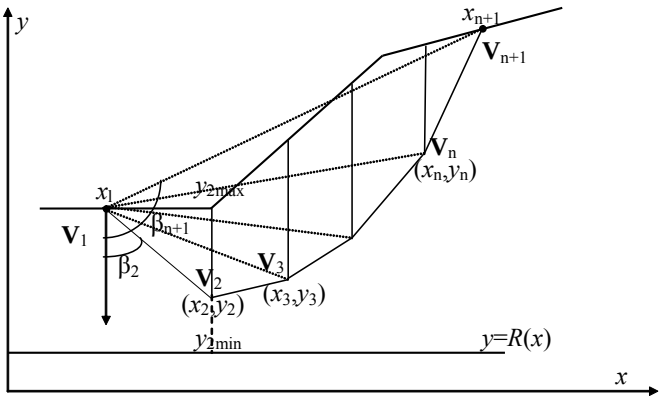


Fig. 3 Procedure for generating trial slip surface

3. PROCEDURES FOR GENERATING TRIAL SLIP SURFACES

The critical solution in slope stability problem needs to consider two major issues: Generation of failure surfaces and the determination of minimum factor of safety. The slip surface generation method used in the present study is developed by Cheng [26,27] which is applicable to both concave and convex slip surface for slope stability problem. The procedures for generating an arbitrary slip surface is shown in Fig. 2, where the ground surface is represented by $y = y_g(x)$, bedrock is represented by $y = R(x)$. The trial slip surface in Fig. 3 is composed of $n + 1$ vertices with coordinates of (x_i, y_i), in which i varies between 1 and $n + 1$, thus there will be n vertical slices with slice angle α_i, in which i varies between 1 and n. The vertices V_1 and V_{n+1} are the exit and entrance points of the slip surface respectively, and the lower and upper bounds for these two vertices can be prescribed by the engineers easily. Usually, the y-coordinates and/or the x-coordinates of these vertices are taken as the variables. The x-coordinates of V_2 to V_n can be obtained by even spacing as given by Eq. (17) while the upper and lower bounds to the y-coordinates $V_{i,\max}$ and $V_{i,\min}$ can be calculated by using Eq. (18).

$$x_i = x_1 + \frac{x_{n+1} - x_1}{n} \times (i-1),\ i = 2,\ ...,\ n \qquad (17)$$

$$\begin{cases} y_{i,\max} = y_g(x_i) \\ y_{i,\min} = R(x_i) \end{cases},\ i = 2,\ ...,\ n \qquad (18)$$

In Fig. 3, a trial slip surface consists of $n + 1$ vertices, each of which is identified by the x- and y- coordinates of x_i, y_i, where i ranges from 1 to $n + 1$. The exit point and entry point, *i.e.* V_1 and V_{n+1} are controlled by using an upper and lower limits given by x_l, x_u and x_L, x_U. The x-coordinates of vertices from 2 to n are calculated by using the following Eq. (19).

$$x_i = x_1 + \frac{x_{n+1} - x_1}{n} \times (i - 1 + \zeta_i),\ i = 2,\ ...,\ n \qquad (19)$$

where ζ_i is a variable used to determine the x-coordinates of V_i and is subjected to $-0.5 < \zeta_i < 0.5$. For a given value of x_2, we can calculate $y_{2\max}$ and $y_{2\min}$ using Eq. (18). A variable χ_2 within the range of 0 to 1 is introduced to represent the y-coordinates of V_2. y_2 can be obtained using Eq. (20).

$$y_2 = y_{2\min} + (y_{2\max} - y_{2\min}) \times \chi_2 \qquad (20)$$

Two lines are obtained by connecting V_1 and V_{n+1}, V_1 and V_2 respectively. The angles of the lines from the

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downward direction V_1 are denoted as β_2, …, β_{n+1} respectively. $n - 2$ values are randomly generated within the range of β_2 to β_{n+1} using $n - 2$ values of χ_3, …, χ_n in the range of 0 to 1.0 as given by Eq. (21), and they are sorted by ascending order and are related to β_3 to β_n respectively. The y-coordinates of V_3 to V_n are then determined by Eq. (22).

$$\beta_i = \chi_i \times (\beta_{n+1} - \beta_2) \tag{21}$$

$$y_i = y_1 + \tan(\beta_i - \pi/2) \times (x_i - x_1) \tag{22}$$

where $i = 3, \ldots, n$.

Each trial slip surface obtained by this procedure can be represented mathematically by the vector $V = (x_1, \zeta_2, \chi_2, \ldots, \zeta_n, \chi_n; x_{n+1})$, in which $2n$ variables are used to determine the slip surface. The lower bound and upper bound to each element in V are denoted as l_i and u_i respectively, where i ranges from 1 to $2n$.

For the generated failure surface as shown in Fig. 4, CD can be considered as unacceptable as this will be a restraint to the slope failure. CD can be kept for analysis if necessary, but it will not be the critical solution in most cases so that a more efficient search is to correct CD so that the failure mechanism is acceptable. D' is determined by extrapolating BC to D', where D' and D have the same x-coordinate. Vertices D should be moved to D', and the same check should be performed for every adjacent three vertices until the whole failure surface (convex surface) is acceptable. If D is higher than D', CD will be acceptable and no modification is required. In view of the above process, the optimization problem associated with the location of the critical slip surface for the present formulation is outline as Eq. (23).

$$\begin{aligned}
&\min \quad f(\mathbf{V}) \\
&s.t. \quad x_l \le x_1 \le x_u \quad x_L \le x_{n+1} \le x_U \\
&-0.5 < \zeta_i < 0.5 \quad i = 2, \ldots, n \\
&\beta_2 < \beta_i < \beta_{n+1} \Longleftrightarrow 0 < \chi_i < 1 \quad i = 3, \ldots, n \\
&y_{2\min} \le y_2 \le y_{2\max} \Longleftrightarrow 0 < \chi_2 < 1
\end{aligned} \tag{23}$$

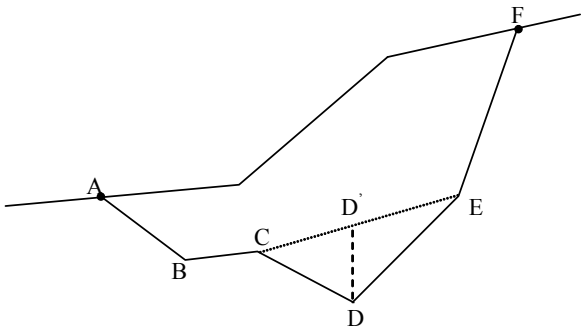


Fig. 4 Correction procedures used in this study

It must be noticed that the trial slip surface of $n + 1$ vertices is generated by using $m = 2n$ variables. The present technique in generating a failure surface has the advantage that every failure surface will be kinematically acceptable, and the number of control variables is kept to a limited value, and these two features have overcome the problems by Zolfaghari *et al*. [23] and Chen [31].

4. ORIGINAL AND NEW HARMONY SEARCH ALGORITHMS

Based on the author's experience [27], the harmony search method is found to be relatively efficient and is adopted by the authors for many geotechnical problems. Harmony search algorithm (HS) was developed by Geem *et al*. [32] and Lee *et al*. [33] which was conceptualized based on the musical process of searching for a perfect state of harmony. HS is a phenomenon-mimicking algorithm inspired by the improvisation process of musicians. In the HS algorithm, each musician plays a note for finding a best harmony. The advantages of the HS include: (1) Gradient of objective function is not required; (2) Discrete or continuous variables can be adopted; (3) Initial trial is not required; (4) May escape from local minima. The HS algorithm uses a stochastic random search that is based on the harmony memory considering rate *HR* and the pitch adjusting rate *PR*. Harmony search algorithm (*HS*) is a population based search method. For readers who are not familiar with this method, the work by Lee and Geem [33] should be consulted first before reading the improved harmony search method in this paper. A harmony memory *HM* of size *M* (usually equals $2m$) is used to generate a new harmony which is probably better than the optimum in the current harmony memory. The harmony memory consists of *M* harmonies (slip surfaces) and the *M* harmonies are usually generated randomly. Consider $HM = \{hm_1 \ hm_2, \ldots, hm_M\}$

$$hm_i = (v_{i1}, v_{i2}, \ldots, v_{im}) \tag{24}$$

where each element of hm_i corresponds to that in vector **V** as described above. The most important procedure in *HS* is the generation of a new harmony named hm_{M+1} as shown in Fig. 5, and its solution procedures are outlined in Fig. 5.

Step 1: Initialize the algorithm parameters: *HR*, *PR*, *M* and randomly generate *M* harmonies (slip surfaces) and evaluate the harmonies ;

Step 2: Generate a new harmony (as shown in Fig. 5) and evaluate it;

Step 3: Update the *HM*; *i.e.* if the new harmony is better than the worst harmony in the *HM* in terms of factor of safety, the worst harmony is replaced with the new harmony.

Step 4: Repeat steps 2 and 3 until the termination criterion as suggested by Cheng *et al*. [28] is achieved.

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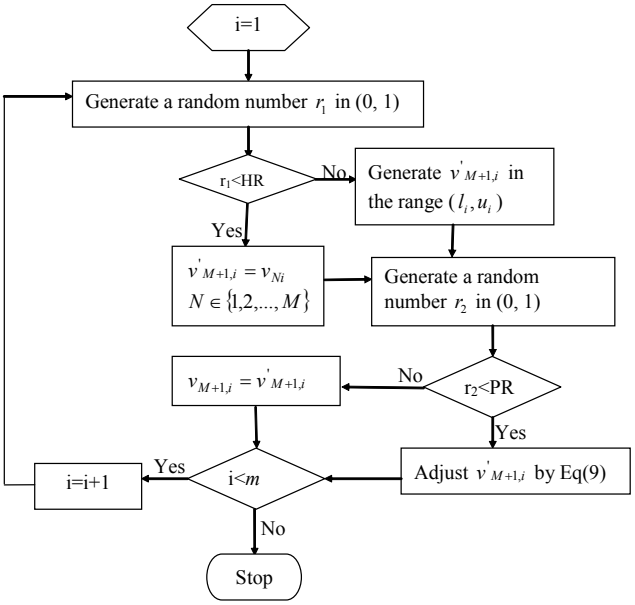


Fig. 5 The procedure for generating a new harmony

The flowchart of generating a new harmony hm_{M+1} is shown in Fig. 6 the adjusted process is given as follows:

Take the ith value of the coarse harmony h'_n, v'_{ni}, for instance, its lower bound and upper bounds are named herein $v_{i\min}$, $v_{i\max}$, a random number r_0 in the range [0,] is generated. If $r_0 > 0.5$, v'_{ni} is adjusted to v_{ni} using the Eq. (9), otherwise, Eq. (10) is used to calculate the new value of v_{ni}.

$$\begin{cases} v_{M+1,i} = v'_{M+1,i} + (u_i - v'_{M+1,i}) \times rd & r_i > 0.5 \\ v_{M+1,i} = v'_{M+1,i} - (v'_{M+1,i} - l_i) \times rd & r_i \le 0.5 \end{cases} \quad (25)$$

where rd and r_i are random numbers in the range [0,1] respectively. In addition, the number of evaluations of objective function during the search for the optimum is denoted as NEOF, and this value can represent the computation time required by the optimization algorithm.

Based on many trials by the authors, it is found that the original simple harmony search algorithm works well for simple optimization problem with less than 25 control variables in slope stability problems. For more complicate problem with a large number of control variables, the authors found that the results from the original harmony search algorithm are not satisfactory for some difficult problems as shown in later section. The authors have developed two modified harmony search algorithms for such difficult cases which differ from the original method in two aspects. The first difference is the probability of each harmony. The better the objective function value of one harmony, the more probable will the harmony be chosen for the generation of a new harmony. The second modification is that instead of a new harmony is generated as in the original method, several new harmonies (*Nhm*) are generated during each iteration step in the modified algorithm. In general, the modified harmony search

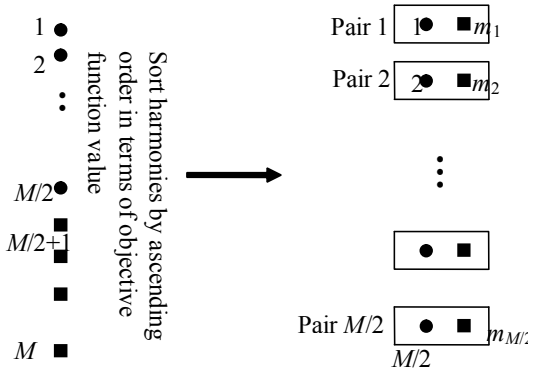


Fig. 6 Rearrangement for harmonies in *HM*

method (NHS) is much more effective than the original harmony search method for large scale optimization but will be slightly less efficient for small scale problem. NHS is also more stable for small to large scale problems and is recommended for use.

In the original harmony search method (OHS), only one harmony is obtained from M harmonies in the current *HM*, and each harmony is used with the same probability. This is the main reason why the original harmony is highly efficient for small scale problem but can be trapped by the local minimum easily for large scale optimization problems. Actually, the M harmonies in *HM* can be classified into groups based on their objective function values, and the probability of the better harmony should be higher than the worse ones. In the new harmony search algorithm (NHS) as proposed in this study, the harmonies are rearranged into M / 2 pairs as illustrated in Fig. 6.

In Fig. 6, m_i, i = 1, …, M/2 is randomly chosen from M/2 + 1 to M. The pairs located at the front of *HM* have greater probability to generate new harmonies. We can introduce a parameter λ ($0 < \lambda \le 1$) to decide which pair is used to generate new harmonies. An array $BB(\ )$ is used to represent the probabilities of all the pairs, where $BB(i) = \lambda(1-\lambda)^{i-1}$, i = 1, …, M/2. The accumulated probability array $AC(\ )$ is calculated as $AC(i) = \sum_{j=1}^{i} BB(j)$. A random number r_a is obtained within the range of 0 to $AC(M/2)$. If $AC(i-1) < r_a \le AC(i)$, then the i-th pair is used to generate one new harmony. The procedures for generating the new harmony by one given pair are outlined in Fig. 7. Suppose i-th pair of harmonies, namely, hm_i and hm_k, are used to obtain a new harmony. A random number δ in the range of 0.2 to 0.8 (the lower and upper bounds to δ are based on the facts that the larger the value of δ, the more probable the better individuals in the current pair is chosen, and vice versa) is first randomly determined and the probabilities of hm_i and hm_k are given by ρ_i and ρ_k respectively using Eq. (26). The accumulated probabilities of hm_i and hm_k are ac_i and ac_k.

$$\begin{cases} \rho_i = \delta \ ; \ \ \rho_k = \delta(1-\delta) \\ ac_i = \rho_i \ ; \ \ ac_k = \rho_i + \rho_k \end{cases} \quad (26)$$

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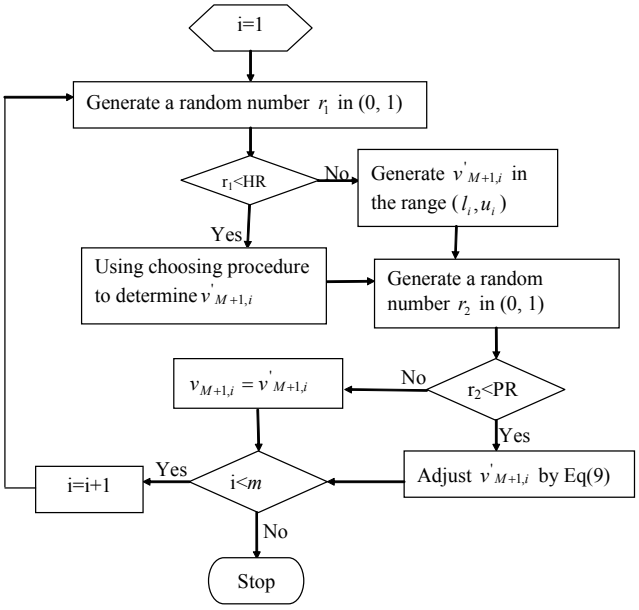


Fig. 7 The procedure for generating new harmony in NHS1

If one random number r_b within the range of 0 to ac_k is smaller than ac_i, then hm_i is used, and if r_b is higher than ac_i and smaller than ac_k, then hm_k is used. This procedure is called the choosing procedure.

Instead of only one new harmony is obtained in the original Harmony search algorithm, several new harmonies are generated in NHS. Two different versions of new harmony search methods are proposed in this paper. In the first method NHS1, the iterative steps for NHS1 are as follows:

1. Step 1: Initialize the algorithm parameters: *HR*, *PR*, *cp*, λ, *M* and randomly generate *M* harmonies (slip surfaces) and evaluate them, the counter $js = 0$, the optima harmony is called hm_o and its objective function value is f_o, its initial value is set to an arbitrary large value and a value of 100 (much greater than normal factors of safety) is taken for slope stability analysis;
2. Step 2: As shown in Fig. 5, the harmonies in *HM* are grouped in to *M*/2 pairs.
3. Step 3: Generate *M*/2 random numbers rn_i, $i = 1, \ldots, M/2$ within the range of 0 to 1. If rn_i is lower than *cp*, a new harmony is obtained as illustrated in Fig. 6, and altogether *D* new harmonies are obtained. One iteration step is finished and $js = js + 1$;
4. Step 4: Evaluate *D* new harmonies and choose *M* harmonies into *HM* from *M* old harmonies and *D* new harmonies. The best harmony in *HM* is called hm_g and its corresponding value is f_g.
5. If $js = N_1$, then check $|f_o - f_g| \le \varepsilon$. If this relation is satisfied, the NHS will terminate, otherwise, set $N_1 = N_1 + N_2$, $hm_o = hm_g$; $f_o = f_g$ and NHS will go to step 2. If $js < N_1$, NHS will go to step 2 then.

It must be noted that *M*/2 random numbers and the parameter *cp* are used to determine the number of new harmonies to be generated within one iteration step in step 3. An alternative procedure to determine the number of new harmonies is the use of parameter N_h which is the number of new harmonies to be obtained within one iteration step. The latter procedure is much more straightforward and is also easier to implement than the former one. In the following example studies, the value of N_h is equal to 5, and $N_1 = 500$, $N_2 = 200$.

The second new harmony search method NHS2 differs from the original method in two aspects. The first difference is the probability of each harmony. Instead of the use of uniform probability in the original harmony search method, the better the objective function value of one harmony, the more probable will it be chosen for the generation of a new harmony. A parameter $\delta(0 < \delta \le 1)$ is introduced and all the harmonies in *HM* are sorted by ascending order, and a probability is assigned to each of them. For instance, $pr(i)$ means the probability to choose the *i-th* harmony
Where

$$pr(i) = \delta \times (1-\delta)^{i-1} \tag{27}$$

for $i = 1, 2, \ldots, M$. From Eq. (27), it can be seen that the larger the value of δ, the more probable is the first harmony being chosen. An array $ST(i)$, $i = 0, 1, 2, \ldots, M$ should be used to implement the above procedure for choosing the harmony.

$$ST(i) = \sum_{j=1}^{i} pr(j) \tag{28}$$

where $ST(i)$ represents the accumulating probability for *i-th* harmony. $ST(0)$ is defined as 0.0 for the sake of implementation. A random number r_c is given form the range $[0, ST(M)]$, and the k^{th} harmony in *HM* is chosen if the following criterion is satisfied.

$$ST(k-1) < r_c \le ST(k)\ ,\ k = 1,\ 2,\ \ldots,\ M \tag{29}$$

The second modification in NHS2 is that instead of a new harmony, a certain number of new harmonies (*Nhm*) are generated during each iteration step in the modified algorithm. The utilization of *HM* is intuitively more exhaustive by generating several new harmonies than by generating one new harmony during one iteration. In order to retain the structure of *HM* unchanged, the *M* harmonies with the lower objective functions (for the minimization optimization problem) from $M + Nhm$ harmonies are included in the *HM* again and *Nhm* harmonies of higher objective functions are rejected.

Based on extensive tests on different types of problems, the authors have found that the two improved harmony search methods usually perform better for more difficult problems (presence of multi local minima, large number of control variables) and the number of evaluations and the results are better than the original harmony search method. On the other hand, for simple problems with smaller number of control variables, the original harmony search method is usually more efficient.

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5. NUMERICAL EXAMPLES

Based on the authors' internal study, the original harmony search method is found to be more efficient for small scale optimization problems. When the number of control variable is large and there is a small region in the solution domain where there is a major change in the soil parameters, it is found that the original harmony search method can be trapped by the presence of local minimum easily. For the present proposals, it is also found to work well for normal problems over wide range of soil parameters where there is no special geotechnical features. For problems with special geotechnical features, the robustness of the present optimization algorithm will be demonstrated by three relatively difficult examples where the precise location of the critical failure surface has a strong influence on the factor of safety and many global optimization methods may be trapped easily by the local minimum.

To illustrate the applicability of the proposed modified harmony search methods, 3 examples will be considered. Example 1 is a slope with four layers of soils as shown in Fig. 8 while the soil parameters are given in Table 2. Soil layer 3 is a thin irregular weak layer and the Spencer method is used to calculate the factor of safety. Since soil layer 3 is thin with poor soil parameters, many control variables should lie within this region which is not easily determined automatically from the OHS. This situation is particularly important when the number of control variables is large which will be demonstrated. The results as shown in Fig. 9 have clearly illustrated that the performance of the OHS is poor under this case, while the improved harmony method give much lower factor of safety with similar critical slip surface as shown in Fig. 8.

All the stochastic optimization methods rely on the use of some parameters which are difficult to be determined for general case. In general, these parameters are based on the statistics from large number of numerical tests. It is surprising to find that the performance of the modern global optimization methods under different optimization parameters in slope stability problems is seldom investigated in the past. Due to the special problems in slope stability analysis (discontinuous function), the authors view that the performance of the algorithm with respect to the use of optimization parameters is very important and a statistical check has been carried out in the present study. Assuming x_l = 11, x_u = 15; x_L = 25, x_U = 35, the randomly generated 50 series of parameters as shown in Table 3 are used to check the robustness of the present algorithm. From Fig. 9, the results corresponding to the 50 series of parameters are mainly located in the range of 1.28 to 1.33, with only one result higher than 1.40 and two results higher than 1.35. The average value of the 50 factors of safety is 1.31 and the average number of NEOF is 11617. If we assume the number of slice is equal to 20 (21 control variables); four levels are set for three parameters: *HR*, *PR*, λ thereby obtaining an orthogonal table given in Table 4. Four levels for *HR* are 0.8, 0.85, 0.9, and 1.0; 0.1, 0.15, 0.2, and 0.25 for *PR*; and

Table 2 Geotechnical parameters of example 1

Layers	γ (kN/m^3)	*c* (kPa)	ϕ (degree)
1	19.0	15.0	20.0
2	19.0	17.0	21.0
3	19.0	5.00	10.0
4	19.0	35.0	28.0

Table 3 50 series of parameters for the investigation on the robustness of NHS1 and NHS2

	1	2	3	4	5	6	7	8	9	10
HR	0.992	0.906	0.948	0.970	0.975	0.988	0.921	0.966	0.921	0.962
PR	0.186	0.133	0.101	0.269	0.187	0.131	0.208	0.192	0.263	0.163
λ	0.441	0.772	0.738	0.331	0.541	0.518	0.664	0.746	0.524	0.374
n	21	15	18	18	21	18	19	24	20	17
	11	12	13	14	15	16	17	18	19	20
HR	0.958	0.909	0.988	0.906	0.969	0.989	0.991	0.946	0.993	0.987
PR	0.205	0.180	0.221	0.295	0.149	0.287	0.180	0.299	0.154	0.195
λ	0.649	0.557	0.476	0.411	0.451	0.759	0.783	0.302	0.346	0.656
n	19	21	15	20	24	17	17	18	15	22
	21	22	23	24	25	26	27	28	29	30
HR	0.989	0.911	0.967	0.917	0.910	0.962	0.990	0.940	0.992	0.977
PR	0.123	0.249	0.236	0.297	0.201	0.166	0.132	0.215	0.200	0.175
λ	0.349	0.309	0.359	0.521	0.785	0.650	0.739	0.447	0.559	0.554
n	22	16	17	23	21	17	20	17	21	22
	31	32	33	34	35	36	37	38	39	40
HR	0.921	0.903	0.928	0.940	0.990	0.950	0.965	0.983	0.993	0.996
PR	0.216	0.196	0.185	0.102	0.153	0.237	0.282	0.122	0.131	0.255
λ	0.667	0.652	0.337	0.363	0.585	0.536	0.755	0.622	0.737	0.638
n	15	22	24	20	20	24	22	23	19	16
	41	42	43	44	45	46	47	48	49	50
HR	0.905	0.941	0.995	0.980	0.918	0.961	0.997	0.976	0.944	0.930
PR	0.293	0.139	0.228	0.127	0.196	0.125	0.288	0.190	0.212	0.199
λ	0.345	0.658	0.371	0.679	0.684	0.726	0.526	0.452	0.759	0.577
n	19	19	24	17	18	15	15	18	18	17

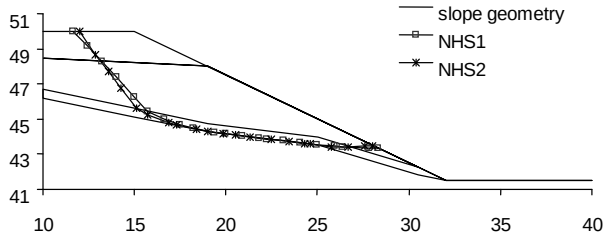


Fig. 8 Geotechnical profile and critical solution for example 1

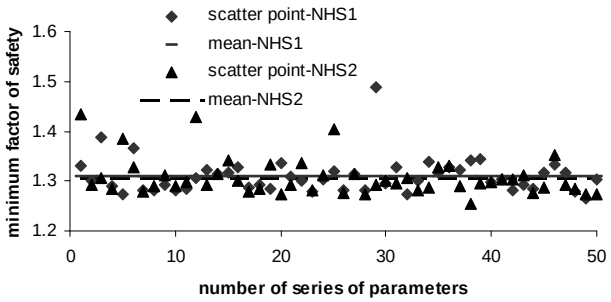


Fig 9 Summary of the results for 50 series of parameters

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0.3, 0.4, 0.5 and 0.6 for λ. The sensitivity of each parameter can be obtained through 16 tests. If the F value (Factorial Analysis of Variance after Fisher [34]) of one parameter is larger than the critical value $F_{0.05}$ and is smaller than $F_{0.01}$, it implies that the calculated result is sensitive to this parameter, otherwise if the F value is smaller than $F_{0.05}$, it shows that the result is insensitive to this parameter. If the F value is larger than $F_{0.01}$, the result is hyper-sensitive to this parameter.

For NHS1, From the result in the 5-th column in Table 4, the F values of the three parameters *HR*, *PR*, λ can be obtained as 0.78,0.90 and 0.92 respectively while $F_{0.05} = 4.8$ and $F_{0.01} = 9.8$. It can be concluded that the three parameters are insensitive to the analysis. Similarly, the F values of the two parameters *HR*, *PR* used in NHS2 can be calculated as 0.47 and 0.59, compared with the standard value $F_{0.05} = 3.9$ and $F_{0.01} = 7.0$ (different number of parameters and levels result in different values of $F_{0.05}$ and $F_{0.01}$, and they can be determined easily from the existing Tables), we can conclude that both parameters are insensitive to the NHS2 analysis. The results obtained by NHS1 and NHS2 are almost identical in this problem.

OHS is trapped by the local minima as shown in Fig. 10 with relatively poor results for the factor of safety, though the NEOF required by OHS is smaller than those required by NHS1 and NHS2. OHS located the factor of safety within the range of 1.40 to 1.60 with different number of control variables. On the other hand, NHS1 and NHS2 find the minimum factor of safety cluster around 1.3 which is much better than the results by OHS. One special advantage of the harmony search is that the number of evaluation does not increase sharply with the increase in the number of control variables (number of slice), and this is a very important reason for the adoption of harmony search method for large scale slope stability analysis.

Example 2 as shown in Fig. 11 is a slope with three layers where an irregular weak layer is sandwiched between two strong layers. The geotechnical properties for layers 1 to 3 respectively are friction angle 20°, 10° and 20°; cohesion 28.73kPa, 0.0kPa, and 28.73kPa; and unit weight 18.84kN/m^3 for all three layers. The Spencer's method is adopted to determine the factor of safety. As described above, the same 50 series of parameters randomly generated within their corresponding ranges are used to test the robustness of the proposed algorithm.

Assuming $x_l = 30$, $x_u = 45$; $x_L = 70$, $x_U = 75$, the same orthogonal test table as used in example 1 is performed to investigate the sensitivities of parameters in the proposed algorithm. The obtained results are listed in Table 5. The corresponding F values for the three parameters are given by $F_{HR} = 1.48$, $F_{PR} = 1.12$, $F_{\lambda} = 1.68$. For NHS2, $F_{HR} = 1.15$, $F_{PR} = 0.57$. Compared with the standard values of $F_{0.05}$, $F_{0.01}$, it can also be conclude that the three parameters are insensitive to the optimization analysis.

The same values for the parameters used in OHS, NHS1, and NHS2 as used in example 1 are adopted to compare the results obtained by the three algorithms with the different number of slices. The results are summarized in Fig. 12. Results in Fig. 12 are basically similar to those in Fig. 10. OHS tends to locate a much higher factor of safety than NHS1 and NHS2 for a given number of slices and the results fluctuates in the range of 1.60 to 1.80 with the number of slices varying from 20 to 80. On the other hand, the results by NHS1 and NHS2 are much more stable than OHS and the numbers of evaluation are also acceptable (most problems are finished within 5 minutes).

Example 3 as shown in Fig. 13 is based on a problem where an irregular weak layer is sandwiched between two strong layers. The geotechnical properties for layers 1 to 3 respectively are friction angle 35°, 25° and 35°; cohesion 20.0kPa, 0.0kPa, and 10.0kPa; and unit weight 19.0kN/m^3 for all three layers. For analysis, the parameters for this problem are $x_l = 3.0$, $x_u = 8.0$;

Table 4 Orthogonal test table of three and two parameters and four levels by NHS1 and NHS2 respectively **

	HR	*PR*	λ	results	
				NHS1	NHS2
1	1 (0.85)	1 (0.10)	1 (0.3)	1.3196	1.2938
2	1 (0.85)	2 (0.15)	2 (0.4)	1.3325	1.3276
3	1 (0.85)	3 (0.20)	3 (0.5)	1.3542	1.3321
4	1 (0.85)	4 (0.25)	4 (0.6)	1.3391	1.3413
5	2 (0.90)	1 (0.10)	2 (0.4)	1.3370	1.3079
6	2 (0.90)	2 (0.15)	1 (0.3)	1.3492	1.2917
7	2 (0.90)	3 (0.20)	4 (0.6)	1.3265	1.3298
8	2 (0.90)	4 (0.25)	3 (0.5)	1.3245	1.3057
9	3 (0.95)	1 (0.10)	3 (0.5)	1.3616	1.3235
10	3 (0.95)	2 (0.15)	4 (0.6)	1.2924	1.3223
11	3 (0.95)	3 (0.20)	1 (0.3)	1.3050	1.3134
12	3 (0.95)	4 (0.25)	2 (0.4)	1.3171	1.2783
13	4 (1.00)	1 (0.10)	4 (0.6)	1.3509	1.3327
14	4 (1.00)	2 (0.15)	3 (0.5)	1.3262	1.3298
15	4 (1.00)	3 (0.20)	2 (0.4)	1.2998	1.3144
16	4 (1.00)	4 (0.25)	1 (0.3)	1.2914	1.2903

** note that the parameter λ is not used in NHS2, so the number of total parameters is equal to 2

Table 5 The orthogonal test table of four parameters and four levels for example 2

	HR	*PR*	λ	results	
				NHS1	NHS2
1	1 (0.85)	1 (0.10)	1 (0.3)	1.4781	1.4578
2	1 (0.85)	2 (0.15)	2 (0.4)	1.4746	1.4379
3	1 (0.85)	3 (0.20)	3 (0.5)	1.4348	1.4849
4	1 (0.85)	4 (0.25)	4 (0.6)	1.4441	1.4707
5	2 (0.90)	1 (0.10)	2 (0.4)	1.3952	1.4034
6	2 (0.90)	2 (0.15)	1 (0.3)	1.4346	1.4041
7	2 (0.90)	3 (0.20)	4 (0.6)	1.4367	1.4684
8	2 (0.90)	4 (0.25)	3 (0.5)	1.4549	1.4092
9	3 (0.95)	1 (0.10)	3 (0.5)	1.3761	1.4690
10	3 (0.95)	2 (0.15)	4 (0.6)	1.3961	1.4154
11	3 (0.95)	3 (0.20)	1 (0.3)	1.4874	1.3912
12	3 (0.95)	4 (0.25)	2 (0.4)	1.4145	1.3789
13	4 (1.00)	1 (0.10)	4 (0.6)	1.4397	1.4071
14	4 (1.00)	2 (0.15)	3 (0.5)	1.4200	1.4052
15	4 (1.00)	3 (0.20)	2 (0.4)	1.4711	1.4654
16	4 (1.00)	4 (0.25)	1 (0.3)	1.4560	1.5514

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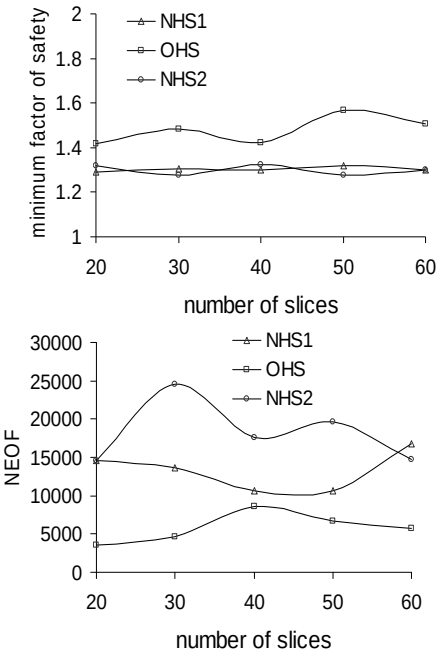


Fig. 10 Comparison of factor of safety and NEOF for different number of control variables between OHS and NHS

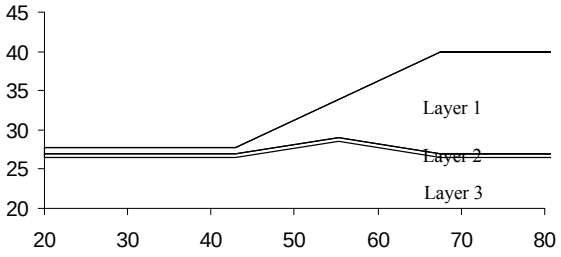


Fig. 11 Cross section of example 2

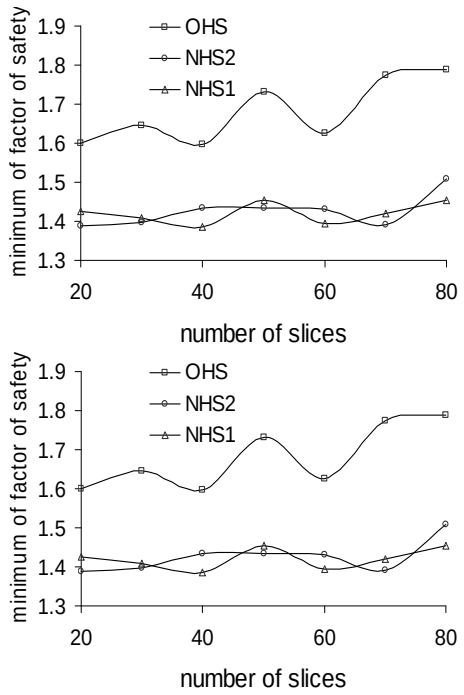


Fig. 12 Comparison of factor of safety and NEOF for different number of slices and between OHS, NHS and OHS

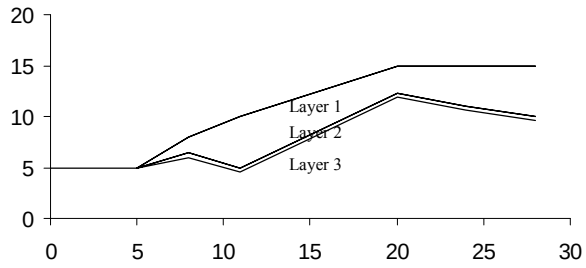


Fig. 13 Cross-section for example 3

$x_L = 20$, $x_U = 28$. As described above, the orthogonal analysis is firstly performed to investigate the sensitivity of the related parameters. The results obtained are listed in Table 6. From the data listed in 5-th column in Table 6, we can calculate the F values of three parameters which are equal to 7.95, 14.80, and 5.32 respectively for NHS1. According to the comparison with the critical values $F_{0.05}$ and $F_{0.01}$, it can be concluded that HR and λ are sensitive and PR is hypersensitive to the analysis. For NHS2, the corresponding F values are 5.82 and 2.28 and HR is sensitive while PR is insensitive to the analysis. It should be noted that the parameters used in the search algorithms are not sensitive for the simple examples, but they can be sensitive or hypersensitive for complicated examples. It is suggested that engineers should perform an orthogonal analysis for complicated problems. The authors have also found similar behaviour for genetic algorithms and ant colony method. Actually, all stochastic global optimization method will face similar problem in the choice of parameters, but the present modified proposals have greatly reduced this limitation which is serious for the original formulation.

The comparisons of results and NEOF between OHS, NHS1 and NHS2 in Fig. 14 show that the OHS locates results much higher than those by NHS1 and NHS2. The NEOF required by OHS is smaller than those used by NHS1 and NHS2. This is mainly because after N_2 iterations, there is no improvement on the factor of safety obtained so far, which is called precocity. On the other hand, the NHS1 and NHS2 can improve the result found so far within N_2 iterations, so the NEOF used by them is always larger than that used by OHS. The largest one is not more than 30000 and this is acceptable as the solution time is less than 5 minutes.

6. RESULTS AND DISCUSSION

In this paper, the variational principle has been demonstrated to be equivalent to the global optimization method, and the limitations of the classical variational principle formulation as mentioned by De Jong [11,12] can be overcome by the adoption of modern heuristic global optimization method. Modern stochastic optimization method is however much more convenient to be used for difficult problems.

The original harmony search algorithm is easily trapped by the local minima due to the procedure that only one new harmony is obtained and equal probability are use for the harmonies within the solution domain. In views of the efficiency of the harmony search

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Table 6 The orthogonal test table of four parameters and four levels for example 3

	HR	*PR*	λ	results	
				NHS1	NHS2
1	1 (0.85)	1 (0.10)	1 (0.3)	1.4843	1.5419
2	1 (0.85)	2 (0.15)	2 (0.4)	1.5074	1.5462
3	1 (0.85)	3 (0.20)	3 (0.5)	1.5207	1.6225
4	1 (0.85)	4 (0.25)	4 (0.6)	1.5461	1.5227
5	2 (0.90)	1 (0.10)	2 (0.4)	1.5239	1.4814
6	2 (0.90)	2 (0.15)	1 (0.3)	1.4752	1.4872
7	2 (0.90)	3 (0.20)	4 (0.6)	1.5047	1.5345
8	2 (0.90)	4 (0.25)	3 (0.5)	1.5267	1.5215
9	3 (0.95)	1 (0.10)	3 (0.5)	1.4831	1.4681
10	3 (0.95)	2 (0.15)	4 (0.6)	1.4817	1.4923
11	3 (0.95)	3 (0.20)	1 (0.3)	1.4711	1.5107
12	3 (0.95)	4 (0.25)	2 (0.4)	1.5415	1.5385
13	4 (1.00)	1 (0.10)	4 (0.6)	1.4579	1.4779
14	4 (1.00)	2 (0.15)	3 (0.5)	1.4702	1.4970
15	4 (1.00)	3 (0.20)	2 (0.4)	1.4857	1.4870
16	4 (1.00)	4 (0.25)	1 (0.3)	1.5047	1.4828

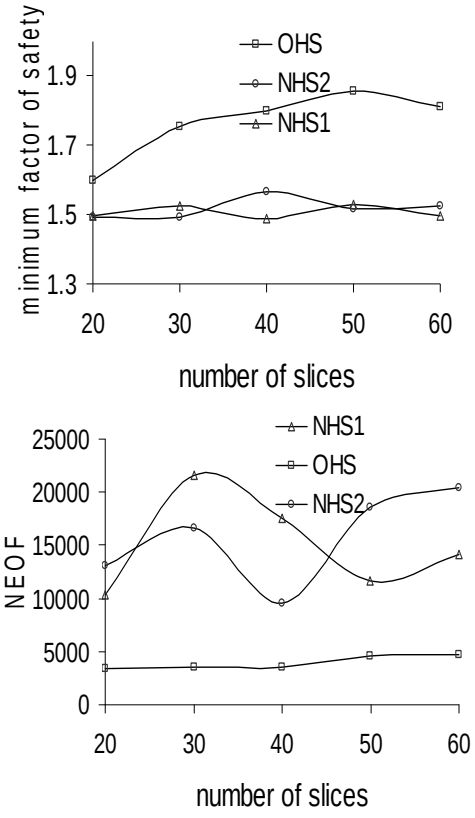


Fig 14 Comparison of factor of safety and NEOF for different number of slices and between OHS, NHS and OHS

method and its limitations, two modified harmony search algorithms are proposed by generating pairs of harmonies and the use of different probabilities to different harmony.

The modified harmony algorithms are found to be highly effective and efficient for difficult problems as shown in this paper. For the three relatively difficult problems with irregular weak layer of soils, water pressure and earthquake loading, it is found that the two proposed modified harmony search algorithms have performed well and are relatively insensitive to the global optimization parameters in most cases. On the other hand, the original harmony search method is trapped by the local minima easily for these types of problems which are not uncommon in practice. From numerous internal tests, it appears that the modified harmony methods are more effective and stable methods over a wide range of problems as compared with the original harmony method, except when the problems under consideration is simple with relatively few control variables.

Stochastic algorithms are approximate and not accurate algorithms. These algorithms usually find a solution close to the best one, and they usually find it fast and easily. Every stochastic algorithm relies on the use of some parameters for analysis but there is no serious method in determining these kinds of parameters. The success of a suitable global optimization algorithm may rely on the use of these parameters, but it is surprising to find that there is not any previous study on the sensitivity of these parameters on the optimization analysis in slope stability problem. Slope stability problem has the special features of widely varying ground and subsoil profile, soil parameters and loading. It is found that the two proposed algorithms are relatively insensitive to the use of these parameters due to the special arrangement in generating pairs of harmonies and this property is highly beneficial in slope stability problem.

7. CONCLUSIONS

The efficiency of a global optimization algorithm is an important factor to be considered. Majority of the global optimization methods will also require tremendous evaluations for large scale optimization analysis. The authors have tested the proposed algorithms to a maximum 160 control variables (sufficient for most of the slope stability analyses) and have found that the proposed algorithms are still efficient and has performed satisfactory. It is true that the original harmony method is usually more efficient than the modified methods for simple and small problems, but the original harmony method suffers from the limitation of being trapped by the presence of local minimum for large scale problem.

The modified harmony algorithms and the trick in generating a random number with weighting to different zones have been proved to be highly effective in geotechnical engineering. The present algorithms are however not limited to geotechnical problems, and such algorithms should be applicable to general optimization problems. The authors are now considering the use of such technique for the back-analysis of pile driving analysis which is another important problem faced by many engineers. The authors are currently working on some plasticity problem using the variational principle, and it is found that it is extremely difficult to solve many problems using the classical calculus of variation. The present paper come from the works in plasticity problem and has been demonstrated to have practical

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advantages over the variational principle in practical applications.

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