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Uniform s-Cross-Intersecting Families

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In this paper we study a question related to the classical Erdős–Ko–Rado theorem, which states that any family of k-element subsets of the set $[n]=\{1,\dots,n\}$ in which any two sets intersect has cardinality at most $\binom{n-1}{k-1}$.

We say that two non-empty families $\mathcal{A},\mathcal{B}\subset\binom{[n]}{k}$ are *s-cross-intersecting* if, for any $A\in\mathcal{A}$, $B\in\mathcal{B}$, we have $|A\cap B|\geqslant s$. In this paper we determine the maximum of $|\mathcal{A}|+|\mathcal{B}|$ for all n. This generalizes a result of Hilton and Milner, who determined the maximum of $|\mathcal{A}|+|\mathcal{B}|$ for non-empty 1-cross-intersecting families.

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1. Introduction

Let $[n]=\{1,\dots,n\}$ be an n-element set, and for $0\leqslant k\leqslant n$ let $\binom{[n]}{k}$ denote the collection of all its k-element subsets. Further, let $2^{[n]}$ denote the power set. Any subset $\mathcal{F}\subset 2^{[n]}$ is called a *set family*, or *family* for short. The following classical result is one of the cornerstones of extremal set theory. If $F\cap F'\neq\emptyset$ for all $F,F'\in\mathcal{F}$, then $\mathcal{F}$ is called *intersecting*.

Theorem 1.1 (Erdős, Ko and Rado [1]). *If $\mathcal{A}\subset\binom{[n]}{k}$ is intersecting and $n\geqslant 2k$, then $|\mathcal{A}|\leqslant\binom{n-1}{k-1}$ holds.*

This is one of the first results in extremal set theory, and probably the first result about intersecting families.

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Numerous results have extended the Erdős–Ko–Rado theorem in different ways. One of the directions was to study *non-trivial* intersecting families, that is, excluding the obvious examples of intersecting families of sets that all contain a fixed element. Note that the family of all sets containing a single element has size matching the bound from the EKR theorem. Probably the most widely known result in this direction is the Hilton–Milner theorem [4], which gives the maximum size of a non-trivial intersecting family.

In the same paper Hilton and Milner dealt with pairs of *cross-intersecting* families. We call $\mathcal{A},\mathcal{B}\subset 2^{[n]}$ *cross-intersecting* if, for every $A\in\mathcal{A}$ and $B\in\mathcal{B}$, we have $A\cap B\neq\emptyset$. They proved the following inequality.

Theorem 1.2 (Hilton and Milner [4]). *Let $\mathcal{A},\mathcal{B}\subset\binom{[n]}{k}$ be non-empty cross-intersecting families with $n\geqslant 2k$. Then*

$$|\mathcal{A}|+|\mathcal{B}|\leqslant\binom{n}{k}-\binom{n-k}{k}+1.$$

This inequality was generalized by Frankl and Tokushige [3] to the case $\mathcal{A}\subset\binom{[n]}{a}$, $\mathcal{B}\subset\binom{[n]}{b}$ with $a\neq b$, and with more general constraints on the sizes of $\mathcal{A},\mathcal{B}$. A simple proof of the theorem above may be found in [2].

We say that two families $\mathcal{A},\mathcal{B}\subset 2^{[n]}$ are *s-cross-intersecting* if, for any $A\in\mathcal{A}$, $B\in\mathcal{B}$, we have $|A\cap B|\geqslant s$. In this paper we prove the following generalization of the Hilton–Milner theorem. Define the family

$$\mathcal{C}=\{C\subset\binom{[n]}{k}:|C\cap[k]|\geqslant s\}.$$

Theorem 1.3. *Let $k>s\geqslant 1$ be integers. Suppose that $\mathcal{A},\mathcal{B}\subset\binom{[n]}{k}$ are non-empty s-cross-intersecting families. Then for $n>2k-s$ we have*

$$\max_{\mathcal{A},\mathcal{B}}|\mathcal{A}|+|\mathcal{B}|=|\mathcal{C}|+1. \tag{1.1}$$

2. Preliminaries

We start with the following auxiliary statement, which is of independent interest. Consider a bipartite graph H with parts V_1,V_2 and a group of automorphisms Γ, where Γ is acting on V_1 and V_2 but respects the parts (that is, $\gamma(v)\in V_i$ holds for all $\gamma\in\Gamma$ and $v\in V_i$). Let $T_1,\dots,T_p$ and $S_1,\dots,S_q$ be the orbits of the action of Γ, with $V_1=T_1\sqcup\ldots\sqcup T_p$ and $V_2=S_1\sqcup\ldots\sqcup S_q$.

Lemma 2.1. *There exists an independent set I in H of maximal cardinality, such that for any i either $I\supset T_i$ or $I\cap T_i=\emptyset$ and for any j either $I\supset S_j$ or $I\cap S_j=\emptyset$.*

Construct an auxiliary bipartite graph W on the set of vertices $\{T_1,\dots,T_p\}$ and $\{S_1,\dots,S_q\}$ and with edges between T_i and S_j if and only if there is at least one edge connecting a vertex of T_i to a vertex of S_j. Put weights $|T_i|,|S_j|$ on the vertices T_i,S_j, correspondingly.

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Let $B \subset V_1 \cup V_2$ be an independent set in H and define

$$\beta_i = |B \cap T_i|/|T_i|, \quad \beta_{p+j} = |B \cap S_j|/|S_j| \quad \text{for } i = 1,\dots,p,\ j = 1,\dots,q.$$

Claim 2.2. *If there is an edge between T_i and S_j in W, then*

$$\beta_i + \beta_{p+j} \leqslant 1. \tag{2.1}$$

Proof. Just note that Γ is transitive on both T_i and S_j. Therefore, T_i, S_j induce a *biregular* bipartite graph (*i.e.*, the degree is constant on each side) with non-zero degrees. Inequality (2.1) follows easily from the aforementioned regularity, since among the neighbours of a β_i-fraction of the vertices from T_i there is at least a β_i-fraction of the vertices from S_j. □

We call any non-negative vector $(\beta_1,\dots,\beta_{p+q})$ satisfying (2.1) a *fractional independent set*. Note that each vector corresponding to an independent set in W is a fractional independent set with coordinates from $\{0,1\}$.

Proof of Lemma 2.1. We take a fractional independent set $\mathbf{v} := (\beta_1,\dots,\beta_{p+q})$ in W with maximal weight

$$\sum_{i=1}^{p} \beta_i |T_i| + \sum_{j=1}^{q} \beta_{p+j} |S_j|.$$

The vector $\mathbf{u} := (1-\beta_1,\dots,1-\beta_{p+q})$ is a minimal weight fractional vertex cover (*i.e.*, for each each edge $(T_i, S_j) \in W$ we have $\mathbf{u}_i + \mathbf{u}_{p+j} \geqslant 1$). It is a standard result in combinatorial optimization that there exists a minimal weight fractional vertex cover $\mathbf{u}$ in W with integral coordinates (see *e.g.* [5]). Thus, there exists a maximal weight fractional independent set $\mathbf{v}$ in W with all coordinates integral. This fractional independent set corresponds to the desired independent set in the graph H. □

We recall the definition of *left shifting* (left compression), which we would simply refer to as *shifting*. For a given pair of indices $i < j \in [n]$ and a set $A \in 2^{[n]}$ we define the (i,j)-shift $S_{i,j}(A)$ of A in the following way. If $i \in A$ or $j \notin A$, then $S_{i,j}(A) = A$. If $j \in A, i \notin A$, then $S_{i,j}(A) := (A - \{j\}) \cup \{i\}$ that is, $S_{i,j}(A)$ is obtained from A by replacing element j with element i.

Next, we define the (i,j)-shift $S_{i,j}(\mathcal{F})$ of a family $\mathcal{F} \subset 2^{[n]}$:

$$S_{i,j}(\mathcal{F}) := \{S_{i,j}(A) : A \in \mathcal{F}\} \cup \{A : A, S_{i,j}(A) \in \mathcal{F}\}.$$

We call a family $\mathcal{F}$ *shifted* if $S_{i,j}(\mathcal{F}) = \mathcal{F}$ for all $1 \leqslant i < j \leqslant n$.

3. Proof of Theorem 1.3

We assume that $s \geqslant 2$, as the case $s = 1$ is covered by Theorem 1.2. Recall that

$$\mathcal{C} = \left\{ C \subset \binom{[n]}{k} : |C \cap [k]| \geqslant s \right\}.$$

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It is not hard to see that if $\mathcal{A},\mathcal{B}$ are s-cross-intersecting, then so are $S_{i,j}(\mathcal{A}),S_{i,j}(\mathcal{B})$. Therefore, we may assume without loss of generality that $\mathcal{A},\mathcal{B}$ are shifted and thus both contain $[k]$. It is obvious that in any case $|\mathcal{A}|+|\mathcal{B}|\leqslant 2|\mathcal{C}|$, since $\mathcal{A},\mathcal{B}\subset\mathcal{C}$. Consider a bipartite graph $G=(\mathcal{C}-\{[k]\},\mathcal{C}-\{[k]\},E)$ with two copies $\mathcal{C}^1,\mathcal{C}^2$ of $\mathcal{C}-\{[k]\}$ as parts and with two sets $C_1,C_2\in\mathcal{C}$ from different parts being connected by an edge if and only if $|C_1\cap C_2|<s$. Our goal is to show that the maximum independent set in G has size $|\mathcal{C}|-1$. Indeed, it is clear that $|\mathcal{A}-\{[k]\}|+|\mathcal{B}-\{[k]\}|\leqslant\alpha(G)$, and in that case we get the desired bound $|\mathcal{A}|+|\mathcal{B}|\leqslant|\mathcal{C}|+1$.

One way to show that G does not have independent sets larger than its parts is to exhibit a perfect matching in G. We were unable to do it in general. We are going to circumvent this problem by doing something different. We cover the vertices of a certain weighted graph directly related to G by disjoint edges and paths of even vertex length in such a way that the *total weight* of any independent set in any of these graphs is at most half of the total weight of the set of vertices of that subgraph.

We consider the following group of isomorphisms Γ, acting on $[n]$ (and on the vertex set of G). Γ is a product of two groups, one consisting of all permutations of $[k]$ and the other consisting of all permutations of $[k+1,n]$. Each of the parts of G splits into orbits $\mathcal{C}_i^j$, where $j=1,2$ and corresponds to the index of the part, and $i=s,\dots,k-1$ and indicates the size of the intersection of sets from $\mathcal{C}_i^j$ with $[k]$:

$$\mathcal{C}_i^j=\left\{C\in\binom{[n]}{k}:|C\cap[k]|=i\right\}.$$

We refer to these orbits as $\mathcal{C}_i$ if we think of them as set families.

Take an independent set B in G of largest size. Using Lemma 2.1, we may assume without loss of generality that for each j,i either $B\supset\mathcal{C}_i^j$ or $B\cap\mathcal{C}_i^j=\emptyset$. Consider an auxiliary *weighted* graph W as in the proof of Lemma 2.1. That is, the vertices of W are $\mathcal{C}_i^j$ and the vertices from different parts are connected if and only if there are two sets from the corresponding orbits that intersect in less than s elements. We also put weights on each vertex of W equal to the cardinality of the corresponding orbit. Thus, in view of Lemma 2.1, it is enough for us to show that one part of W has the same weight as the heaviest independent set in W.

Put $n=2k-s+1+l$, where $l\geqslant 0$. It is easy to check that $\mathcal{C}_i^1$ is connected to $\mathcal{C}_t^2$ with

$$t\in\{k-i-l,\dots,k-i+s-1\}\cap\{s,\dots,k-1\}.$$

Next we study the weights of the interconnected vertices. We call the value of i *meaningful* if the resulting indices of all families depending on i lie in the set $\{s,\dots,k-1\}$. We need the following technical lemma.

Lemma 3.1.

(1) $|\mathcal{C}_i|\geqslant|\mathcal{C}_{k-i+s-1}|$ *if and only if* $s\leqslant i\leqslant(k+s-1)/2$.
(2) $|\mathcal{C}_{\lfloor(k-l)/2\rfloor-i}|\leqslant|\mathcal{C}_{\lfloor(k+s-1)/2\rfloor+i}|$ *for all meaningful positive integer values of* i.

Proof. The proof of the lemma is just a careful algebraic manipulation. We remark that for the first reading one may omit all the integer parts in the computations below to make the verification easier.

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(1) Assume $i \leqslant (k+s-1)/2$. We have

$$|\mathcal{C}_i| = \binom{k}{i}\binom{n-k}{k-i}, \quad |\mathcal{C}_{k-i+s-1}| = \binom{k}{k-i+s-1}\binom{n-k}{i-s+1}.$$

Therefore

$$\begin{aligned}\frac{|\mathcal{C}_i|}{|\mathcal{C}_{k-i+s-1}|} &= \frac{(i-s+1)!^2(k-i+s-1)!(n-k-i+s-1)!}{(k-i)!^2 i!(n-2k+i)!}\\ &= \frac{(k-i+s-1)\cdots(k-i+1)}{i\cdots(i-s+2)} \cdot \frac{(n-k-i+s-1)\cdots(n-2k+i+1)}{(k-i)\cdots(i-s+2)}\\ &\geqslant 1,\end{aligned}$$

because $k-i+s-1 \geqslant i$ since $i \leqslant (k+s-1)/2$, and $n-k-i+s-1 \geqslant k-i$ since $n \geqslant 2k-s+1$.

(2) Assume first that $\lfloor (k-l)/2 \rfloor \geqslant \lceil (k-s+1)/2 \rceil$. We have

$$\begin{aligned}\frac{|\mathcal{C}_{\lfloor (k+s-1)/2 \rfloor + i}|}{|\mathcal{C}_{\lfloor (k-l)/2 \rfloor - i}|} &= \frac{\left(\left\lceil \frac{k+l}{2} \right\rceil + i\right)!^2 \left(\left\lfloor \frac{k-l}{2} \right\rfloor - i\right)! \left(n-k-\left\lceil \frac{k+l}{2} \right\rceil - i\right)!}{\left(\left\lceil \frac{k-s+1}{2} \right\rceil - i\right)!^2 \left(\left\lfloor \frac{k+s-1}{2} \right\rfloor + i\right)! \left(n-k-\left\lceil \frac{k-s+1}{2} \right\rceil + i\right)!}\\ &= \frac{\left(\left\lfloor \frac{k-l}{2} \right\rfloor - i\right)\cdots\left(\left\lceil \frac{k-s+1}{2} \right\rceil - i + 1\right)}{\left(\left\lfloor \frac{k+s-1}{2} \right\rfloor + i\right)\cdots\left(\left\lceil \frac{k+l}{2} \right\rceil + i + 1\right)}\\ &\quad \times \frac{\left(\left\lceil \frac{k+l}{2} \right\rceil + i\right)\cdots\left(\left\lceil \frac{k-s+1}{2} \right\rceil - i + 1\right)}{\left(n-k-\left\lceil \frac{k-s+1}{2} \right\rceil + i\right)\cdots\left(n-k-\left\lceil \frac{k+l}{2} \right\rceil - i + 1\right)} =: (*).\end{aligned}$$

We note that $n = 2k-s+1+l$, and therefore

$$n-k-\left\lceil \frac{k-s+1}{2} \right\rceil + i = \left\lfloor \frac{k-s+1}{2} \right\rfloor + l + i \leqslant \left(\left\lceil \frac{k+l}{2} \right\rceil + i\right) - \left\lceil \frac{s-1-l}{2} \right\rceil.$$

Then

$$\begin{aligned}(*) &\geqslant \frac{\left(\left\lfloor \frac{k-l}{2} \right\rfloor - i\right)\cdots\left(\left\lceil \frac{k-s+1}{2} \right\rceil - i + 1\right)}{\left(\left\lfloor \frac{k+s-1}{2} \right\rfloor + i\right)\cdots\left(\left\lceil \frac{k+l}{2} \right\rceil + i + 1\right)}\\ &\quad \times \frac{\left(\left\lceil \frac{k+l}{2} \right\rceil + i\right)\cdots\left(\left\lceil \frac{k+l}{2} \right\rceil + i - \left\lceil \frac{s-1-l}{2} \right\rceil + 1\right)}{\left(n-k-\left\lceil \frac{k+l}{2} \right\rceil + \left\lceil \frac{s-1-l}{2} \right\rceil - i\right)\cdots\left(n-k-\left\lceil \frac{k+l}{2} \right\rceil - i + 1\right)}.\end{aligned}$$

In the first fraction the number of factors in both denominator and numerator is the same and is $\lfloor (k-l)/2 \rfloor - \lceil (k-s+1)/2 \rceil$, which is at most $\lceil (s-1-l)/2 \rceil$. The ratio of the $(j+1)$st factors in the first fraction, that is, the expression

$$\frac{\left\lfloor \frac{k-l}{2} \right\rfloor - i - j}{\left\lfloor \frac{k+s-1}{2} \right\rfloor + i - j}, \quad \text{for } j \in \{0, \ldots, \lceil (s-1-l)/2 \rceil - 1\},$$

is at most 1 and thus does not increase as j increases.

In the second fraction the number of factors in both the numerator and the denominator is $\lceil (s-1-l)/2 \rceil$. Similarly to the first fraction, the expression

$$\frac{\left\lceil \frac{k+l}{2} \right\rceil + i - j}{n-k-\left\lceil \frac{k+l}{2} \right\rceil + \left\lceil \frac{s-1-l}{2} \right\rceil - i - j}, \quad \text{for } j \in \{0, \ldots, \lceil (s-1-l)/2 \rceil - 1\},$$

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is at least 1 and thus does not decrease as j increases. Therefore, the product of these two fractions is at least

$$\left(\frac{\left\lceil\frac{k-s+1}{2}\right\rceil-i}{\left\lceil\frac{k+l}{2}\right\rceil+i}\right)^{\lceil(s-1-l)/2\rceil}\left(\frac{\left\lceil\frac{k+l}{2}\right\rceil+i}{n-k-\left\lceil\frac{k+l}{2}\right\rceil+\left\lceil\frac{s-1-l}{2}\right\rceil-i}\right)^{\lceil(s-1-l)/2\rceil}$$
$$=\left(\frac{\left\lceil\frac{k-s+1}{2}\right\rceil-i}{k-s+l+1-\left\lceil\frac{k+l}{2}\right\rceil+\left\lceil\frac{s-1-l}{2}\right\rceil-i}\right)^{\lceil(s-1-l)/2\rceil}\geqslant 1.$$

The last inequality holds since without integer parts the denominator is equal to the numerator, and the possible gain because of the integer parts in the denominator is at most $1/2$. However, both numbers are integers, so the denominator is not bigger than the numerator.

Assume now that $\lfloor(k-l)/2\rfloor<\lceil(k-s+1)/2\rceil$. Then

$$\frac{|\mathcal{C}_{\lfloor(k+s-1)/2\rfloor+i}|}{|\mathcal{C}_{\lfloor(k-l)/2\rfloor-i}|}=\frac{\left(\left\lceil\frac{k+l}{2}\right\rceil+i\right)!^2\left(\left\lfloor\frac{k-l}{2}\right\rfloor-i\right)!\left(n-k-\left\lceil\frac{k+l}{2}\right\rceil-i\right)!}{\left(\left\lceil\frac{k-s+1}{2}\right\rceil-i\right)!^2\left(\left\lfloor\frac{k+s-1}{2}\right\rfloor+i\right)!\left(n-k-\left\lceil\frac{k-s+1}{2}\right\rceil+i\right)!}$$
$$=\frac{\left(\left\lceil\frac{k+l}{2}\right\rceil+i\right)\cdots\left(\left\lfloor\frac{k+s-1}{2}\right\rfloor+i+1\right)}{\left(\left\lceil\frac{k-s+1}{2}\right\rceil-i\right)\cdots\left(\left\lfloor\frac{k-l}{2}\right\rfloor-i+1\right)}$$
$$\times\frac{\left(\left\lceil\frac{k+l}{2}\right\rceil+i\right)\cdots\left(\left\lceil\frac{k-s+1}{2}\right\rceil-i+1\right)}{\left(n-k-\left\lceil\frac{k-s+1}{2}\right\rceil+i\right)\cdots\left(n-k-\left\lceil\frac{k+l}{2}\right\rceil-i+1\right)}=:(*).$$

We note that $n=2k-s+1+l$, and therefore

$$n-k-\left\lceil\frac{k-s+1}{2}\right\rceil+i=\left\lfloor\frac{k-s+1}{2}\right\rfloor+l+i\leqslant\left(\left\lceil\frac{k+l}{2}\right\rceil+i\right)+\left\lfloor\frac{l-s+1}{2}\right\rfloor.$$

Then

$$(*)\geqslant\frac{\left(\left\lceil\frac{k+l}{2}\right\rceil+i\right)\cdots\left(\left\lfloor\frac{k+s-1}{2}\right\rfloor+i+1\right)}{\left(\left\lceil\frac{k-s+1}{2}\right\rceil-i\right)\cdots\left(\left\lfloor\frac{k-l}{2}\right\rfloor-i+1\right)}$$
$$\times\frac{\left(\left\lceil\frac{k-s+1}{2}\right\rceil+\left\lfloor\frac{l-s+1}{2}\right\rfloor-i\right)\cdots\left(\left\lceil\frac{k-s+1}{2}\right\rceil-i+1\right)}{\left(n-k-\left\lceil\frac{k-s+1}{2}\right\rceil+i\right)\cdots\left(n-k-\left\lceil\frac{k-s+1}{2}\right\rceil-\left\lfloor\frac{l-s+1}{2}\right\rfloor+i+1\right)}.$$

Bounding the two fractions similarly to how it was done in the previous case, one can see that this expression is at least

$$\left(\frac{\left\lceil\frac{k+l}{2}\right\rceil+i+1}{\left\lceil\frac{k-s+1}{2}\right\rceil-i+1}\right)^{\lfloor(l-s+1)/2\rfloor}\left(\frac{\left\lceil\frac{k-s+1}{2}\right\rceil-i+1}{n-k-\left\lceil\frac{k-s+1}{2}\right\rceil-\left\lfloor\frac{l-s+1}{2}\right\rfloor+i+1}\right)^{\lfloor(l-s+1)/2\rfloor}$$
$$=\left(\frac{\left\lceil\frac{k+l}{2}\right\rceil+i+1}{k-s+l+1-\left\lceil\frac{k-s+1}{2}\right\rceil-\left\lfloor\frac{l-s+1}{2}\right\rfloor+i+1}\right)^{\lfloor(l-s+1)/2\rfloor}\geqslant 1.$$

As in the previous case, in the last fraction the denominator is equal to the enumerator if one removes all integer parts, and as in the previous case we conclude that the last inequality is valid. □

The next crucial step is to decompose the graph W into symmetric chains of even length. Let $(j,\bar{j})=(1,2)$ or $(2,1)$. We know that $\mathcal{C}_i^j$ is connected to $\mathcal{C}_{k+s-1-i}^{\bar{j}}$ for all i. We call these *edges of the first type*. Next, $\mathcal{C}_i^j$ is connected to $\mathcal{C}_i^{\bar{j}}$ if $\lceil(k-l)/2\rceil\leqslant i<(k+s-1)/2$. We call these *edges*

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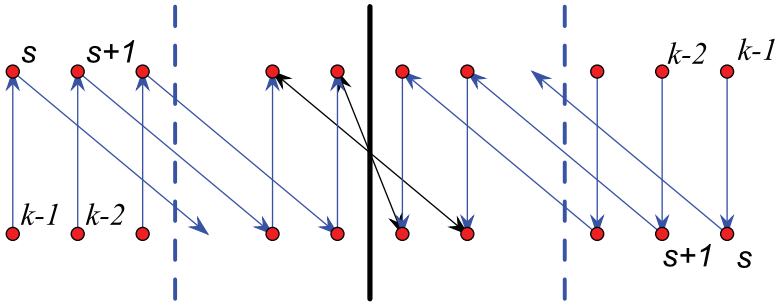


Figure 1. Decomposition of the graph W into symmetric chains.

of the second type. Finally, $\mathcal{C}^{j}_{\lfloor (k-l)/2 \rfloor - i}$ is connected to $\mathcal{C}^{\bar{j}}_{\lfloor (k+s-1)/2 \rfloor + i}$ for each meaningful $i \geqslant 1$ and $s \geqslant 2$. We call these *edges of the third type*.

We claim that the graph W is decomposed into paths of even length using the three types of edges above. See Figure 1, where the two parts are represented by two horizontal lines of points, and the values near the points indicate the size of the intersection of the sets of the corresponding family with $[k]$. The arrows on the edges indicate the direction from the vertex of smaller weight to the vertex of larger weight. The black two-directional edges are the edges of the second type and connect the vertices of the same weight. The vertical edges are the edges of the first type, and the other edges are the edges of the third type.

The solid vertical line corresponds to the size of the intersection with $[k]$ equal to $(k+s-1)/2$ in both parts. The two dashed lines correspond to the intersection size $\lceil (k-l)/2 \rceil$ for one of the parts. Between these two lines each vertex has an edge of the second type. Note that the starting vertex of the edges of the third type is always outside the region bounded between the two dashed lines. The edges of the third type may or may not intersect the dashed lines, but they never intersect the black line.

In the middle of each path P there is an edge of the second type, connecting two vertices of the same weight. Due to Lemma 3.1, for each path the weight of the vertex does not decrease as we move along the path towards the middle edge of the second type. Therefore, it is easy to see that any independent set in P has weight less than or equal to half of the weight of P. This means that any independent set in W cannot have weight bigger than the half of the weight of W. This concludes the proof of Theorem 1.3.

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