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Decay of correlations for piecewise smooth maps with indifferent fixed points

HUYI HU

Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA
(e-mail: hu@math.psu.edu)

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Abstract. We consider a piecewise smooth expanding map f on the unit interval that has the form $f(x) = x + x^{1+\gamma} + o(x^{1+\gamma})$ near 0, where $0 < \gamma < 1$. We prove by showing both lower and upper bounds that the rate of decay of correlations with respect to the absolutely continuous invariant probability measure μ is polynomial with the same degree $1/\gamma - 1$ for Lipschitz functions. We also show that the density function h of μ has the order $x^{-\gamma}$ as $x \to 0$. Perron–Frobenius operators are the main tool used for proofs.

0. *Introduction*

Let $f : I \to I$ be a piecewise smooth map on the unit interval. It is well known that if f is uniformly expanding, then it admits an absolutely continuous invariant probability measure μ and (f, μ) has an exponential decay of correlations. If f has an indifferent fixed point, then f still admits an absolutely continuous invariant measure. In addition, if at the indifferent fixed point the map has local expression $f(x) \approx x + Cx^{1+\gamma}$, $0 < \gamma < 1$, then the measure μ is finite (see e.g. [**P**]). In this paper we prove that the decay of correlations for such systems is polynomial by showing that both upper and lower bounds of the decay rates are of the same order $n^{-\beta+1}$, $\beta = \gamma^{-1}$.

It was believed for a long time that there are some dynamical systems with a polynomial decay of correlations. There are several results related to this. For the upper bounds, the first result was obtained for piecewise linear maps with an indifferent fixed point (Takahashi model) by Mori [**M**] and Lambert *et al* [**LaSV**] independently in 1993. They have proved that the upper bounds for decay rates are of order $n^{-\beta+1}$. For nonlinear maps similar to ours, the same polynomial upper bounds were proved by Young [**Y**] in 1997 using probabilistic coupling. At about the same time, Liverani *et al* [**LiSV**] obtained slightly weaker upper bounds, i.e. $Cn^{-\beta+1}(\log n)^{\beta}$, for a one parameter family of maps that have the form $f(x) = x(1 + 2^{\gamma}x^{\gamma})$ as $x \in [0, \frac{1}{2})$ by using perturbations and expectations of the Perron–Frobenius operators. Polynomial upper bounds were also obtained for similar maps by Pollicott and Yuri [**PY**] and their methods are more straightforward than those used in [**LiSV**]. Using the generating functions of a renewal

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process, the exact upper bounds were also claimed by Isola [**I**]. On the other hand, there are fewer estimates for lower bounds. Fisher and Lopes published a paper [**FL**] in 2001 for the exact lower and upper bounds for piecewise linear maps studied in [**M**] and [**LaSV**] by working the potentials. Recently, using the renewal sequences of bounded operators, Sarig [**S**] obtained both bounds for the maps studied in [**LiSV**] with test functions supported on the interval $[\frac{1}{2}, 1]$. It seems that with some additional work, his results can be applied in the maps we study as well.

The work in this paper was done at the end of 1997. It is one of the first results for the polynomial decay of correlations for nonlinear maps. The arguments in this paper are essentially for the lower bounds. As a byproduct, we obtain the upper bounds as well, and they have the same degree (Theorem B). In addition to the decay rates of correlations, we have also obtained the increasing rates of the density functions as x tends to 0 (Theorem A) and the upper and lower bounds for the rates of mixing of the cylinders (Corollary of Theorem B).

Our approach gives a clear geometric picture of how a test function converges to its equilibrium under the Perron–Frobenius operator and why the convergence cannot be too fast for some test functions. This is mainly based on the fact that we know exactly how the system behaves dynamically near the indifferent fixed point 0, in addition to the fact that the system is uniformly expanding away from 0. So we can obtain precise information for the density function of the invariant measure, including the speed of the density function approaching infinity as x goes to 0. Hence, we can get an accurate description for the Perron–Frobenius operator and, therefore, understand the way a test function changes under the operator. This makes it possible to estimate the rates of convergence of a test function under iterations of the Perron–Frobenius operator and, therefore, the rates of decay of correlations.

We assume that f has an indifferent fixed point 0 and $fx = x + x^{1+\gamma} + o(x^{1+\gamma})$ near 0. We use the Perron–Frobenius operator $\mathcal{L} = \mathcal{L}_\phi$ with the potential function $\phi = -\log|f'|$ to reprove the existence of the absolutely continuous invariant measure μ with a density function $h(x)$, and then to show that $x^{-\gamma}h(x)$ tends to a constant as $x \to 0$. The latter is of interest itself (see e.g. [**CF1**] and [**CF2**] for similar works for some C^2 maps). Next, we use the normalized Perron–Frobenius operator $\tilde{\mathcal{L}} = \mathcal{L}_{\log\eta}$, i.e. the operator with potential function $\log\eta(x) = \phi(x) + \log h(x) - \log h(fx)$. Since the operator preserves L^1 norms and leaves constant functions invariant, $\tilde{\mathcal{L}}^n g \to \mu(g)$ in L^1 for any continuous function g. Moreover, if higher order terms are ignored, then near 0 we have $\tilde{\mathcal{L}}g(x) \approx (1 - x_1^\gamma)g(x_1) + x_1^\gamma \bar{g}(x_1)$, where x_1 is the preimage of x near 0 and $\bar{g}$ is the average of g with weight η at the rest of preimages (see (4.3) for details). It states that the change from g to $\tilde{\mathcal{L}}g$ at x is proportional to x^γ. Also, note that when restricted to a neighborhood of 0, backward orbits approach 0 in a polynomial rate. Combining these facts we see that the convergence $\tilde{\mathcal{L}}^n g \to \mu(g)$ is at most at a polynomial rate. In our proof, we show that for some functions g, whose values are very large near 0 and smaller than $\mu(g)$ away from 0, the convergence is at a rate between $C'n^{-\beta+1}$ and $Cn^{-\beta+1}$, $C' < C$. Clearly, the upper bound part implies that for any Lipschitz function g, the convergence rates of $\tilde{\mathcal{L}}^n g$ are also bounded by $Cn^{-\beta+1}$. Since all the rates can be passed to the rates of decay of correlations by a standard procedure, we get the results.

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We state assumptions and the main results in §1. Theorem A, which concerns existence and the properties of density functions of invariant measures, is proved in §3. Theorem B and its corollary, which deal with decay rates of correlations of Lipschitz functions and mixing rates of cylinders, respectively, are proved in §7. To obtain Theorem B, we prove Proposition 5.2 in §§5 and 6, which asserts that the rate of convergence $\tilde{\mathcal{L}}^n g \to \mu(g)$ is polynomial.

1. *Assumptions, statements of results and notation*

Let $I = [0, 1]$ be the unit interval and $f : I \to I$ be a piecewise smooth map.

A fixed point p of f is called *indifferent* if $\lim_{x \to p} f'(x) = 1$.

ASSUMPTIONS. *Let $f : I \to I$ such that:*

(I) *there is a finite partition $\xi = \{I_0, I_1, \ldots, I_Q\}$ into subintervals such that for each q, restricted to I_q, $f|_{\operatorname{int} I_q}$ is twice differentiable and $f|_{\operatorname{int} I_q}$ maps $\operatorname{int} I_q$ to $(0, 1)$ diffeomorphically;*

(II) *0 is an indifferent fixed point of f;*

(III) *$|f'| > 1$ on $(0, 1]$ and f'' is bounded on $[\tau, 1]$ for all $\tau > 0$.*

Moreover, we need the following assumption for technical reasons.

(IV) *Near $x = 0$, f and its derivative have the form*

$$f(x) = x + x^{1+\gamma} + x^{1+\gamma}\delta_0(x), \tag{1.1}$$

$$f'(x) = 1 + (1 + \gamma)x^{\gamma} + x^{\gamma}\delta_1(x), \tag{1.2}$$

$$f''(x) = \frac{\gamma(1 + \gamma) + \delta_2(x)}{x^{1-\gamma}}, \tag{1.3}$$

where $\delta_i(x) \to 0$ as $x \to 0$ for $i = 0, 1, 2$.

The last assumption states that f is equal to $x + x^{1+\gamma}$ plus higher order terms and the first and second derivative of the higher order terms are still of higher orders.

We denote by I_0 the element of the partition ξ that contains 0.

THEOREM A. *Suppose that $f : I \to I$ satisfies Assumptions (I)–(IV). Then f has an absolutely continuous invariant probability measure μ whose density function $h(x)$ satisfies:*

(i) *$0 < h(x) < \infty$ for all $x \in (0, 1]$;*

(ii) *h is Lipschitz on $[\tau, 1]$ for all $0 < \tau < 1$;*

(iii) *there exists $R > 0$ such that*

$$|x^{\gamma}h(x) - \sigma_0| \le R \max\{x^{\gamma}, \delta_1(x)\},$$

where

$$\sigma_0 = \lim_{x \to 0} \sum_{\bar{x}_1 \in f^{-1}x \setminus I_0} \frac{h(\bar{x}_1)}{f'(\bar{x}_1)}$$

is a constant. In particular

$$\lim_{x \to 0} x^{\gamma}h(x) = \sigma_0.$$

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The part of existence of absolutely continuous invariant measures was first proved by Pianigiani [**P**] in a more general setting by using the first return map. For (iii), a similar result can be seen in [**CF1**] and [**CF2**] for a map with the form

$$fx = \frac{x}{1-x} \quad \text{as } 0 \le x \le \frac{1}{2},$$

which admits a σ-finite absolutely continuous invariant measure.

For a Lipschitz function F, denote by $\|F\|$ the C^0 norm.

THEOREM B. *Suppose that $f : I \to I$ satisfies Assumptions (I)–(IV). Let μ be the absolutely continuous invariant probability measure and let $\beta = \gamma^{-1}$. Then:*

(i) *for any Lipschitz function G, there is a constant $C = C(G) > 0$ such that for any Lipschitz function F,*

$$\left| \int (F \circ f^n) G \, d\mu - \int F \, d\mu \int G \, d\mu \right| \le \frac{C}{n^{\beta-1}} \|F\| \quad \forall n > 0;$$

(ii) *there exist Lipschitz functions G and F, and a constant $C' > 0$ such that*

$$\left| \int (F \circ f^n) G \, d\mu - \int F \, d\mu \int G \, d\mu \right| \ge \frac{C'}{n^{\beta-1}} \quad \forall n > 0.$$

Remark. By the proof of the theorem, we see that there are C^∞ functions F and G satisfying the inequality in (ii).

Denote

$$\xi_m = \bigvee_{i=0}^{m-1} f^{-i}\xi.$$

So if $E \in \xi_m$, then

$$E = \bigcap_{i=0}^{m-1} f^{-i} I_{q_i}$$

for some $q_0, \dots, q_{m-1}$. Also we denote by $E^{(m)}$ the element of ξ_m containing 0.

COROLLARY. *Under the supposition of Theorem B, there exist constants $C > C' > 0$ and $l > 0$ such that for any $m \ge 0$, $E \in \xi_m$ and for any measurable set $E' \subset [0, 1]$,*

$$|\mu(f^{-n-m}E' \cap E) - \mu E \cdot \mu E'| \le \frac{Cm^{\beta-1}}{(n-l)^{\beta-1}} \mu E \cdot \mu E' \quad \forall n > l,$$

and if, in addition, $m \ge l$ and $E = E^{(m)}$, then

$$|\mu(f^{-n-m}E' \cap E) - \mu E \cdot \mu E'| \ge \frac{C'm^{\beta-1}}{(n+m)^{\beta-1}} \mu E \cdot \mu E' \quad \forall n > 0.$$

We introduce some notation.

Recall that I_0 is the element of the partition ξ containing 0. For $x \in I_0$, we denote $x_0 = x$ and $x_{i+1} = f^{-1}x_i \cap I_0$ for all $i > 0$. Choose a small neighborhood $P_0 \subset I_0$ of 0.

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For any function g, if $x \in I_0$, then we denote

$$\sigma_g(x) = \sum_{\bar{x}_1 \in f^{-1}(fx)\setminus I_0} \frac{g(\bar{x}_1)}{f'(\bar{x}_1)}. \tag{1.4}$$

We should note that $\sigma_g(x)$ depends on values of g at $f^{-1}(fx)\setminus\{x\}$.

Take non-decreasing functions $\rho_\pm(x) \geq 0$ and denote $B(x, \rho(x)) = (x - \rho_-(x), x + \rho_+(x))$. We require that $\rho_\pm(x)$ are chosen in such a way that $\rho_\pm(x) = O(x^{1+\gamma})$ on P_0, $fB(x, \rho(x)) \supset B(fx, \rho(fx))$ for all $x \in I$, and $y \in B(x, \rho(x))$ if and only if $x \in B(y, \rho(y))$. The latter implies $\rho_+(x) > \rho_-(x)$ on P_0. So $B(x, \rho(x))$ is not a ball in the Euclidean metric. Since $\rho_\pm(x)$ are non-decreasing, we have $\rho(x) \geq \bar{\rho}$ for some $\bar{\rho} > 0$ on $I\setminus I_0$.

For any $n \geq 0$, denote $B_n(x, \rho) = \{y \in I : d(f^i y, f^i x) \leq \rho(f^i x)\ \forall 0 \leq i \leq n\}$.

We always denote $\beta = \gamma^{-1}$. Choose $\beta_- < \beta < \beta_+$ such that $\beta_+ - \beta$ and $\beta - \beta_-$ are small, for example, less than 0.1 and $0.1(\beta - 1)$ respectively.

2. *Preliminary*

LEMMA 2.1. *Let $x \in I_0$. For any $\theta \geq 0$,*

$$\frac{x^\theta}{x_1^\theta} \cdot \frac{d(x_1, y_1)}{d(x, y)} = \frac{x^\theta}{x_1^\theta} \cdot \frac{1}{f'(x)} + o(x^\gamma) = 1 - (1 + \gamma - \theta x^\gamma + o(x^\gamma)), \quad x \to 0,$$

where $y \in B(x, \rho(x))$ and $y_1 \in B(x_1, \rho(x_1))$.

Proof. This is because by Assumption (IV), $x = fx_1 = x_1(1 + x_1^\gamma + o(x^\gamma))$ and $d(x, y) = (f'(x_1) + o(x^\gamma))\, d(x_1, y_1) = (1 + (1 + \gamma)x^\gamma + o(x^\gamma))\, d(x_1, y_1)$. □

LEMMA 2.2. *Given $\beta_- < \beta < \beta_+$, we can choose P_0 small enough such that for any $x \in P_0$:*

(i) *if*

$$x = x_0 \geq \left(\frac{\beta_-}{r}\right)^\beta$$

for some $r > 0$, then

$$x_n \geq \left(\frac{\beta_-}{r + n}\right)^\beta;$$

(ii) *if*

$$x = x_0 \leq \left(\frac{\beta_+}{r}\right)^\beta$$

for some $r > 0$, then

$$x_n \leq \left(\frac{\beta_+}{r + n}\right)^\beta.$$

Proof. If x is small, then we can find $1 < \lambda < \beta/\beta_-$ such that $f(x) \leq x(1 + \lambda x^\gamma)$.

Suppose that

$$x \leq \left(\frac{\beta_-}{r}\right)^\beta.$$

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We have

$$f(x) \le \left(\frac{\beta_-}{r}\right)^\beta \left(1 + \frac{\lambda\beta_-}{r}\right) = \beta_-^\beta \frac{r + \lambda\beta_-}{r^{\beta+1}}.$$

Note that $\lambda\beta_- < \beta$. If r is large enough, then

$$\left(1 - \frac{1}{r}\right)^\beta \left(1 + \frac{\lambda\beta_-}{r}\right) \le 1 \quad \text{or} \quad (r-1)^\beta (r + \lambda\beta_-) \le r^{\beta+1}.$$

So we get that

$$f(x) \le \left(\frac{\beta_-}{r-1}\right)^\beta.$$

This implies the result in (i).

Part (ii) can be proved similarly. □

Define

$$\Delta(x, y) = \begin{cases} 1 + \dfrac{J_0}{x}\, d(x, y), & \forall x \in P_0,\ y \in B(x, \rho(x)), \\ 1 + J\, d(x, y), & \forall x \in I \backslash P_0,\ y \in B(x, \rho(x)), \end{cases}$$

where $J, J_0 > 0$ are constants satisfying the following proposition.

PROPOSITION 2.3. (Distortion estimates) *There exist constants $J, J_0 > 0$ such that for all $x \in I$, $y \in B(x, \rho(x))$:*

(i) *if $x_1 \in f^{-1}x$, $y_1 \in f^{-1}y \cap B(x_1, \rho(x_1))$, then*

$$\Delta(x_1, y_1) \cdot \frac{f'(x_1)}{f'(y_1)} \le \Delta(x, y);$$

(ii) *for all $n > 0$, if $x_n \in f^{-n}x$, $y_n \in f^{-n}y \cap B_n(x_n, \rho)$, then*

$$\frac{(f^n)'(x_n)}{(f^n)'(y_n)} \le \Delta(x, y).$$

Proof. (i) First we suppose that $x \in P_0$. By (1.3) and the fact that $f'(y) > 1$, there is a constant $c > 0$ such that

$$\frac{f'(x)}{f'(y)} < 1 + (f'(x) - f'(y)) \le 1 + cx^\gamma \frac{d(x, y)}{x}.$$

Note that $x^{-1}\, d(x, y)$ is of order x^γ. So by Lemma 2.1 with $\theta = 1$ we have

$$\begin{aligned} \Delta(x_1, y_1) \cdot \frac{f'(x_1)}{f'(y_1)} &\le \left(1 + J_0 \frac{d(x_1, y_1)}{x_1}\right) \cdot \left(1 + cx_1^\gamma \frac{d(x_1, y_1)}{x_1}\right) \\ &= 1 + J_0 \left[1 + \frac{cx_1^\gamma}{J_0} + O(x_1^{2\gamma})\right] \frac{d(x_1, y_1)}{x_1} \\ &= 1 + J_0 \left(1 + \frac{cx_1^\gamma}{J_0} + O(x_1^{2\gamma})\right) (1 - \gamma x^\gamma + o(x^\gamma)) \frac{d(x, y)}{x}. \end{aligned}$$

If J_0 is large enough, then the right-hand side is less than $1 + J_0 x^{-1}\, d(x, y)$.

For the case $x \notin P_0$, the result is clear since f is uniformly expanding outside P_0.

(ii) This can be obtained from (i) by induction. □

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Remark. The ratio $(f^n)'(x_n)/(f^n)'(y_n)$ only depends on preimages of x and y. So if $f^{n-1}x_n \in I\backslash I_0$, then we still have

$$\frac{(f^n)'(x_n)}{(f^n)'(y_n)} \le 1 + J\,d(x, y)$$

for some $J > 0$, even if $x \in P_0$.

Recall the definition (1.4) of σ_g.

COROLLARY 2.4. *Let $x, y \in P_0$. If g satisfies $g(\bar{y}_1) \le g(\bar{x}_1)\Delta(\bar{x}_1, \bar{y}_1)$ for all $\bar{x}_1 \in f^{-1}x\backslash I_0$, $\bar{y}_1 \in f^{-1}y \cap B(\bar{x}_1, \bar{\rho})$, then*

$$\sigma_g(y_1) \le \sigma_g(x_1)(1 + J\,d(x, y)).$$

Proof. By (1.4) and Proposition 2.3(i),

$$\frac{\sigma_g(y_1)}{\sigma_g(x_1)} = \frac{\sum_{\bar{y}_1 \in f^{-1}y\backslash I_0} g(\bar{y}_1)/f'(\bar{y}_1)}{\sum_{\bar{x}_1 \in f^{-1}x\backslash I_0} g(\bar{x}_1)/f'(\bar{x}_1)} \le \max\left\{\frac{g(\bar{y}_1)}{g(\bar{x}_1)} \cdot \frac{f'(\bar{x}_1)}{f'(\bar{y}_1)}\right\}$$
$$\le \max\left\{\Delta(\bar{x}_1, \bar{y}_1)\frac{f'(\bar{x}_1)}{f'(\bar{y}_1)}\right\}$$

where max is taken over all pairs $\bar{x}_1 \in f^{-1}x\backslash I_0$ and $\bar{y}_1 \in f^{-1}y \cap B(\bar{x}_1, \bar{\rho})$. Since $\bar{x}_1$, $\bar{y}_1 \notin I_0$,

$$\Delta(\bar{x}_1, \bar{y}_1)\frac{f'(\bar{y}_1)}{f'(\bar{x}_1)} \le (1 + Jd(\bar{x}_1, \bar{y}_1))\frac{f'(\bar{y}_1)}{f'(\bar{x}_1)} \le 1 + J\,d(x, y). \qquad \square$$

3. *The density function*

In this section we prove Theorem A and then prove a result (Lemma 3.5) which implies that $h(x)$ is strictly decreasing with x.

Proof of Theorem A. Define the Perron–Frobenius operator $\mathcal{L} = \mathcal{L}_\phi$, $\phi = -\log f'$ from the set of continuous functions on $(0, 1]$ to itself by

$$\mathcal{L}g(x) = \sum_{\hat{x}_1 \in f^{-1}x} \frac{g(\hat{x}_1)}{f'(\hat{x}_1)}.$$

Let υ denote the Lebesgue measure on I. Clearly, $\upsilon(\mathcal{L}g) = \upsilon(g)$ for any integrable function g on $(0, 1]$.

Also, it is well known that for any fixed point h of $\mathcal{L}$, a measure μ given by $\mu(g) = \upsilon(g \cdot h)$ is an invariant measure of f. In fact, we can directly check that $\mathcal{L}(h \cdot (g \circ f)) = g \cdot (\mathcal{L}h)$, then we have $\mu(g \circ f) = \upsilon(h \cdot (g \circ f)) = \upsilon(\mathcal{L}(h \cdot (g \circ f))) = \upsilon((\mathcal{L}h) \cdot g) = \upsilon(h \cdot g) = \mu(g)$. (See e.g. [**B**] for more details.)

Let $\mathcal{B}$ denote the set of continuous functions g on $(0, 1]$ with the norm

$$\|g\| = \sup_{x \in (0,1]} \{xg(x)\}. \tag{3.1}$$

It is easy to check that $\mathcal{B}$ is a Banach space and $\mathcal{L}$ is a Linear operator on $\mathcal{B}$. Lemma 3.1 below implies that the operator $\mathcal{L}$ is continuous.

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Put

$$\mathcal{G} = \{g \in \mathcal{B} : g > 0,\ \upsilon(g) = 1,\ g(y) \le g(x)\Delta(x, y)\ \forall x \in I, \\ y \in B(x, \rho(x)), x^{\gamma} g(x) \le H_0\ \forall x \in P_0\},$$

where H_0 is a constant to be determined later.

$\mathcal{G}$ is not empty since $(1-\gamma)x^{-\gamma} \in \mathcal{G}$. It is clear that $\mathcal{G}$ is a convex set. By Lemmas 3.2 and 3.3, $\mathcal{G}$ is compact and $\mathcal{L}\mathcal{G} \subset \mathcal{G}$ if H_0 is large enough. So by the Schauder–Tychonoff fixed point theorem (see e.g. [**DS**]), $\mathcal{L}$ has a fixed point $h \in \mathcal{G}$ and, therefore, (i) and (ii) follow from the definition of $\mathcal{G}$. Part (iii) can be obtained from Lemma 3.4 and the fact that $\phi(x) = (1+\gamma)x^{\gamma} + o(x^{\gamma})$. □

LEMMA 3.1. *$\mathcal{L}$ is a bounded linear operator.*

Proof. Since $\mathcal{L}$ is a positive operator and x^{-1} is the maximal element in the unit ball with respect to the norm in (3.1), we only need to prove that $x\mathcal{L}(x^{-1})$ is bounded.

Note that $f'(x) \ge 1$. We have

$$\mathcal{L}\left(\frac{1}{x}\right) < \sum_{\hat{x}_1 \in f^{-1}x} \frac{1}{\hat{x}_1} \le \frac{1}{x_1} + \sum_{\bar{x}_1 \in f^{-1}x \setminus I_0} \frac{1}{\bar{x}_1} \le \frac{1}{x} \max_{z\in[0,1]}\{f'(z)\} + \sum_{\bar{x}_1 \in f^{-1}x \setminus I_0} \frac{1}{\bar{x}_1},$$

where the last inequality follows from the fact that

$$x = f(x_1) \le x_1 \max_{z\in[0,1]}\{f'(z)\}.$$

Since the second term is bounded,

$$\|\mathcal{L}\| = \sup_{x\in(0,1]} \{x\mathcal{L}(x^{-1})\}$$

is finite. □

LEMMA 3.2. *The set $\mathcal{G}$ is compact.*

Proof. First, $\mathcal{G}$ is a bounded set. In fact, for any $g \in \mathcal{G}$, if $x \notin P_0$, then

$$1 \ge \int_{B(x,\rho(x))} g(y)\,dy \ge g(x)\frac{1}{1+J\rho(x)} \cdot 2\rho(x).$$

That is,

$$xg(x) \le \frac{x}{2\rho(x)}(1+J\rho(x)) \le \sup_{x\notin P_0}\left\{\frac{x(1+J\rho(x))}{2\rho(x)}\right\}.$$

If $x \in P_0$, then $xg(x) \le H_0 x^{1-\gamma} \le H_0$.

Using the facts that $g(y) \le \Delta(x, y)g(x)$ for all $y \in B(x, \rho(x))$ and $xg(x) \le H_0 x^{1-\gamma}$ for all $x \in P_0$, we know that $\mathcal{G}$ is also an equicontinuous set. □

LEMMA 3.3. *If H_0 is large enough, then $\mathcal{L}\mathcal{G} \subset \mathcal{G}$.*

Proof. Take $g \in \mathcal{G}$. We prove $\mathcal{L}g \in \mathcal{G}$.

It is clear that $\mathcal{L}g > 0$ and $\upsilon(\mathcal{L}g) = \upsilon(g) = 1$.

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If $x, y \in I$ with $d(x, y) \le \rho(x)$, then Proposition 2.3(i) and the same arguments as in the proof of Corollary 2.4 give

$$\frac{\mathcal{L}g(y)}{\mathcal{L}g(x)} = \frac{\sum_{\hat{y}_1 \in f^{-1}y} g(\hat{y}_1)/f'(\hat{y}_1)}{\sum_{\hat{x}_1 \in f^{-1}x} g(\hat{x}_1)/f'(\hat{x}_1)} \le \max\left\{\Delta(\hat{x}_1, \hat{y}_1)\frac{f'(\hat{y}_1)}{f'(\hat{x}_1)}\right\} \le \Delta(x, y),$$

where max is taken over all pairs $\hat{x}_1 \in f^{-1}x$ and $\hat{y}_1 \in f^{-1}y \cap B(\hat{x}_1, \rho(\hat{x}_1))$.

Suppose that $x \in P_0$. Using Lemma 2.1 with $\theta = \gamma$ and using the fact that $x^\gamma g(x) \le H_0$ for all $x \in P_0$, we get

$$\begin{aligned} x^\gamma \mathcal{L}g(x) &= x_1^\gamma g(x_1)\frac{x^\gamma}{x_1^\gamma}\frac{1}{f'(x_1)} + x^\gamma \sum_{\bar{x}_1 \in f^{-1}x \setminus I_0} \frac{g(\bar{x}_1)}{f'(\bar{x}_1)} \\ &\le H_0\left[1 - x^\gamma + o(x^\gamma) + \frac{x^\gamma}{H_0}\sum_{\bar{x}_1 \in f^{-1}x \setminus I_0} \frac{g(\bar{x}_1)}{f'(\bar{x}_1)}\right]. \end{aligned}$$

Since all elements g in $\mathcal{G}$ are uniformly bounded on $I \setminus I_0$, the sum in the last term is bounded. So if we take H_0 large enough, then the right-hand side of the inequality is less than H_0. □

LEMMA 3.4. *There exists $R > 0$ such that*

$$|h(x)\phi(x) - (1 + \gamma)\sigma_0| \le R \max\{x^\gamma, \delta_1(x)\},$$

where $\phi(x) = f'(x) - 1 = (1 + \gamma)x^\gamma + x^\gamma\delta_1(x)$.

Proof. Denote $\alpha(x) = \max\{x^\gamma, |\delta_1(x)|\}$ for $x \in P_0$. We may assume that $\alpha(x)$ is non-decreasing on P_0, otherwise we use $\max_{0 \le y \le x}\{\alpha(y)\}$ instead.

First we claim that there exists $R > 0$ such that if

$$h(x)\phi(x) \ge (1 + \gamma + c + R\alpha(x_1))\sigma_0$$

for some $c \ge 0$ and $x \in P_0$, then

$$h(x_1)\phi(x_1) \ge (1 + \gamma + c(1 + \tfrac{1}{2}x_1^\gamma) + R\alpha(x_2))\sigma_0.$$

In fact, since $\mathcal{L}h = h$, we have that for $x \in P_0$,

$$h(x_1) = (1 + \phi(x_1))(h(x) - \sigma_h(x_1)) \ge (1 + \phi(x_1))(h(x) - \sigma_0 - J\sigma_0 x_1), \tag{3.2}$$

where $\sigma_h(x_1) \le \sigma_0(1 + Jx_1)$ follows from Corollary 2.4. Also, it is easy to check by (1.1) and the definition of $\delta_1(x)$ that

$$(1 + \phi(x_1)) \cdot \frac{\phi(x_1)}{\phi(x)} = 1 + x_1^\gamma + x_1^\gamma\delta_1^*(x_1)$$

for some $\delta_1^*(x)$ which is bounded by $\delta_1(x)$ multiplied by a constant coefficient. So by (3.2)

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we get

$$\begin{aligned}
&\frac{\phi(x_1)h(x_1)}{\sigma_0}\\
&\quad\geq \frac{\phi(x)h(x)}{\sigma_0}\cdot(1+\phi(x_1))\frac{\phi(x_1)}{\phi(x)} - (1+Jx_1)\phi(x_1)(1+\phi(x))\\
&\quad\geq (1+\gamma+c+R\alpha(x_1))(1+x_1^\gamma+x_1^\gamma\delta_1^*(x_1))\\
&\qquad -((1+\gamma)x_1^\gamma+x_1^\gamma\delta_1(x_1))(1+\phi(x_1)) - Jx_1\phi(x_1)(1+\phi(x_1))\\
&\quad= (1+\gamma+c+R\alpha(x_1)) + (c+R\alpha(x_1))(x_1^\gamma+x_1^\gamma\delta_1^*(x_1)) + (1+\gamma)x_1^\gamma\delta_1^*(x_1)\\
&\qquad -(1+\gamma)x_1^\gamma\phi(x_1) - x_1^\gamma\delta_1(x_1)(1+\phi(x_1)) - Jx_1\phi(x_1)(1+\phi(x_1)).
\end{aligned}$$

If P_0 is small enough, then $|\delta_1^*(x_1)|\leq\frac{1}{2}$ and therefore $cx_1^\gamma(1+\delta_1^*(x_1))\geq\frac{1}{2}cx_1^\gamma$. Note that $\alpha(x)$ is greater than or equal to $\delta_1(x)$ and x^γ. So

$$\tfrac{1}{2}R\alpha(x_1)+(1+\gamma)\delta_1^*(x_1)-(1+\gamma)\phi(x_1)-(\delta_1(x_1)+Jx_1^{1-\gamma}\phi(x_1))(1+\phi(x_1))>0$$

if R is sufficiently large. Hence, we have

$$\frac{\phi(x_1)h(x_1)}{\sigma_0}\geq(1+\gamma+c+R\alpha(x_1))+\frac{1}{2}cx_1^\gamma\geq(1+\gamma+c+R\alpha(x_2))+\frac{1}{2}cx_1^\gamma.$$

This means that the claim is true.

Using this claim we can see that

$$\phi(x)h(x)\leq(1+\gamma)\sigma_0+2R\sigma_0\alpha(x_1)\quad\forall x\in P_0.$$

Otherwise, we may have

$$\phi(x)h(x)\geq(1+\gamma)\sigma_0+2R\sigma_0\alpha(x_1)=(1+\gamma)\sigma_0+c\sigma_0+R\sigma_0\alpha(x_1)$$

for some $x\in P_0$, where $c=R\alpha(x_1)>0$. Then by using the claim repeatedly, and using the fact

$$c\cdot\left(1+\frac{1}{2}\sum_{i=1}^{n-1}x_i^\gamma\right)\left(1+\frac{1}{2}x_n^\gamma\right)\geq c\cdot\left(1+\frac{1}{2}\sum_{i=1}^{n}x_i^\gamma\right)$$

we get that

$$\phi(x_n)h(x_n)\geq(1+\gamma)\sigma_0+R\sigma_0\alpha(x_{n+1})+c\sigma_0\cdot\left(1+\frac{1}{2}\sum_{i=1}^{n}x_i^\gamma\right).$$

By Lemma 2.2,

$$x_i^\gamma\geq\frac{\beta_-}{r+i}\quad\forall i\geq0$$

for some $r>0$ and therefore $\sum_{i=1}^\infty x_i^\gamma$ diverges. This contradicts the fact that $x^\gamma h(x)$ is bounded for all $x\in P_0$.

Since $\phi(x)h(x)>0$, the inequality of the other direction can be proved similarly. □

LEMMA 3.5. *For any $0<\gamma'<1$, we can choose P_0 small enough such that*

$$h(y)\geq h(x)\cdot\left(1+\frac{J_0'}{x^{1-\gamma'}}d(x,y)\right)\quad\forall x\in P_0,\ x-\rho(x)\leq y\leq x$$

for some $J_0'>0$.

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Proof. Denote

$$\tau = \inf_{x\in(0,1]} \left\{ \frac{1-\gamma}{x^\gamma h(x)} \right\}.$$

By Lemma 2.1, there exists $c > 0$ such that

$$\frac{x^\gamma}{x_1^\gamma} \cdot \frac{1}{f'(x)} = 1 - x^\gamma + o(x^\gamma) > 1 - cx^\gamma$$

for all $x \in P_0$. We take

$$H_0' \le \min_{x\in P_0} \left\{ \frac{\tau\sigma_h(x)}{c} \right\}$$

and then define

$$\mathcal{G}_1 = \Bigg\{ g \in \mathcal{G} : g(x) \ge \tau h(x)\ \forall x \in (0,1],\ x^\gamma g(x) \ge H_0'\ \forall x \in P_0,$$
$$g(y) \ge g(x)\left(1 + \frac{J_0'}{x^{1-\gamma'}}\, d(x,y)\right)\ \forall x \in P_0,\ x - \rho(x) \le y \le x \Bigg\}.$$

$\mathcal{G}_1$ is not empty because $(1-\gamma')x^{-\gamma'} \in \mathcal{G}_1$. Clearly, $\mathcal{G}_1$ is compact since it is closed in $\mathcal{G}$. We will prove $\mathcal{L}\mathcal{G}_1 \subset \mathcal{G}_1$. Then we can take h as a fixed point of $\mathcal{L}$ in $\mathcal{G}_1$ and, therefore, h has the required property.

Let $g \in \mathcal{G}_1$. First, we have

$$\mathcal{L}g(x) \ge \mathcal{L}\tau h(x) = \tau\mathcal{L}h(x) = \tau h(x).$$

Secondly, since $\sigma_g(x) \ge \tau\sigma_h(x)$ and $H_0' \le c^{-1}\tau\sigma_h(x)$ for all $x \in P_0$, we get

$$x^\gamma \mathcal{L}g(x) = x_1^\gamma g(x_1)\frac{x^\gamma}{x_1^\gamma}\frac{1}{f'(x_1)} + x^\gamma\sigma_g(x_1) \ge H_0'\left(1 - cx^\gamma + \frac{x^\gamma}{H_0'}\tau\sigma_h(x_1)\right) \ge H_0'.$$

Now it remains to check

$$\mathcal{L}g(y) \ge \mathcal{L}g(x)\left(1 + \frac{J_0'}{x^{1-\gamma'}}\, d(x,y)\right).$$

That is,

$$\frac{g(y_1)}{f'(y_1)} + \sigma_g(y_1) \ge \left(\frac{g(x_1)}{f'(x_1)} + \sigma_g(x_1)\right)\left(1 + \frac{J_0'}{x^{1-\gamma'}}\, d(x,y)\right). \tag{3.3}$$

By (1.3), if $x \in P_0$, then

$$\frac{f'(x)}{f'(y)} \ge 1 + f'(x) - f'(y) \ge 1 + \frac{c'}{x^{1-\gamma}}\, d(x,y)$$

for some $c' > 0$. Also, using Lemma 2.1 for $\theta = 1-\gamma'$ we have

$$\begin{aligned}
\frac{g(y_1)}{f'(y_1)} &\ge \frac{g(x_1)}{f'(x_1)}\left(1 + \frac{J_0'}{x_1^{1-\gamma'}}\, d(x_1,y_1)\right)\left(1 + \frac{c'}{x^{1-\gamma}}\, d(x,y)\right) \\
&\ge \frac{g(x_1)}{f'(x_1)}\left(1 + J_0'(1 - (\gamma+\gamma')x^\gamma + o(x^\gamma))\frac{d(x,y)}{x^{1-\gamma'}} + c'\frac{d(x,y)}{x^{1-\gamma}}\right) \\
&\ge \frac{g(x_1)}{f'(x_1)}\left(1 + J_0'\frac{d(x,y)}{x^{1-\gamma'}} + \frac{c'}{2}\frac{d(x,y)}{x^{1-\gamma}}\right),
\end{aligned}$$

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if P_0 is small enough. Therefore, using Lemma 3.4 and interchanging the roles of x and y, we can see that (3.3) holds if we show

$$\frac{g(x_1)}{f'(x_1)}\frac{c'}{2}\frac{d(x,y)}{x^{1-\gamma}} \geq J\sigma_g(y_1)\,d(x,y) + \sigma_g(x_1)\frac{J_0'}{x^{1-\gamma'}}\,d(x,y),$$

or

$$x^{\gamma-\gamma'}\frac{c'g(x_1)}{2f'(x_1)} \geq Jx^{1-\gamma'}\sigma_g(y_1) + J_0'\sigma_g(x_1).$$

However, this is true if P_0 is small, because $x^{\gamma-\gamma'}g(x) \geq x^{\gamma-\gamma'}x^{-\gamma}H_0' = x^{-\gamma'}H_0' \to \infty$, while all other quantities are bounded as $x \to 0$. □

4. *The operator $\tilde{\mathcal{L}}$*

Take

$$\eta(x) = \frac{h(x)}{f'(x)h(fx)}$$

if $x > 0$ and $\eta(0) = 1$. By Lemma 4.4 below, $\eta(x)$ is continuous on each I_q.

Take the normalized Perron–Frobenius operator $\tilde{\mathcal{L}} = \mathcal{L}_{\log\eta}$ from the set of continuous functions on [0, 1] to itself by

$$\tilde{\mathcal{L}}g(x) = \sum_{\hat{x}_1 \in f^{-1}x} \eta(\hat{x}_1)g(\hat{x}_1),$$

or, equivalently,

$$\tilde{\mathcal{L}}g(x) = \frac{1}{h}\mathcal{L}(hg) = \frac{1}{h(x)}\sum_{\hat{x}_1 \in f^{-1}x}\frac{h(\hat{x}_1)}{f'(\hat{x}_1)}g(\hat{x}_1).$$

Recall that the measure μ, defined by $\mu(g) = \upsilon(hg)$, is an f invariant measure, where υ is the Lebesgue measure on I.

LEMMA 4.1. *The operator $\tilde{\mathcal{L}}$ has the following properties:*

(i) $\tilde{\mathcal{L}}c = c$ *for any constant function* c;

(ii) $\mu(\tilde{\mathcal{L}}g) = \mu(g)$ *for any integrable function* g.

Proof. (i) Using $\mathcal{L}h = h$, we get

$$\tilde{\mathcal{L}}c = \frac{1}{h}\mathcal{L}(ch) = \frac{c}{h}\mathcal{L}h = \frac{c}{h}h = c.$$

(ii) Since $\upsilon(\mathcal{L}g) = \upsilon(g)$,

$$\mu(\tilde{\mathcal{L}}g) = \upsilon\left(h\cdot\frac{1}{h}\mathcal{L}(hg)\right) = \upsilon(\mathcal{L}(hg)) = \upsilon(hg) = \mu(g).$$ □

LEMMA 4.2. *We have*

$$\lim_{w\to 0}\frac{\mu[0,w]}{w^{1-\gamma}} = \frac{\sigma_0}{1-\gamma}.$$

Consequently, there exists

$$a > \frac{\sigma_0}{1-\gamma} > a' > 0$$

such that

$$a'w^{1-\gamma} \leq \mu[0,w] \leq aw^{1-\gamma}.$$

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Proof. We use the fact that $\mu[0, w] = \int_0^w h(x)\,dx$ and then use Theorem A(iii). □

LEMMA 4.3. *Let $w \in I_0$. Then*

$$\int_0^w \prod_{i=1}^n \eta(x_i)\,d\mu(x) = \mu[0, w_n].$$

Proof. Note that $f^n : [0, w_n] \to [0, w]$ is a one-to-one map. We have

$$\tilde{\mathcal{L}}^n \chi_{[0,w_n]}(x) = \prod_{i=1}^n \eta(x_i) \quad \text{as } x \in [0, w]$$

and $\tilde{\mathcal{L}}^n \chi_{[0,w_n]}(x) = 0$ as $x \in [w, 1]$. So, by Lemma 4.1(ii),

$$\int_0^w \prod_{i=1}^n \eta(x_i)\,d\mu(x) = \mu(\tilde{\mathcal{L}}^n \chi_{[0,w_n]}) = \mu[0, w_n].$$ □

Take $\psi(x)$ such that $\eta(x) = 1 - \psi(x)$. Recall the definition (1.4) of σ_g.

LEMMA 4.4. *η and ψ have the following properties:*

(i) $\psi(x) = (1/h(fx))\sigma_h(x)$ *if* $x \in I_0$;
(ii) $\lim_{x\to 0}(\psi(x)/x^\gamma) = 1$;
(iii) $\lim_{x\to 0} \eta(x) = 1$ *and, therefore, η is continuous on each I_q;*
(iv) $0 \le \eta(x) \le 1$, *and* $\eta(x) = 1$ *if and only if* $x = 0$;
(v) $\psi(x)$ *is strictly increasing and* $\eta(x)$ *is strictly decreasing on* P_0;
(vi) *for all* $x \in P_0$ *and* $\bar{x} \in I \setminus P_0$, $\eta(x) > \eta(\bar{x})$, *if* P_0 *is small enough.*

Proof. Since $\mathcal{L}h = h$,

$$h(fx) = \frac{h(x)}{f'(x)} + \sigma_h(x)$$

for $x \in P_0$. So,

$$\eta(x) = \frac{h(x)}{f'(x)h(fx)} = 1 - \frac{1}{h(fx)}\sigma_h(x).$$

This implies (i).

Part (ii) follows from (i) and Theorem A(iii).

By (ii), $\eta = 1 - x^\gamma + o(x^\gamma)$. So η is continuous at 0. By the definition, it is continuous at all other points.

By Lemma 4.1(i),

$$\sum_{\hat{x}_1 \in f^{-1}x} \eta(\hat{x}_1) = 1,$$

so $0 \le \eta(\hat{x}_1) \le 1$. Then (iv) is clear.

To get (v), we use (i) and then compare Lemma 3.5 and Corollary 2.4, from which we see that $h(x)$ changes at a faster rate than $\sigma_h(x)$.

Part (vi) simply follows from (iv) and (v). □

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By (i) of the lemma, for $x \in I_0$ we can write $\tilde{\mathcal{L}}g(x)$ as

$$\tilde{\mathcal{L}}g(x) = \eta(x_1)g(x_1) + \psi(x_1)\bar{g}(x_1) = (1 - \psi(x_1))g(x_1) + \psi(x_1)\bar{g}(x_1), \tag{4.1}$$

where $\bar{g}(x_1)$ is the average of $\{g(\bar{x}_1)\}$ with weight $\{\eta(\bar{x}_1)\}$, $\bar{x}_1 \in f^{-1}x \setminus I_0$, i.e.

$$\bar{g}(x_1) = \frac{\sum_{\bar{x}_1 \in f^{-1}x \setminus I_0} \eta(\bar{x}_1)g(\bar{x}_1)}{\sum_{\bar{x}_1 \in f^{-1}x \setminus I_0} \eta(\bar{x}_1)} = \frac{\sum_{\bar{x}_1 \in f^{-1}x \setminus I_0} (h(\bar{x}_1)/f'(\bar{x}_1))g(\bar{x}_1)}{\sum_{\bar{x}_1 \in f^{-1}x \setminus I_0} (h(\bar{x}_1)/f'(\bar{x}_1))}. \tag{4.2}$$

The second part of the lemma states that if higher order terms are ignored, then $\psi(x_1) \approx x_1^\gamma$ and therefore $\tilde{\mathcal{L}}$ has the form

$$\tilde{\mathcal{L}}g(x) \approx (1 - x_1^\gamma)g(x_1) + x_1^\gamma \bar{g}(x_1). \tag{4.3}$$

LEMMA 4.5. *Let $g_n(x) = \tilde{\mathcal{L}}^n g(x)$. Then for any $x \in P_0$,*

$$g_n(x) = g(x_n)\prod_{i=1}^{n} \eta(x_i) + g_n^*(x),$$

where

$$g_n^*(x) = \sum_{j=1}^{n} \bar{g}_{n-j}(x_j)\psi(x_j)\prod_{i=1}^{j-1} \eta(x_i).$$

Proof. Use induction. By (4.1) the result is true for $n = 1$. Suppose it is true for some n. Then

$$\tilde{\mathcal{L}}g_n(x) = \eta(x_1)g(x_{n+1})\prod_{i=2}^{n+1} \eta(x_i) + \eta(x_1)g_n^*(x_1) + \psi(x_1)\bar{g}_n(x_1).$$

Since

$$\begin{aligned}\eta(x_1)g_n^*(x_1) &= \eta(x_1)\sum_{j=1}^{n} \bar{g}_{n-j}(x_{j+1})\psi(x_{j+1})\prod_{i=2}^{j} \eta(x_i) \\ &= \eta(x_1)\sum_{j=2}^{n+1} \bar{g}_{n+1-j}(x_j)\psi(x_j)\prod_{i=2}^{j-1} \eta(x_i) = \sum_{j=2}^{n+1} \bar{g}_{n+1-j}(x_j)\psi(x_j)\prod_{i=1}^{j-1} \eta(x_i),\end{aligned}$$

we get

$$\eta(x_1)g_n^*(x_1) + \psi(x_1)\bar{g}_n(x_1) = \sum_{j=1}^{n+1} \bar{g}_{n+1-j}(x_j)\psi(x_j)\prod_{i=1}^{j-1} \eta(x_i),$$

which is equal to $g_{n+1}^*(x)$. This completes the proof. □

LEMMA 4.6. *Let $x, y \in P_0$ with $x > y$. If $\bar{g}_i(x) \geq 0$ for all $0 \leq i \leq n-1$, then*

$$\begin{aligned}\tilde{\mathcal{L}}^n g(x) - \tilde{\mathcal{L}}^n g(y) \geq\ & g(x_n)\prod_{i=1}^{n} \eta(x_i) - g(y_n)\prod_{i=1}^{n} \eta(y_i) \\ &+ \sum_{j=1}^{n} (\bar{g}_{n-j}(x_j) - \bar{g}_{n-j}(y_j))\psi(x_j)\prod_{i=1}^{j-1} \eta(x_i).\end{aligned}$$

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Proof. By Lemma 4.5, we only need prove

$$g_n^*(x) - g_n^*(y) \geq \sum_{j=1}^{n} (\bar{g}_{n-j}(x_j) - \bar{g}_{n-j}(y_j))\psi(x_j) \prod_{i=1}^{j-1} \eta(x_i).$$

Note that

$$g_n^*(x) - g_n^*(y) = \sum_{j=1}^{n} (\bar{g}_{n-j}(x_j) - \bar{g}_{n-j}(y_j))\psi(x_j) \prod_{i=1}^{j-1} \eta(x_i) + \sum_{j=1}^{n} \bar{g}_{n-j}(y_j) \left(\psi(x_j) \prod_{i=1}^{j-1} \eta(x_i) - \psi(y_j) \prod_{i=1}^{j-1} \eta(y_i) \right).$$

We only need to prove that

$$\psi(x_j) \prod_{i=1}^{j-1} \eta(x_i) = \frac{\sigma_h(x_j)}{h(x_{j-1})} \cdot \frac{h(x_{j-1})}{(f^{j-1})'(x_{j-1})h(x)} = \frac{\sigma_h(x_j)}{(f^{j-1})'(x_{j-1})h(x)}$$

is increasing, where the first inequality follows from Lemma 4.4(i) and the definition of $\eta(x)$. By Proposition 2.3 and Corollary 2.4, both $(f^{j-1})'(y_{j-1})/(f^{j-1})'(x_{j-1})$ and $\sigma_h(x_j)/\sigma_h(x_j)$ are bounded by $1 + J\, d(x, y)$. Hence, by Lemma 3.5, we see that $h(x)$ is decreasing faster than $\sigma_h(x_j)$ and $(f^{j-1})'(x_{j-1})$ if P_0 is small enough. □

PROPOSITION 4.7. *Given $\beta_- < \beta < \beta_+$, we can choose P_0 sufficiently small such that for any $x \in P_0$:*

(i) *if*

$$x = x_0 \leq \left(\frac{\beta}{r}\right)^{\beta}$$

for some $r > 0$, then

$$\prod_{i=1}^{n} \eta(x_i) \geq \left(\frac{r}{r+n}\right)^{\beta_+};$$

(ii) *if*

$$x = x_0 \geq \left(\frac{\beta}{r}\right)^{\beta}$$

for some $r > 0$, then

$$\prod_{i=1}^{n} \eta(x_i) \leq \left(\frac{r}{r+n}\right)^{\beta_-}.$$

Proof. Take $\beta_+ > \beta'_+ > \beta''_+ > \beta$. Let P_0 be small enough such that for any

$$x = \left(\frac{\beta'_+}{r}\right)^{\beta} \in P_0, \quad 1 - \psi(x) \geq 1 - \frac{\beta'_+}{\beta''_+} x^{\gamma} \quad \text{and} \quad 1 - \left(\frac{\beta'_+}{r}\right) \geq \left(1 - \frac{1}{r}\right)^{\beta_+}.$$

Hence, using Lemma 2.2 for β''_+, we have

$$1 - \psi(x_i) \geq 1 - \frac{\beta'_+}{\beta''_+} x_i^{\gamma} \geq 1 - \frac{\beta'_+}{r+i} \geq \left(1 - \frac{1}{r+i}\right)^{\beta_+} = \left(\frac{r+i-1}{r+i}\right)^{\beta_+}.$$

Taking the product we get the result of (i).

Part (ii) can be proved in a similar way. □

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We denote

$$\tilde{\Delta}(x, y) = \begin{cases} 1 + \dfrac{\tilde{J}_0}{x}\, d(x, y), & \forall x \in P_0,\ y \in B(x, \rho(x)); \\ 1 + \tilde{J}\, d(x, y), & \forall x \in I \backslash P_0,\ y \in B(x, \rho(x)), \end{cases}$$

where $\tilde{J}, \tilde{J}_0 > 0$ are constants to be determined by the following lemma.

LEMMA 4.8. *There exist constants $\tilde{J}, \tilde{J}_0 > 0$ such that for all $x \in I$, $y \in B(x, \rho(x))$:*

(i) *if $x_1 \in f^{-1}x$, $y_1 \in f^{-1}y \cap B(x_1, \rho(x_1))$, then*

$$\tilde{\Delta}(x_1, y_1) \cdot \frac{\eta(y_1)}{\eta(x_1)} \le \tilde{\Delta}(x, y);$$

(ii) *if $x_n \in f^{-n}x$, $y_n \in f^{-n}y \cap B_n(x_n, \rho)$, then*

$$\prod_{i=1}^{n} \frac{\eta(f^i y_n)}{\eta(f^i x_n)} \le \tilde{\Delta}(x, y) \quad \forall n > 0;$$

(iii) *if a function g satisfies $g(\hat{y}_1) \le g(\hat{x}_1)\tilde{\Delta}(\hat{x}_1, \hat{y}_1)$ for all $\hat{x}_1 \in f^{-1}x$, $\hat{y}_1 \in f^{-1}y \cap B(\hat{x}_1, \rho(\hat{x}_1))$, then $\tilde{\mathcal{L}}g(y) \le \tilde{\mathcal{L}}g(x) \cdot \tilde{\Delta}(x, y)$.*

Proof. Note that

$$\frac{\eta(y)}{\eta(x)} = \frac{h(y)}{h(x)} \cdot \frac{f'(x)}{f'(y)} \cdot \frac{h(fx)}{h(fy)}.$$

So, if we take $\tilde{J}, \tilde{J}_0 > 0$ such that

$$\tilde{\Delta}(x, y) \ge \Delta(x, y)^2 \Delta(fx, fy),$$

then the rest is the same as in the proof of Proposition 2.3 and Lemma 3.3. □

Remark. Recall the remark after Proposition 2.3. We also have that if $f^{n-1}x_n \in I \backslash I_0$, then

$$\prod_{i=1}^{n} \frac{\eta(f^i y_n)}{\eta(f^i x_n)} \le 1 + \tilde{J}\, d(x, y)$$

for some $\tilde{J} > 0$ even if $x \in P_0$.

Recall the definition of $\bar{g}(x)$ in (4.2).

LEMMA 4.9. *There exists a constant $\bar{J} > 0$ such that for all $x, y \in I_0$, with $d(x, y) \le \bar{\rho}$, if $g(\bar{y}) \le g(\bar{x})\Delta(\bar{x}, \bar{y})$ for all $\bar{x} \in f^{-1}(fx) \backslash I_0$, $\bar{y} \in f^{-1}(fy) \cap B(\bar{x}, \rho(\bar{x}))$, then*

$$\bar{g}(y) \le \bar{g}(x)(1 + \bar{J}\, d(x, y)).$$

Proof. Clearly, $\eta(\bar{y})g(\bar{y}) \le \eta(\bar{x})g(\bar{x})\tilde{\Delta}(\bar{x}, \bar{y})^2$. Hence, by (4.2),

$$\begin{aligned} \bar{g}(y) &= \frac{\sum_{\bar{y} \in f^{-1}(fy) \backslash I_0} \eta(\bar{y})g(\bar{y})}{\sum_{\bar{y} \in f^{-1}(fy) \backslash I_0} \eta(\bar{y})} \le \frac{\sum_{\bar{x} \in f^{-1}(fx) \backslash I_0} \eta(\bar{x})g(\bar{x})\tilde{\Delta}(x, y)^2}{\sum_{\bar{x} \in f^{-1}(fx) \backslash I_0} \eta(\bar{x})\tilde{\Delta}(\bar{x}, \bar{y})^{-1}} \\ &= \bar{g}(x) \max\{\tilde{\Delta}(\bar{x}, \bar{y})^3\}, \end{aligned}$$

where max is taken over all pairs $\bar{x} \in f^{-1}(fx) \backslash I_0$ and $\bar{y} \in f^{-1}(fy) \cap B(\bar{x}, \rho(\bar{x}))$. So the result follows by choosing $\bar{J} > 0$ such that $1 + \bar{J}d(x, y) \ge (1 + \tilde{J}d(\bar{x}, \bar{y}))^3$. □

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5. *Convergent rate*

The main result in this section is Proposition 5.2, which shows that the rate of convergence $\tilde{\mathcal{L}}^n g \to \mu(g)$ is polynomial. This proposition plays a key role for the proof of Theorem B. Since the proof is long, we put some lemmas in the next section.

From now on we denote $g_n(x) = \tilde{\mathcal{L}}^n g(x)$.

For any $b_+ \in (0, 1)$, define a function $\Gamma(x, y) = \Gamma_{b_+}(x, y)$ by

$$\Gamma(x, y) = \begin{cases} 1 + \dfrac{K_0}{x}\, d(x, y), & \forall x \in P_0,\ y \in B(x, \rho(x)); \\ 1 + K\, d(x, y), & \forall x \in I \backslash P_0,\ y \in B(x, \rho(x)), \end{cases}$$

where $K, K_0 > 0$ are constants chosen as in the following lemma.

LEMMA 5.1. *There exist constants $K, K_0 > 0$ such that for any $x \in I$, $y \in B(x, \rho(x))$:*

(i) *if $g(x) \le b_+$, $g(x) \le g(y)\tilde{\Delta}(y, x)$ and $g(y) \le g(x)\tilde{\Delta}(x, y)$, then*

$$1 - g(y) \le (1 - g(x))\Gamma(x, y);$$

(ii) *if $1 - g(\bar{y}_1) \le (1 - g(\bar{x}_1))\Gamma(\bar{x}_1, \bar{y}_1)$ for all $\bar{x}_1 \in f^{-1}x$, $\bar{y}_1 \in f^{-1}y \cap B(\bar{x}_1, \rho(\bar{x}_1))$, then*

$$1 - \tilde{\mathcal{L}}g(y) \le (1 - \tilde{\mathcal{L}}g(x))\Gamma(x, y);$$

(iii) *there exists a constant $\bar{K} > 0$ such that for all $x, y \in I_0$ with $d(x, y) \le \bar{\rho}$, if $1 - g(\bar{y}) \le (1 - g(\bar{x}))\Gamma(\bar{x}, \bar{y})$ for all $\bar{x} \in f^{-1}(fx) \backslash I_0$, $\bar{y} \in f^{-1}(fy) \cap B(\bar{x}, \rho(\bar{x}))$, then*

$$1 - \bar{g}(y) \le (1 - \bar{g}(x))(1 + \bar{K}\, d(x, y)).$$

Proof. Since $g(x) \le g(y)\tilde{\Delta}(y, x) = g(y) + g(y)(\tilde{\Delta}(y, x) - 1)$, we have

$$\begin{aligned} 1 - g(y) &\le 1 - g(x) + g(y)(\tilde{\Delta}(y, x) - 1) \le (1 - g(x)) \left(1 + \frac{g(y)}{1 - b_+}(\tilde{\Delta}(y, x) - 1)\right) \\ &\le (1 - g(x))(1 + \tilde{\Delta}(y, x) - 1)^{\max\{1, g(y)/(1-b_+)\}} \\ &\le (1 - g(x))(\tilde{\Delta}(y, x))^{\max\{1, g(y)/(1-b_+)\}}. \end{aligned}$$

Note that for $x \in P_0$, $d(x, y) \le \rho(x) = O(x^{1+\gamma})$ and $g(y) \le g(x)\tilde{\Delta}(x, y) \le b_+\tilde{\Delta}(x, y)$. So $g(y)$ is bounded. Hence, it is clear that K_0 and K exist. This is (i). Parts (ii) and (iii) follow from the same arguments as in the proof of Lemma 4.8 and Lemma 4.9. □

PROPOSITION 5.2. *For any $0 < b_- \le b_+ < 1$, we can find arbitrarily small $v \in P_0$ such that for any continuous functions $g_+ \ge g_- > 0$ of the form*

$$g_\pm(x) = \begin{cases} A_\pm \displaystyle\prod_{i=0}^{k-1} \eta(x_i), & x \in [0, v], \\ b_\pm, & x \in [v, 1], \end{cases} \tag{5.1}$$

where $A_+ \ge A_- > 1$ and $k > 0$ are constants that make $\mu(g_+) \ge 1$ and $\mu(g_-) \le 1$, if a function g satisfies:

(a) *$g(x) \le g_+(x)$ for all $x \in I$ and $g(x) \ge g_-(x)$ for all $x \le v$;*

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(b) $\mu(g) = 1$;
(c) $g(y) \le g(x)\tilde{\Delta}(x, y)$ *for all* $x \in I$, $y \in B(x, \rho(x))$; *and*
(d) g *is decreasing on* $[0, v]$;
then for all $n \ge 0$,
(i)

$$1 - g_n(x) \ge \frac{D' A_-}{(n+k)^{\beta-1}} \quad \forall x \in I \backslash I_0;$$

(ii)

$$1 - g_n(x) \le \frac{D A_+}{n^{\beta-1}} \quad \forall x \in I;$$

(iii)

$$\frac{\bar{D}' A_-}{(n+k)^{\beta-1}} \le \int |g_n(x) - 1| \, d\mu(x) \le \frac{\bar{D} A_+}{n^{\beta-1}};$$

where D, D', $\bar{D}$, $\bar{D}' > 0$ *are constants only depending on* f.

Proof. We divide the proof into three steps.

Step I. We choose v and construct functions $g_\pm(x)$.

Take $0 < b_- \le b_+ < 1$.

Take $u \in I_0$ with $u \le \bar{\rho}$, where $\bar{\rho} = \inf\{\rho(x) : x \in I \backslash I_0\}$, such that for all $x > u$, $\eta(x) \le \eta(u)$, and for all $x \in [u, fu]$, $y \in B(x, \rho(x))$, $\Gamma(x, y) \ge 1 + 3\bar{K} d(\bar{x}, \bar{y})$ for all $\bar{x}, \bar{y} \in I \backslash I_0$ with $d(\bar{x}, \bar{y}) \le \rho(\bar{x})$. This is possible because of the definition of $\Gamma(x, y)$.

Take $v = u_m \in P_0$ for some $m > 0$, and write $v = (\beta/s)^\beta$. We first assume that $s \ge m$, otherwise we can choose a smaller u. Then we assume that m is large enough such that

$$\prod_{i=1}^{m} \eta(x_i) \le \frac{1}{2} \quad \forall x \in I_0 \backslash [0, u]. \tag{5.2}$$

Since

$$\prod_{i=1}^{n} \eta(x_i) + \sum_{j=1}^{n} \psi(x_j) \prod_{i=1}^{j-1} \eta(x_i) = 1,$$

it implies that for any $n \ge m$,

$$\sum_{j=1}^{n} \psi(x_j) \prod_{i=1}^{j-1} \eta(x_i) \ge \frac{1}{2} \quad \forall x \in I_0 \backslash [0, u]. \tag{5.3}$$

Lastly, we assume that s is large enough such that

$$c' s^{(\beta-1)/(\beta_+ - \beta + 1)} \ge \max\left\{ \left(\frac{2^{\beta_+}}{b_-}\right)^{1/\beta_-}, \left(\frac{2a\beta^{\beta-1}}{a' b_- \beta_-^{\beta-1}}\right)^{1/(\beta_- - \beta + 1)} \right\}, \tag{5.4}$$

and

$$s^{\beta_- - \beta + 1} \ge \frac{4 C_1 C_2 b_+}{C_3' b_-}, \tag{5.5}$$

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where c' is given in (5.13), a and a' are given in Lemma 4.2, C_2 and C_3' are as in Lemmas 6.1 and 6.2, respectively, and $C_1 \geq (1-b_+)^{-1}$ and satisfies that if

$$1-g(y) \leq 2(1-g(x))(1+\bar{K}\,d(x,y)) \quad \forall x \in [u, fu],\ 0 < y \leq x;$$
$$1-g(y) \leq (1-g(x))\Gamma(x,y) \quad \forall x \geq u,\ y \in B(x, \rho(x)),$$

then

$$\max\{1-g(x),\ x \in I\} \leq C_1 \min\{1-g(x),\ x \geq u\}. \tag{5.6}$$

Now we choose $A_+ \geq A_- \geq 1$ and $k > 0$ such that

$$A_\pm \prod_{i=0}^{k-1} \eta(v_i) = b_\pm, \tag{5.7}$$

$$A_+\mu[0, v_k] + b_+\mu[v, 1] \geq 1 \quad \text{and} \quad A_-\mu[0, v_k] + b_-\mu[v, 1] \leq 1. \tag{5.8}$$

This is possible. In fact, by Proposition 4.7 and Lemma 4.2 we have

$$\left(\frac{s}{s+k}\right)^{\beta_+} \leq \prod_{i=1}^{k} \eta(v_i) \leq \left(\frac{s}{s+k}\right)^{\beta_-}, \tag{5.9}$$

$$a'\left(\frac{\beta_-}{s+k}\right)^{\beta-1} \leq \mu[0, v_k] \leq a\left(\frac{\beta_+}{s+k}\right)^{\beta-1}. \tag{5.10}$$

These imply

$$\lim_{k\to 0} \frac{\prod_{i=0}^{k-1} \eta(v_i)}{\mu[0, v_k]} = 0.$$

So we can take k such that

$$\frac{1}{b_+} - \mu[v, 1] \leq \frac{\mu[0, v_k]}{\prod_{i=0}^{k-1} \eta(v_i)} \leq \frac{1}{b_-} - \mu[v, 1], \tag{5.11}$$

and then take $A_\pm$ such that (5.7) is satisfied.

Now we define $g_\pm(x)$ by using (5.1). Lemma 4.3 and (5.8) give

$$\mu(g_+) = A_+\mu[0, v_k] + b_+\mu[v, 1] \geq 1, \quad \mu(g_-) = A_-\mu[0, v_k] + b_-\mu[v, 1] \leq 1.$$

Note that by (5.9)–(5.11) we can obtain

$$\frac{s^{\beta_+}}{a\beta_+^{\beta-1}}\left(\frac{1}{b_+} - \mu[v, 1]\right) \leq (s+k)^{\beta_+-\beta+1}, \quad (s+k)^{\beta_--\beta+1} \leq \frac{s^{\beta_-}}{a'\beta_-^{\beta-1}}\left(\frac{1}{b_-} - \mu[v, 1]\right),$$

and, therefore,

$$c's^{\beta_+/(\beta_+-\beta+1)} \leq k+s, \quad k \leq cs^{\beta_-/(\beta_--\beta+1)}, \tag{5.12}$$

where $c > c' > 0$ are constants satisfying

$$c'^{\beta_+-\beta+1} = \frac{1}{a\beta_+^{\beta-1}}\left(\frac{1}{b_+} - \mu[v, 1]\right), \quad c^{\beta_--\beta+1} = \frac{1}{a'\beta_-^{\beta-1}}\left(\frac{1}{b_-} - \mu[v, 1]\right). \tag{5.13}$$

Hence by (5.4) we have

$$k+s \geq c's^{(\beta-1)/(\beta_+-\beta+1)} \cdot s \geq \left(\frac{2^{\beta_+}}{b_-}\right)^{1/\beta_-} \cdot s \geq 2s, \quad \text{i.e. } k \geq s. \tag{5.14}$$

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Moreover, by (5.7) and (5.9),

$$b_{\pm}\left(\frac{s+k}{s}\right)^{\beta_-} \le A_{\pm} \le b_{\pm}\left(\frac{s+k}{s}\right)^{\beta_+}, \tag{5.15}$$

and, therefore, by (5.14),

$$A_- \ge b_-\left(\frac{s+k}{s}\right)^{\beta_-} \ge 2^{\beta_+} > 1. \tag{5.16}$$

Step II. We prove that any function g satisfying conditions (a)–(d) has the following properties:
$(\mathcal{A}_n)$ $1 - g_n(x) > 0$ for all $x \ge u$;
$(\mathcal{B}_n)$ $\max\{1 - g_n(x),\ x \in I\} \le C_1 \min\{1 - g_n(x),\ x \ge u\}$.

First we consider the case $0 \le n \le m$. Since $1 - g(x) \ge 1 - b_+ > 0$ for all $x \ge v = u_m$, by Lemma 4.1(i), $1 - g_n(x) \ge 1 - b_+ > 0$ for all $x \ge u \ge f^n v$. We get $(\mathcal{A}_n)$. Since $C_1 \ge (1 - b_+)^{-1}$, $(\mathcal{B}_n)$ follows.

Now we consider the cases $n > m$. We only need to prove the following:
$(\mathcal{A}_n^*)$ $1 - g_n(x) > 2g(x_n)\prod_{i=1}^n \eta(x_i)$ for all $x \in [u, fu]$;
$(\mathcal{B}_n')$ $1 - g_n(y) \le 2(1 - g_n(x))(1 + \bar{K}\, d(x, y))$ for all $x \in [u, fu]$, $0 < y \le x$;
$(\mathcal{B}_n'')$ $1 - g_n(y) \le (1 - g_n(x))\Gamma(x, y)$ for all $x \ge u,\ y \in B(x, \rho(x))$.
In fact, by the definition of $\tilde{\mathcal{L}}$, we know that $(\mathcal{A}_{n-1})$ and $(\mathcal{A}_n^*)$ imply $(\mathcal{A}_n)$. Also, by $(\mathcal{B}_n')$, $(\mathcal{B}_n'')$ and (5.6), we can get $(\mathcal{B}_n)$.

To prove $(\mathcal{A}_n^*)$, $(\mathcal{B}_n')$ and $(\mathcal{B}_n'')$, we use induction. Assume $(\mathcal{B}_j)$ are true for all $0 \le j \le n - 1$. Then by Lemma 6.3 and the choice of C_1, $(\mathcal{A}_n^*)$ is true.

Note that $(\mathcal{B}_j'')$ holds for any $j = 0, 1, \ldots, m$ because of Lemma 5.1(i), (ii) and the fact that $1 - g(x) \ge 1 - b_+ > 0$ for all $x \ge v$. So we may assume $(\mathcal{A}_j)$ and $(\mathcal{B}_j'')$ for all $j = 0, 1, \ldots, n - 1$. Hence, if we assume $(\mathcal{A}_n^*)$ in addition, then by Lemmas 6.4 and 6.5, $(\mathcal{B}_n')$ and $(\mathcal{B}_n'')$ hold, respectively.

Step III. We prove that g_n satisfies (i)–(iii).

Since $\mu(g_n) = \mu(g) = 1$,

$$\int_{\{g_n>1\}} (g_n(x) - 1)\, d\mu(x) = \int_{\{g_n<1\}} (1 - g_n(x))\, d\mu(x) = \frac{1}{2}\int |g_n(x) - 1|\, d\mu(x). \tag{5.17}$$

Then the first inequality of (iii) follows immediately from Lemma 6.2 with $\bar{D}' = 2C_3'$. By using $(\mathcal{A}_n)$ for all $n > 0$, we get that the upper bound estimate in (iii) follows from Lemma 6.6 with $\bar{D} = 2C_4$.

If we use $(\mathcal{B}_n)$, then

$$\begin{aligned}\int_{\{g_n<1\}} (1 - g_n(x))d\mu(x) &\le \max\{1 - g(x) : x \in I\} \\ &\le C_1 \min\{1 - g(x) :\ x \ge u\} \\ &\le \frac{C_1}{\mu(I\backslash I_0)}\int_{\{g_n<1\}} (1 - g_n(x))\, d\mu(x). \end{aligned} \tag{5.18}$$

Considering (5.17) and the results in (iii), we get (i) with $D' = (2C_1)^{-1}\bar{D}' = C_1^{-1}C_3'$, and get (ii) with $D = (2\mu(I\backslash I_0))^{-1}C_1\bar{D} = (\mu(I\backslash I_0))^{-1}C_1C_4$. □

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6. *Some supplementary lemmas*

In this section we prove lemmas which are used for the proof of Proposition 5.2.

LEMMA 6.1. *There exists $C_2 > 0$ such that for any $x > u$,*

$$\prod_{i=1}^{k+n} \eta(x_i) \le C_2 \frac{1}{k^{\beta_- - \beta + 1}} \cdot \frac{1}{(n+k)^{\beta - 1}} \quad \forall n, k > 0.$$

Proof. By Proposition 4.7, for $x^* = (\beta_-/r^*)^\beta \in P_0$ fixed,

$$\prod_{i=1}^{k+n} \eta(x_i^*) \le \left(\frac{r^*}{r^* + k + n} \right)^{\beta_-}.$$

So the result is clear for this x^*. Since by Lemma 4.4(iv) $\eta(x)$ is smaller outside P_0 than inside, the result holds for all $x \in I_0 \backslash P_0$ as well. □

LEMMA 6.2. *Let $C_3' = 2^{-\beta} a' \beta_-^{\beta - 1}$, where a' is given in Lemma 4.2. Then*

$$\int_{\{g_n > 1\}} (g_n(x) - 1)\, d\mu(x) \ge C_3' A_- \frac{1}{(n+k)^{\beta - 1}} \quad \forall n > 0.$$

Proof. Take $t > 0$ such that

$$\left(\frac{t}{t+k} \right)^{\beta_+} = \frac{1}{b_-} \left(\frac{s}{s+k} \right)^{\beta_-}. \tag{6.1}$$

Clearly, $s \le t$. Also, by (5.16) the right-hand side is no more than $1/2^{\beta_+}$. So,

$$\frac{t}{t+k} \le \frac{1}{2}.$$

We get $t \le k$.

Take

$$z^{(n)} = \left(\frac{\beta}{t(1 + n/k)} \right)^{\beta}. \tag{6.2}$$

We claim

$$[0, z^{(n)}] \subset \{x : g_n(x) \ge 1\} \quad \forall n \ge 0. \tag{6.3}$$

In fact, for any $x \le z^{(n)}$, by Proposition 4.7 and (5.9),

$$\prod_{i=1}^{k+n} \eta(x_i) \ge \left(\frac{t(1+n/k)}{t(1+n/k) + k + n} \right)^{\beta_+} = \left(\frac{t}{t+k} \right)^{\beta_+} = \frac{1}{b_-} \left(\frac{s}{s+k} \right)^{\beta_-} \ge \frac{1}{b_-} \prod_{i=1}^{k} \eta(v_i).$$

Then by (5.1), (4.1) and (5.7),

$$g_n(x) \ge A_- \prod_{i=1}^{k+n} \eta(x_i) \ge \frac{A_-}{b_-} \prod_{i=1}^{k} \eta(v_i) = 1.$$

Now using Lemma 4.3 we have

$$\begin{aligned} \int_{\{g_n > 1\}} (g_n(x) - 1)\, d\mu(x) &\ge A_- \int_0^{z^{(n)}} \prod_{i=1}^{k+n} \eta(x_i)\, d\mu(x) - \mu[0, z^{(n)}] \\ &= A_- \mu[0, z_{n+k}^{(n)}] - \mu[0, z^{(n)}]. \end{aligned}$$

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Since $k > t$, by (6.2) and Lemma 2.2,

$$z_{n+k}^{(n)} \geq \left(\frac{\beta_-}{t(1+\frac{n}{k})+k+n}\right)^{\beta} \geq \left(\frac{\beta_-}{2(k+n)}\right)^{\beta}.$$

Then by Lemma 4.2,

$$A_-\mu[0, z_{n+k}^{(n)}] \geq A_- a'(z_{n+k}^{(n)})^{\beta-1} \geq \frac{A_- a' \beta_-^{\beta-1}}{2^{\beta-1}} \left(\frac{1}{k+n}\right)^{\beta-1}.$$

So the result follows if we show $A_-\mu[0, z_{n+k}^{(n)}] \geq 2\mu[0, z^{(n)}]$.

Note that

$$\frac{z_{n+k}^{(n)}}{z^{(n)}} \geq \left(\frac{\beta_- t}{\beta(t+k)}\right)^{\beta}.$$

Using (5.16) and the fact $t > s$, we can get

$$\begin{aligned} A_- \frac{\mu[0, z_{n+k}^{(n)}]}{\mu[0, z^{(n)}]} &\geq b_- \left(\frac{s+k}{s}\right)^{\beta_-} \cdot \frac{a'}{a} \left(\frac{\beta_- t}{\beta(t+k)}\right)^{\beta-1} \\ &\geq \frac{a' b_-}{a} \left(\frac{\beta_-}{\beta}\right)^{\beta-1} \left(\frac{s+k}{s}\right)^{\beta_- - \beta+1}. \end{aligned}$$

The right-hand side is greater than or equal to two because by (5.12) and (5.4),

$$\left(\frac{k+s}{s}\right)^{\beta_- - \beta+1} \geq (c' s^{(\beta-1)/(\beta_+ - \beta+1)})^{\beta_- - \beta+1} \geq \frac{2a}{a' b_-} \left(\frac{\beta}{\beta_-}\right)^{\beta-1}. \qquad \square$$

LEMMA 6.3. *Let $n > m$. Suppose for all $0 \leq j \leq n-1$,*

$$\max\{1 - g_j(x),\ x \in I\} \leq C_1 \min\{1 - g_j(x),\ x \geq u\}.$$

Then

$$1 - g_n(x) > 2g(x_n) \prod_{i=1}^{n} \eta(x_i) \quad \forall x \in [u, fu].$$

Proof. By (5.17), (5.18) and Lemma 6.2, we have that for all $1 \leq j \leq n$,

$$\begin{aligned} 1 - g_j(x) &\geq \frac{1}{C_1} \int_{\{g_j > 1\}} (g_j(x) - 1)\, d\mu(x) \\ &\geq \frac{C_3' A_-}{C_1} \frac{1}{(k+j)^{\beta-1}} \geq \frac{C_3' A_-}{C_1} \frac{1}{(k+n)^{\beta-1}} \quad \forall x \geq u. \end{aligned}$$

So the same inequality is true for $1 - \bar{g}_j(x)$. By Lemma 4.5 and (5.3),

$$1 - g_n(x) > \sum_{j=1}^{n} (1 - \bar{g}_{n-j}(x_j)) \psi(x_j) \prod_{i=1}^{j-1} \eta(x_i) \geq \frac{C_3' A_-}{2C_1} \frac{1}{(k+n)^{\beta-1}}.$$

On the other hand, since $n > m$, $g(x_n) \leq A_+ \prod_{i=1}^{k} \eta(x_i)$. So by Lemma 6.1,

$$g(x_n) \prod_{i=1}^{n} \eta(x_i) \leq A_+ \prod_{i=1}^{k+n} \eta(v_i) \leq A_+ C_2 \frac{1}{k^{\beta_- - \beta+1}} \frac{1}{(k+n)^{\beta-1}}.$$

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Now, considering (5.7) and (5.14) we have

$$\frac{1-g_n(x)}{2g(x_n)\prod_{i=1}^n \eta(x_i)} \geq \frac{C_3' A_-}{2^2 C_1 C_2 A_+} k^{\beta_- - \beta + 1} \geq \frac{C_3' b_-}{4 C_1 C_2 b_+} s^{\beta_- - \beta + 1}.$$

By (5.5) it is greater than or equal to one. □

LEMMA 6.4. *Let $n > m$. Suppose $g(x)$ is decreasing on $[0, v]$. Suppose further that:*
(i) $1 - g_j(x) > 0$ *for all* $0 \leq j \leq n-1$, $x \geq u$;
(ii) $1 - g_j(y) \leq (1 - g_j(x))\Gamma(x, y)$, *for all* $0 \leq j \leq n-1$, $x \geq u$, $y \in B(x, \rho(x))$; *and*
(iii)

$$1 - g_n(x) \geq 2g(x_n) \prod_{i=1}^n \eta(x_i) \quad \forall x \in [u, fu].$$

Then for all $x \in [u, fu]$ with $1 - g_n(x) > 0$,

$$1 - g_n(y) \leq 2(1 - g_n(x))(1 + \bar{K}\, d(x, y)) \quad \forall 0 < y \leq x.$$

Proof. By supposition (ii) and Lemma 5.1(iii),

$$(1 - \bar{g}_{n-j}(x_j)) - (1 - \bar{g}_{n-j}(y_j)) \geq -\bar{K}\, d(x, y) \cdot (1 - \bar{g}_{n-j}(x_j)). \tag{6.4}$$

Using Lemma 4.5 for the function $1 - g(x)$ and then using supposition (iii), we have

$$\begin{aligned} \sum_{j=1}^n (1 - \bar{g}_{n-j}(x_j))\psi(x_j) \prod_{i=1}^{j-1} \eta(x_i) &= 1 - g_n(x) - (1 - g(x_n)) \prod_{i=1}^n \eta(x_i) \\ &< 1 - g_n(x) + g(x_n) \prod_{i=1}^n \eta(x_i) \\ &\leq \frac{3}{2}(1 - g_n(x)) \leq 2(1 - g_n(x)). \end{aligned} \tag{6.5}$$

Therefore, by using Lemma 4.6 for the function $1 - g(x)$, we obtain

$$\begin{aligned} (1 - g_n(x)) - (1 - g_n(y)) \geq (1 - g(x_n)) \prod_{i=1}^n \eta(x_i) - (1 - g(y_n)) \\ \times \prod_{i=1}^n \eta(y_i) - 2\bar{K}(1 - g_n(x))\, d(x, y). \end{aligned} \tag{6.6}$$

If $1 - g(y_n) \leq 0$, then either $1 - g(x_n) \geq 0$ or $0 \geq 1 - g(x_n) \geq 1 - g(y_n)$. Since $\eta(x)$ is decreasing, (6.6) becomes

$$(1 - g_n(x)) - (1 - g_n(y)) \geq -2\bar{K}(1 - g_n(x))\, d(x, y). \tag{6.7}$$

Then the result follows.

If $1 - g(y_n) \geq 0$, then $0 \leq 1 - g(y_n) \leq 1 - g(x_n) \leq 1 - g_n(x)$. Since $\eta(x_i) > 0$ and $\eta(y_i) < 1$, (6.6) becomes

$$(1 - g_n(x)) - (1 - g_n(y)) \geq -(1 - g_n(x)) - 2\bar{K}(1 - g_n(x))\, d(x, y).$$

This is the result of the lemma. □

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LEMMA 6.5. *Suppose all the conditions in Lemma 6.4 are satisfied. Then*

$$1 - g_n(y) \le (1 - g_n(x))\Gamma(x, y) \quad \forall x \in [u, fu],\ y \in B(x, \rho(x)).$$

Proof. First we assume $y \le x$. The same arguments as in the proof of above lemma tell us that (6.5) holds. Further, if $1 - g(y_n) \le 0$, then (6.7) follows as well and therefore the result is true. So we consider the case $1 - g(y_n) \ge 0$. Note that $g(y_n) \ge g(x_n) \ge g_n(x)$. By Lemma 4.8(ii) and (5.2),

$$\begin{aligned}(1 - g(x_n))\prod_{i=1}^{n}\eta(x_i) - (1 - g(y_n))\prod_{i=1}^{n}\eta(y_i) &\ge (1 - g(y_n))\left(\prod_{i=1}^{n}\eta(x_i) - \eta(y_i)\right)\\ &\ge -(1 - g(y_n))\prod_{i=1}^{n}\eta(x_i)(\tilde{\Delta}(x, y) - 1)\\ &\ge -(1 - g_n(x))\frac{\tilde{\Delta}(x, y) - 1}{2}.\end{aligned}$$

So, by (6.6),

$$\begin{aligned}(1 - g_n(x)) - (1 - g_n(y)) &\ge -(1 - g_n(x))\left(\frac{\tilde{\Delta}(x, y) - 1}{2} + \frac{3\bar{K}\, d(x, y)}{2}\right)\\ &\ge -(1 - g_n(x))(\Gamma(x, y) - 1),\end{aligned}$$

where the last step follows from the choice of u. This is the result.

Now we assume $y \ge x$. We use Lemma 4.6 for the function $g(x)$, while interchanging the roles of x and y and replacing $\bar{g}_{n-j}(x_j) - \bar{g}_{n-j}(y_j)$ by $(1 - \bar{g}_{n-j}(y_j)) - (1 - \bar{g}_{n-j}(y_j))$, to get

$$\begin{aligned}g_n(x) - g_n(y) \le{}& g(x_n)\prod_{i=1}^{n}\eta(x_i) - g(y_n)\prod_{i=1}^{n}\eta(y_i)\\ &+ \sum_{j=1}^{n}[(1 - \bar{g}_{n-j}(y_j)) - (1 - \bar{g}_{n-j}(x_j))]\psi(x_j)\prod_{i=1}^{j-1}\eta(x_i). \qquad (6.8)\end{aligned}$$

Since $g(y_n) \le g(x_n)$ and $\eta(y_n) \le \eta(x_n)$, by Lemma 4.8(i) and supposition (iii),

$$\begin{aligned}&g(x_n)\prod_{i=1}^{n}\eta(x_i) - g(y_n)\prod_{i=1}^{n}\eta(y_i)\\ &\le \left[g(y_n)\prod_{i=1}^{n}\eta(y_i)\right](\tilde{\Delta}(x, y) - 1)\\ &\le \left[g(x_n)\prod_{i=1}^{n}\eta(x_i)\right](\tilde{\Delta}(x, y) - 1) \le \frac{1 - g_n(x)}{2}(\Gamma(x, y) - 1).\end{aligned}$$

Note that the arguments for (6.4) and (6.5) still hold. So (6.8) becomes

$$\begin{aligned}(1 - g_n(y)) - (1 - g_n(x)) &= g_n(x) - g_n(y)\\ &\le (1 - g_n(x))\left(\frac{\Gamma(x, y) - 1}{2} + \frac{3\bar{K}\, d(x, y)}{2}\right)\\ &\le (1 - g_n(x))(\Gamma(x, y) - 1).\end{aligned}$$

This completes the proof. □

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LEMMA 6.6. *Let* $C_4 = a\beta_+^{\beta-1}$. *Suppose* $1 - g_j(x) > 0$ *for all* $0 \le j \le n$, $x \ge u$. *Then*

$$\int_{\{g_n>1\}} (g_n(x) - 1)\, d\mu(x) \le \frac{C_4 A_+}{n^{\beta-1}}.$$

Proof. The supposition implies that for all $x \in I_0$,

$$\sum_{j=1}^{n} [(\bar{g}_{n-j}(x_j) - 1)]\psi(x_j) \prod_{i=1}^{j-1} \eta(x_i) \le 0.$$

If $g_n(x) \ge 1 > b_+$, then $g(x_n) \ge 1 > b_+$ and therefore $x_n \le v$. So, by Lemma 4.5,

$$g_n(x) - 1 \le (g(x_n) - 1) \prod_{i=1}^{n} \eta(x_i) < g(x_n) \prod_{i=1}^{n} \eta(x_i) \le A_+ \prod_{i=1}^{n+k} \eta(x_i).$$

Note that $\{x : g_n(x) > 1\} \subset [0, u]$. Also, note that $k \ge s \ge m$ and therefore $u_{n+k} = v_{n+k-m} \le v_n$. We have

$$\begin{aligned}
&\int_{\{g_n>1\}} (g_n(x) - 1)\, d\mu(x) \\
&\quad \le A_+ \int_0^u \prod_{i=1}^{n+k} \eta(x_i)\, d\mu(x) = A_+ \mu[0, u_{n+k}] \\
&\quad \le A_+ \mu[0, v_n] \le A_+ \cdot a v_n^{1-\gamma} \le a A_+ \left(\frac{\beta_+}{s+n}\right)^{\beta-1} \le a A_+ \left(\frac{\beta_+}{n}\right)^{\beta-1}. \qquad \square
\end{aligned}$$

7. *Proofs of Theorem B and its corollary*

PROPOSITION 7.1. *There exist* $B, \bar{B} > 0$ *such that for any Lipschitz function* g *with* $\mu(g) = 1$ *and for all* $n > 0$*:*

(i)

$$|1 - \tilde{\mathcal{L}}^n g(x)| \le \frac{B}{\epsilon n^{\beta-1}} \quad \forall x \in I \backslash I_0,$$

(ii)

$$\int |\tilde{\mathcal{L}}^n g(x) - 1|\, d\mu(x) \le \frac{\bar{B}}{\epsilon n^{\beta-1}},$$

where $\epsilon > 0$ *only depends on the Lipschitz constant of* g.

Proof. Take $0 < b_- < b_+ < 1$, and take $v \in P_0$, $k > 0$, and functions $g_\pm$ with $\mu(g_+) > 1$ and $\mu(g_-) < 1$ as in Proposition 5.2. Then we choose A and b such that $A \prod_{i=1}^{k} \eta(v_i) = b$ and such that the function $\hat{g}$ defined by

$$\hat{g}(x) = \begin{cases} A \prod_{i=0}^{k-1} \eta(x_i), & x \in [0, v], \\ b, & x \in [v, 1], \end{cases}$$

satisfies $\mu(\hat{g}) = 1$. Then we write

$$1 - \tilde{\mathcal{L}}^n g = \frac{1}{2\epsilon}[1 - \tilde{\mathcal{L}}^n(\hat{g} - \epsilon[1 - g])] - \frac{1}{2\epsilon}[1 - \tilde{\mathcal{L}}^n(\hat{g} + \epsilon[1 - g])].$$

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Suppose that we can find $\epsilon > 0$ such that both functions $\hat{g}(x) + \epsilon[1 - g(x)]$ and $\hat{g}(x) - \epsilon[1 - g(x)]$ satisfy the requirements (a), (c) and (d) in Proposition 5.2. By using the proposition for these functions, we can get

$$|1 - \tilde{\mathcal{L}}^n g(x)| \le \frac{1}{2\epsilon} \cdot \frac{DA_+}{n^{\beta-1}} + \frac{1}{2\epsilon} \cdot \frac{DA_+}{n^{\beta-1}} = \frac{DA_+}{\epsilon n^{\beta-1}} \quad \forall x \in I \backslash I_0,$$

and

$$\int |\tilde{\mathcal{L}}^n g(x) - 1| \, d\mu(x) \le \frac{1}{2\epsilon} \cdot \frac{\bar{D}A_+}{n^{\beta-1}} + \frac{1}{2\epsilon} \cdot \frac{\bar{D}A_+}{n^{\beta-1}} = \frac{\bar{D}A_+}{\epsilon n^{\beta-1}}.$$

Therefore, the result follows with $B = DA_+$ and $\bar{B} = \bar{D}A_+$.

Clearly we can find $\epsilon > 0$ such that (a) and (d) in Proposition 5.2 hold for functions $\hat{g}(x) \pm \epsilon(1 - g(x))$. It remains to show that there exists $\epsilon > 0$ such that

$$\frac{\hat{g}(y) \pm \epsilon(1 - g(y))}{\hat{g}(x) \pm \epsilon(1 - g(x))} \le \tilde{\Delta}(x, y) \quad \forall x \in I, y \in B(x, \rho(x)).$$

That is, we need

$$\frac{\tilde{\Delta}(x, y)\hat{g}(x) - \hat{g}(y)}{|\tilde{\Delta}(x, y)(1 - g(x)) - (1 - g(y))|} \ge \epsilon > 0 \tag{7.1}$$

for all $x \in I$ and $y \in B(x, \rho(x))$.

First, we consider the case $x \in [0, v]$.

Recalling the definition of $\tilde{\Delta}(x, y)$ and Lemma 4.8(i), we have

$$\left(1 + \frac{\tilde{J}_0 \, d(x_k, y_k)}{x_k}\right) \cdot \frac{\prod_{i=0}^{k-1} \eta(y_i)}{\prod_{i=0}^{k-1} \eta(x_i)} = \tilde{\Delta}(x_k, y_k) \cdot \frac{\prod_{i=0}^{k-1} \eta(y_i)}{\prod_{i=0}^{k-1} \eta(x_i)} \le \tilde{\Delta}(x, y).$$

Hence, by the definition of $\hat{g}$,

$$\begin{aligned}\tilde{\Delta}(x, y)\hat{g}(x) - \hat{g}(y) &\ge A\left(\tilde{\Delta}(x, y) \cdot \frac{\prod_{i=0}^{k-1} \eta(x_i)}{\prod_{i=0}^{k-1} \eta(y_i)} - 1\right) \prod_{i=0}^{k-1} \eta(y_i) \\ &\ge A\left(1 + \frac{\tilde{J}_0 \, d(x_k, y_k)}{x_k} - 1\right) \prod_{i=0}^{k-1} \eta(y_i) = A\tilde{J}_0 \frac{d(x_k, y_k)}{x_k} \prod_{i=0}^{k-1} \eta(y_i).\end{aligned}$$

Also, we have

$$\begin{aligned}|\tilde{\Delta}(x, y)(1 - g(x)) - (1 - g(y))| &\le (\tilde{\Delta}(x, y) - 1)|1 - g(x)| + |g(y) - g(x)| \\ &\le \left(|1 - g(x)| + \frac{xL_g}{\tilde{J}_0}\right) \cdot \frac{\tilde{J}_0 \, d(x, y)}{x},\end{aligned}$$

where L_g is a Lipschitz constant of g.

Now we get that the left-hand side of (7.1) is greater than or equal to

$$\frac{A \prod_{i=0}^{k-1} \eta(y_i)}{|1 - g(x)| + xL_g \tilde{J}_0^{-1}} \cdot \frac{d(x_k, y_k)}{d(x, y)} \cdot \frac{x}{x_k}.$$

It is bounded from below for all $x \in [0, v]$ and $y \in B(x, \rho(x))$ because $(f^k)'(x) \to 1$ and $\eta(y) \to 1$ as $x \to 0$.

The case $x \in [v, 1]$ can be considered similarly. □

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Proof of Theorem B. First, we note that by the definition of $\tilde{\mathcal{L}}$, for any functions F and G defined on I, $\tilde{\mathcal{L}}((F \circ f) \cdot G) = F \cdot (\tilde{\mathcal{L}}G)$. Hence,

$$\tilde{\mathcal{L}}^n((F \circ f^n) \cdot G) = F \cdot (\tilde{\mathcal{L}}^n G).$$

So, by using Lemma 4.1(ii) we have that

$$\begin{aligned}\mu((F \circ f^n) \cdot G) - \mu(F)\mu(G) &= \mu(\tilde{\mathcal{L}}^n((F \circ f^n) \cdot G)) - \mu(F \cdot \mu(G)) \\ &= \mu(F \cdot (\tilde{\mathcal{L}}^n G)) - \mu(F \cdot \mu(G)) \\ &= \mu(F \cdot (\tilde{\mathcal{L}}^n G - \mu(G))). \qquad (7.2)\end{aligned}$$

To prove (i), we take Lipschitz functions F and G on $[0, 1]$. The above formula gives

$$\mu((F \circ f^n) \cdot G) - \mu(F)\mu(G) \le \|F\|\mu(|\tilde{\mathcal{L}}^n G - \mu(G)|).$$

By Proposition 7.1, there exist $\bar{B} > 0$ and $\epsilon = \epsilon(G) > 0$ such that

$$\mu(|\tilde{\mathcal{L}}^n G - \mu(G)|) = \mu(|\tilde{\mathcal{L}}^n(G - \mu(G) + 1) - 1|) \le \frac{\bar{B}}{\epsilon n^{\beta-1}}.$$

So we can take $C = \bar{B}\epsilon^{-1}$.

Now we prove (ii). Let G be any Lipschitz function satisfying the requirements (a)–(d) in Proposition 5.2 for some functions $g_-(x) \le g_+(x)$. In particular, $\mu(G) = 1$. Then we know that there exists $D' > 0$ such that for all $n > 0$,

$$1 - \tilde{\mathcal{L}}^n G(x) \ge \frac{D' A_-}{(n + k)^{\beta-1}} \quad \forall x \in I \backslash I_0,$$

where A_- and k are described in the same proposition.

Take a Lipschitz function $F(x) \ge 0$ such that $F(x) = 0$ on I_0 and $\mu(F) > 0$. Then by (7.2) we have

$$\begin{aligned}|\mu((F \circ f^n) \cdot G) - \mu(F)\mu(G)| &= |\mu(\chi_{I \backslash I_0} \cdot F \cdot (\tilde{\mathcal{L}}^n G - 1))| \\ &\ge \mu(F) \min_{x \in I \backslash I_0} \{1 - \tilde{\mathcal{L}}^n G(x)\} \ge \mu(F) \frac{D' A_-}{(n + k)^{\beta-1}}.\end{aligned}$$

Now the result follows with $C' = (k + 1)^{-(\beta-1)} D' A_- \mu(F)$. □

Recall that $E^{(j)}$ is the element of ξ_j containing 0.

LEMMA 7.2. *There exist $l > 0$ such that for all $j \ge l$, if a function g satisfies:*
(a) $g(x) > 0$ *as* $x \in E^{(j)}$ *and* $g(x) = 0$ *as* $x \notin E^{(j)}$;
(b) $\int_{E^{(j)}} g \, d\mu = 1$; *and*
(c) $g(y) \le g(x)(1 + \tilde{J}\, d(x, y))$ *for all* $x, y \in E^{(j)}$,
then for all $n > 0$;
(i)

$$1 - \tilde{\mathcal{L}}^{n+j} g(x) \ge \frac{D' A_-}{(n + j)^{\beta-1}} \quad \forall x \in I \backslash I_0,$$

(ii)

$$1 - \tilde{\mathcal{L}}^{n+j} g(x) \le \frac{D A_+}{n^{\beta-1}} \quad \forall x \in I,$$

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where D, D' are as in Proposition 5.2, and $A_+ = \sup\{g(x) : x \in E^{(j)}\}$ and $A_- = \inf\{g(x) : x \in E^{(j)}\}$.

Proof. Take $0 < b_- \le b_+ < 1$ such that

$$\frac{b_+}{b_-} = \frac{A_+}{A_-}.$$

Let $v = (\beta/s)^\beta$ be the point given in Proposition 5.2.

For each $j > 0$, consider the function $g_j(x) = \tilde{\mathcal{L}}^j g(x)$. Since $f^j : E^{(j)} \to I$ is a one-to-one map,

$$g_j(x) = g(x_j) \prod_{i=1}^{j} \eta(f^i x_j) \le A_+ \prod_{i=1}^{j} \eta(f^i x_j),$$

for all $x \in I$, where $x_j = f^{-j}x \cap E^{(j)}$.

Note that if $x \le y$, then

$$\frac{g(x)\eta(x)}{g(y)\eta(y)} \ge \frac{1}{1+\tilde{J}\,d(x,y)} \cdot \frac{1-\psi(x)}{1-\psi(y)} \ge \frac{1+\psi(y)-\psi(x)}{1+\tilde{J}\,d(x,y)}.$$

By Lemmas 4.4(i), 3.5 and Corollary 2.4 it is easy to see that the right-hand side is greater than or equal to one if x is small. It means that $g(x)\eta(x)$ and therefore $g_j(x)$ is decreasing on $[0, v]$ if $E^{(j)}$ is small enough.

By Lemma 2.2, the length of $E^{(j)}$ is between $(\beta_-/(r+j))^\beta$ and $(\beta_+/(r+j))^\beta$ for some $r > 0$. So, if j is large enough, then by (c), $g(y) \le 2g(x)$ for any $x, y \in E^{(j)}$. Hence, by (b) and Lemma 4.2, we have

$$\frac{(r+j)^{\beta-1}}{2a\beta_+^{\beta-1}} \le \frac{1}{2\mu E^{(j)}} \le A_- \le A_+ \le \frac{2}{\mu E^{(j)}} \le \frac{2(r+j)^{\beta-1}}{a'\beta_-^{\beta-1}}. \tag{7.3}$$

On the other hand, by Proposition 4.7,

$$\prod_{i=1}^{j} \eta(v_i) \le \left(\frac{s}{s+j}\right)^\beta.$$

So if j is large enough, then $g(v) \le b_+$ and therefore $g(x) \le b_+$ for all $x > v$.

Now we see that g_j satisfies all conditions in Proposition 5.2 with $j = k$. Therefore, the results of the lemma follow. □

LEMMA 7.3. *There exist $C > 0$ and $l > 0$ such that for any $m \ge 0$, if $E \in \xi_m$, then for all $n > 0$,*

$$|\mu E - \tilde{\mathcal{L}}^{n+m+l}\chi_E(x)| \le \frac{Cm^{\beta-1}}{n^{\beta-1}}\mu E \quad \forall x \in I \backslash I_0.$$

Proof. Note that $f^m : E \to I$ is a one-to-one map and $f^{m-1}E = I_q$ for some q.

First we consider the case $f^{m-1}E = I_q \ne I_0$. Put

$$g(x) = \frac{1}{\mu E}\tilde{\mathcal{L}}^m \chi_E(x) = \frac{1}{\mu E} \prod_{i=1}^{m} \eta(f^i x_m),$$

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where $x_m = f^{-m}x \cap E$. By the remark after Lemma 4.8, we know that $g(y) \le g(x)(1 + \tilde{J}\, d(x, y))$ for any $x \in I$, $y \in B(x, \bar{\rho})$. Since $\mu(g) = 1$, by similar arguments as in the proof of Lemma 3.2, we know that g is bounded and the bounds are independent of m and E provided that $f^{m-1}E \neq I_0$. Consequently, g is a Lipschitz function and the Lipschitz constant is independent of m and E. So by Proposition 7.1, we have

$$|\mu E - \tilde{\mathcal{L}}^{n+m}\chi_E(x)| \le \frac{C}{n^{\beta-1}}\mu E \le \frac{Cm^{\beta-1}}{n^{\beta-1}}\mu E \quad \forall x \in I\backslash I_0$$

for all $n > 0$, where $C \ge B\epsilon^{-1}$.

Secondly, we consider the case that there exists $l \le j \le m$ such that $f^{m-j}E = E^{(j)} \subset I_0$, where l is as in Lemma 7.2. We may assume that j is the largest number with this property. Take

$$g(x) = \frac{1}{\mu E}\tilde{\mathcal{L}}^{m-j}\chi_E(x) = \frac{1}{\mu E}\prod_{i=1}^{m-j}\eta(f^i x_{m-j}),$$

where $x_{m-j} = f^{-m+j}x \cap E$. Clearly $g(x)$ satisfies all the requirements in Lemma 7.2. So we get that, for all $n > 0$,

$$1 - \frac{1}{\mu E}\tilde{\mathcal{L}}^{n+m}\chi_E(x) = 1 - \tilde{\mathcal{L}}^{n+j}g(x) \le \frac{DA_+}{n^{\beta-1}} \quad \forall x \in I\backslash I_0.$$

Recall (7.3), and note that r only depends on f. We may assume $l > r$. Since $j \le m$, we have

$$A_+ \le \frac{2^\beta m^{\beta-1}}{a'\beta_-^{\beta-1}}.$$

So the result follows with

$$C \ge \frac{2^\beta D}{a'\beta_-^{\beta-1}}.$$

Finally, we consider the case that $f^{m-j}E = E^{(j)} \subset I_0$ only holds for $j < l$. We take a partition

$$E = E_{l-j} \cup \left(\bigcup_{i=1}^{l-j-1}\bigcup_{q=1}^{Q} E_{i,q}\right)$$

such that $f^{m+i-1}E_{i,q} = I_q$ and $f^{m+l-j-1}E_{l-j} = I_0$. For each $E_{i,q}$, we use arguments similar to the first case for the function $g(x) = (\mu E_{i,q})^{-1}\tilde{\mathcal{L}}^{m+i}\chi_{E_{i,q}}(x)$ to get that, for all $n > 0$,

$$|\mu E_{i,q} - \tilde{\mathcal{L}}^{n+m+i}\chi_{E_{i,q}}(x)| \le \frac{Cm^{\beta-1}}{n^{\beta-1}}\mu E_{i,q} \quad \forall x \in I\backslash I_0. \tag{7.4}$$

Also, we have $f^{m-j}E_{l-j} = E^{(l)}$. So by taking $g(x) = (\mu E_{l-j})^{-1}\tilde{\mathcal{L}}^{m-j}\chi_{E_{l-j}}(x)$, the same reasons as in the second case imply that, for all $n > 0$,

$$\mu E_{l-j} - \tilde{\mathcal{L}}^{n+m+l-j}\chi_{E_{l-j}}(x) \le \frac{Cm^{\beta-1}}{n^{\beta-1}}\mu E_{l-j} \quad \forall x \in I\backslash I_0. \tag{7.5}$$

Since $i \le l$ and $l - j \le l$, (7.4) and (7.5) still hold if we use $\tilde{\mathcal{L}}^{n+m+l}$ instead of $\tilde{\mathcal{L}}^{n+m+i}$ and $\tilde{\mathcal{L}}^{n+m+l-j}$, respectively. Hence, the result follows if we take summation. □

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Proof of Corollary of Theorem B. Using (7.2) and taking $F = \chi_{E'}$ and $G = \chi_E$, we get

$$\mu(f^{-n-m}E' \cap E) - \mu E' \cdot \mu E = \mu(\chi_{E'} \cdot (\tilde{\mathcal{L}}^{n+m}\chi_E - \mu E)).$$

Since $E' \subset I \backslash I_0$,

$$\mu E' \cdot \min_{x \in I \backslash I_0} (\mu E - \tilde{\mathcal{L}}^{n+m}\chi_E(x)) \leq \mu(\chi_{E'} \cdot (\mu E - \tilde{\mathcal{L}}^{n+m}\chi_E)) \leq \mu E' \cdot \max_{x \in I \backslash I_0} (\mu E - \tilde{\mathcal{L}}^{n+m}\chi_E(x)).$$

Therefore, the first inequality follows from Lemma 7.3 with $n + l$ replaced by n. For the second one, we take $g(x) = (\mu E)^{-1}\chi_E(x)$ and then apply Lemma 7.2(i) with $j = m$ to get

$$\mu E - \tilde{\mathcal{L}}^{n+m}\chi_E(x) \geq \frac{D'A_-}{(n+m)^{\beta-1}}\mu E \quad \forall x \in I \backslash I_0$$

for all $n > 0$. By (7.3), $A_- \geq (2a)^{-1}\beta_+^{1-\beta}m^{\beta-1}$. So we get the inequality by taking $C' \leq (2a)^{-1}\beta_+^{1-\beta} \cdot D'$. □

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References

[B] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms (Lecture Notes in Mathematics, 470)*. Springer, New York, 1975.

[CF1] P. Collet and P. Ferrero. Absolutely continuous invariant measure for expanding pick maps of the interval expect at a marginal fixed point. *Instabilities and Nonequilibrium Structures*. Eds. E. Tirapehui and D. Villaroel. Kluwer Academic, New York, 1989.

[CF2] P. Collet and P. Ferrero. Some limit ratio theorem related to a real endomorphism in case of neutral fixed point. *Ann. Inst. Henri Poicaré* **52** (1990), 283–301.

[DS] N. Dunford and J. Schwartz. *Linear Operator Part I.* Interscience Publishers, New York, 1958.

[FL] A. Fisher and A. Lopes. Exact bounds for the Polynomial decay of correlation, $1/f$ noise and the CLT for the equilibrium states of a non-Hölder potential. *Nonlinearity* **14** (2001), 1071–1104.

[I] S. Isola. On systems with finite ergodic degree. *Far East J. Dyn. Syst.* **5** (2003), 1–62.

[LaSV] A. Lambert, S. Siboni and S. Vaienti. Statistical properties of a nonuniformly hyperbolic map of the interval. *J. Statist. Phys.* **72** (1993), 1305–1330.

[LiSV] C. Liverani, B. Saussol and S. Vaienti. A probabilistic approach to intermittency. *Ergod. Th. & Dynam. Sys.* **19** (1999), 671–685.

[M] M. Mori. On the intermittency of a piecewise linear map. *Tokyo J. Math.* **16** (1993), 411–428.

[P] G. Pianigiani. First return map and invariant measures. *Israel J. Math.* **35** (1980), 32–48.

[PY] M. Pollicott and M. Yuri. Statistical Properties of maps with indifferent periodic points. *Comm. Math. Phys.* **217** (2001), 503–520.

[S] O. Sarig. Subexponential decay of correlations. *Invent. Math.* **150** (2002), 629–653.

[Y] L.-S. Young. Recurrence time and rate of mixing. *Israel J. Math.* **110** (1999), 153–188.

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