MORE ON HYPERGEOMETRIC LÉVY PROCESSES

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Abstract

Kuznetsov and co-authors in 2011–14 introduced the family of *hypergeometric Lévy processes*. They appear naturally in the study of fluctuations of stable processes when one analyses stable processes through the theory of positive self-similar Markov processes. Hypergeometric Lévy processes are defined through their characteristic exponent, which, as a complex-valued function, has four independent parameters. In 2014 it was shown that the definition of a hypergeometric Lévy process could be taken to include a greater range of the aforesaid parameters than originally specified. In this short article, we push the parameter range even further.

Keywords: Hypergeometric Lévy process; Wiener–Hopf factorisation; stable process; self-similar Markov process

2010 Mathematics Subject Classification: Primary 60G51
Secondary 60G18; 60G52; 47A68

1. Introduction

Recall that a (killed) general one-dimensional Lévy process is a stochastic process issuing from the origin with stationary and independent increments and almost surely right-continuous paths. We write $X = \{X_t : t \geq 0\}$ for its trajectory and $\mathbb{P}$ for its law. The law $\mathbb{P}$ of a Lévy process is characterised by its one-time transition probabilities. In particular, there always exists a quadruple (q, a, σ, Π), where $q \geq 0$ is the killing rate, $a \in \mathbb{R}$ is the linear coefficient, $\sigma \geq 0$ is the Gaussian coefficient, and Π is a measure on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$, which gives the rate at which jumps of different sizes arrive, such that $\mathbb{E}[\mathrm{e}^{\mathrm{i}zX_t}] = \mathrm{e}^{t\psi(\mathrm{i}z)}$ for $z \in \mathbb{R}$, where the Laplace exponent $\psi(z)$ is given by the Lévy–Khintchine formula

$$\psi(z) = -q + az + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (\mathrm{e}^{zx} - 1 - zx\mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x).$$

The (spatial) Wiener–Hopf factorisation of a Lévy process ξ with Laplace exponent ψ consists of the equation $\psi(z) = -\kappa(-z)\hat{\kappa}(z)$ for $z \in \mathrm{i}\mathbb{R}$, where κ and $\hat{\kappa}$ are the Laplace exponents of subordinators H and $\hat{H}$, respectively, this time in the sense that $\mathbb{E}[\mathrm{e}^{-\lambda H_t}] = \mathrm{e}^{-t\kappa(\lambda)}$ and $\mathbb{E}[\mathrm{e}^{-\lambda \hat{H}_t}] = \mathrm{e}^{-t\hat{\kappa}(\lambda)}$ for $\mathrm{Re}\,\lambda \geq 0$. The subordinators H and $\hat{H}$ are respectively known as the ascending and descending ladder-height processes, and are related via a time change to the running maximum and running minimum of the process ξ. The Wiener–Hopf factorisation has long been valued as the insight into a great many fluctuation identities and, accordingly, has proved to be enormously important in the general theory of Lévy processes. See the books [6] and [1] for more on the factorisation and the central role it plays in the analysis of a rich variety of problems.

There are relatively few instances where one finds concrete examples of Lévy processes for which explicit identities exist for the characteristic exponent, the underlying quadruple

doi:10.1017/apr.2016.47 © *Applied Probability Trust* 2016

https://doi.org/10.1017/apr.2016.47 Published online by Cambridge University Press

(q, a, σ, Π) as well as for the Wiener–Hopf factors. This is especially the case for Lévy processes with two-sided jumps. One family of Lévy processes for which this degree of tractability is available is the four-parameter family of so-called *hypergeometric Lévy processes*.

We say that ξ with parameters $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ belongs to the hypergeometric class of Lévy processes if it has Laplace exponent which satisfies

$$\psi(z) = -\frac{\Gamma(1-\beta+\gamma-z)}{\Gamma(1-\beta-z)}\frac{\Gamma(\hat{\beta}+\hat{\gamma}+z)}{\Gamma(\hat{\beta}+z)}, \qquad z \in \mathrm{i}\mathbb{R}. \tag{1.1}$$

It is known that the hypergeometric class of Lévy processes admits parameter combinations in the domain $\mathcal{A}_1 \cup \mathcal{A}_2$, where

$$\mathcal{A}_1 = \{\beta \le 1,\ \gamma \in (0,1),\ \hat{\beta} \ge 0,\ \hat{\gamma} \in (0,1)\}$$

and

$$\mathcal{A}_2 = \{\beta \in [1,2],\ \gamma, \hat{\gamma} \in (0,1),\ \hat{\beta} \in [-1,0];\ 1-\beta+\hat{\beta}+\gamma \ge 0,\ 1-\beta+\hat{\beta}+\hat{\gamma} \ge 0\}.$$

These processes were introduced by Kuznetsov *et al.* [5] and Kuznetsov and Pardo in [3], and given a slightly more inclusive definition in [7] by extending the parameter range. The authors in [3] also showed that, in the parameter regime $\mathcal{A}_1$, the density of the Lévy measure can be written explicitly in terms of the ${}_2\mathcal{F}_1$ hypergeometric function (which motivates the name of this class). Moreover, it was shown that, for $\lambda \ge 0$, the Wiener–Hopf factors of a hypergeometric Lévy process are given by

$$\kappa(\lambda) = \frac{\Gamma(1-\beta+\gamma+\lambda)}{\Gamma(1-\beta+\lambda)}, \qquad \hat{\kappa}(\lambda) = \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\lambda)}{\Gamma(\hat{\beta}+\lambda)}. \tag{1.2}$$

In [7] it was shown that, for the parameter regime $\mathcal{A}_2$, whilst the Lévy density remains the same as in the regime $\mathcal{A}_1$, the Wiener–Hopf factors can be written, for $\lambda \ge 0$, as

$$\kappa(\lambda) = (\lambda - \hat{\beta})\frac{\Gamma(1-\beta+\gamma+\lambda)}{\Gamma(2-\beta+\lambda)}, \qquad \hat{\kappa}(\lambda) = (\beta-1+\lambda)\frac{\Gamma(\hat{\beta}+\hat{\gamma}+\lambda)}{\Gamma(1+\hat{\beta}+\lambda)}.$$

Our objective in this short article is to show that the parameter range for which (1.1) represents the characteristic exponent of a Lévy process may be extended further.

2. Extending the parameter domain further

We first define the two sets of admissible parameters that we would like to include in the definition of hypergeometric Lévy processes:

$$\mathcal{A}_3 = \bigcup_{n=0}^{\infty}\{\beta \in [0,1],\ \gamma, \hat{\gamma} \in (0,1),\ \hat{\beta} \in [-(n+1), -n];\ 1-\beta+\hat{\beta}+\hat{\gamma}+n \le 0,\ 1-\beta+\hat{\beta}+\gamma+n \ge 0\}$$

and

$$\mathcal{A}_4 = \bigcup_{n=1}^{\infty}\{\hat{\beta} \in [0,1],\ \gamma, \hat{\gamma} \in (0,1),\ \beta \in [n, n+1];\ n-\beta+\hat{\beta}+\hat{\gamma} \ge 0,\ n-\beta+\hat{\beta}+\gamma \le 0\}.$$

https://doi.org/10.1017/apr.2016.47 Published online by Cambridge University Press

Theorem 2.1. *There exists a Lévy process ξ with Laplace exponent ψ as given in (1.1), where $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ take values in $\mathcal{A}_3 \cup \mathcal{A}_4$. Let $\eta = 1 - \beta + \hat{\beta} + \gamma + \hat{\gamma}$.*

(i) *In the parameter regime $\mathcal{A}_3$, for $\lambda \geq 0$, the Wiener–Hopf factors are given by*

$$\kappa(\lambda) = \frac{\Gamma(1-\beta+\gamma+\lambda)}{\Gamma(1-\beta+\lambda)} \prod_{j=0}^{n} \frac{-\hat{\beta}-j+\lambda}{-\hat{\beta}-\hat{\gamma}-j+\lambda}$$

$$\text{and} \quad \hat{\kappa}(\lambda) = \frac{\Gamma(n+1+\hat{\beta}+\hat{\gamma}+\lambda)}{\Gamma(n+1+\hat{\beta}+\lambda)}.$$

Its Lévy density is given by

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(-\hat{\gamma})\Gamma(\eta-\gamma)} \mathrm{e}^{(\hat{\beta}+\hat{\gamma})x} \displaystyle\sum_{k=0}^{n} \frac{(1+\hat{\gamma})_k (\eta)_k}{(\eta-\gamma)_k} \frac{\mathrm{e}^{kx}}{k!} \\ -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\hat{\gamma})\Gamma(-\gamma)} \mathrm{e}^{-(1-\beta+\gamma)x} {}_2\mathcal{F}_1(1+\gamma, \eta; \eta-\hat{\gamma}; \mathrm{e}^{-x}), & x > 0, \\ -\dfrac{\Gamma(\eta)}{\Gamma(-\hat{\gamma})\Gamma(\eta-\gamma)} \mathrm{e}^{(\hat{\beta}+\hat{\gamma})x} \displaystyle\sum_{k\geq n+1} \frac{(1+\hat{\gamma})_k (\eta)_k}{(\eta-\gamma)_k} \frac{\mathrm{e}^{kx}}{k!}, & x < 0 \end{cases}$$

(*using the notation* $(z)_k = z(z+1)\cdots(z+k-1)$).

(ii) *In the parameter regime $\mathcal{A}_4$, for $\lambda \geq 0$, the Wiener–Hopf factors are given by*

$$\kappa(\lambda) = \frac{\Gamma(1+n-\beta+\gamma+\lambda)}{\Gamma(1+n-\beta+\lambda)} \quad \text{and} \quad \hat{\kappa}(\lambda) = \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\lambda)}{\Gamma(\hat{\beta}+\lambda)} \prod_{j=1}^{n} \frac{\beta-j+\lambda}{\beta-\gamma-j+\lambda}.$$

Its Lévy density is given by

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(-\gamma)\Gamma(\eta-\hat{\gamma})} \mathrm{e}^{-(1-\beta+\gamma)x} \displaystyle\sum_{k\geq n} \frac{(1+\gamma)_k (\eta)_k}{(\eta-\hat{\gamma})_k} \frac{\mathrm{e}^{-kx}}{k!}, & x > 0, \\ -\dfrac{\Gamma(\eta)}{\Gamma(-\gamma)\Gamma(\eta-\hat{\gamma})} \mathrm{e}^{-(1-\beta+\gamma)x} \displaystyle\sum_{k=0}^{n-1} \frac{(1+\gamma)_k (\eta)_k}{(\eta-\hat{\gamma})_k} \frac{\mathrm{e}^{-kx}}{k!} \\ -\dfrac{\Gamma(\eta)}{\Gamma(-\hat{\gamma})\Gamma(\eta-\gamma)} \mathrm{e}^{(\hat{\beta}+\hat{\gamma})x} {}_2\mathcal{F}_1(1+\hat{\gamma}, \eta; \eta-\gamma; \mathrm{e}^{x}), & x < 0. \end{cases}$$

For all parameter sets, the process is killed at rate

$$q = \frac{\Gamma(1-\beta+\gamma)\Gamma(\hat{\beta}+\hat{\gamma})}{\Gamma(1-\beta)\Gamma(\hat{\beta})}.$$

Furthermore, the process ξ has no Gaussian component. When $\gamma + \hat{\gamma} \in (0,1)$ (respectively $\gamma + \hat{\gamma} \in [1,2)$) ξ has paths of bounded variation and no drift (respectively paths of unbounded variation).

Remark 2.1. Note that the parameter ranges $\mathcal{A}^{\hat{\beta}}_{EHG}$ and $\mathcal{A}^{\beta}_{EHG}$ defined in [7, Remark 3] are included in the parameter ranges $\mathcal{A}_3$ and $\mathcal{A}_4$, respectively.

Remark 2.2. Let ξ be a Lévy process as defined in (2.1) whose parameters lie in $\mathcal{A}_4$. Note that the ascending Wiener–Hopf factor belongs to the class of β-subordinators. The same objects for the descending ladder exponent $\hat{\kappa}$ are somewhat more complicated. Similar remarks can be made for κ and $\hat{\kappa}$ for the parameter regime $\mathcal{A}_3$.

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3. Proof of Theorem 2.1

We prove (i), first working under the assumption that $1-\beta+\hat{\beta}+\hat{\gamma}+n<0$ and $1-\beta+\hat{\beta}+\gamma+n>0$. The function ψ has simple poles at the points $\{1-\beta+\gamma+k, -\hat{\beta}-\hat{\gamma}-k : k\in\mathbb{N}_0\}$ and simple roots at the points $\{1-\beta+k, -\hat{\beta}-k : k\in\mathbb{N}_0\}$ (where $\mathbb{N}_0=\{0,1,2,\dots\}$).

For a given choice of parameters $(\beta,\gamma,\hat{\beta},\hat{\gamma})\in\mathcal{A}_3$, there are $n+1$ positive poles, say ρ_j^*, of the form $-\hat{\beta}-\hat{\gamma}-j$ for $j=0,\dots,n$ with the rest of the positive poles, say ρ_l^{**}, being of the form $-\beta+\gamma+l$ for $l\in\mathbb{N}$. Similarly, there are $n+1$ positive roots, say ζ_j^*, of the form $-\hat{\beta}-j$ for $j=0,\dots,n$ with the rest of the positive roots, say ζ_l^{**}, being of the form $-\beta+l$ for $l\in\mathbb{N}$. The conditions $1-\beta+\hat{\beta}+\hat{\gamma}<0$, $1-\beta+\hat{\beta}+\gamma>0$ mean that the positive poles and roots interlace as follows:

$$0<\zeta_1^{**}<\rho_0^*<\zeta_0^*<\rho_1^{**}<\zeta_2^{**}<\rho_1^*<\cdots<\zeta_n^*<\rho_{n+1}^{**}<\zeta_{n+2}^{**}<\rho_{n+2}^{**}<\cdots. \tag{3.1}$$

For $k\in\mathbb{N}$, denote the ordered (i.e. in order of interlacement) positive poles and roots by ρ_k and ζ_k, respectively. Then, for $z\in\mathbb{C}$ such that the left-hand side is well defined,

$$\begin{aligned}\prod_{k\geq1}\frac{1+z/\zeta_k}{1+z/\rho_k}&=\prod_{l\geq1}\frac{1+z/\zeta_l^{**}}{1+z/\rho_l^{**}}\prod_{j=0}^{n}\frac{1+z/\zeta_j^*}{1+z/\rho_j^*}\\&=\prod_{l\geq1}\frac{1+z/(-\beta+l)}{1+z/(-\beta+\gamma+l)}\prod_{j=0}^{n}\frac{1+z/(-\hat{\beta}-j)}{1+z/(-\hat{\beta}-\hat{\gamma}-j)}\\&\approx\frac{\Gamma(1-\beta+\gamma+z)}{\Gamma(1-\beta+z)}\prod_{j=0}^{n}\frac{-\hat{\beta}-j+z}{-\hat{\beta}-\hat{\gamma}-j+z}\\&=:\kappa(z),\end{aligned} \tag{3.2}$$

where '$\approx$' means 'up to a constant'. Thanks to the representation of κ on the left-hand side of (3.2) and the interlacement in (3.1), it follows from Lemma 1 of [4] (see also Equations (16) and (17) in that paper) that $\kappa(\lambda)$ is a special Bernstein function.

Now denote the negative poles by $\hat{\rho}_k$. For $k\geq1$, they take the form $\hat{\beta}+\hat{\gamma}+k+n$. Similarly, the roots $\hat{\zeta}_k$ take the form $\hat{\beta}+k+n$. Note that these poles and roots interlace in the following way: $\cdots<-\hat{\rho}_2<-\hat{\zeta}_2<-\hat{\rho}_1<-\hat{\zeta}_1<0$. Therefore,

$$\prod_{k\geq1}\frac{1+z/\hat{\zeta}_k}{1+z/\hat{\rho}_k}=\prod_{k\geq1}\frac{1+z/(n+\hat{\beta}+\hat{\gamma}+k)}{1+z/(n+\hat{\beta}+k)}\approx\frac{\Gamma(n+1+\hat{\beta}+\hat{\gamma}+z)}{\Gamma(n+1+\hat{\beta}+z)}=:\hat{\kappa}(z).$$

A similar argument to that used in the previous paragraph shows that $\hat{\kappa}(\lambda)$ is also a special Bernstein function.

Next, using the relation $\Gamma(x+1)=x\Gamma(x)$ repeatedly, one easily verifies that, for the expression given in (1.1), $\psi(z)=-\kappa(-z)\hat{\kappa}(z)$. We can now appeal to Vigon's theory of philanthropy [8, Chapter 7] (see also Section 6.6 of [6]), which shows that $\psi(z)$ is the Laplace exponent of a Lévy process. Specifically, Vigon's theory of philanthropy states that if two subordinators have Laplace exponents $\phi_i(\lambda)$, $\lambda\geq0$, $i=1,2$, each of which has absolutely continuous and nonincreasing Lévy density, then they belong together in a Wiener–Hopf factorisation (they become 'friends'). That is, $\phi_1(-\mathrm{i}\theta)\phi_2(\mathrm{i}\theta)$, $\theta\in\mathbb{R}$, is the characteristic exponent of a Lévy process.

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To show that ξ is a meromorphic Lévy process, apply [4, Theorem 1(v)] in the killed case and [4, Corollary 2] in the un-killed case. For Lévy processes in the meromorphic class, it is known that their Lévy measure has a density, π, of the form

$$\pi(x) = \mathbf{1}_{\{x>0\}} \sum_{n\geq 1} a_n \rho_n \mathrm{e}^{-\rho_n x} + \mathbf{1}_{\{x<0\}} \sum_{n\geq 1} \hat{a}_n \hat{\rho}_n \mathrm{e}^{\hat{\rho}_n x}, \qquad x \in \mathbb{R}. \tag{3.3}$$

We will now compute the coefficients $a_k\rho_k$ and $\hat{a}_k\hat{\rho}_k$ in representation (3.3) and accordingly prove that π is equivalent to the expression given in the statement of the theorem.

First, note that the poles $(\rho_k\colon 1 \leq k \leq n+1)$ derive from the first ratio of gamma functions in (1.2) and the remaining poles $(\rho_k\colon k \geq n+2)$ derive from the second ratio of gamma functions in (1.2). Using the relation $a_k\rho_k = -\mathrm{Res}(\psi(z)\colon z = \rho_k)$, we find that, for $1 \leq k \leq n+1$,

$$a_k\rho_k = -\frac{(-1)^{k-1}}{(k-1)!} \frac{1}{\Gamma(1-\hat{\gamma}-k)} \frac{\Gamma(\eta-1+k)}{\Gamma(\eta-\gamma-1+k)},$$

and, for $k \geq n+2$,

$$a_k\rho_k = -\frac{(-1)^{k-n-2}}{(k-n-2)!} \frac{1}{\Gamma(2-\gamma-k+n)} \frac{\Gamma(\eta-2-n+k)}{\Gamma(\eta-2-\hat{\gamma}-n+k)}.$$

A similar analysis of negative poles allows us to compute the coefficients $\hat{a}_k\hat{\rho}_k$ in a similar fashion. Substituting these expressions into (3.3) and, recalling the series representation of the ${}_2\mathcal{F}_1$ hypergeometric function, we obtain the density of the Lévy measure as stated in the theorem.

Now suppose that at least one of the following two equalities holds:

$$1-\beta+\hat{\beta}+\hat{\gamma}+n=0, \qquad 1-\beta+\hat{\beta}+\gamma+n=0.$$

Then a finite number of the roots and poles are equal, so cancel each other out in the product representation of the Wiener–Hopf factors. The remaining roots and poles still interlace in the required fashion, so all of the expressions previously calculated still hold. However, this means that the coefficients corresponding to the poles that have vanished are no longer present in the calculation for π.

One can prove part (ii) of the theorem in a similar way to part (i); we omit the details for the sake of brevity. The expression for the killing rate follows by noting that $q = -\psi(0)$. The proof of the claim about bounded and unbounded variation follows along the same lines as the proof of Proposition 1 of [7].

Remark 3.1. A fuller version of the proof, with diagrams to illustrate the interlacing of zeros and poles, is posted as [2].

Acknowledgements

The idea for this work was proposed by Alexey Kuznetsov and it was conducted whilst ELH was in receipt of an EPSRC-funded SAMBa CDT undergraduate summer academic internship scholarship for which she is grateful.

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https://doi.org/10.1017/apr.2016.47 Published online by Cambridge University Press