COMPOSITIO MATHEMATICA

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Compositio Math. **156** (2020), 881–907.

doi:10.1112/S0010437X20007083





https://doi.org/10.1112/S0010437X20007083 Published online by Cambridge University Press

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Classification of algebraic solutions of irregular Garnier systems

Karamoko Diarra and Frank Loray

In the memory of Tan Lei

Abstract

We prove that algebraic solutions of Garnier systems in the irregular case are of two types. The classical ones come from isomonodromic deformations of linear equations with diagonal or dihedral differential Galois group; we give a complete list in the rank-2 case (two indeterminates). The pull-back ones come from deformations of coverings over a fixed degenerate hypergeometric equation; we provide a complete list when the differential Galois group is $\mathrm{SL}_2(\mathbb{C})$. As a byproduct, we obtain a complete list of algebraic solutions for the rank-2 irregular Garnier systems.

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1. Introduction

Painlevé equations and Garnier systems are particular cases of isomonodromic differential equations: local solutions parametrize deformations of linear differential equations with fixed monodromy representation. This has been proved by Fuchs [Fuc07], Garnier [Gar12] and Okamoto [Oka87] for deformations of Fuchsian equations and by Jimbo *et al.* [JMU81] and Kimura [Kim89] in the irregular case. Solutions of isomonodromic equations are expected to be very transcendental in general, as proved by Umemura in the Painlevé case [Ume90].

Received 5 July 2019, accepted in final form 14 November 2019, published online 14 April 2020.
2010 Mathematics Subject Classification 34M55, 34M56, 34M03 (primary).
Keywords: ordinary differential equations, isomonodromic deformations, Hurwitz spaces.

We thank CNRS, Université de Rennes 1, Henri Lebesgue Center and ANR-16-CE40-0008 project 'Foliage' for financial support. We also thank the Simons Foundation's project NLAGA which invited the two of us to Dakar, where we started working on this subject. We finally thank Hiroyuki Kawamuko and Yousuke Ohyama for helpful discussions on the subject.
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https://doi.org/10.1112/S0010437X20007083 Published online by Cambridge University Press

However, for special isomonodromic equations there arise classical solutions that can be expressed by means of solutions of linear differential equations (see Watanabe [Wat98] for instance) or algebraic solutions. Although there is a long history about algebraic or classical solutions of Painlevé equations (see the introduction of [DM00]), it is only recently that all of them have been classified and described. The last and most difficult case was the Painlevé VI equation, for which the exhaustive list has been found by Boalch [Boa06a, Boa06b, Boa07a, Boa07b, Boa10] following Hitchin's idea [Hit95b] considering deformations of Fuchsian systems with finite monodromy group. However, that the list was complete has been proved later, by Lisovyy and Tykhyy in [LT14]: they used the approach of Dubrovin and Mazzocco in [DM00, Maz01a, Maz01b] based on the finite braid group orbits. Algebraic solutions of Painlevé equations have connections with solutions of the Witten–Dijkgraaf–Verlinde–Verlinde equation in 2-dimensional topological field theories [Dub95], anti-self-dual Einstein metrics [Hit95a], Hilbert modular varieties [CS08, Cou14] and Frobenius manifolds [Dub95, KMS18]. More recently, algebraic solutions of classical Garnier systems (in the Fuchsian case) have been investigated by several authors [Dia13, Cou17, CoMo18, CH16, Gir16, CaMa18, Kom18] but we are far from a complete classification. In this paper, we give a complete classification of algebraic solutions for degenerate Garnier systems (i.e. in the irregular case).

Garnier systems can be derived as isomonodromic equations for deformations of rank-2 and trace-free systems of linear differential equations with rational coefficients on the Riemann sphere (see [JMU81, Boa12]):

$$\mathcal{E}: \frac{dY}{dx} = A\cdot Y \quad \text{with } Y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{and} \quad A=\begin{pmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{pmatrix}, \quad a,b,c\in\mathbb{C}(x). \tag{1}$$

After rational gauge transformation, we can minimize the order of poles, following Hukuhara–Levelt–Turrittin. Then the local structure of the system at any pole can be described as follows. In a convenient local coordinate $x=\varphi(z)$ and after local holomorphic gauge transformation, the system $\varphi^*\mathcal{E}$ reduces to one of the following models:

$$\begin{array}{lllll}
\textbf{Log} & A=\begin{pmatrix} \frac{\theta}{2} & 0 \\ 0 & -\frac{\theta}{2} \end{pmatrix}\frac{1}{x}, & \theta\in\mathbb{C}\setminus\mathbb{Z} & \text{logarithmic nonresonant} & \begin{pmatrix} 0 \\ \theta \end{pmatrix} \\
\textbf{Log}^{\textbf{res}} & A=\begin{pmatrix} \frac{n}{2} & x^n \\ 0 & -\frac{n}{2} \end{pmatrix}\frac{1}{x}, & n\in\mathbb{Z}_{\geqslant 0} & \text{logarithmic resonant} & \begin{pmatrix} 0 \\ n \end{pmatrix} \\
\textbf{Irr}^{\textbf{un}} & A=\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\left(\frac{1}{x^{k+1}}+\frac{\theta}{x}\right)+\tilde{A}, & \begin{cases} k\in\mathbb{Z}_{>0} \\ \theta\in\mathbb{C} \end{cases} & \text{irregular unramified case} & \begin{pmatrix} k \\ \theta \end{pmatrix} \\
\textbf{Irr}^{\textbf{ram}} & A=\begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix}\frac{1}{x^{k+1}}+\tilde{A}, & k\in\mathbb{Z}_{>0} & \text{irregular ramified case} & \begin{pmatrix} k-\frac{1}{2} \\ 0 \end{pmatrix}
\end{array} \tag{2}$$

where $\tilde{A}$ is holomorphic. The *formal data* at each pole is encoded by the matrix $\binom{\kappa}{\theta}$ appearing on the right. The Poincaré–Katz irregularity index $\kappa\in\frac{1}{2}\mathbb{Z}_{\geqslant 0}$ corresponds to k or $k-\frac{1}{2}$, where $k+1$ is the order of the pole, depending on the nature of the singular point. The exponent $\theta\in\mathbb{C}$ is the difference of eigenvalues of the residual matrix; it is zero for half-integers $\kappa\notin\mathbb{Z}_{\geqslant 0}$. Once we have fixed the formal data at each pole $t_1,\dots,t_n$:

$$\Sigma=\begin{pmatrix} \kappa_1 & \cdots & \kappa_n \\ \theta_1 & \cdots & \theta_n, \end{pmatrix} \tag{3}$$

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there remain $3T$ extra parameters to describe the system up to linear change of variable x and unknown Y, where $N = T + 3$ is the total order of poles counted with multiplicity. This can be decomposed as follows. The first T parameters stand for the *irregular curve*, namely the data $X = (\varphi_1, \dots, \varphi_n)$ for each pole (of order $k+1$) of the k-jet of local coordinate $\varphi : (\mathbb{C}_z, 0) \to \mathbb{P}^1_x$ in which the system can be normalized to models (2) (see [Mal83, Heu10, CMR17, ChMa18, AB18]). We denote by $\mathcal{T}^\Sigma$ the corresponding moduli space up to Moebius transformation on the target space $\mathbb{P}^1_x$ (which does not depend on the exponents θ_i). In the Fuchsian (or logarithmic) setting $\kappa_i = 0$, the 0-jet of φ is just the position of the pole and we get the usual moduli space $M_{0,n}$. There remain $2T$ so-called 'accessory parameters' which stand for *monodromy data*, including monodromy representation and Stokes matrices (see [MR90, Kri02, BMM06, vdPS09, Boa14]). Although computation of monodromy data is very transcendental, the *isomonodromic foliation*, whose leaves are deformations with constant monodromy data, is of algebraic nature: the moduli space of systems (or more generally connections) with fixed local formal type is an irreducible quasi-projective variety $\mathcal{S}ys^\Sigma$ of dimension $3T$, and the isomonodromic foliation $\mathcal{I}so^\Sigma$ on it is defined by the kernel

$$\ker\big(\underbrace{\Omega \wedge \cdots \wedge \Omega}_{T \text{ times}}\big)$$

of a rational 2-form Ω of rank T on $\mathcal{S}ys^\Sigma$ (see [Gar12, Kim89, Boa12, IS13, Ina16]). In the case $T = 1$, this leads to Painlevé equations. In the case $T = 2$, explicit differential equations are listed in [Kim89, Kaw09]. For $T > 2$, the explicit form of the isomonodromic differential equation is not known in general except for the classical Garnier system corresponding to the Fuchsian case (where all the $\kappa_i = 0$).

Algebraic solutions correspond to isomonodromic leaves whose closure are N-dimensional algebraic subvarieties. It is a very difficult (and widely open) problem, coming back from Poincaré, to determine all leaves with algebraic closure of a given foliation (see for instance [Lin02]). However, the Painlevé property for isomonodromic equations (see [Mal83]) can be reinterpreted as the complete transversality of the isomonodromic foliation $\mathcal{I}so$ to the natural projection $\pi : \mathcal{S}ys \to \mathcal{T}$ onto the moduli space of irregular curves. This allows us to define a monodromy representation for $\mathcal{I}so$ (see [DM00, IIS06]), namely a representation of the fundamental group of $\mathcal{T}$ into the analytic automorphism group of a π-fiber. This can be reinterpreted as an action of the mapping class group of the irregular curve onto the irregular character variety, the space of monodromy data (see [DM00, IIS06, vdPS09, PR15, Kli16]). From this point of view, algebraic solutions lead to finite orbits for this action (see [DM00, CL09]) and, in the Fuchsian case, we even get a one-to-one correspondence [Cou17, CH16]. However, in the irregular case, the monodromy is not enough to detect algebraic solutions. For instance, Painlevé I and II equations have trivial monodromy while almost all solutions are transcendental.

The starting point of our classification result is an irregular version of the Corlette–Simpson theorem [CS08] recently proved by Pereira *et al.* [LPT16]. Considering isomonodromic deformations as a flat completion of a partial connection on the total space of the deformation of the curve, this allows us to prove (see Corollary 12) the following result.

Theorem 1. *Any algebraic solution of an irregular Garnier system is of one of the following types.*

- *Classical: comes from the deformation of a system with diagonal or dihedral differential Galois group.*
- *Pull-back: comes from the deformation obtained by pulling back a fixed system by an algebraic family of ramified covers.*

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This result has been proved by Ohyama and Okumura in [OO13] for algebraic solutions of irregular Painlevé equations; they checked *a posteriori* by a case-by-case study of the known list of algebraic solutions. It is used here to find all algebraic solutions in the irregular Garnier case. The main result of the paper is the classification of solutions of pull-back type. Let us describe more precisely the idea of this construction, which has been used by Doran, Kitaev, Andreev, Vidunas and the first author (see [Dor01, AK02, AK03, Kit03, Kit06a, Kit06b, Kit06c, VK07, VK09, Dia13]).

We consider a fixed linear differential equation $\mathcal{E}_0$ on $\mathbb{P}^1$, which can be a system (1), or equivalently a second-order scalar equation (which if often more convenient for computations) or a linear meromorphic connection. Then we consider an algebraic family of ramified covers $(\phi_t : \mathbb{P}^1 \to \mathbb{P}^1)_t$, where $t \in P$ is a projective variety. Clearly, the deformation $t \mapsto \mathcal{E}_t := \phi_t^* \mathcal{E}_0$ preserves the monodromy data; more precisely, thinking of $\mathcal{E}_0$ as a system, or a connection (E_0, ∇_0), we get a flat connection $(\Phi^* E_0, \Phi^* \nabla_0)$ over the total space $t : \boldsymbol{C} = \sqcup_t C_t \to P$ defined by pull-back via the total ramified cover $\Phi := (\phi_t, t) : \boldsymbol{C} \to C_0 \times P$. This deformation gives rise to a partial solution of a (possibly irregular) Garnier system; moreover, if the family $(\phi_t)_t$ is algebraic, we get an algebraic partial solution. Here, 'partial solution' means that, to each differential equation $\mathcal{E}_t$ on $\mathbb{P}^1$, we can associate the irregular curve $X_t \in \mathcal{T}$ which is the data of positions of poles and jets of local coordinates, and the map $t \mapsto X_t$ may not be locally onto, and it will not be for general $\mathcal{E}_0$ and $(\phi_t)_t$. When the deformation reaches the maximal dimension $T = \dim(\mathcal{T})$, then we get a complete algebraic solution. Sections 4, 5, 6 and 7 are devoted to the classification of such solutions. Inspired by the similar classification in the logarithmic case established by the first author in [Dia13], we define the irregular analogue of the Euler characteristic and the Riemann–Hurwitz formula for irregular curves. Then we prove that, assuming $\mathcal{E}_0$ irregular with differential Galois group not reduced to the diagonal or dihedral group (to avoid classical solutions), $\mathcal{E}_0$ is of degenerate hypergeometric type (at most three poles counted with multiplicity) and the cover degree of ϕ_t is bounded by 6. Finally, the list of solutions is obtained by scattering poles to reduce to the list of [Dia13]. In the pure Garnier case $T > 1$ (i.e. excluding Painlevé equations), we obtain three solutions for Garnier systems and they occur for ranks $T = 2$ or 3 (see Tables 2 and 3). They all come from pull-back of the degenerate (or ramified) Kummer equation $u'' + (2/3x)u' - (1/x)u = 0$ by coverings of degrees 4 and 6. Consequently: *irregular Garnier systems of rank $T > 3$ admit only classical algebraic solutions.*

Canonical (or Schlesinger) transformations (see [DM07]) induce biregular isomorphisms between foliated moduli spaces $(\mathcal{S}ys^{\Sigma}, \mathcal{I}so^{\Sigma}) \xrightarrow{\sim} (\mathcal{S}ys^{\Sigma'}, \mathcal{I}so^{\Sigma'})$ with different formal data. Apart from the fact that permutation of poles and changing signs of the exponents θ_i give rise to the same objects and moduli spaces, one can shift the exponents θ_i by arbitrary integers by means of birational gauge transformations of the systems, without affecting the monodromy data. We obviously classify algebraic solutions up to these isomorphisms.

Theorem 2. *Up to canonical transformations, there are exactly three nonclassical algebraic solutions for irregular Garnier systems of rank $T > 1$. The list of corresponding formal data is as follows:*

$$\begin{pmatrix} 0 & 1 & 1 \\ \frac{1}{3} & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the rank $T = 2$ case, the list of explicit Garnier systems is provided in [Kim89, Kaw09]. The two first algebraic solutions of Theorem 2 are as follows under Kimura's notation [Kim89].

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- $H(1,2,2;2)$ with parameters $\varkappa_0 = \varkappa_1 = 0$ and $\varkappa = \frac{2}{9}$ (and $\eta_0 = \eta_1 = 1$): q_1 is implicitly defined by[1]
$$\left(\frac{q_1(q_1+1)}{(q_1-1)(q_1-2)}\right)^3 = \left(\frac{t_2}{t_1}\right)^2$$
and the other variables are given by
$$q_2 = \frac{q_1+1}{2q_1-1}, \quad p_1 = -\frac{1}{2}\frac{t_1}{(q_1-1)^2} - \frac{1}{2}\frac{t_2}{q_1^2} - \frac{1}{6}\frac{2q_1-1}{q_1(q_1-1)}$$
and
$$p_2 = -\frac{1}{2}\frac{(2q_1-1)^2 t_1}{(q_1-2)^2} - \frac{1}{2}\frac{(2q_1-1)^2 t_2}{(q_1+1)^2} + \frac{1}{2}\frac{2q_1-1}{(q_1-2)(q_1+1)}.$$
- $H(2,3;2)$ with parameters $\varkappa_0 = 0$ and $\varkappa_\infty = -\frac{1}{2}$ (and $\eta = 1$): q_1 is a solution of
$$\left(\frac{q_1(3q_1+2t_1)}{3}\right)^3 = 2t_2^2$$
and the other variables are given by
$$q_2 = -q_1 - \frac{2}{3}t_1, \quad p_1 = \frac{q_1}{4} + \frac{t_1}{2} - \frac{1}{6q_1} - \frac{t_2}{2q_1^2}$$
and
$$p_2 = -\frac{q_1}{4} + \frac{t_1}{3} + \frac{1}{2(3q_1+2t_1)} - \frac{9t_2}{2(3q_1+2t_1)^2}.$$

In each case, the solutions $(t_1,t_2) \mapsto (p_1,p_2,q_1,q_2)$ satisfy the Hamiltonian system
$$\frac{dq_j}{dt_i} = \frac{\partial H_i}{\partial p_j} \quad \text{and} \quad \frac{dp_j}{dt_i} = -\frac{\partial H_i}{\partial q_j} \quad \forall i,j = 1,2 \tag{4}$$
for explicit Hamiltonians H_i given in § 9.

The second solution coincides with one of the two solutions found by Kawamuko in [Kaw13]. For the third solution, we are able to compute the algebraic isomonodromic deformation of the pull-back linear differential equation, but we do not know the explicit form of the Garnier system in that case.

Theorem 3. *Up to canonical transformations, classical algebraic solutions of irregular Garnier systems of rank $T = 2$ occur exactly for the following formal data:*

- *infinite discrete family*
$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix};$$
- *two-parameter families*
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \theta_1 & \theta_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \theta_1 & \theta_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \theta_1 & \theta_2 & -\theta_1-\theta_2 \end{pmatrix};$$
- *one-parameter families*
$$\begin{pmatrix} 0 & 0 & 2 \\ \frac{1}{2} & \frac{1}{2} & \theta \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & \theta \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \theta \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{3}{2} & 0 \\ \frac{1}{2} & 0 & \theta \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 2 \\ 0 & \theta & -\theta \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & \theta & -\theta \end{pmatrix};$$

[1] In Kimura's paper, canonical coordinates are denoted λ_i and μ_i instead of variables q_i and p_i, respectively, and Hamiltonians K_i instead of H_i.

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- *sporadic solutions*

$$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{5}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}.$$

In the first case, there are countably many distinct algebraic solutions of unbounded degree. In any other case, there is exactly one algebraic solution for each formal data.

Kawamuko already discovered the fourth one-parameter family of solutions in [Kaw13] and the third sporadic solution in [Kaw03]. Suzuki already discovered the first and third two-parameter families of solutions in [Suz06]. We compute the first solution of Theorem 2 in § 9.

Corollary 4. *In the rank $N = 2$ case, irregular Garnier systems with the following formal data:*

$$\begin{pmatrix} \frac{1}{2} & 2 \\ 0 & \theta \end{pmatrix}, \quad \begin{pmatrix} 1 & \frac{3}{2} \\ \theta & 0 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ \theta \end{pmatrix}, \quad \begin{pmatrix} \frac{7}{2} \\ 0 \end{pmatrix}$$

admit no algebraic solution.

The structure theorem is presented in § 3. Sections 4, 5, 6 and 7 are devoted to the classification of pull-back-type solutions; in these sections, the irregular Euler characteristic is introduced and the irregular Riemann–Hurwitz formula is established. Classical solutions are classified in § 8 for the case $T = 2$. Finally, explicit Hamiltonians are given in § 9 for the above explicit algebraic solutions. A long version of the paper is presented in [DL18] with more details, including tables of hypergeometric and Painlevé equations, their algebraic solutions, Garnier systems, the relationship between different types of linear differential equations (systems, connections, Riccati and scalar equations), Stokes matrices and differential Galois group. We warmly thank the referees who helped us to improve the presentation.

2. Classical algebraic solutions of irregular Garnier systems

In this section, we describe one of the two methods we use to produce algebraic isomonodromic deformations, namely those that are characterized by their Galois group. It is convenient here to work with a holomorphic vector bundle E equipped with a linear meromorphic connection ∇ instead of dealing with systems, as these latter ones can be viewed as connections on the trivial bundle, and some arguments need to deal with modified bundles. Also, most of the arguments and results remain valid for arbitrary curves C instead of $\mathbb{P}^1$.

2.1 Classical solutions: the Galois group is C_∞ or D_∞

The rough idea is as follows. Since the differential Galois group of a linear differential equation can be determined from its coefficients by algebraic operations, it follows that iso-Galois deformations are of algebraic nature: we have an algebraic stratification of each moduli space $\mathcal{S}ys^\Sigma$ where strata are defined in terms of the Galois group. In fact, this is not exactly true since there might be infinitely many strata corresponding to finite groups in the dihedral case. However, the locus of each finite group is algebraic and coincides with a finite number of isomonodromic leaves. It follows that the leaf associated to a finite linear group is algebraic. This is the approach followed by Hitchin in [Hit95b] and Boalch in [Boa06a, Boa07a], where they considered deformations of Fuchsian systems with finite linear group, and therefore finite Galois group. Recall that in the Fuchsian case, the differential Galois group of the system is the Zariski closure of the

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monodromy group in $\mathrm{SL}_2(\mathbb{C})$. However, in the irregular case, there are additional entries. The local monodromy at an irregular pole admits a decomposition

$$\underbrace{\begin{pmatrix} e^{i\pi\theta} & 0 \\ 0 & e^{-i\pi\theta} \end{pmatrix}}_{\text{formal monodromy}} \underbrace{\begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}\cdots\begin{pmatrix} 1 & s_k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t_k & 1 \end{pmatrix}}_{\text{Stokes matrices}} \quad \text{(unramified case)}, \tag{5}$$

$$\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\text{formal monodromy}} \underbrace{\begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}\cdots\begin{pmatrix} 1 & s_k \\ 0 & 1 \end{pmatrix}}_{\text{Stokes matrices}} \quad \text{(ramified case)}, \tag{6}$$

tracing back the local analytic structure of the pole. Then the differential Galois group contains all Stokes matrices as well as the 'exponential torus', which is the diagonal group in the normalization (5) and (6) (see [MR90]). These observations give us the following possible differential Galois groups in the irregular case:

$$\begin{cases} C_\infty = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}; \lambda \in \mathbb{C}^* \right\}, \\ D_\infty = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}; \lambda \in \mathbb{C}^* \right\} = \left\langle C_\infty, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle, \\ P_\infty = \left\{ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}; \mu \in \mathbb{C} \right\}, \\ T_\infty = \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix}; \lambda \in \mathbb{C}^*, \mu \in \mathbb{C} \right\}, \\ \mathrm{SL}_2(\mathbb{C}). \end{cases} \tag{7}$$

As we shall see, Galois groups P_∞ and T_∞ do not occur in algebraic solutions of Garnier systems (see § 3). However, the locus of C_∞ and D_∞ is algebraic, and is a finite union of isomonodromy leaves in many cases when $C = \mathbb{P}^1$.

Let us first explain the diagonal case C_∞. The poles can only be of types **Log** and $\mathbf{Irr^{un}}$ and, in this latter case, the Galois group C_∞ coincides with all local exponential torii at irregular singular points and all Stokes matrices are trivial. The two eigendirections of the Galois group correspond to two ∇-invariant line bundles $L, L^{-1} \subset E$ of the vector bundle for the normalized equation, and we have $E = L \oplus L^{-1}$. The connection ∇ restricts as meromorphic connections on $(L, \nabla|_L)$ and $(L, \nabla|_L)^{\otimes(-1)}$ and, at each pole t_i, the corresponding residues are $\pm\theta_i/2$: they are opposite for L and L^{-1}. The Fuchs relation yields

$$\sum_{i=1}^{n} \epsilon_k \theta_k \in \mathbb{Z}, \quad \epsilon_i = \pm 1.$$

There are finitely many such relations for each formal data and, given one relation, the connection $(L, \nabla|_L)$ (and therefore (E, ∇)) can be uniquely determined by Mittag–Leffler's theorem from the data of the irregular curve ($C = \mathbb{P}^1$ + principal parts).

Something similar occurs when the Galois group is D_∞. The poles must be of type **Log**, $\mathbf{Irr^{un}}$ or $\mathbf{Irr^{ram}}$ and the normal subgroup $C_\infty \subset D_\infty$ must coincide with all local exponential torii at irregular singular points. In that case, Stokes matrices are trivial and, if $C = \mathbb{P}^1$, the global monodromy group is generated by matrices $M_i \in D_\infty$ (local monodromy at t_i) satisfying

$$M_1 \cdots M_n = I.$$

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Precisely, for a normalized equation, we have

$$\begin{aligned} M_i \text{ diagonal} &\Rightarrow \kappa_i \in \mathbb{Z}_{\geqslant 0}, \\ M_i \text{ anti-diagonal} &\Rightarrow \begin{pmatrix} \kappa_i \\ \theta_i \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \text{ or } \begin{pmatrix} \frac{1}{2}+\nu \\ 0 \end{pmatrix}, \quad \nu \geqslant 0. \end{aligned}$$

The number of anti-diagonal matrices among $M_1, \ldots, M_n$ is even.

Remark 5. One can check that, if $C = \mathbb{P}^1$ and there are only two anti-diagonal matrices among $M_1, \ldots, M_n$, say M_{n-1} and M_n, then again the (algebraic) locus of D_∞ consists in a finite number of isomonodromic leaves. Indeed, the monodromy representation is determined by the matrices M_i; for $i = 1, \ldots, n-2$, we have

$$M_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix} \quad \text{with } \lambda_i = e^{\pm\sqrt{-1}\pi\theta_i};$$

we can rescale $M_n = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ by diagonal conjugacy, and M_{n-1} is determined by the relation. The connection is also determined by the irregular curve in this case (principal parts).

2.2 Garnier algebraic solutions and apparent singular points

So far, we have neglected to consider apparent singular points in isomonodromic deformations since we were dealing with normalized equations. However, Garnier systems are derived from isomonodromic deformations of scalar differential equations $d^2u/dx^2 + f(x)(du/dx) + g(x)u = 0$ with N poles (counted with multiplicity) and $N-3$ apparent singular points (see [Kim89, Kaw09]). This can be reinterpreted as the data of a connection ∇ on the bundle $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ with N poles, and we recover the scalar equation by taking $\mathcal{O}_{\mathbb{P}^1} \subset E$ as a cyclic vector. Equivalently, and closer to our point of view, one can consider a SL_2-connection (E, ∇) with $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$, and the cyclic vector (with the right number of apparent singular points) is chosen to be the constant line bundle $L \subset E$ fitting with one of the eigendirections over one pole (typically at ∞ when normalizing the position of poles on $\mathbb{P}^1$). Here, the poles are of type **Log**, $\mathbf{Log^{res}}$, $\mathbf{Irr^{un}}$ or $\mathbf{Irr^{ram}}$, but in the case **Log**, we also allow $\theta \in \mathbb{Z}$ as a degenerate case of $\mathbf{Log^{res}}$ when the singular point becomes apparent. These considerations lead us to the following facts.

Proposition 6. *Let $(E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}, \nabla)$ be an irregular SL_2-connection and $L \subset E$ be a cyclic vector like above. The (local) isomonodromic deformation of (E, ∇) provides a (local) solution of the Garnier system if the line bundle L is not ∇-invariant. If $\mathrm{Gal}(E, \nabla) = C_\infty$ and there is no apparent singular point, then L is ∇-invariant and the deformation fails to provide a solution of the corresponding Garnier system.*

Proof. The condition that L is not ∇-invariant is equivalent to the fact that it defines a cyclic vector, which allows us to define the scalar system with N poles and $N-3$ apparent singular points. This condition is preserved under isomonodromic deformations and the deformation of the scalar system leads to a Garnier solution. In the case where (E, ∇) is normalized (i.e. without apparent singular points) and the Galois group is C_∞, then $E = L_0 \oplus L_0^{-1}$ with L_0, L_0^{-1} two ∇-invariant line bundles. Automatically, we have $L_0 \simeq \mathcal{O}_{\mathbb{P}^1}$ (a constant line bundle) and L_0 (respectively L_0^{-1}) coincides with an eigendirection over each pole. It follows that L_0 or L_0^{-1} coincides with L everywhere and L is therefore ∇-invariant. □

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In the presence of *apparent singular points*, the condition for a deformation to be isomonodromic and giving rise to a Garnier solution is more subtle. Let (E_0, ∇_0) be a meromorphic SL_2-connection, which is normalized except at an apparent singular point $p_0 \in C$: it is of type **Log** with $\theta = \nu \in \mathbb{Z}_{>0}$. Then, after a birational bundle transformation $\phi : E_0 \dashrightarrow E_0'$, we can erase the singular point; moreover, we can assume that ϕ is supported by p_0, i.e. inducing a biholomorphic bundle transformation outside of p_0. Generic local sections of E_0 at p_0 are transformed into sections of E_0', all tangent at the order $\nu - 1$ to a given ∇-invariant analytic subbundle $L_0 \subset E_0'$. Then we have the following result.

Proposition 7. *Under the notation above, given a deformation of (E_0, ∇_0) induced by a flat connection (E, ∇), the following are equivalent:*

- *(E, ∇) is logarithmic near p;*
- *the deformation p of the singular point p_0 remains apparent and the corresponding deformation of the line bundle $L_0 \subset E_0'$ is induced by a local analytic ∇-invariant line bundle $L \subset E'$.*

In this case we say that the deformation is isomonodromic.

Consequently, we will encode the data of such an equation (E_0, ∇_0) by the data of the normalized equation (E_0', ∇_0') together with the data of p_0 and the local ∇_0'-invariant analytic line bundle L_0, or better by its fiber $l_0 := L_0|_{p_0} \subset E_0'|_{p_0}$ (initial condition) that is usually called *parabolic data*. This method to deal with apparent singular points is used in [IIS06].

Remark 8. The case $\nu = 0$ also occurs in solutions of Garnier systems as a degenerate case of singular points of type $\mathbf{Log}^{\mathbf{res}}$. Garnier solutions can also be interpreted as before as deformation of a normalized equation (E_0, ∇_0) together with the parabolic data $(p_0 \in C, l_0 \in E_0|_{p_0})$ and there are as many Garnier solutions as choices of initial condition l_0.

If we have several apparent singular points, the condition above must be imposed for each of them to get an isomonodromic solution, and therefore giving rise to Garnier solutions. By the way, we can have Garnier solutions corresponding to an equation with Galois group C_∞ in the presence of apparent singular points. This is the case for the Laguerre polynomial solution of the Painlevé V equation (see [OO13]).

Proposition 9. *The isomonodromic deformation of an irregular normalized connection (E_0, ∇_0) with an apparent singular point (p_0, l_0) (notation above) is algebraic if and only if:*

- *the deformation of (E_0, ∇_0) is algebraic with Galois group C_∞ or D_∞;*
- *the local ∇_0-invariant analytic line bundle L_0 defined by l_0 has algebraic closure, i.e. corresponds to one of the two ∇_0 invariant line bundles in the diagonal case C_∞, or of the two multivalued line bundles in the dihedral case D_∞.*

Proof. The global deformation space of the irregular curve including the apparent singular point may be viewed as a fiber bundle over the deformation space of the strict irregular curve, with fiber C corresponding to the position of the apparent singular point for a fixed normalized equation. Clearly, the deformation of (E_0, ∇_0) is algebraic if and only if it is algebraic in each of these two directions. But to be algebraic along C-fibers implies, due to Proposition 7, that the local analytic line bundle L_0 has algebraic closure. And, this implies that the Galois group must be C_∞ or D_∞ (the third possibility SL_2 in the irregular case is excluded here). □

Proposition 9 can immediately be generalized to the case of several apparent singular points.

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Remark 10. In the dihedral case D_∞, an algebraic solution of a Garnier system with apparent singular point, like in Proposition 9, always arises in a family as a limit of algebraic solutions with linear monodromy D_∞ and no apparent singular points like in § 2.1. Indeed, from the monodromy side, this can be written as

$$M_1 \cdots M_n = I \quad \text{with } M_{n-2} = \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad M_{n-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\underbrace{M_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{apparent}} \rightsquigarrow \tilde{M}_{n-2} = \begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1^{-1} & 0 \end{pmatrix}, \quad \tilde{M}_{n-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \underbrace{\tilde{M}_n = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}}_{\text{nonapparent}},$$

where $\lambda_1\lambda_2 = \lambda$. For the differential equation, the apparent point is replaced by a nonapparent logarithmic singular point.

3. Structure theorem

The key result for our classification is the following theorem.

Theorem 11 [LPT16]. *Let (E, ∇) be a flat meromorphic $\mathfrak{sl}_2$-connection on a projective manifold X. Then at least one of the following assertions holds true.*

(1) *Maybe after passing to a (possibly ramified) 2-fold cover $f : X' \to X$, the connection (E, ∇) is birationally equivalent to a diagonal connection on the trivial bundle:*

$$\nabla = d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$$

with ω a rational closed 1-form on X.

(2) *There exists a rational map $\Phi : X \dashrightarrow C$ to a curve and a meromorphic connection (E_0, ∇_0) on C such that (E, ∇) is equivalent to $\Phi^*(E_0, \nabla_0)$ by bundle transformation.*

The following result is a direct consequence.

Corollary 12. *Any algebraic solution of an irregular Garnier system comes from:*

(1) *the deformation pull-back from a fixed connection (C_0, E_0, ∇_0) with Galois group $\mathrm{SL}_2(\mathbb{C})$ via an algebraic family of ramified covers;*

(2) *or the deformation of a connection having Galois group D_∞;*

(3) *or the deformation of a connection having Galois group C_∞ with at least one apparent singular point, like in Proposition 9.*

This result has been recently proved in the Painlevé case in [OO13] by checking *a posteriori* the known list of algebraic solutions. As we shall see, the result above also holds for isomonodromy equations for curves of genus $g > 0$; however, we do not know the corresponding form of polynomial differential equation, i.e. higher genus Garnier systems. Some analytic form is described in [Kri02]; we knew from [IS13] that we should have an algebraic differential equation though.

To prove Corollary 12, we just notice that any solution (algebraic of not) comes from an isomonodromic deformation, or equivalently a flat connection over the total space $\boldsymbol{C}$ of the deformation curve. If the solution is algebraic, then $\boldsymbol{C}$ is algebraic, as well as the flat connection. We can therefore apply Theorem 11 and deduce Corollary 12.

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4. Ramified covers and differential equations

Given a ramified cover $\phi : C \to C_0$ of degree d, and given a point $c_0 \in C_0$, we can associate the pull-back divisor $\phi^*[c_0] = m_1[t_1] + \cdots + m_s[t_s]$ with $t_i \in C$ pair-wise distinct for $i = 1, \ldots, s$. This means that $\{t_1, \ldots, t_s\}$ is the fiber of ϕ and, through convenient local coordinates x_i near t_i, $\phi(x_i) = c_0 + (x_i)^{m_i}$; we have $m_1 + \cdots + m_s = d$ and, for generic c_0, we have all $m_i = 1$. We can therefore associate to ϕ its *passport*

$$\left(\begin{bmatrix} m_{1,1} \\ \vdots \\ m_{1,s_1} \end{bmatrix} \quad \cdots \quad \begin{bmatrix} m_{\nu,1} \\ \vdots \\ m_{\nu,s_\nu} \end{bmatrix} \right), \quad \sum_{l=1}^{s_k} m_{k,l} = d \quad \text{for } k = 1, \ldots, \nu. \tag{8}$$

It is the data, for each critical value $c_1, \ldots, c_\nu \in C_0$ of ϕ, of the corresponding partition $m_{k,1} + \cdots + m_{k,s_k} = d$: there are s_k points in the fiber $\phi^{-1}(c_k)$ with multiplicities $m_{k,l}$, $l = 1, \ldots, s_k$. The *Riemann–Hurwitz* formula writes

$$\underbrace{2 - 2g(C)}_{\chi(C)} = d\underbrace{(2 - 2g(C_0))}_{\chi(C_0)} - R, \quad \text{where } R = \text{total ramification} := \sum_{k=1}^{n} (d - s_k). \tag{9}$$

Given a differential equation (E_0, ∇_0) with polar divisor D on C_0, we can consider the pull-back $(E, \nabla) := \phi^*(E_0, \nabla_0)$. It is easy to deduce the local formal data of (E, ∇) from that one of (E_0, ∇_0) and the passport. Precisely, let $p \in C$ be a point of multiplicity m for ϕ (ramification $r = m - 1$); then the local formal data of (E_0, ∇_0) at $\phi(p)$ and (E, ∇) at p are related by

$$\begin{pmatrix} \kappa \\ \theta \end{pmatrix} = \begin{pmatrix} m\kappa_0 \\ m\theta_0 \end{pmatrix}$$

except when $\kappa = 0$, $\theta_0 \in \mathbb{Q} \setminus \mathbb{Z}$ and $m\theta_0 \in \mathbb{Z}$, where the singular point becomes apparent, i.e. can be deleted by bundle transformation; in that latter case, we indeed delete the pole. We deduce, for the respective order of poles, that

$$\operatorname{ord}_p(\nabla) \leqslant m \cdot \operatorname{ord}_{\phi(p)}(\nabla_0) + 1 - m$$

with strict inequality in the special case above, and the ramified case $\kappa_0 \notin \mathbb{Z}$ and $m > 1$. On the other hand, when $\phi(p)$ is not a pole for ∇_0, then p is also nonsingular for ∇.

In the following, we still fix the differential equation (C_0, E_0, ∇_0) and we want to deform the ramified cover and pull-back equation

$$\phi_t : C_t \to C_0, \quad (E_t, \nabla_t) := \phi_t^*(E_0, \nabla_0).$$

We consider an irreducible algebraic family, parametrized by say $P \ni t$ (irreducible and projective), and there is a Zariski open subset $U \subset P$ where the passport of ϕ_t is locally constant, as well as the number B of its critical values outside of the (fixed) polar locus of ∇_0. Note that B bounds the dimension of P, and maybe switching to a larger family, i.e. with a larger parameter space P, we can assume that $B = \dim(P)$. The deformation $t \mapsto (C_t, E_t, \nabla_t)$, being automatically isomonodromic, locally factors through the universal isomonodromic deformation (see [Heu10]). In general, it defines a projective subvariety contained in the transcendental isomonodromy leaf $\mathcal{L}$ of the corresponding isomonodromy foliation $\mathcal{F}$. But, if $P \to \mathcal{L}$ is locally dominant, then it is globally dominant and the entire leaf $\mathcal{L}$ itself is algebraic, giving rise to an algebraic solution of the corresponding isomonodromy equation. A necessary condition for this is that the dimension T of $\mathcal{L}$ is bounded by the dimension B of P. We will call *admissible* the data of a differential

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equation (C_0, E_0, ∇_0) and a passport (8) such that $B \geqslant T$. In the following, it is convenient to add in the passport all trivial fibers

$$\left.\begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix}\right\} d \text{ times}$$

appearing over poles, so that we may assume that $\nu = n + B$ in (8) with entries $k = 1, \ldots, n$ corresponding to fibers over the poles of ∇_0, and entries $k = n+1, \ldots, \nu$ corresponding to free critical points that are deformed along the family. Our aim now is to classify those admissible data such that the pull-back differential equation has irregular Teichmüller dimension $T \leqslant B$. We will see in the next section that this inequality gives very strong constraints.

5. Scattering ramifications

Suppose that we are given a normalized differential equation (C_0, E_0, ∇_0) with poles $p_1, \ldots, p_n \in C_0$ and local formal data $(\kappa_i, \theta_i)_{i=1,\ldots,n}$ like (3). Suppose that we are given a ramified covering $\phi : C \to C_0$ with passport $(d = m_{k,1} + \cdots + m_{k,s_k})_{k=1,\ldots,\nu}$ over $c_1, \ldots, c_\nu \in C_0$ like (8), where:

- $(m_{k,l})_l$ for $k = 1, \ldots, n$ correspond to the multiplicities of ϕ along fibers over the poles of ∇_0 ($c_k = p_k$), some of them being possibly unbranched, i.e. $m_{k,l} = 1$ for all l;
- $(m_{k,l})_l$ for $k = n+1, \ldots, n+b$ correspond to fibers of ϕ over nonsingular points of ∇_0, all of which are branching, i.e. $m_{k,l} > 1$ for at least one l.

Denote by $R_k := d - s_k$ the total ramification number of the fiber over c_k. We denote by (C, E, ∇) the normalized equation that can be deduced from the pull-back $\phi^*(E_0, \nabla_0)$ by bundle transformation. Let N_k denote the number of poles counted with multiplicity in the fiber over c_k. The irregular Teichmüller dimension of (C, E, ∇) is given by

$$T = 3g - 3 + N = 3g - 3 + N_1 + \cdots + N_n,$$

where g is the genus of C, given by the Riemann–Hurwitz formula (9) with $R := \sum_{k=1}^{\nu} R_k$. We now explain how to compute N_k by means of (κ_k, θ_k) and $(m_{k,l})_l$.

- If $\kappa_k = 0$ and $\theta_k \notin \mathbb{Q} \setminus \mathbb{Z}$, then
$$N_k = s_k = d - R_k.$$
- If $\kappa_k = 0$ and θ_k has order $m > 1$ modulo $\mathbb{Z}$, then
$$N_k = d - R_k - \#\{l = 1, \ldots, s_k \mid m \text{ divides } m_{k,l}\}.$$
- If $\kappa_k \in \mathbb{Z}_{>0}$, then
$$N_k = \sum_{l=1}^{s_k} (m_{k,l} + 1) = d(\kappa_k + 1) - R_k.$$
- If $\kappa_k \in \mathbb{Z}_{>0} - \frac{1}{2}$, then
$$N_k = d(\kappa_k + 1) - R_k + \#\{l = 1, \ldots, s_k \mid 2 \text{ does not divide } m_{k,l}\}.$$

We assume that ϕ admissible, i.e. that $T \leqslant B$.

In this section, we show that we can replace ϕ by another ramified cover ϕ' with more critical points but less multiplicity in fibers in such a way that $T - B$ can only decrease. The total

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ramification will be unchanged, but will be scattered outside of the polar locus. This will allow us to replace the deformation of ϕ by the wider deformation of ϕ', so that we will be able to recover ϕ (and its deformation) by confluence of critical values. By the way, the passport of ϕ' will be as simple as possible and it will be easy to classify such covers.

The first step consists in scattering the branching points over critical values outside of the polar locus. We call *simple branching* a fiber of the form

$$\begin{bmatrix} m_{k,1} \\ m_{k,2} \\ \vdots \\ m_{k,s_k} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

i.e. with $R_k = 1$.

Lemma 13. *Let $n < k \leqslant \nu$, i.e. c_k is not a pole of ∇. We can deform $\phi \rightsquigarrow \phi'$ over a neighborhood of c_k so that the single fiber of ϕ with total ramification R_k is replaced by R_k simple branching fibers for ϕ'. We have increased B without changing R or T.*

Proof. Fix a disc $\Delta \subset C_0$ in which c_k is the unique critical value of ϕ. The monodromy of the ramified covering is given by the permutation that decomposes into the product of s_k cyclic permutations of orders $m_{k,1}, \ldots, m_{k,s_k}$ with disjoint support. In order to construct ϕ', it is enough to define its monodromy in Δ, namely the data of R_k transpositions whose product is the monodromy of ϕ. But it suffices to decompose each cyclic permutation of length $m_{k,l}$ above as the product of $m_{k,l} - 1$ transpositions, which indeed gives

$$\sum_{l=1}^{s_k} (m_{k,l} - 1) = d - s_k = R_k$$

transpositions making the job. □

After applying Lemma (13) to each critical fiber of ϕ outside the polar locus, we can now assume that all fibers $\phi^{-1}(c_k)$ are simple branching for $k > n$. In this way, we have maximized B without touching fibers over poles of ∇_0 so far, so N has not changed. We now discuss how to simplify fibers over poles of ∇ by putting some of their branch points out, in additional simple branching fibers, without increasing $N - B$; this will however increase B. We do this in successive lemmas discussing the type of poles.

Lemma 14. *Let $0 \leqslant k \leqslant n$, i.e. $c_k = p_k$ is a pole of ∇, and assume that $\kappa_k = 0$ and $\theta_k \notin \mathbb{Q} \setminus \mathbb{Z}$. We can deform $\phi \rightsquigarrow \phi'$ over a neighborhood of c_k so that the single fiber of ϕ with total ramification R_k is replaced by a nonbranching fiber over p_k (i.e. $R'_k = 0$) and R_k simple branching fibers nearby. We have $N'_k = d = N_k + R_k$ and $B' = B + R_k$.*

Proof. We proceed like in the proof of Lemma 13 by replacing the monodromy of $\phi^{-1}(p_k)$ by the product of the identity and R_k transpositions. Here, the identity stands for the trivial monodromy of the nonbranching fiber over p_k. □

Lemma 15. *Let $0 \leqslant k \leqslant n$, i.e. $c_k = p_k$ is a pole of ∇, and assume that $\kappa_k \in \mathbb{Z}_{>0}$. We can deform $\phi \rightsquigarrow \phi'$ over a neighborhood of c_k so that the single fiber of ϕ with total ramification R_k is replaced by a nonbranching fiber over p_k (i.e. $R'_k = 0$) and R_k simple branching fibers nearby. We have $N'_k = d(\kappa_k + 1) = N_k + R_k$ and $B' = B + R_k$.*

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The proof is the same as before.

Lemma 16. *Let $0 \leqslant k \leqslant n$, i.e. $c_k = p_k$ is a pole of ∇, and assume that $\kappa_k = 0$ and θ_k has order $m > 1$ modulo $\mathbb{Z}$. We can deform $\phi \rightsquigarrow \phi'$ over a neighborhood of p_k so that the single fiber of ϕ with total ramification R_k is replaced by the fiber with passport*

$$\begin{bmatrix} m'_{k,1} \\ \vdots \\ m'_{k,s'_k} \end{bmatrix} = \begin{bmatrix} m \\ \vdots \\ m \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

and only simple branching fibers nearby. We have $N'_k \geqslant N_k$, $B' \geqslant B$ and $N'_k - B' \leqslant N_k - B$.

Proof. Here, we cannot just proceed as before. Indeed, if m divides $m_{k,l}$, then there is no pole on the preimage (or an apparent one that disappears after normalization); however, replacing by $m_{k,l}$ nonbranching points would increase N_k by $m_{k,l}$, but increase B only by $m_{k,l}-1$, so that $N - B$ increases by 1. We thus have to take care of those points with m dividing $m_{k,l}$.

Consider the euclidean division $m_{k,l} = s^0 \cdot m + s^1$. Then we can replace the point with multiplicity $m_{k,l}$ by:

- s^0 points of multiplicity m in the fiber $\phi^{-1}(p_k)$ (contributing to no pole);
- s^1 nonbranching points in the fiber $\phi^{-1}(p_k)$ (contributing to s^1 poles); and
- $s^0 + s^1 - 1$ additional simple branching fibers around it.

To realize the deformation of ϕ, we have to realize the corresponding monodromy representation, which is easy in this case. Indeed, the concatenation of a cyclic permutation (or a cycle inside a permutation) runs as follows:

$$\underbrace{(1 \dots \mu)}_{\text{over } p_k} = \underbrace{(1 \dots \mu')(\mu'+1 \dots \mu)}_{\text{over } p_k \text{ after concatenation}} \cdot \underbrace{(1, \mu'+1)}_{\text{simple branching fiber}},$$

where $1 < \mu' < \mu$. We just have to repeat this procedure $s_0 + s_1 - 1$ times. By the way, we get $N'_k = N_k + s^1 - 1$ (or $N'_k = N_k = 0$ if m divides $m_{k,l}$) and $B' = B + s^0 + s^1 - 1$. We proceed similarly with all $m_{k,l} \neq m, 1$ in the fiber. □

Lemma 17. *Let $0 \leqslant k \leqslant n$, i.e. $c_k = p_k$ is a pole of ∇, and assume that $\kappa_k \in \mathbb{Z}_{>0} - \frac{1}{2}$. We can deform $\phi \rightsquigarrow \phi'$ over a neighborhood of p_k so that the single fiber of ϕ with total ramification R_k is replaced by the fiber with passport*

$$\begin{bmatrix} m'_{k,1} \\ \vdots \\ m'_{k,s'_k} \end{bmatrix} = \begin{bmatrix} 2 \\ \vdots \\ 2 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

and only simple branching fibers nearby. We have $N'_k - B' = N_k - B$.

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The proof is similar to that of Lemma 16. We call a *scattered admissible covering* (with respect to (C_0, E_0, ∇_0)) an admissible covering ϕ satisfying the conclusion for ϕ' in Lemmas 13–17. We assume from now on that ϕ is scattered. We now deform the differential equation (E_0, ∇_0) on C_0 into a logarithmic one (E_0', ∇_0') without changing the ramified cover, in such a way that $N - B$ does not increase. This will allow us to conclude with the classification established by the first author [Dia13] in the logarithmic case.

Lemma 18. *Let p_k be an irregular unramified pole of ∇, i.e. $\kappa_k \in \mathbb{Z}_{>0}$. We can deform the differential equation $(E_0, \nabla_0) \rightsquigarrow (E_0', \nabla_0')$ over a neighborhood of p_k so that the deformed equation is also normalized with $\kappa_k + 1$ simple poles instead of a single pole of multiplicity $\kappa_k + 1$. If ϕ is a scattered covering, then $N' = N$ (and $B' = B$).*

Proof. It is similar to the previous proofs: instead of dealing with the monodromy of the covering, we use the monodromy of the differential equation. We can write the monodromy M of ∇_0 around p_k as the product of $\kappa_k + 1$ nontrivial linear transformations:

$$M = M_0 \cdot M_1 \cdots M_{\kappa_k}, \quad M_i \in \mathrm{SL}_2(\mathbb{C}), \quad M_i \neq \pm I.$$

By standard arguments *à la Riemann–Hilbert*, we can first realize these matrices as local monodromy of a differential equation over a disc with $(\kappa_k + 1)$ simple poles in normal form and total monodromy M. Next, by surgery over the disc, we replace the single pole p_k in (E_0, ∇_0) by this new differential equation and get (E_0', ∇_0') with the desired properties. □

Lemma 19. *Let p_k be an irregular and ramified pole of ∇, i.e. $\kappa_k \in \mathbb{Z}_{>0} - \frac{1}{2}$. We can deform the differential equation $(E_0, \nabla_0) \rightsquigarrow (E_0', \nabla_0')$ over a neighborhood of p_k so that the deformed equation is also normalized with $\bar\kappa_k + 1 = \kappa_k + \frac{3}{2}$ simple poles instead of a single pole of multiplicity $\bar\kappa_k + 1$, one of which is at the critical point p_k for ϕ, with $\theta_k' = \frac{1}{2}$. If ϕ is a scattered covering, then $N' = N$ (and $B' = B$).*

Proof. The proof is similar, except that we need $M = M_0 \cdot M_1 \cdots M_{\bar\kappa_k}$ with $\mathrm{trace}(M_0) = 0$. Then we can realize M_0 as the monodromy of a logarithmic differential equation with exponent $\frac{1}{2}$. □

6. Irregular Euler characteristic

Following Poincaré, we can associate, to a fuchsian differential equation, an orbifold structure on the base curve C_0 (see also [Dia13]) and get the notion of orbifold Euler characteristic. Here, we define an irregular version of it. Given (C_0, E_0, ∇_0) in normal form, we define the *orbifold order* ν_k at a logarithmic singular point as the order of the local monodromy:

- $\nu_k \in \mathbb{Z}_{>1}$ is the order of $[\theta_k \mod \mathbb{Z}]$ if $\theta_k \in \mathbb{Q} \setminus \mathbb{Z}$;
- $\nu_k = \infty$ if not.

We define the *irregular Euler characteristic* of the differential equation as

$$\chi^{\mathrm{irr}}(C_0, E_0, \nabla_0) := 2 - 2g_0 - \sum_{k=1}^{n} (1 + \kappa_k) + \sum_{\kappa_k = 0} \frac{1}{\nu_k}.$$

In the logarithmic case, this notion coincides with the orbifold Euler characteristic $\chi^{\mathrm{irr}} = \chi^{\mathrm{orb}}$. We note that the two operations of Lemmas 18 and 19, replacing (E_0, ∇_0) by a logarithmic equation (E_0', ∇_0'), do not change the irregular Euler characteristic. Moreover, like as in [Dia13, Proposition 2.5], we have the following characterization.

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Proposition 20. *If $\chi^{\mathrm{irr}}(C_0, E_0, \nabla_0) \geqslant 0$, then the Galois group $\mathrm{Gal}(E_0, \nabla_0)$ is virtually abelian. In particular, in the irregular case, ∇_0 has only trivial Stokes and the Galois group is one dimensional: it is diagonal or dihedral.*

Proof. In the logarithmic case, the curve C_0 with its orbifold structure is a finite quotient of the sphere or the torus. The differential equation lifts as a differential equation with trivial or abelian monodromy group, respectively. Since the Galois group is the Zariski closure of the monodromy group in the logarithmic case, we get that $\mathrm{Gal}(E_0, \nabla_0)$ is virtually abelian. In the irregular case, $\chi^{\mathrm{irr}} \geqslant 0$ gives us

$$-\chi^{\mathrm{irr}} = 2g_0 - 2 + \underbrace{\sum_{\kappa_k=0} \left(1 - \frac{1}{\nu_k}\right)}_{\geqslant 0} + \underbrace{\sum_{\kappa_k>0} (1+\kappa_k)}_{\geqslant 1+1/2} \leqslant 0.$$

We promptly see that $g_0 = 0$, like as in the logarithmic case, and we have the following possible formal local data up to bundle transformation:

$$\begin{pmatrix} \kappa_1 \cdots \kappa_n \\ \theta_1 \cdots \theta_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

In the first case, we assume that we have an unramified irregular singular point, and the inequality for χ^{irr} gives no place for any other singular point; moreover, $\kappa = 1$. The monodromy around the unique singular point decomposes as

$$M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} \lambda(1+st) & \lambda s \\ \lambda^{-1} t & \lambda^{-1} \end{pmatrix},$$

which must be trivial, implying that $\lambda = e^{i\pi\theta} = 1$ and $s = t = 0$. This means that $\theta \in \mathbb{Z}$, or equivalently $\theta = 0$ after bundle transformation, and we have trivial Stokes matrices. The local (and therefore global) Galois group is diagonal, like the differential equation.

In the second and third cases, the irregular point is ramified and there is a place for a single logarithmic pole with orbifold order $\nu = 2$. The local monodromy decomposes as

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -s \end{pmatrix},$$

which is never the identity. The second case is therefore impossible. In the third case, M is also the local monodromy at the logarithmic pole: the trace must be zero, $s = 0$, implying trivial Stokes again. □

Proposition 21. *Let (C_0, E_0, ∇_0) be a normalized differential equation with local formal data $(\kappa_k, \theta_k)_k$. Let $\phi : C \to C_0$ be a degree-d ramified cover and (E, ∇) be the pull-back equation. Let T be the dimension of the irregular Teichmüller deformation space for (C, E, ∇). Let B be the number of critical values of ϕ outside the poles of ∇_0. Then we have*

$$T - B \geqslant g - 1 - d \cdot \chi^{\mathrm{irr}}(C_0, E_0, \nabla_0),$$

where g is the genus of C.

Proof. Let us decompose

$$N = N_1 + \cdots + N_n,$$
$$R = R_1 + \cdots + R_n + B,$$

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where N_k is the number of poles of ∇ (counted with multiplicity) along the fiber $\phi^{-1}(p_k)$ and R_k is the total ramification along $\phi^{-1}(p_k)$. Now we have

$$T = 3g - 3 + N = g - 1 + \underbrace{2g - 2}_{=d(2g_0-2)+R} + N$$

(by Riemann–Hurwitz), which gives

$$T - B = g - 1 + d(2g_0 - 2) + \sum_{k=1}^{n} (N_k + R_k).$$

Let us lower bound $N_k + R_k$ in a function of the type of pole p_k for ∇_0. We note that, along scattering in Lemmas 13–17, the value of $N_k + R_k$ can only decrease, so that it is enough to estimate a lower bound for a scattered covering ϕ.

If $\kappa_k = 0$ and $\nu_k \in \mathbb{Z}_{>0} \cup \{\infty\}$ is the orbifold order, then we have

$$d = N_k + m \cdot \nu_k \quad \text{and} \quad R_k = m \cdot (\nu_k - 1),$$

so that

$$N_k + R_k = N_k + \frac{d - N_k}{\nu_k}(\nu_k - 1) = d\left(1 - \frac{1}{\nu_k}\right) + \frac{N_k}{\nu_k} \geqslant d\left(1 - \frac{1}{\nu_k}\right).$$

If $\kappa_k \in \mathbb{Z}_{>0}$, then we find (after scattering) that

$$N_k = d(1 + \kappa_k) \quad \text{and} \quad R_k = 0.$$

If $\kappa_k \in \mathbb{Z}_{>0} - \frac{1}{2}$, then we have

$$d = m_0 + 2m_1, \quad N_k = m_0(1 + \kappa_k + \tfrac{1}{2}) + m_1(1 + 2\kappa_k) \quad \text{and} \quad R_k = m_1,$$

so that $m_1 = (d - m_0)/2$ and, after substitution, we find that

$$N_k + R_k = d(1 + \kappa_k) + \frac{m_0}{2} \geqslant d(1 + \kappa_k).$$

After summing for $k = 1, \ldots, n$, we find the expected lower bound. □

Corollary 22. *Under the assumptions of Proposition 21, if ∇_0 is irregular with nontrivial Stokes matrices and $T - B \leqslant 0$, then we have $\chi^{\mathrm{irr}} < 0$, $g_0 = g = 0$ and*

$$d|\chi^{\mathrm{irr}}| \leqslant 1.$$

Proof. The inequality $\chi^{\mathrm{irr}} < 0$ directly follows from Proposition 20 and the fact that we are assuming nontrivial Stokes matrices. Then Proposition 21 gives

$$0 \geqslant T - B \geqslant g - 1 + \underbrace{d|\chi^{\mathrm{irr}}|}_{>0},$$

which implies that $g = 0$ (and therefore $g_0 = 0$). Then we deduce the expected inequality. □

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Table 1. Logarithmic classification.

Local formal data θ_k	Degree	Covering passport
$(\frac{1}{2},\frac{1}{3},\theta)$	6	$\left(\begin{bmatrix}2\\2\\2\end{bmatrix}\quad\begin{bmatrix}3\\3\end{bmatrix}\quad\begin{bmatrix}1\\\vdots\\1\end{bmatrix}\quad 3\right)$
$(\frac{1}{2},\frac{1}{3},\theta)$	4	$\left(\begin{bmatrix}2\\2\end{bmatrix}\quad\begin{bmatrix}3\\1\end{bmatrix}\quad\begin{bmatrix}1\\\vdots\\1\end{bmatrix}\quad 2\right)$
$(\frac{1}{2},\frac{1}{3},\theta)$	3	$\left(\begin{bmatrix}2\\1\end{bmatrix}\quad[3]\quad\begin{bmatrix}1\\1\\1\end{bmatrix}\quad 1\right)$
$(\frac{1}{2},\frac{1}{4},\theta)$	4	$\left(\begin{bmatrix}2\\2\end{bmatrix}\quad[4]\quad\begin{bmatrix}1\\\vdots\\1\end{bmatrix}\quad 1\right)$
$(\frac{1}{2},\theta_1,\theta_\infty)$	2	$\left([2]\quad\begin{bmatrix}1\\1\end{bmatrix}\quad\begin{bmatrix}1\\1\end{bmatrix}\quad 1\right)$

7. Classification of covers

In this section, we classify pull-back algebraic solutions of irregular Garnier systems (see § 2). In other words, we list all differential equations (C_0, E_0, ∇_0) and ramified coverings $\phi : C \to C_0$ such that, by deforming $\phi \rightsquigarrow \phi_t$, we get a complete isomonodromic deformation $\phi_t^*(E_0, \nabla_0)$. In fact, we omit classical solutions that will be discussed in § 2.1 and therefore assume that $\chi^{\mathrm{irr}}(C_0, E_0, \nabla_0) < 0$ (see Proposition 20). Moreover, the equations (E_0, ∇_0) are listed up to bundle transformation; in particular, we can assume without loss of generality that (E_0, ∇_0) is a normalized equation. In the following, we use the notation of previous sections. In particular, T is the dimension of the irregular Teichmüller space of the irregular curve given by $\phi^*(C_0, E_0, \nabla_0)$, and B is the dimension of deformation of ϕ, obtained by moving the critical values of ϕ outside the poles of ∇_0. To get a complete deformation, we need $T \leqslant B$ and we will assume that $T > 0$ (otherwise there is no deformation).

Proposition 23 [Dia13]. *Assume that (C_0, E_0, ∇_0) is a (normalized) logarithmic connection with at least one pole p_k having infinite orbifold order $\nu_k = \infty$ and assume that $\phi : C \to C_0$ is a scattered ramified cover of degree $d \geqslant 2$. If $T \leqslant B$ and $\chi^{\mathrm{irr}}(C_0, E_0, \nabla_0) < 0$, then (C_0, E_0, ∇_0) is hypergeometric and, up to bundle transformation, we are in the list of Table 1.*

Proof. This proposition is proved in [Dia13, second table, p. 142]. It can be proved directly by using the inequality of Lemma 21. □

By confluence of the poles of the differential equation, we deduce the following result.

Proposition 24. *Assume that (C_0, E_0, ∇_0) is a (normalized) differential equation with at least one irregular pole p_k having nontrivial Stokes matrices and assume that $\phi : C \to C_0$ is a scattered ramified cover of degree $d \geqslant 2$. If $T \leqslant B$, then (C_0, E_0, ∇_0) is a degenerate hypergeometric equation and, up to bundle transformation, we are in the list of Table 2.*

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TABLE 2. Irregular classification with scattered cover.

Local formal data for (E_0, ∇_0)	Degree	Covering passport	Local formal data for (E, ∇)	Isomonodromy equation
$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}$	6	$\begin{pmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\mathrm{Gar}^3(1,1,1)$
$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}$	4	$\begin{pmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 \\ 2 \end{bmatrix} & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \frac{1}{3} & 0 & 1 \end{pmatrix}$	$\mathrm{Gar}^2(0,1,1)$
$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}$	3	$\begin{pmatrix} [3] & \begin{bmatrix} 2 \\ 1 \end{bmatrix} & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 0 \end{pmatrix}$	$P_{III}^{D_7}$
$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{4} & 0 \end{pmatrix}$	4	$\begin{pmatrix} [4] & \begin{bmatrix} 2 \\ 2 \end{bmatrix} & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$P_{III}^{D_6}$
$\begin{pmatrix} 0 & \frac{1}{2} \\ \theta & 0 \end{pmatrix}$	2	$\begin{pmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & [2] & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ \theta - 1 & 0 & \theta \end{pmatrix}$	P_V
$\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \theta \end{pmatrix}$	2	$\begin{pmatrix} [2] & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ \theta - 1 & \theta \end{pmatrix}$	$P_{III}^{D_6}$
$\begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$	2	$([2] \quad 1)$	$\begin{pmatrix} 3 \\ 0 \end{pmatrix}$	P_{II}

Proof. From Lemmas 18 and 19, the irregular poles can be scattered as several logarithmic poles with all of them having exponent θ with infinite orbifold order, except for one in the ramified case, having exponent $\frac{1}{2}$. We are led to the list of Table 1. We then deduce the list of Table 2 by confluence of poles with infinite order θ, and possibly one of them $\frac{1}{2}$. In the first four entries of Table 1, we have no other choice than make the two poles with exponent $\frac{1}{2}$ and θ confluing into a ramified irregular pole with $\kappa = \frac{1}{2}$. In the last entry, however, we have several possible confluences, namely $\{\frac{1}{2}, \theta_1\} \rightsquigarrow \kappa = \frac{1}{2}$, $\{\theta_1, \theta_2\} \rightsquigarrow \kappa = 1$ and $\{\frac{1}{2}, \theta_1, \theta_2\} \rightsquigarrow \kappa = \frac{3}{2}$. □

Finally, by confluence of ramification fibers of ϕ, we complete the list for pull-back solutions.

PROPOSITION 25. *Assume that (C_0, E_0, ∇_0) is a (normalized) differential equation with at least one irregular pole p_k having nontrivial Stokes matrices and assume that $\phi : C \to C_0$ is a nonscattered ramified cover of degree $d \geqslant 2$. If $T \leqslant B$, then (C_0, E_0, ∇_0) is a degenerate hypergeometric equation and, up to bundle transformation, we are in the list of Table 3.*

Proof. We now invert the scattering process of Lemmas 13–17. To do this, we replace simple branching fibers outside the poles of ∇_0 by additional ramifications over poles. Note that in Table 2, each entry satisfies $T = B$, so that we cannot add ramifications over logarithmic poles with finite orbifold order, otherwise $T - B$ becomes > 0 (see Lemma 16). The only possibility is therefore to add ramifications over irregular poles of ∇_0, or logarithmic poles with exponent θ having infinite orbifold order. Only the first two lines give examples with $T > 0$. □

Remark 26. We observe that the algebraic solution of $P_{II}(0)$ can be constructed from two pull-back constructions (see Table 2 last line and Table 3 line 2). This comes from the fact that the

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Table 3. Irregular classification with confluent cover.

Local formal data for (E_0, ∇_0)	Degree	Covering passport	Local formal data for (E, ∇)	Isomonodromy equation
$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}$	6	$\begin{pmatrix} [3] & [4] & 2 \\ 3 & 2 & \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$\mathrm{Gar}^2(1,2)$
$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}$	6	$\begin{pmatrix} [3] & [6] & 1 \\ 3 & & \end{pmatrix}$	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$	P_{II}
$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}$	4	$\begin{pmatrix} [3] & [4] & 1 \\ 1 & & \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ \frac{1}{3} & 1 \end{pmatrix}$	P_{IV}

Airy equation $\begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$ is itself the pull-back from the Kummer equation $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}$ by a 3-fold ramified cover. Similarly, the algebraic solution of $P_{III}^{D_6}$ appears twice in Table 2 for $\theta = 0$ due to the fact that the Kummer equation $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \theta \end{pmatrix}$ is the double cover of $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{4} & 0 \end{pmatrix}$ in that case.

8. Classification of classical solutions in the rank-2 case

In § 7, we have given a complete classification of algebraic solutions of irregular Garnier systems whose linear Galois group is SL_2. Indeed, we have classified those solutions of type (1) in Corollary 12. It does not make sense to do the same for solutions of type (2) or (3) in Corollary 12, since there are infinitely many, for arbitrary large rank N. However, for a given rank, it makes sense to classify, and we do this in this section for the case $N = 2$, which is the first open case after the Painlevé equations. Recall that we have two solutions of type (1) in the case $N = 2$, one for each formal data:

$$\begin{pmatrix} 0 & 1 & 1 \\ \frac{1}{3} & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

8.1 Without apparent singular point

They correspond to the type (2) of Corollary 12 and have Galois group D_∞.

$$\begin{cases} \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} & \text{infinite discrete family,} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \theta_1 & \theta_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \theta_1 & \theta_2 \end{pmatrix} & \text{two-parameter families,} \\ \begin{pmatrix} 0 & 0 & 2 \\ \frac{1}{2} & \frac{1}{2} & \theta \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & \theta \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \theta \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{3}{2} & 0 \\ \frac{1}{2} & 0 & \theta \end{pmatrix} & \text{one-parameter families,} \\ \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \frac{5}{2} \\ \frac{1}{2} & 0 \end{pmatrix} & \text{sporadic solutions.} \end{cases} \tag{10}$$

These formal data correspond to those for which the dihedral group D_∞ occurs as a Galois group of the linear equation, up to bundle transformation. In order to find this list, we have to take into account the following constraints:

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- there are two or four poles where the local monodromy (or Galois group) is anti-diagonal, and the local formal type must be
$$\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{k}{2} \\ 0 \end{pmatrix} \quad \text{with } k \in \mathbb{Z}_{>0} \text{ odd};$$
- other poles are of local formal type
$$\begin{pmatrix} k \\ \theta \end{pmatrix} \quad \text{with } k \in \mathbb{Z}_{\geqslant 0},\ \theta \in \mathbb{C}.$$

We would like to insist that it is not necessary to consider particular values for θ as normalized equations with differential Galois group D_∞ having poles with diagonal local monodromy occurring in a family where each exponent θ can be deformed arbitrarily. This comes from the fact that the monodromy representation itself can be deformed as well.

The first entry corresponds to the unique case with four poles having local anti-diagonal monodromy. It is an irregular version of the Picard–Painlevé equation (see [Maz01b, LPU08]): there are infinitely many algebraic solutions, in bijection with the orbits of $\mathbb{Q} \times \mathbb{Q}$ under the standard action of $\mathrm{SL}_2(\mathbb{Z})$. In fact, if $\tilde{C} \to C \simeq \mathbb{P}^1$ denotes the elliptic curve given by the 2-fold cover ramifying over the four poles of (E, ∇), then Picard solutions are related with torsion points on $\tilde{C}$ and how they vary when deforming the poles, and the curve $\tilde{C}$. Here, the story is the same. Indeed, the locus of the D_∞ Galois group in the moduli space is closed algebraic, and a differential equation in this closed set has trivial Stokes matrices and is characterized by its monodromy representation.

For all other cases, recall that the differential equation with dihedral Galois group can be determined by means of the exponents θ_i, as for its monodromy, once we know the irregular curve. Therefore, for each exponent θ_i, we get exactly one algebraic solution.

8.2 With apparent singular point

They correspond to the type (3) of Corollary 12 and have Galois group C_∞ or D_∞. However, as noticed in Remark 10, algebraic solutions with Galois group D_∞ and apparent singular points always arise as particular cases of more general algebraic solutions with Galois group D_∞ and arbitrary singular points as listed in § 8.1. It just remains to complete the list with those algebraic solutions with Galois group C_∞ and (at least one) apparent singular point:

$$\begin{cases} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \theta_1 & \theta_2 & \theta_3 \end{pmatrix}, \quad \theta_1 + \theta_2 + \theta_3 = 0 & \text{two-parameter family,} \\ \begin{pmatrix} 0 & 0 & 2 \\ 0 & \theta & -\theta \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & \theta & -\theta \end{pmatrix} & \text{one-parameter families,} \\ \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} & \text{sporadic solution.} \end{cases} \tag{11}$$

For each value of the exponents θ_i, there is exactly one normalized equation once the irregular curve is fixed, and therefore exactly one algebraic solution. Solutions with two or more apparent singular points arise as particular cases of these ones by specifying the exponents θ_i.

9. Explicit Hamiltonians for some irregular Garnier systems

Here, we provide the linear differential equation and the Hamiltonians for some particular formal types (the complete list comes from [Kim89, Kaw09]). We translate our notation with Kimura and Kawamuko's.

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9.1 $\mathbf{Kim_4(1,2,2)}$

The general linear differential equation with (nonapparent) poles $x=0,1,\infty$ and corresponding formal type $\left(\begin{smallmatrix}2&2&1\\ \theta_0&\theta_1&\theta_\infty\end{smallmatrix}\right)$ can be normalized into the form[2]

$$L(1,2,2):\begin{cases} u''+f(x)u'+g(x)u=0 \quad \text{with}\\ f(x)=\dfrac{t_2}{x^2}+\dfrac{2-\theta_0}{x}+\dfrac{t_1}{(x-1)^2}+\dfrac{2-\theta_1}{x-1}-\displaystyle\sum_{k=1,2}\frac{1}{x-q_k},\\ g(x)=\dfrac{(\theta_0+\theta_1-1)^2-\theta_\infty^2}{4x(x-1)}-\dfrac{t_1H_1}{x(x-1)^2}+\dfrac{t_2H_2}{x^2(x-1)}+\displaystyle\sum_{k=1,2}\frac{q_k(q_k-1)p_k}{x(x-1)(x-q_k)}. \end{cases} \tag{12}$$

Singular points $x=q_1,q_2$ are apparent if and only if the coefficients H_1,H_2 are given by

$$H(1,2,2):\begin{cases} H_1=-\dfrac{q_1^2(q_1-1)^2(q_2-1)}{t_1(q_1-q_2)}\left(p_1^2-\left(\dfrac{\theta_0}{q_1}-\dfrac{t_2}{q_1^2}+\dfrac{\theta_1-1}{q_1-1}-\dfrac{t_1}{(q_1-1)^2}\right)p_1+\dfrac{(\theta_0+\theta_1-1)^2-\theta_\infty^2}{4q_1(q_1-1)}\right)\\ \qquad+\dfrac{(q_1-1)q_2^2(q_2-1)^2}{t_1(q_1-q_2)}\left(p_2^2-\left(\dfrac{\theta_0}{q_2}-\dfrac{t_2}{q_2^2}+\dfrac{\theta_1-1}{q_2-1}-\dfrac{t_1}{(q_2-1)^2}\right)p_1+\dfrac{(\theta_0+\theta_1-1)^2-\theta_\infty^2}{4q_2(q_2-1)}\right),\\ H_2=-\dfrac{q_1^2(q_1-1)^2q_2}{t_2(q_1-q_2)}\left(p_1^2-\left(\dfrac{\theta_0-1}{q_1}-\dfrac{t_2}{q_1^2}+\dfrac{\theta_1}{q_1-1}-\dfrac{t_1}{(q_1-1)^2}\right)p_1+\dfrac{(\theta_0+\theta_1-1)^2-\theta_\infty^2}{4q_1(q_1-1)}\right)\\ \qquad+\dfrac{q_1q_2^2(q_2-1)^2}{t_2(q_1-q_2)}\left(p_2^2-\left(\dfrac{\theta_0-1}{q_2}-\dfrac{t_2}{q_2^2}+\dfrac{\theta_1}{q_2-1}-\dfrac{t_1}{(q_2-1)^2}\right)p_1+\dfrac{(\theta_0+\theta_1-1)^2-\theta_\infty^2}{4q_2(q_2-1)}\right). \end{cases} \tag{13}$$

A deformation of (12) is isomonodromic if and only if the parameters satisfy the Hamiltonian system (4),

$$\frac{dq_j}{dt_i}=\frac{\partial H_i}{\partial p_j}\quad\text{and}\quad\frac{dp_j}{dt_i}=-\frac{\partial H_i}{\partial q_j}\quad\forall i,j=1,2. \tag{14}$$

To construct the pull-back solution (second line of Table 2), we start with the differential equation

$$\frac{d^2u}{dz^2}+\frac{2}{3z}\frac{du}{dz}-\frac{1}{z}u=0$$

and consider its pull-back by the branch cover

$$z=\phi(x)=\frac{t_2^2(2x-4q_1x+q_1^2+q_1)^3}{16q_1^3(q_1+1)^3x^2(x-1)^2}.$$

Comparing with (12), we get the first solution of Theorem 2.

9.2 $\mathbf{Kim_6(2,3)}$

The general linear differential equation with (nonapparent) poles $x=0,\infty$ and corresponding formal type $\left(\begin{smallmatrix}2&3\\ \theta_0&\theta_\infty\end{smallmatrix}\right)$ can be normalized into the form[3]

$$L(2,3):\begin{cases} u''+f(x)u'+g(x)u=0 \quad \text{with}\\ f(x)=\dfrac{t_2}{x^2}+\dfrac{2-\theta_0}{x}-t_1-\dfrac{x}{2}-\displaystyle\sum_{k=1,2}\frac{1}{x-q_k},\\ g(x)=\dfrac{\theta_0+\theta_\infty-1}{8}-\dfrac{H_1}{2x}+\dfrac{t_2H_2}{x^2}+\displaystyle\sum_{k=1,2}\frac{q_k(q_k-1)p_k}{x(x-1)(x-q_k)}. \end{cases} \tag{15}$$

[2] From Kimura's formulae [Kim89], set $\eta_i=1$, $\varkappa_i=\theta_i$, $\varkappa=\frac{1}{4}[(\theta_0+\theta_1-1)^2-\theta_\infty^2]$ and $K_i=H_i$.

[3] From Kimura's formulae [Kim89], set $\eta_0=1$, $\varkappa_0=\theta_0$, $\varkappa_\infty=(\theta_0+\theta_\infty-1)/4$ and $K_i=H_i$.

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Singular points $x = q_1, q_2$ are apparent if and only if the coefficients H_1, H_2 are given by

$$H(2,3): \begin{cases} H_1 = \dfrac{2q_1^2}{q_1-q_2}\left(p_1^2 - \left(\dfrac{\theta_0}{q_1} - \dfrac{t_2}{q_1^2} + \dfrac{q_1}{2} + t_1\right)p_1 + \dfrac{\theta_0+\theta_\infty-1}{8}\right) \\ \qquad - \dfrac{2q_2^2}{q_1-q_2}\left(p_2^2 - \left(\dfrac{\theta_0}{q_2} - \dfrac{t_2}{q_2^2} + \dfrac{q_2}{2} + t_1\right)p_2 + \dfrac{\theta_0+\theta_\infty-1}{8}\right), \\ H_2 = -\dfrac{q_1^2 q_2}{t_2(q_1-q_2)}\left(p_1^2 - \left(\dfrac{\theta_0-1}{q_1} - \dfrac{t_2}{q_1^2} + \dfrac{q_1}{2} + t_1\right)p_1 + \dfrac{\theta_0+\theta_\infty-1}{8}\right) \\ \qquad + \dfrac{q_1 q_2^2}{t_2(q_1-q_2)}\left(p_2^2 - \left(\dfrac{\theta_0-1}{q_2} - \dfrac{t_2}{q_2^2} + \dfrac{q_2}{2} + t_1\right)p_2 + \dfrac{\theta_0+\theta_\infty-1}{8}\right) \end{cases} \tag{16}$$

and isomonodromic deformations are defined by (14). To construct the pull-back solution (first line of Table 3), we start with the differential equation

$$\frac{d^2u}{dz^2} + \frac{2}{3z}\frac{du}{dz} - \frac{1}{z}u = 0$$

and consider its pull-back by the branch cover

$$z = \phi(x) = \frac{(3x^2 + 8t_1x + 4q_1t_1 + 6q_1^2)^3}{6912x^2}.$$

Comparing with (15), we get the second solution of Theorem 2.

9.3 $\mathbf{Kaw_4(5/2, 3/2)}$

The general linear differential equation with (nonapparent) poles $x = 0, \infty$ and corresponding formal type $\left(\begin{smallmatrix} \frac{3}{2} & \frac{1}{2} \\ 0 & 0 \end{smallmatrix}\right)$ can be normalized into the form[4] $u'' = g(x)u$ with

$$g(x) = \frac{t_2^2}{4x^3} + \frac{H_1}{x^2} + \frac{H_2}{x} + \frac{t_1}{2} + \frac{x}{4} + \sum_{k=1,2}\left(\frac{3}{4(x-q_k)^2} - \frac{p_k}{x-q_k}\right). \tag{17}$$

Then set

$$u_1 = q_1 + q_2, \quad u_2 = q_1q_2, \quad v_1 = \frac{q_1+q_2}{2(q_1-q_2)^2} + \frac{p_1q_1 - p_2q_2}{q_1-q_2}$$
$$v_2 = -\frac{1}{(q_1-q_2)^2} - \frac{p_1-p_2}{q_1-q_2}$$

and

$$\begin{cases} K_1 = 2u_1v_1^2 + 4u_2v_1v_2 - 4v_1 - \dfrac{u_1^2}{2} - t_1u_1 + \dfrac{u_2}{2} + \dfrac{t_2^2}{2u_2}, \\ t_2K_2 = -2u_2v_1^2 + 2u_2^2v_2^2 - 2u_2v_2 + \dfrac{u_1u_2}{2} + t_1u_2 - \dfrac{t_2^2u_1}{2u_2}. \end{cases}$$

Deformation of (17) is isomonodromic if and only if

$$\frac{du_j}{dt_i} = \frac{\partial K_i}{\partial v_j} \quad \text{and} \quad \frac{dv_j}{dt_i} = -\frac{\partial K_i}{\partial u_j} \quad \forall i,j = 1,2.$$

The first classical sporadic solution of Theorem 3, with dihedral linear Galois group, can be constructed by pulling back the differential equation $d^2u/dz^2 = (1/z - 3/16x^2)u$ by the ramified covering

$$z = \phi(x) = \frac{(x^2 + 3t_1x - 3t_2)^2}{36x};$$

[4] From Kawamuko's formulae [Kaw09], set $\lambda_i = q_i$, $\mu_i = p_i$ and $h_i^{04} = H_i$.

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after normalizing, and comparing with (17), we get the rational solution

$$(t_1, t_2) \mapsto (u_1, u_2, v_1, v_2) := \left(-t_1, t_2, 0, \frac{3}{4t_2}\right).$$

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