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Computable elements and functions in effectively enumerable topological spaces

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This paper is a part of the ongoing program of analysing the complexity of various problems in computable analysis in terms of the complexity of the associated index sets. In the framework of effectively enumerable topological spaces, we investigate the following question: given an effectively enumerable topological space whether there exists a computable numbering of all its computable elements. We present a natural sufficient condition on the family of basic neighbourhoods of computable elements that guarantees the existence of a principal computable numbering. We show that weakly-effective ω–continuous domains and the natural numbers with the discrete topology satisfy this condition. We prove weak and strong analogues of Rice's theorem for computable elements. Then, we construct principal computable numberings of partial majorant-computable real-valued functions and co-effectively closed sets and calculate the complexity of index sets for important problems such as root verification and function equality. For example, we show that, for partial majorant-computable real functions, the equality problem is Π^1_1-complete.

1. Introduction

This research is motivated by rapidly increasing interest in continuous data representations with suitable computational properties. It turns out that many problems in numerical computations, such as real number equality, set equality and etc., are undecidable (non-computable). Therefore, studying degrees of non-computability became recently an active research area in computable analysis (see among others Brattka and Gherardi 2009, 2011; Brodhead and Cenzer 2008; Cenzer and Remmel 1998; Selivanov and Schröder 2014).

In this paper we concentrate on studying properties of computable elements, majorant-computable functions and computable closed sets in the framework of effectively enumerable topological spaces. We use numbering theory and index sets as promising techniques merging classical recursion theory and computable analysis in topological spaces. There are several reasons for doing this. One of them is that the theory of index sets provides methods for encoding problems in an effective way by natural numbers, i.e., generate the corresponding index sets which can be used for the analysis of their complexity in the settings of arithmetical and analytical hierarchies. Another reason is that the theory of index sets has been already successfully employed in many areas in mathematics and computer science. In recursion theory, index sets have been applied to obtain both

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new results and new elegant proofs of classical theorems such as the Post's theorem and the density theorem (Ershov 1977; Rogers 1967; Shoenfield 1971). Many recent advancements in computable model theory are also closely related to index sets (Calvert *et al.* 2007, 2006; Downey *et al.* 2003; Downey and Montalban 2008). In computer science, the Rice–Shapiro theorem provides a simple description of effectively enumerable properties of program languages and the complexity of some decision problems in programming have been studied in Spreen (1995) using index sets. While some important results have been obtained in Brodhead and Cenzer (2008), Cenzer and Remmel (1998, 1999), Spreen (1984, 1995, 1998) and Weihrauch and Deil (1980) many methods and techniques of numbering theory have not been fully employed in computable analysis.

A class of effectively enumerable topological spaces has been proposed in Korovina and Kudinov (2008). This is a wide class containing weakly effective ω–continuous domains, computable metric spaces and positive predicate structures (Korovina and Kudinov 2014) that retains certain natural effectivity requirements which allow us to introduce reasonable concepts of (partial) computable functions. The main results of this paper are as follows.

We propose a natural condition on the family of basic neighbourhoods of computable elements that guarantees the existence of a principal computable numbering. We show that weakly effective ω–continuous domains and the natural numbers with the discrete topology satisfy this condition. For such spaces, we address the problem whether there exists a nontrivial subset of the computable elements with a recursive index set. We give an example which demonstrates the main difficulties in straightforward transfers of Rice's and the Rice–Shapiro theorems. Nevertheless, employing the Branching Lemma (Berger 1993), for effectively enumerable T_0-spaces satisfying the condition that the family of basic neighbourhoods of the computable elements are *wn*-families, we prove a generalisation of Rice's theorem for sets of computable elements. The question of whether an analogue of the Rice–Shapiro theorem holds for such spaces is still an open problem.

We transfer the notion of majorant-computability from structures (Korovina and Kudinov 2005) to effectively enumerable topological spaces and calculate the complexity of index sets of important classes. In Korovina and Kudinov (2014), we have shown strong relations between effectively enumerable topological spaces and positive predicate structures, i.e., predicate structures such that their positive $\exists$-theories are computably enumerable. Thus, for a positive predicate structure A, the topological space $\mathcal{A} = (A, \tau_\Sigma^A)$, where a base of topology τ_Σ^A consisting of the subsets of A defined by existential formulas, is effectively enumerable. In this paper, we construct principal computable numberings for effectively open sets and majorant-computable real-valued functions, and calculate the complexity of index sets for important problems such as root verification, set equality and function equality. For example, we show that, for majorant-computable real functions, the equality problem is Π_1^1-complete. For majorant-computable real-valued functions, we give an analogue of Rice's theorem which shows that for majorant-computable real-valued functions any non-trivial property is not limit decidable while for recursive functions there are limit decidable properties, for example, the halting problem.

In a natural way, we construct a principal computable numbering of the co-effectively closed subsets of an effectively enumerable topological space. We prove that, with respect to this numbering, the index sets for the computable closed sets of a Euclidean space

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and for the computably compact sets are Σ^0_3–complete. Similar results for the Cantor and Baire spaces can be found in Brodhead and Cenzer (2008).

2. Preliminaries

We refer the reader to Rogers (1967) and Soare (1987) for basic definitions and fundamental concepts of recursion theory. We recall that, in particular, φ_e denotes the partial computable (recursive) function with an index e in the Kleene numbering, φ^s_e denotes the computation of φ_e for s steps such that the function φ^s_e is uniformly primitive recursive. In this paper we also use notations $W_e = \mathrm{dom}(\varphi_e)$, $W^s_e = \mathrm{dom}(\varphi^s_e)$, and $\pi_e = \mathrm{im}(\varphi_e)$. A sequence $\{V_i\}_{i\in\omega}$ of computably enumerable (c.e.) sets is *computable* if $\{(n,i)|\, n \in V_i\}$ is computably enumerable. It is worth noting that this is equivalent to existence of a computable function $f : \omega \to \omega$ such that $V_i = W_{f(i)}$.

In classical computability (recursion) theory (Welch 1984; Yates 1965, 1969) and its applications in computable model theory (Downey *et al.* 2003; Downey and Montalban 2008), computable analysis (Ceitin 1971; Weihrauch 2000) a common situation is the following. We start with the class of computable real numbers $\mathbb{R}_c$ as an example. It is well-known that a real number r is computable if and only if the constraint

$$*(RD, LD; r) \rightleftharpoons \big(RD = \{i \mid q_i \in \mathbb{Q} \text{ and } q_i > r\} \text{ and } LD = \{i \mid q_i \in \mathbb{Q} \text{ and } q_i < r\} \text{ are c.e.}\big)$$

holds. Therefore, a criterion for r to belong to $\mathbb{R}_c$ is based on the existence of computably enumerable sets RD, LD satisfying the constraint $*(RD, LD; r)$. This could be generalised to another class $\mathcal{K}$, i.e., a criterion for an element k to belong to the class $\mathcal{K}$ is based on the existence of computably enumerable sets $V_1, \ldots, V_s$ satisfying a particular constraint $*(\bar{V}; k)$ such that, for any $\bar{V}$ and different k_1, k_2, the constraints $*(\bar{V}; k_1)$ and $*(\bar{V}; k_2)$ do not hold simultaneously. We call such $\mathcal{K}$ a *constrained class*. In the case when $\mathcal{K}$ is a constrained class we say that $\{k^m\}_{m\in\omega}$ is a *computable sequence* of elements of K if there exist computable sequences $\{V^m_1\}_{m\in\omega}, \ldots, \{V^m_s\}_{m\in\omega}$ of computably enumerable sets such that for every $m \in \omega$ the constraint $*(V^m_1, \ldots, V^m_s; k^m)$ holds. In other words, one can say that the property $(\exists \bar{V}) * (\bar{V}; k)$ holds uniformly on the elements of the sequence (Downey *et al.* 2003). For example, $\{r^m\}_{m\in\omega}$ is a computable sequence of computable real numbers if there exist computable sequences $\{LD^m\}_{m\in\omega}$ and $\{RD^m\}_{m\in\omega}$ of computably enumerable sets such that $*(RD^m, LD^m; r^m)$ holds for every $m \in \omega$, i.e., the property $(\exists RD)(\exists LD) * (RD, LD; r)$ holds uniformly on $r \in \mathbb{R}_c$.

For background on numbering theory we refer to Ershov (1999). Let A be a set. A *numbering* is a surjection function $\xi : \omega \to A$. Assume additionally that A is a constrained class. A numbering $\xi : \omega \to A$ is called *computable* if $\{\xi(i)\}_{i\in\omega}$ is the computable sequence of all elements of A. A numbering $\alpha : \omega \to A$ is called *principal computable* if it is computable and every computable numbering ξ is computably reducible to α, i.e., there exists a computable function $f : \omega \to \omega$ such that $\xi(i) = \alpha(f(i))$. It is worth noting that these definitions agree with the notions of computable and principal computable numberings of $A \subseteq \mathcal{P}(\omega)$ (Ershov 1999).

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Definition 2.1. Let $\mathcal{K}$ be a constrained class, $\alpha : \omega \to \mathcal{K}$ be a principal computable numbering and $L \subseteq \mathcal{K}$. The set $Ix(L) = \{n \mid \alpha(n) \in L\}$ is called an *index set* for the subclass L.

In order to calculate complexity of index sets we use arithmetical and analytical hierarchies which classifies certain sets of the natural numbers based on the complexity of formulas defining them (Rogers 1967; Soare 1987).

Below we recall the notion of multiple *m*-reducibility that is one of the central concepts in the classical theory of computability and introduce the notion of multiple *hm*-reducibility.

Definition 2.2. Let $A, B, C, D \subseteq \omega$. The pair (A, B) is *multiple hm-reducible* to (C, D), denoted by $(A, B) \leqslant_{hm} (C, D)$, if there exists a computable function $f : \omega \to \omega$ such that

$$x \in A \to f(x) \in C \text{ and } x \in B \to f(x) \in D.$$

The pair (A, B) is *multiple m-reducible* to (C, D), denoted by $(A, B) \leqslant_m (C, D)$, if there exists a computable function $f : \omega \to \omega$ such that

$$x \in A \leftrightarrow f(x) \in C \text{ and } x \in B \leftrightarrow f(x) \in D.$$

The set A is *m-reducible* to D, denoted by $A \leqslant_m D$, if there exists a computable function $f : \omega \to \omega$ such that

$$x \in A \leftrightarrow f(x) \in D.$$

The function f is called a reduction function.

Let $\overline{A} = \omega \setminus A$. It is easy to see that $A \leqslant_m B$ if and only if $(A, \overline{A}) \leqslant_{hm} (B, \overline{B})$.

Definition 2.3. Let Γ be a class of arithmetical (analytical) complexity. A set $A \subseteq \omega$ is called Γ*–hard* if $B \leqslant_m A$ for any $B \in \Gamma$. A set A is called Γ*–complete* if $A \in \Gamma$ and A is Γ–hard.

The following theorem summarises results on complexity of well-known problems in the classical computability theory (see, e.g., Rogers (1967); Soare (1987)) which we will use further on. Let $\mathcal{N} = \omega \setminus \{0\}$. Taking a standard agreement (Rogers 1967) that a downward closed nonempty $T \subseteq \mathcal{N}^{<\omega}$ is a tree, we denote the Baire space $\mathcal{N}^{\omega}$ of all paths in $\mathcal{N}^{<\omega}$ as $\mathcal{B}$.

Proposition 2.1.

- $\text{Inf} = \{e \mid W_e \text{ is infinite}\}$ is Π^0_2-complete.
- $\text{Fin} = \{e \mid W_e \text{ is finite}\}$ is Σ^0_2-complete.
- $\text{Cofin} = \{e \mid W_e \text{ is cofinite}\}$ is Σ^0_3-complete.
- $\text{R} = \{e \mid W_e \text{ is computable}\}$ is Σ^0_3-complete.
- There exists a computable sequence of computable trees $\{T_m\}_{m \in \omega}$ such that the set $I = \{m \mid T_m \text{ has an infinite path}\}$ is Σ^1_1-complete.

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3. Basic background

3.1. *Logic for the reals*

In this paper we work with the ordered structure of the real numbers in the finite predicate language without equality, $(\mathbb{R}, \sigma_P) = \left(\mathbb{R}, \mathcal{M}_E^*, \mathcal{M}_H^*, \mathcal{P}_E^+, \mathcal{P}_H^+, <\right)$, where $\mathcal{M}_E^*, \mathcal{M}_H^*$ are interpreted as the open epigraph and the open hypograph of multiplication respectively, and $\mathcal{P}_E^+, \mathcal{P}_H^+$ are interpreted as the open epigraph and the open hypograph of addition respectively, e.g., $\mathcal{P}_H^+(x,y,z) \rightleftharpoons x + y < z$ and $\mathcal{P}_E^+(x,y,z) \rightleftharpoons x + y > z$.

The *atomic formulas* are the atomic formulas in the predicate language σ_P. For simplicity we use standard notations, e.g., $x+y < z$, $x+y > z$, $x \cdot y < z$, $x \cdot y > z$ and $x > 0 \rightleftharpoons x+x > x$.

The set of $\exists$-*formulas* is the closure of the set of atomic formulas under $\wedge, \vee$ and existential quantifiers.

Definition 3.1.
A set $B \subseteq \mathbb{R}^n$ is $\exists$–*definable* if there exists a $\exists$-formula φ such that $\bar{x} \in B \leftrightarrow \mathbb{R} \models \varphi(\bar{x})$.

Remark 3.1. We recall that all predicates $Q_i \in \sigma_P$ occur only positively in $\exists$-formulas. Hence all $\exists$-definable subsets of $\mathbb{R}^n$ are open sets and equality on the reals is not representable by a $\exists$-formula.

Proposition 3.1. Let γ_0 be a Gödel numbering of $\exists$-formulas and Sent denote the set of all $\exists$-sentences. Then the following statements hold.

1. The set $\{n \mid \gamma_0(n) \in \text{Sent}\}$ is computable;
2. The set $\{n \mid \gamma_0(n) \in \text{Sent and } \mathbb{R} \models \gamma_0(n)\}$ is computably enumerable.

Proof. The first clam follows from properties of a Gödel numbering. The second claim follows from the density of $\mathbb{Q}$ in $\mathbb{R}$ and Remark 3.1. It also can be deduced from quantifier elimination in the strictly ordered field of the reals (Delzell 1982). □

Proposition 3.2 (Korovina and Kudinov 2009). There is an algorithm which by a computable sequence of $\exists$–formulas $\{\psi_i\}_{i\in\omega}$ produces a computable sequence of quantifier free formulas $\{\varphi_i\}_{i\in\omega}$ such that $\mathbb{R} \models (\forall y \in [a,b]) \bigvee_{i\in\omega} \psi_i(y,\bar{x}) \leftrightarrow \mathbb{R} \models \bigvee_{i\in\omega} \varphi_i(a,b,\bar{x})$.

Proposition 3.3 (Korovina and Kudinov 2009). There is an algorithm which by a computable sequence of $\exists$–formulas $\{\psi_i\}_{i\in\omega}$, computable real functions $\bar{f}$, computable real numbers $\bar{r}$ produces a computable sequence of quantifier free formulas $\{\varphi_i\}_{i\in\omega}$ such that $\mathbb{R} \models \bigvee_{i\in\omega} \psi_i(\bar{f}(\bar{y}), \bar{r}, \bar{x}) \leftrightarrow \mathbb{R} \models \bigvee_{i\in\omega} \varphi_i(\bar{y}, \bar{x})$.

3.2. *Effectively enumerable topological spaces*

Now we recall the notion of effectively enumerable topological space. Let (X, τ, α) be a topological space, where X is a non-empty set, $B \subseteq 2^X$ is a base of the topology τ and $\alpha : \omega \to B$ is a numbering.

Definition 3.2 (Korovina and Kudinov 2008). A topological space (X, τ, α) is *effectively enumerable* if the following conditions hold.

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1. There exists a computable function $g : \omega \times \omega \times \omega \to \omega$ such that
$$\alpha(i) \cap \alpha(j) = \bigcup_{n \in \omega} \alpha(g(i, j, n)).$$
2. The set $\{i | \alpha(i) \neq \varnothing\}$ is computably enumerable.

Definition 3.3 (Korovina and Kudinov 2008). An effectively enumerable topological space (X, τ, α) is *strongly effectively enumerable* if there exists a computable function $g : \omega \times \omega \to \omega$ such that
$$X \setminus cl(\alpha(i)) = \bigcup_{j \in \omega} \alpha(g(i, j)).$$

In Korovina and Kudinov (2008) it has been shown that the class of effectively enumerable topological spaces is a natural proper extension of the weakly effective ω–continuous domains (Gierz *et al.* 2003) and the class of strongly effectively enumerable topological spaces is a natural proper extension of the computable metric spaces (Weihrauch 2000). In particular, the real numbers and the continuous real functions belong to this class.

Let us note that when a base numbering is changed, computability properties may also change. Therefore we need to formalise reducibility and equivalence between base numberings.

Definition 3.4. Let (X, τ, α) and (X, τ, β) be topological spaces. We don't assume that $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$. We say that β is *computably reducible* to α, denoted $\beta \leqslant \alpha$, if there exists a computable sequence $\{A_n\}_{n \in \omega}$ of c.e. sets such that $\beta(n) = \bigcup_{i \in A_n} \alpha(i)$ for all $n \in \omega$. Numberings α and β are equivalent, denoted $\alpha \equiv \beta$, if $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$.

Remark 3.2. It is worth noting that if $\alpha \equiv \beta$ then (X, τ, α) is an effectively enumerable topological space if and only if (X, τ, β) is an effectively enumerable topological space. As we will see below, any change of base numberings from the same equivalence class preserves the classes of effectively open sets, computable elements, majorant-computable functions, etc.

We can relativise the notions from Definition 3.4 to an appropriate oracle or a Turing degree **a**. For the definitions of Turing degree and Turing jump we refer to Soare (1987).

Definition 3.5. Let **a** be a Turing degree. Let (X, τ, α) and (X, τ, β) be topological spaces. We do not assume that $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$. We say that α is **a**-*computably reducible* to β, denoted $\alpha \leqslant^{\mathbf{a}} \beta$, if there exists a **a**-computable sequence $\{A_n\}_{n \in \omega}$ of **a**-c.e. sets such that $\beta(n) = \bigcup_{i \in A_n} \alpha(i)$ for all $n \in \omega$. Numberings α and β are **a**-*equivalent* if $\alpha \leqslant^{\mathbf{a}} \beta$ and $\beta \leqslant^{\mathbf{a}} \alpha$.

Below we use a straightforward observation that if $\varnothing \in \mathrm{im}(\alpha)$ and $\varnothing \in \mathrm{im}(\beta)$ then $\beta \leqslant^{\mathbf{a}} \alpha$ iff there exists an **a**-computable function $h : \omega^2 \to \omega$ such that $\beta(n) = \bigcup_{i \in \omega} \alpha(h(n, i))$ for all $n \in \omega$.

Definition 3.6. Let (X, τ, α) be a topological space. We say that α is **a**-*decidable* if the set $\{(n, m) \mid \alpha(n) = \alpha(m)\}$ is **a**-computable.

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Remark 3.3. Similar notions can be defined even for partial numberings. If we consider partial numberings with **a**-computable domains then it is easy to see that any partial numbering is **a**-equivalent to some total numbering and any **a**-decidable partial numbering is **a**-equivalent to some total **a**-decidable numbering.

Proposition 3.4. For every Turing degree **a**, there exists an effectively enumerable T_1-space (X, τ, α) such that any base numbering β **a**-equivalent to α (and even any $\beta \leqslant^{\mathbf{a}} \alpha$) is not **a**-decidable.

Proof. Let $\{D_n\}_{n\in\omega}$ be the canonical numbering of finite subsets of ω, $A \subset \omega$ be infinite and coinfinite. Let $X = \omega \setminus A$ and the topology τ on X consist of the cofinite subsets of X and $\varnothing$. A natural numbering α of the topology base is defined as $\alpha(0) = \varnothing$ and $\alpha(n) = X \setminus D_{n-1}$ for $n > 0$. It is clear, that (X, τ, α) is an effectively enumerable T_1-space.

We start with some analysis of a possibility to find β, the final choice of A will be done later. It is clear, that if β is **a**-decidable, then there exists an **a**-equivalent one-to-one numbering (we just keep precisely one number for each element of the topological base, omitting others). So, suppose that $\beta \leqslant^{\mathbf{a}} \alpha$ is one-to-one numbering. Moreover, without loss of generality we assume that $\beta(n) \neq X$ for all $n \in \omega$ and $\beta(0) = \varnothing$. For $n > 0$, since $\beta(n) = \bigcup_{i\in\omega}(X \setminus D_{g(n,i)})$ for an **a**-computable function $g : \omega^2 \to \omega$, we have $X \setminus D_{g(n,0)} \subseteq \beta(n)$.

Our goal is to built a strongly **a**-computable table of finite sets for the set A (see Soare (1987) for non-relativised case), i.e., a strongly **a**-computable sequence $\{F_n\}_{n\in\omega}$ of finite sets such that $F_n \cap F_m = \varnothing$ and $F_n \cap (\omega \setminus A) \neq \varnothing$ for all $n \neq m$. We are going to built $\{F_n\}_{n\in\omega}$ and some monotonic **a**-computable $f : \omega \to \omega$ simultaneously.

At stage 0, we put $F_0 = D_{g(0,0)}$ (since it is nonempty), $f(0) = 0$.

Suppose that F_s and $f(s)$ have been constructed. At stage $s+1$, using some oracle from **a**, we compute numbers $N = \sum_{i:i\leqslant s} \mathrm{card}(F_i)$, $M = 2^N$ and finally put

$$f(s+1) = f(s) + M,\ F_{s+1} = \bigcup_{n:f(s)<n\leqslant f(s+1)} D_{g(n,0)} \setminus \bigcup_{i\leqslant s} F(i).$$

The construction is complete. It is clear, that $\{F_n\}_{n\in\omega}$ forms a disjoint strongly **a**-computable family of finite sets. If, for some s, $F_{s+1} \cap X = \varnothing$ then, for $f(s) < i \leqslant f(s)+2^N$, all 2^N sets of the kind $\beta(i)$ are located between X and $X \setminus \bigcup_{i\leqslant s} F(i)$. However, since each $\beta(i) \neq X$, there exist only $2^N - 1$ possibilities for a location. This is a contradiction by the Dirichlet principle. So, the existence of an **a**-decidable one-to-one numbering β implies the existence of strongly **a**-computable table of finite sets for the set A. Now we choose any **a**-computably enumerable **a**-hyper-simple set A. One way to construct A is the relativisation of any classical construction of c.e. hyper-simple set to an oracle from **a**. By definition, there is no such table. Therefore, the corresponding one-to-one numbering $\beta \leqslant^{\mathbf{a}} \alpha$ does not exist. The proposition is proven. □

Recently, various notions of computable topological space have been proposed (see among others Grubba and Weihrauch 2007, Spreen 1998 and Weihrauch 2000). The following results illustrate differences of our approach and the approaches mentioned above.

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Proposition 3.5.
There exists an effectively enumerable topological space $\mathcal{X}$ which is not a T_0-space.

Proof. Let $\mathcal{X}$ be an algebraically closed field with the Zariski topology, e.g., $X = \mathbb{C}$ with the base of closed sets which are solutions of rational polynomial equations of one variable. By quantifier elimination, it is an effectively enumerable topological space. However, there exists inseparable elements, e.g., $\sqrt{2}$ and $-\sqrt{2}$. Therefore, it is not T_0-space. □

Definition 3.7 (Weihrauch 2000). Let $\mathcal{X} = (X, \tau, \alpha)$ be a T_0-topological space and α be a partial base numbering. The space $\mathcal{X}$ is *computable* if $\{(u, v) | \, u, v \in \mathrm{dom}(\alpha) \text{ and } \alpha(u) = \alpha(v)\}$ is c.e.

Proposition 3.6. There exists an effectively enumerable T_1-space which is not a computable topological space in the sense of Weihrauch (2000).

Proof. Let $\mathbf{a} = \mathbf{0}'$. We fix the effectively enumerable T_1-space $\mathcal{X} = (X, \tau, \alpha)$ constructed in Proposition 3.4. Assume that there exists a partial numbering β such that (X, τ, β) is a computable topological space. Since any c.e. set is $\mathbf{a}$-computable, β is a-equivalent to some total $\mathbf{a}$-decidable numbering β_1. This contradicts Proposition 3.4. Therefore, $\mathcal{X}$ is required. □

Definition 3.8 (cf. Grubba and Weihrauch 2007). Let $\mathcal{X} = (X, \tau, \alpha)$ be a T_0-topological space and α be a partial base numbering such that $\mathrm{dom}(\alpha)$ is computable set and $\varnothing \notin \mathrm{im}(\alpha)$. The space $\mathcal{X}$ is *computable* if there exists a computable sequence $\{A_{ij}\}_{i,j \in \omega}$ of c.e. sets such that

$$\alpha(i) \cap \alpha(j) = \bigcup_{n \in A_{ij}} \alpha(n).$$

Remark 3.4. Sometimes it makes sense to consider $\varnothing \in \mathrm{im}(\alpha)$. In this case it is natural to assume that $\alpha^{-1}(\varnothing)$ is computable.

Proposition 3.7. There exists an effectively enumerable T_2-space which is not a computable topological space in the sense of Grubba and Weihrauch (2007).

Proof. We consider $(\omega, +, *, 0, 1)$. Assume that γ is a Gödel numbering of $\exists$-formulas. Let $\mathcal{X} = (\omega, \tau, \alpha)$, where τ is the discrete topology and $\alpha(n) = \{x | \omega \models \gamma(n)[x]\}$. It is easy to see that $\mathcal{X}$ is an effectively enumerable T_2-space. Since $\varnothing \in \mathrm{im}(\alpha)$ and, moreover, $\{n | \alpha(n) = \varnothing\}$ is not c.e., the space $\mathcal{X}$ is not computable. □

In this paper we use the following characterisation of total computable functions between two effectively enumerable topological spaces (Korovina and Kudinov 2008).

Proposition 3.8 (Korovina and Kudinov 2008). Let $\mathcal{X} = (X, \tau, \alpha)$ be an effectively enumerable topological space and $\mathcal{Y} = (Y, \lambda, \beta)$ be an effectively enumerable T_0-space. For a total function $f : X \to Y$ the following are equivalent.

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1. f is computable;
2. f is effectively continuous, i.e., there exists a computable function $h : \omega \times \omega \to \omega$ such that $f^{-1}(\beta(j)) = \bigcup_{i\in\omega} \alpha(h(i, j))$.

4. Computable elements of the lattice $\mathcal{O}_X$

In this section we consider effectively open sets of an effectively enumerable space as analogues of computable elements.

Definition 4.1 (Korovina and Kudinov 2008). Let (X, τ, α) be an effectively enumerable topological space. A set $A \subseteq X$ is *effectively open* if there exists a computably enumerable set V such that

$$A = \bigcup_{n\in V} \alpha(n).$$

If A and V satisfy the requirement above we tell that the constraint $*(V; A)$ holds.

Remark 4.1. It is clear that a set $A \subseteq X$ is effectively open if there exists a computable function $h : \omega \to \omega$ such that $A = \bigcup_{n\in\omega} \alpha(h(n))$.

The complement of an effectively open set is called *co-effectively closed*. Let $\mathcal{O}_X$ denote the set of all open subsets of X and $\mathcal{O}^e_X$ denote the set of all effectively open subsets of X. It is easy to see that $\mathcal{O}^e_X$ is a constrained class. The following propositions are natural corollaries of the definition.

Proposition 4.1 (Korovina and Kudinov 2008). A set $A \subseteq \mathbb{R}^n$ is effectively open if and only if there exists a computable sequence of quantifier free formulas $\{\varphi_i\}_{i\in\omega}$ such that $\bar{a} \in A \leftrightarrow \mathbb{R} \models \bigvee_{i\in\omega} \varphi_i(\bar{a})$.

Proposition 4.2. Let (X, τ, α) be an effectively enumerable topological space. A sequence $\{A_k\}_{k\in\omega}$ of effectively open subsets of X is computable if there exists a computable function $H : \omega \times \omega \to \omega$ such that $A_k = \bigcup_{n\in\omega} \alpha(H(k, n))$.

Theorem 4.1. For every effectively enumerable topological space X there exists a principal computable numbering α^e_X of $\mathcal{O}^e_X$.

Proof. Define $\alpha^e_X(e) = \bigcup\{\alpha(k) \mid k \in \pi_e\}$. From the classical computability theory (Soare 1987) it follows that $\{\pi_e\}_{e\in\omega}$ is a computable sequence as well as $\{B_e\}_{e\in\omega}$, where $B_e = \pi_e \cup \{m\}$ and $\alpha(m) = \varnothing$. Since every B_e is nonempty, $B_e = \pi_{f(e)}$ for some computable f and every $\varphi_{f(e)}$ is a computable function. So, $\alpha^e_X(e) = \bigcup_{i\in\omega} \varphi_{f(e)}(i)$. By Proposition 4.2, α^e_X is a computable numbering of $\mathcal{O}^e_X$. To show that α^e_X is principal we assume that $F : \omega \to \mathcal{O}^e_X$ is a computable numbering of $\mathcal{O}^e_X$. By Proposition 4.2, there exists a computable function H such that $F(k) = \bigcup_{n\in\omega} \alpha(H(k, n))$. For some computable function $\beta : \omega \to \omega$, $H(x, y) = \varphi_{\beta(x)}(y)$. Therefore, $F(k) = \alpha^e_X(\beta(k))$.

□

Theorem 4.2. Let $(\mathcal{O}^e_X, \alpha^e_X)$ be such that $\mathcal{O}^e_X$ does not coincide with the upper semi-lattice generated by the basic elements, i.e., there exists $A \in \mathcal{O}^e_X$ which cannot be represented as

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a finite union of basic open sets. Then, the problem of set equality and the problem of set inclusion are Π^0_2–hard.

Proof. Let $A \in \mathcal{O}^e_X$ do not belong to the upper semi-lattice generated by the basic elements. Let $A = \bigcup_{k\in\omega} \alpha^e_X(h(k))$. In order to prove Π^0_2–hardness of the problems we show that the Π^0_2-complete set $\mathrm{Inf} = \{n \mid W_n \text{ is infinite}\}$ is m–reducible to $Ix(A) = Ix(\{B \mid B \in \mathcal{O}^e_X \text{ and } B = A\})$. For that we define a reduction function g such that $\alpha^e_X(g(m)) = \bigcup_{k\in J_m} \alpha^e_X(h(k))$, where $J_m = \{i \mid (\exists k \in W_m)\, k \geqslant i\}$. It is easy to see that

$$A \subseteq \alpha^e_X(g(m)) \leftrightarrow A = \alpha^e_X(g(m)) \leftrightarrow m \in \mathrm{Inf}.$$

Therefore the problems of set equality and inclusion are Π^0_2–hard. □

Proposition 4.3. For the effectively open subsets of Euclidean space, the problem of set equality and the problem of set inclusion are Π^0_2–complete.

Proof. For proof simplicity we assume $n = 1$. Let us show that $Ix(=_e) = Ix(\{(A, B) \mid A \in \mathcal{O}^e_{\mathbb{R}}, B \in \mathcal{O}^e_{\mathbb{R}} \text{ and } B = A\}) \in \Pi^0_2$. We fix a computable numbering θ of all compact intervals with rational endpoints. Then

$$(n,k) \in Ix(=_e) \leftrightarrow (\forall m)\left(\theta(m) \subseteq \alpha^e_{\mathbb{R}}(n) \leftrightarrow \theta(m) \subseteq \alpha^e_{\mathbb{R}}(k)\right) \rightleftharpoons (\forall m)\Phi(m,n) \leftrightarrow \Phi(m,k).$$

By Proposition 3.2, $\Phi(i, j)$ is equivalent to an effective sequence of $\exists$-sentences. So, by Proposition 3.1, it defines a computably enumerable set. Therefore, $Ix(=_e) \in \Pi^0_2$. By similar arguments, $Ix(\subseteq_e) \in \Pi^0_2$.

□

5. Computable elements of effectively enumerable topological spaces

The notion of computable element is based on the following observation.

Proposition 5.1. For an effectively enumerable T_0–space (X, τ, α) and a point $x \in X$ the following assertions are equivalent.

1. The function $f : \{0\} \to X$, defined as $f(0) = x$, is computable.
2. The set $A_x = \{n \mid x \in \alpha(n)\}$ is computably enumerable.
3. For any effectively enumerable T_0–space $\mathcal{Y}$ the function $f_Y : Y \to X$, defined as $f_Y(y) = x$, is computable.

Proof. (1) → (2). First, it is worth noting that, for total functions, computability is equivalent to effective continuity (Korovina and Kudinov 2008). So, we assume that f is effectively continuous. Therefore there exists a computable function $h : \omega \times \omega \to \omega$ such that $f^{-1}(\alpha(n)) = \bigcup_{i\in\omega} \beta(h(n,i))$, where β is a numbering of the base of the space $\{0\}$. So, $x \in \alpha(n) \leftrightarrow f^{-1}(\alpha(n)) \neq \varnothing \leftrightarrow \exists i\, \beta(h(n,i)) \neq \varnothing$. Therefore A_x is computably enumerable.
(2) → (3). Let us show that f_Y is effectively continuous. It is easy to see that

$$f_Y^{-1}(\alpha(n)) = \begin{cases} Y, & x \in \alpha(n) \\ \varnothing, & \text{otherwise.} \end{cases}$$

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Let β be a numbering of a topology base of the space Y such that $\beta(m) = Y$ and $\beta(k) = \varnothing$ and

$$S_n = \begin{cases} \{k\}, & n \notin A_x \\ \{k, m\}, & \text{otherwise.} \end{cases}$$

It is easy to see that $f_Y^{-1}(\alpha(n)) = \bigcup_{s \in S_n} \beta(s)$ is computably enumerable. So, f_Y is effectively continuous, i.e., computable.
(3) $\rightarrow$ (1). It is trivial. □

For an effectively enumerable T_0–space, if one of the assertions (1)–(3) holds, x is called a *computable element (point)*. For an arbitrary effectively enumerable topological space, x is called a *computable element* if the assertions (2) holds and, for any $y \in X$, $A_x \neq A_y$. Let $*(V; x) \rightleftharpoons (x \in X$ and $V = \{n \mid x \in \alpha(n)\}$ is computably enumerable). Let X_c be the set of all computable elements of an effectively enumerable space X. It is clear that X_c is a constrained class.

It is easy to see that the definition above agrees with the notions of computable real number, computable element of a computable metric space (Brattka 2001), and computable element of a weakly–effective ω-continuous domain (Spreen 1984; Weihrauch and Deil 1980).

Remark 5.1. It is worth noting that there are effectively enumerable topological spaces without computable elements. As an example we can consider the real numbers without the computable points. It is an effectively enumerable T_0-space however there are no computable elements.

An element $x \in X$ is called *co-effective* if $X \setminus \{x\}$ is an effectively open set. Let X_{co} be the set of all co-effective elements. We investigate relations between the computable and co-effective elements. It turns out that, in general, even for some complete computable metric spaces $X_c \neq X_{co}$. For instance, let us consider the Baire space ω^ω. There exists a non-computable element $f \in \omega^\omega$ which is Π_1^0. Therefore, f is non-computable but co-effective.

Let $D = (D; B, \beta, \leqslant, \perp)$ be a ω-continuous domain where B is a basis, $\beta : \omega \rightarrow B$ is a numbering of the basis (see Gierz *et al.* 2003). We recall that (D, β) is a weakly effective if the relation $\beta(i) \ll \beta(j)$ is computably enumerable.

Proposition 5.2. Let (X, τ, α) be an effectively enumerable topological space. Define $\alpha^*(m) = \bigcup_{k \in D_m} \alpha(k)$ and $\mathcal{B}_X = \mathrm{im}(\alpha)$. Suppose $\left(\mathcal{O}_X, \mathcal{B}_X, \alpha^*, \subseteq, \varnothing\right)$ is a weakly effective ω-continuous domain, where the ordering is the set inclusion and $\perp = \varnothing$. Then, $X_{co} \subseteq X_c$.

Proof. Let $x \in X$ be co-effective. This means that $X \setminus \{x\} = \bigcup_{n \in \omega} \alpha(\chi(n))$ for a computable function $\chi : \omega \rightarrow \omega$. Let us fix some n such that $x \in \alpha(n)$ and $\alpha(n) \ll X$. Then,

$$x \in \alpha(k) \leftrightarrow (\exists s \in \omega)(\exists l_0 \in \omega) \ldots (\exists l_s \in \omega) \left(\alpha(n) \ll \bigcup_{i \leqslant s} \alpha(\chi(l_i)) \cup \alpha(k) \right).$$

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By assumption, $\alpha^*(n) \ll \alpha^*(j)$ is computably enumerable. Therefore, the set A_x is computably enumerable. By definition, x is computable. □

Proposition 5.3. Let (X, τ, α) be an strongly effectively enumerable T_2-space. Then, $X_c \subseteq X_{co}$.

Proof. Let $x \in X$ be computable. Then, from Definition 3.3 it follows that $X \setminus \{x\} = \bigcup_{m \in \omega : x \in \alpha(m)} \left(X \setminus \mathrm{cl}(\alpha(m))\right)$ is effectively open. So, x is co-effective. □

Corollary 5.1. For every Euclidean space, X, $X_c = X_{co}$.

Now we address the natural question whether for an effectively enumerable space there exists a computable numbering of the computable elements. First, we observe that while for the computable real numbers there is no computable numbering (Ceitin 1971; Martin-Löf 1970) as well as there is none for the computable points of a computable complete metric space without isolated points (Brattka 2001), for a weakly–effective ω–continuous domain there is a computable numbering of the computable elements (Weihrauch and Deil 1980). Below we point out a natural sufficient condition on the family of basic neighborhoods of computable elements that guarantees the existence of a principal computable numbering. We show that weakly-effective ω–continuous domains and ω with the discrete topology satisfy this condition.

Definition 5.1 (Ershov 1999). Let S be a family of computable enumerable subsets of ω. S is called a *wn-family* if there exists a partial computable function $\sigma : \omega \to \omega$ such that

i. if $\sigma(n) \downarrow$ then $W_{\sigma(n)} \in S$ and
ii. if $W_n \in S$ then $n \in \mathrm{dom}(\sigma)$ and $W_n = W_{\sigma(n)}$.

Theorem 5.1. Let (X, τ, α) be an effectively enumerable T_0-space and $S_X = \{A_a | a \in X_c\}$. If S_X is a *wn*-family, then there exists an algorithm to construct a principal computable (canonical) numbering $\bar{\alpha} : \omega \to X_c$.

Proof. From Ershov (1999), it follows that the set S_X has a standard principal computable numbering $\gamma : n \mapsto W_{\sigma(h_0(n))}$, where $h_0 : \omega \to \omega$ is a total computable function such that $\mathrm{im}(h_0) = \mathrm{dom}(\sigma)$. Let us define $\bar{\alpha}(n) = a \leftrightarrow A_a = W_{\gamma(n)}$. □

Proposition 5.4. Let $D = (D; B, \beta, \leqslant, \perp)$ be a weakly effective ω–continuous domain. Then, the set S_D of all sets $\{n \mid \beta(n) \ll a\}$, where $a \in D_c$, is a *wn*-family.

Proof. Let us define

$$a_n^s = \begin{cases} a, & \{a\} = W_n^{s+1} \setminus W_n^s \\ -1, & W_n^{s+1} = W_n^s. \end{cases}$$

We use the notation $\beta(n) \ll^t \beta(m)$ to say that the validity of $\beta(n) \ll \beta(m)$ is verified in at most t steps and the notation $\beta(n) \not\ll^t \beta(m)$ to say that the validity of $\beta(n) \ll \beta(m)$ is

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not verified in t steps. By double recursion, we construct $g : \omega^3 \to \omega \cup \{-1\}$ as follows:

$$g(n,s+1,t+1) = \begin{cases} -1, & W_n^{s+1} = \varnothing; \\ g(n,s,t+1), & g(n,s,t+1) \geqslant 0 \wedge (a_n^s \leqslant 0 \vee \beta(a_n^s) \not\gg^{t+1} \beta(g(n,s,t+1)) \wedge \\ & (\{k \in W_n^{t+1} | \beta(k) \gg^{t+1} \beta(a_n^s) \wedge \\ & \beta(k) \gg^{t+1} \beta(g(n,s,t+1))\} = \varnothing \vee \\ & \beta(g(n,s,t+1)) \gg^{t+1} \beta(a_n^s)); \\ a_n^s, & a_n^s \geqslant 0 \wedge (g(n,s,t+1) = -1 \vee \beta(a_n^s) \gg^{t+1} \beta(g(n,s,t+1))); \\ c_n^{s,t} & \text{otherwise,} \end{cases}$$

where $c_n^{s,t} = \min\{k \in W_n^{t+1} | \beta(k) \gg^{t+1} \beta(a_n^s) \wedge \beta(k) \gg \beta(g(n,s,t+1))\}$.

By construction,

— g is a total computable function,
— if $g(n,s,t) \geqslant 0$ then $g(n,s,t) \in W_n$ and
— $G(n,s) = \lim_{t\to\infty} g(n,s,t)$ is well-defined.

It is worth noting that for all $s \in \omega$, $G(n,s+1) = G(n,s)$ or $\beta(G(n,s+1)) \gg \beta(G(n,s))$. Finally, one can define a total computable function $\sigma : \omega \to \omega$ such that

$$W_{\sigma(n)} = \{m \in \omega | \exists s \exists t (\beta(m) \ll \beta(g(n,s,t)) \wedge g(n,s,t) \geqslant 0)\}.$$

Let us show that σ is a required function for $S = S_D \cup \{\varnothing\}$. By properties noted above, for all $n \in \omega$ either $W_n = \varnothing$, so $W_{\sigma(n)} = \varnothing$, or $W_n \neq \varnothing$ so $W_{\sigma(n)} = A_a$, where $a = \sup\{\beta(G(n,s)) | s \in \omega\}$. Therefore, $W_{\sigma(n)} \in S$. If $W_n = A_a$ for some $a \in D_c$ then for all $m \in W_n$ and for some $s \in \omega$, $m \in W_n^s$ holds. Therefore, $\beta(m) \ll \beta(G(n,s))$. It is clear that $W_{\sigma(n)} \subseteq W_n$. So, $W_{\sigma(n)} = A_a$ and S is a *wn*-family. From observations in Ershov (1999) it follows that S_D is a *wn*-family as well. □

Below we assume that X is an effectively enumerable T_0-space satisfying the requirements of Theorem 5.1 and $\bar{\alpha} : \omega \to X_c$ is defined as in Theorem 5.1. Now we address the problem whether there is a nontrivial subset of the computable elements with a computable index set. The following example is important for understanding.

Example 5.1. We consider (ω, τ, α), where τ is the discrete topology and α is defined as follows: $\alpha(0) = \varnothing$, $\alpha(k) = \{k-1\}$. It is clear that $S_\omega = \{\{n\} | n > 0\}$ is a *wn*-family. By Theorem 5.1 there is a principal computable numbering of the set of computable elements which coincides with ω. There exists $K \subset \omega$, e.g., $K = 2 \cdot \omega$, such that $Ix(K)$ is computable.

Proposition 5.5. Let $K \subseteq X_c$. If $Ix(K)$ is computable then K is clopen in X_c.

The claim follows from the following proposition.

Proposition 5.6. Let $K \subseteq X_c$. If $Ix(K)$ is computable then K is open in X_c.

In order to prove the proposition, for $K \subseteq X_c$, we define $\hat{K} = \{A_a \mid a \in K\}$, $W = \{n \mid n \in \mathrm{dom}(\sigma) \text{ and } W_{\sigma(n)} \in \hat{K}\}$, and $W^1 = \{k \mid k \in \mathrm{dom}(\sigma) \text{ and } W_{\sigma(k)} \in S_X \setminus \hat{K}\}$. Below we use the observation that $Ix(K) = \{n \mid \gamma(n) \in \hat{K}\} = \{n \mid h_0(n) \in W\}$, $\omega \setminus Ix(K) = \{n \mid \gamma(n) \in S_X \setminus \hat{K}\} = \{n \mid h_0(n) \in W^1\}$, $W = h_0(Ix(K))$, and $W^1 = h_0(\omega \setminus Ix(K))$ where h_0 and γ are defined in Theorem 5.1. It is clear that W and W^1 are computably enumerable. We give a proof using the following lemmas.

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Lemma 5.1 (Branching lemma). (Berger 1993). Let V and W be computable enumerable sets such that W contains all computably enumerable indices of V. Let $\lambda p.V^p$ be an enumeration of V and $r : \omega \to \omega$ be a total computable function. Then, there is $e \in W$ and $p \in \omega$ such that $W_e = V^p \cup W_{r(p)}$. Furthermore, such e and p are computed uniformly from an computably enumerable index of W and computable indices of the functions $\lambda p.V^p$ and r.

Lemma 5.2. If W is computably enumerable then $\hat{K}$ is monotone, i.e., if $A \subseteq B$, $A \in \hat{K}$ and $B \in S_X$ then $B \in \hat{K}$.

Proof. Let $A = W_{\sigma(m)}$ and $B = W_{\sigma(n)}$. Define $V = A$ and $r(p) \equiv \sigma(n)$. It is worth noting that if $W_a = W_{\sigma(m)} \in \hat{K}$ then $W_{\sigma(a)} = W_a \in \hat{K}$, so that $a \in W$. Therefore, we can use Lemma 5.1 to find $e \in W$ and $p \in \omega$ such that $W_e = V^p \cup W_{r(p)} = W^p_{\sigma(m)} \cup W_{\sigma(n)} = W_{\sigma(n)}$. Since $B \in S_X$ and $e \in W$, $W_{\sigma(e)} = W_e \in \hat{K}$. Therefore, $B \in \hat{K}$. □

Lemma 5.3. Assume $K \subseteq X_c$, $Ix(K)$ is computable and K is not open in X_c. Then, there exist $a \in K$, $m \in \omega$, a computable function $h : \omega \to \omega$ such that $W_{\sigma(m)} = A_a$ and, for all $p \in \omega$, $W^p_{\sigma(m)} \subseteq W_{\sigma(h(p))}$ and $h(p) \in W^1$.

Proof. Assume contrary: for all $a \in K$ ($A_a = W_{\sigma(m)}$) there exists $p \in \omega$ such that for all $b \in X_c$ such that $A_b = W_{\sigma(n)}$ and $W_{\sigma(n)} \supseteq W^p_{\sigma(m)}$ we have $n \in W$. So, $\bigcap_{k \in W_{\sigma(m)}} (\alpha(k) \cap X_c) \subseteq K$. This means that with every point K contains its open neighbourhood. So, K is open in X_c. □

Proof. (Proposition 5.6) Assume contrary: $K \subseteq X_c$ and $Ix(K)$ is computable and K is not open in X_c. Then there exist a, m and h satisfying Lemma 5.3. Let $V = W_{\sigma(m)}$ and $r(p) \equiv \sigma(h(p))$, so, for all $p \in \omega$, $W_{r(p)} \in S_X \setminus \hat{K}$. Using Lemma 5.1 we find $e \in W$ and $p \in \omega$ such that $W_e = W^p_{\sigma(m)} \cup W_{\sigma(h(p))} = W_{\sigma(h(p))} \in S_X \setminus \hat{K}$. Since, $W_e = W_{\sigma(e)}$, $e \notin W$. This contradicts to $e \in W$. □

The following proposition is a weak analogue of Rice's theorem.

Corollary 5.2. If X_c is connected then there is no nontrivial $K \subset X_c$ such that $Ix(K)$ is computable.

Proof. The claim follows from Theorem 5.5 and the definition of connected sets. □

It is worth noting that in a weakly effective ω-continuous domain with the bottom, e.g., $\mathcal{P}(\omega)$ with Scott topology, the set of computable elements is connected. In order to prove a strong analogue of Rice's theorem we use notations and results from Proposition 5.6 and the following observations.

Let $S = \{W_{\sigma(n)}\}_{n \in \omega}$ be a *wn*-family with the standard principal computable numbering $\gamma : n \mapsto W_{\sigma(h_0(n))}$, where $h_0 : \omega \to \omega$ is a total computable function such that $\mathrm{im}(h_0) = \mathrm{dom}(\sigma)$. For $\hat{K} \subseteq S$ we define $Ix_S(\hat{K}) = \{m | \gamma(m) \in \hat{K}\}$.

Definition 5.2 (Rogers 1967). We say that $\hat{K}$ is *effectively discrete* in S if there exists a strongly computable family $\{F_n\}_{n \in \omega}$ of finite subsets of ω such that

$$(\forall A \in S)\left(A \in \hat{K} \leftrightarrow (\exists n \in \omega) A \supseteq F_n\right).$$

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In other words, $\hat{K}$ is effectively discrete in S if it is effectively open in S which is considered as a subspace of $\mathcal{P}(\omega)$ with the Scott topology.

Remark 5.2. It is easy to see that if $\hat{K}$ is effectively discrete in S then $Ix_S(\hat{K})$ is computably enumerable.

Proposition 5.7. If $Ix_S(\hat{K})$ is computable then $\hat{K}$ and $S \setminus \hat{K}$ are effectively discrete in S.

Proof. We use the notations $W = \{n \mid n \in \mathrm{dom}(\sigma) \text{ and } W_{\sigma(n)} \in \hat{K}\}$, and $W^1 = \{k \mid k \in \mathrm{dom}(\sigma) \text{ and } W_{\sigma(k)} \in S_X \setminus \hat{K}\}$ as in Proposition 5.6. Let $W = \mathrm{im}(g)$ for a total computable $g : \omega \to \omega$. First, using uniformisation we construct a partial computable function $h : \omega \times \omega \to \omega$ such that

$$(\forall p \in \omega)(\forall m \in W)\left((\exists k \in W^1)\, W_{\sigma(k)} \supseteq W^p_{\sigma(m)} \leftrightarrow \left(W_{\sigma(h(m,p))} \supseteq W^p_{\sigma(m)} \wedge h(m,p) \in W^1\right)\right).$$

In particular, $h(m,p)\uparrow$ if $\sigma(m)\uparrow$. Given $m \in W$ and $V = W_{\sigma(m)}$, using completeness of the numbering $\{W_n\}_{n\in\omega}$, we construct a total computable function $r : \omega \to \omega$ such that, for all $p \in \omega$,

$$W_{r(p)} = \begin{cases} W_{\sigma(h(m,p))}, & \text{if } h(m,p)\downarrow \\ \varnothing, & \text{if } h(m,p)\uparrow. \end{cases}$$

Using Lemma 5.1, from an computably enumerable index of W and computable indices of the functions $\lambda p.V^p$ and r, we uniformly compute e and p such that $W_e = W^p_{\sigma(m)} \cup W_{r(p)}$. It is worth noting that if $h(m,p)\downarrow$ then $W_e = W_{\sigma(h(m,p))} \in S \setminus \hat{K}$, so $W_{\sigma(e)} = W_e$. Since $e \in W$, $W_{\sigma(e)} \in \hat{K}$, we have a contradiction. It follows that $h(m,p)\uparrow$. Using uniform computability of e and p we construct a partial computable function $p : \omega \to \omega$ such that, for all $m \in \omega$, if $\sigma(m)\downarrow$ then $h(m,p(m))\uparrow$ and $\mathrm{im}(p) = \mathrm{im}(\sigma)$. Then, for $m \in W$, we obtain that $W_{\sigma(k)} \supseteq W^{p(m)}_{\sigma(m)} \to k \in W$ by the choice of p. So, the family $\{F_n\}_{n\in\omega}$, defined as $F_n = W^{p(g(n))}_{\sigma(g(n))}$, testifies that $\hat{K}$ is effectively discrete in S. □

Theorem 5.2. Let $K \subseteq X_c$. The set $Ix(K)$ is computable if and only if K and $X_c \setminus K$ are effectively open in X_c.

Proof. $\leftarrow$). If K is effectively open in X_c then, by definition, $K = \bigcup_{i\in I}(\alpha(i) \cap X_c)$, where I is computably enumerable. Since $\bar{\alpha}$ is computable, we have

$$n \in Ix(K) \leftrightarrow (\exists i \in I)\, i \in A_{\bar{\alpha}(n)}.$$

So, $Ix(K)$ is computably enumerable. Replacing K by $X_c \setminus K$ in the previous reasoning we obtain that $\omega \setminus Ix(K)$ is computably enumerable.
$\rightarrow$). We apply the Proposition 5.7 to $\hat{K} = \{A_a | a \in K\}$ and $S = S_X$. We immediately obtain that, for any $m \in W$,

$$\bigcap_{i \in W^{p(m)}_{\sigma(m)}} (\alpha(i) \cap X_c) \subseteq K.$$

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Therefore,

$$K = \bigcup_{m \in W} \left(\bigcap_{i \in W^{p(m)}_{\sigma(m)}} (\alpha(i) \cap X_c) \right).$$

This means that K is effectively open in X_c. Replacing K by $X_c \setminus K$ in the previous reasoning we obtain that $X_c \setminus K$ is effectively open in X_c. The proof is complete. □

6. Computable functions in effectively enumerable topological spaces

6.1. *Strongly computable functions*

In this section we work with an effectively enumerable topological space $\mathcal{X} = (X, \tau, \alpha)$ and $\mathbb{R} = (\mathbb{R}, \lambda, \alpha^{\mathbb{R}})$, where $\alpha^{\mathbb{R}}$ is a computable numbering of the open intervals with rational endpoints, $\varnothing$ and $\mathbb{R}$. Now we give the notion of strongly computable function on an effectively enumerable topological space.

Definition 6.1. A partial function $f : X \to \mathbb{R}$ is called *strongly computable* if there exist two effectively open sets $U, V \subseteq X \times \mathbb{R}$ such that

1. for all $x \in X$, $U(x)$ is closed downward and $V(x)$ is closed upward;
2. if $x \in \mathrm{dom} f$ then $f(x) = y \leftrightarrow \{y\} = \mathbb{R} \setminus (U(x) \cup V(x))$;
3. if $x \notin \mathrm{dom} f$ then the $\mathbb{R} \setminus (U(x) \cup V(x))$ is not a singleton.

If we fix a canonical numbering $\tilde{\alpha}$ of the base of the topology on $X \times \mathbb{R}$ then $U = \bigcup_{i \in V_1} \tilde{\alpha}(i)$ and $V = \bigcup_{i \in V_2} \tilde{\alpha}(i)$ for some computably enumerable sets V_1 and V_2. We say that the pair (U, V) *defines* a partial strongly computable function $f : X \to \mathbb{R}$ if U and V satisfy the conditions of Definition 6.1 and we say that the constraint $*(V_1, V_2; f)$ holds. For an effectively enumerable space X we denote the set of partial strongly computable functions $f : X \to \mathbb{R}$ as $\mathcal{SC}_X$. It is clear that $\mathcal{SC}_X$ is a constrained class.

By analogy to Definition 4.2 one can define the notion of computable sequence of pairs of effectively open subsets of an effectively enumerable space and the notion of computable numbering of all such pairs.

Lemma 6.1. Assume that $\gamma : \omega \to \mathcal{SC}_X$ is a numbering and a pair (U_n, V_n) of effectively open sets defines the function $\gamma(n)$. Then, γ is computable if $\{(U_n, V_n)\}_{n \in \omega}$ is a computable sequence of pairs of effectively open sets.

Proof. The claim follows from the definitions. □

Theorem 6.1. For every effectively enumerable space X there exists a principal computable numbering of $\mathcal{SC}_X$.

Proof. Let α_{ep} be the principal computable numbering of the set of pairs of effectively open subsets of $X \times \mathbb{R}$ induced by the numbering $\alpha^e_{X \times \mathbb{R}}$ from Theorem 4.1 and $\alpha_{ep}(n) = (\widetilde{U}_n, \widetilde{V}_n)$. We can effectively construct a new pair which satisfies the conditions of

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Definition 6.1. Indeed, by definition,

$$\widetilde{U}_n = \bigcup_{i\in\omega} \alpha(h_1(n,i)) \times \alpha^{\mathbb{R}}(h_2(n,i))$$

for some computable functions h_1 and h_2. Then put

$$U_n = \bigcup_{i\in\omega} \alpha(h_1(n,i)) \times \alpha^{\mathbb{R}}(F(n,i)),$$

where F is a computable function such that $r \in \alpha^{\mathbb{R}}(F(n,i))$ iff there exists $r_1 \in \alpha^{\mathbb{R}}(h_1(n,i))$ such that $r_1 > r$.

By analogy we construct V_n. By definition, the pair (U_n, V_n) defines a partial strongly computable function $f : X \to \mathbb{R}$ such that $f(x) = y \leftrightarrow U_n(x) \cup V_n(x) = \mathbb{R} \setminus \{y\}$. Put $\gamma(n) = f$. By construction, the sequence $\{(U_n, V_n)\}_{n\in\omega}$ is computable and every partial strongly computable function $f : X \to \mathbb{R}$ gets an index. Therefore, γ is a computable numbering of $\mathcal{SC}_X$. To show that γ is principal we assume that $\nu : \omega \to \mathcal{SC}_X$ is a computable numbering and a pair (U_m, V_m) defines $\nu(m)$. Since α_{ep} is principal, there exists a computable function $h : \omega \to \omega$ such that $\alpha_{ep}(h(m)) = (U_m, V_m)$. Therefore, $\nu(m) = \gamma(h(m))$. So, γ is a required numbering. □

6.2. *Majorant–computable functions*

In this subsection we introduce the notion of majorant-computable function on an effectively enumerable topological space which agrees with majorant-computability on the predicate structures (Korovina and Kudinov 2005). Let $U(x) < V(x)$ denote that, for all $z_1 \in \mathbb{R}$ and $z_2 \in \mathbb{R}$, if $z_1 \in U(x)$ and $z_2 \in V(x)$ then $z_1 < z_2$.

Definition 6.2. Let $\mathcal{X}$ be an effectively enumerable topological space. A partial function $f : X \to \mathbb{R}$ is called *majorant–computable* if there exist two effectively open sets $U, V \subseteq X \times \mathbb{R}$ satisfying the following requirements:

1. $(\forall x \in X)\,(U(x) < V(x))$;
2. for all $x \in X$, $U(x)$ is closed downward and $V(x)$ is closed upward;
3. $f(x) = y \leftrightarrow \{y\} = \mathbb{R} \setminus (U(x) \cup V(x))$.

If we fix a canonical numbering $\tilde{\alpha}$ of the base of the topology on $X \times \mathbb{R}$ then $U = \bigcup_{i\in V_1} \tilde{\alpha}(i)$ and $V = \bigcup_{i\in V_1} \tilde{\alpha}(i)$ for some computable enumerable sets V_1 and V_2. We say that the pair (U, V) *defines* a partial majorant-computable function $f : X \to \mathbb{R}$ if U and V satisfy the conditions of Definition 6.2 and we say that the constraint $*(V_1, V_2; f)$ holds. For an effectively enumerable space X we denote the set of partial majorant-computable functions $f : X \to \mathbb{R}$ as $\mathcal{MC}_X$. It is clear that $\mathcal{MC}_X$ is a constrained class.

From the first glance it looks that the notion of majorant-computability is stronger than the notion of strong computability. However, the following theorem shows that $\mathcal{SC}_X = \mathcal{MC}_X$.

Theorem 6.2. A partial function $f : X \to \mathbb{R}$ is majorant-computable if and only if it is strongly computable.

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Proof. $\rightarrow$) The claim is a direct consequence.
$\leftarrow$) Assume $f : X \to \mathbb{R}$ is strongly computable. By definition, there exist two effectively open sets $U, V \subseteq X \times \mathbb{R}$ such that

1. for all $x \in X$, $U(x)$ is closed downward and $V(x)$ is closed upward;
2. if $x \in \mathrm{dom} f$ then $U(x) = \{y | y < f(x)\}$ and $V(x) = \{y | y > f(x)\}$;
3. if $x \notin \mathrm{dom} f$ then $\mathbb{R} \setminus (U(x) \cup V(x))$ is not a singleton.

We construct new effectively open sets U' and V' that satisfy the properties (1)–(3) and the additional requirement: $U'(x) < V'(x)$ for all $x \in X$.

First, one can effectively defines sequences $\{V_s\}_{s\in\omega}$ and $\{U_s\}_{s\in\omega}$ with the following properties:

i. $U_s(x) = (-\infty, r_s)$ and $V_s(x) = (q_s, +\infty)$, where $r_s, q_s \in \mathbb{Q} \cup \{+\infty, -\infty\}$, $r_s < q_s$;
ii. $U_0(x) \subset \cdots \subset U_s(x) \subset \cdots \subset U(x)$ and $\bigcup_{n\in\omega} U_n(x) = U(x)$;
iii. $V_0(x) \subset \cdots \subset V_s(x) \subset \cdots \subset U(x)$ and $\bigcup_{n\in\omega} V_n(x) = V(x)$;
iv. $U_s(x) < V_s(x)$ if $cl(U_s) \cap cl(V_s) = \varnothing$.

Put

$$U^*(x) = \bigcup_{\{s | cl(U_s(x)) \cap cl(V_s(x)) = \varnothing\}} U_s(x),$$

$$V^*(x) = \bigcup_{\{s | cl(U_s(x)) \cap cl(V_s(x)) = \varnothing\}} V_s(x).$$

Then, we take the downward closure $U'(x) = \{y | (\exists z \in U^*(x))\, y < z\}$ and the upward closure $V'(x) = \{y | (\exists z \in V^*(x))\, y > z\}$. As in the proof of Theorem 6.1, one can show that U' and V' are effectively open. By construction, for x such that $U(x) < V(x)$, $U'(x)$ and $V'(x)$ coincide with $U(x)$ and $V(x)$ respectively. For x such that $U(x) \not< V(x)$, $U'(x) \subseteq U_{s-1}(x)$ and $V'(x) \subseteq V_{s-1}(x)$, where s is minimal with the property $cl(U_s(x)) \cap cl(V_s(x)) \neq \varnothing$. By construction, $cl(U_{s-1}(x)) = (-\infty, r]$ and $cl(V_{s-1}(x)) = [q, +\infty)$ for some $r < q$. Then, $[r, q] \subseteq \mathbb{R} \setminus (U'(x) \cup V'(x))$. Therefore, U' and V' satisfy all required properties. So, f is majorant–computable. □

Corollary 6.1. For any effectively enumerable topological space X, there exists a principal computable numbering $\gamma : \omega \to \mathcal{MC}_X$.

Proof. From Theorem 6.2, it follows that $\mathcal{SC}_X = \mathcal{MC}_X$ and, moreover, every sequence $\{f_m\}_{m\in\omega}$ is a computable sequence of strongly computable functions if and only if it is a computable sequence of majorant-computable functions. Therefore, the numbering γ constructed in Theorem 6.1 is a required one. □

Let $\mathrm{Tot}_X \rightleftharpoons \{f \mid f \in \mathcal{MC}_X \text{ and total}\}$ and $\perp_X$ denote the function which is undefined everywhere.

Theorem 6.3. Let K be a non-empty class of majorant-computable functions such that $\perp \notin K$. Then $Ix(K)$ is Π^0_2-hard.

The proof of the theorem is based on the following lemmas, for short $\perp_X \leftrightharpoons \perp$.

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Lemma 6.2. Suppose $A \in \Pi^0_2$, $f \in \mathcal{MC}_X$ and $f \neq \perp$. Then the following holds:

$$(A, \overline{A}) \leqslant_{hm} (Ix(\{f\}), Ix(\{\perp\})).$$

Proof. It is sufficient to show that the pair of the Π^0_2–complete set $\{n | W_n \text{ is infinite}\}$ and its complement is multiple *hm*-reducible to the pair $(Ix(\{f\}), Ix(\{\perp\}))$. For that we construct a reduction function $g : \omega \to \omega$. By definition, there exist effectively open sets U, V such that $f(x) = y \leftrightarrow U(x) \cup V(x) = \mathbb{R} \setminus \{y\}$.

Let

$$U = \bigcup_{i \in \omega} \alpha(h_1(i)) \times \alpha^{\mathbb{R}}(h_2(i))$$

for some computable functions h_1 and h_2. Define

$$(x, y) \in \widetilde{U}_n \leftrightarrow (\exists i \in \omega) \left(x \in \alpha(h_1(i)) \wedge (\exists m \in W_n)(\exists z \in \alpha^{\mathbb{R}}(h_2(i)))\, y < z - \frac{1}{m+1} \right) \rightleftharpoons$$
$$(\exists i \in \omega) \left(x \in \alpha(h_1(i)) \wedge \Phi_{n,i}(y) \right).$$

It is easy to see that for every n, $i \in \omega$ $\Phi_{n,i}(y) \rightleftharpoons \bigvee_{m \in W_n} (\exists z \in \alpha^{\mathbb{R}}(h_2(i)))\, y < z - \frac{1}{m+1}$ is an effective disjunction of existential formulas over the reals. By Proposition 4.1, $\mathbb{R} \models \Phi_{n,i}(y) \leftrightarrow y \in \alpha^e_{\mathbb{R}}(F(n,i))$, where $\alpha^e_{\mathbb{R}}$ is a principal computable numbering of $\mathcal{O}^e_{\mathbb{R}}$ and $F : \omega \times \omega \to \omega$ is a computable function. So,

$$\widetilde{U}_n = \bigcup_{i \in \omega} \alpha(h_1(i)) \times \alpha^{\mathbb{R}}_e(F(n,i)).$$

By analogy we construct $\widetilde{V}_n$. It is easy to see that $(\widetilde{U}_n, \widetilde{V}_n)_{n \in \omega}$ is a computable sequence of effectively open sets that satisfy the conditions of Definition 6.1. Using Theorem 6.2 we can effectively construct a new computable sequence $(U'_n, V'_n)_{n \in \omega}$ of effectively open sets such that (U'_n, V'_n) defines a majorant-computable function $\gamma(g(n))$ for a computable function $g : \omega \to \omega$.

If W_n is infinite then $\gamma(g(n))(x) = f(x)$, i.e., $g(n) \in Ix(\{f\})$. If W_n is finite then $\gamma(g(n)) = \perp$, i.e., $g(n) \in Ix(\{\perp\})$. So, g is a required reduction function. □

Lemma 6.3. Suppose $K \subseteq \mathcal{MC}_X$, $f \in K$ and $f \neq \perp$. Then the following holds:

$$(Ix(\{f\}), Ix(\{\perp\})) \leqslant_{hm} (K, \overline{K}).$$

Proof. obviously, $g \rightleftharpoons \mathrm{id}$ is a reduction function. □

Proof. (Theorem) It follows from Lemmas 6.2 and 6.3. □

Below we prove an analog of Rice's theorem for majorant-computable real functions.

Theorem 6.4 (Generalised Rice theorem). The index set for $K \subseteq \mathcal{MC}_X$ is Δ^0_2 if and only if K is empty or coincides with $\mathcal{MC}_X$.

Proof. Assume K is non-empty and $\perp \notin K$. From Theorem 6.3 it follows that $Ix(K)$ is Π^0_2–hard. Therefore, $Ix(K) \notin \Sigma^0_2$. Assume $K \neq \mathcal{MC}_X$ and $\perp \in K$. Then, $\perp \notin \overline{K}$. From Theorem 6.3 it follows that $Ix(\overline{K})$ is Π^0_2–hard. Therefore, $Ix(\overline{K}) \notin \Sigma^0_2$. □

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Now we take a close look at index sets in the case when $X = \mathbb{R}$.

Theorem 6.5. The index set $Ix(\mathrm{Tot}_{\mathbb{R}})$ is Π^0_2–complete.

Proof. Let us prove that $Ix(\mathrm{Tot}_{\mathbb{R}}) \in \Pi^0_2$. We fix a canonical computable numbering θ of all compact intervals with rational endpoints. Let $f : \mathbb{R} \to \mathbb{R}$ be a majorant-computable function definable by (U, V). Then to say that $\theta(i) \subseteq \mathrm{dom}\, f$ we write that

$$\bigwedge_{n\in\omega} (\forall x \in \theta(i))(\exists y_1)(\exists y_2)((|y_1 - y_2| < \frac{1}{n} \wedge (x, y_1) \in V \wedge (x, y_2) \in U).$$

This has the form Π^0_2 by Proposition 3.1 and Proposition 3.2. It is easy to note that

$$n \in Ix(\mathrm{Tot}_{\mathbb{R}}) \leftrightarrow (\forall m)\, \theta(m) \subseteq \mathrm{dom}\, f.$$

By the previous observation, this has the form $(\forall)\Pi^0_2 \wedge \Pi^0_1$. Since $\Pi^0_2 \supseteq \Pi^0_1$ and the intersection of Π^0_2–relations is Π^0_2, the index set $Ix(\mathrm{Tot}_{\mathbb{R}})$ is Π^0_2. From Theorem 6.3 it follows that $Ix(\mathrm{Tot}_{\mathbb{R}})$ is Π^0_2-complete.

□

Proposition 6.1. For any non-empty co-effectively closed set $A \subset \mathbb{R}$, the index set

$$Ix(\mathcal{Z}ero_A) = Ix(\{f | f \in \mathrm{Tot}_{\mathbb{R}} \text{ and } f^{-1}(0) = A\})$$

is Π^0_2–complete.

Proof. To say that $n \in Ix(\mathcal{Z}ero_A)$ we write that $n \in Ix(\mathrm{Tot}_{\mathbb{R}})$ and

$$(\forall m)(\forall x \in \theta(m)) \left(x \notin A \leftrightarrow (x, 0) \in V_{\beta(n)} \vee (x, 0) \in U_{\beta(n)}\right),$$

where $V_{\beta(n)}$ and $U_{\beta(n)}$ are the corresponding epigraph and hypograph of $\gamma(n)$. This has the form $\Pi^0_2 \wedge (\forall)\Sigma^0_1$ by Proposition 3.1 and Proposition 3.2. Therefore, $Ix(\mathcal{Z}ero_A) \in \Pi^0_2$. From Theorem 6.3 it follows that $Ix(\mathcal{Z}ero_A)$ is Π^0_2-complete.

□

Theorem 6.6. For majorant-computable functions $f : \mathbb{R} \to \mathbb{R}$, $Ix(\{f | f \neq \perp_{\mathbb{R}}\})$ is Σ^1_1-complete.

Our proof is based on the following lemmas.

Lemma 6.4. There exists an effective procedure which given any computable sequence $\{A_n\}_{n\in\omega}$ of effectively open subsets of $\mathcal{B}$ produces a computable sequence $\{\tilde{E}_n\}_{n\in\omega}$ of effectively open subsets of $(0, 1)$ such that $\bigcap_{n\in\omega} A_n = \varnothing \leftrightarrow \bigcap_{n\in\omega} \tilde{E}_n = \varnothing$.

Proof. The well-known algorithm producing continuous fractions can be represented as a function $F : \mathcal{B} \to (0, 1)$ with the following properties:

1. $F(n_0, n_1, \dots) = \cfrac{1}{n_0 + \cfrac{1}{n_1 + \ddots}}$;
2. $F : \mathcal{B} \to (0, 1) \setminus \mathbb{Q}$ is a homeomorphism;
3. F and F^{-1} are computable.

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Assume $\{A_n\}_{n\in\omega}$ is a computable sequence of effectively open subsets of $\mathcal{B}$. The set $F(A_n)$ is effectively open in $(0,1)\setminus\mathbb{Q}$. By computability of F^{-1}, we can effectively find the index of a set E_n which is effectively open in $(0,1)$ and $F(A_n) = E_n \setminus \mathbb{Q}$. Let $\gamma : \omega \to \mathbb{Q}\cap(0,1)$ be a computable numbering. We define $\widetilde{E}_n \rightleftharpoons E_n \setminus \{\gamma(i) | i \leqslant n\}$. It is easy to see that $\{\widetilde{E}_n\}_{n\in\omega}$ is a computable sequence of effectively open subsets of $(0,1)$ such that $\widetilde{E}_n \setminus \mathbb{Q} = F(A_n)$ and $\bigcap_{n\in\omega} A_n = \varnothing \leftrightarrow \bigcap_{n\in\omega} \widetilde{E}_n = \varnothing$. □

Lemma 6.5. Let $\{\widetilde{E}_n\}_{n\in\omega}$ be a computable sequence of effectively open subsets of an effectively enumerable topological space $\mathcal{X} = (X, \tau, \alpha)$. There is an algorithm producing a majorant computable function $f : X \to \mathbb{R}$ such that $\mathrm{dom}(f) = \bigcap_{n\in\omega} \widetilde{E}_n$. Moreover, sets of the kind $\bigcap_{n\in\omega} \widetilde{E}_n$ are precisely domains of majorant-computable functions.

Proof. Let $\widetilde{E}_i = \bigcup_j \alpha(h(i,j)) \times \alpha^{\mathbb{R}}(g(i,j))$, then

$$U = \bigcup_{n\in\omega} \left(\bigcap_{i\leqslant n} \widetilde{E}_i \right) \times (-\infty, -\frac{1}{n}),$$
$$V = \bigcup_{n\in\omega} \left(\bigcap_{i\leqslant n} \widetilde{E}_i \right) \times (\frac{1}{n}, +\infty).$$

It is clear that U and V define a majorant-computable function such that if $x \in \bigcap_{n\in\omega} \widetilde{E}_n$ then $f(x) = 0$, otherwise $f(x)\uparrow$.

Now assume that $f : X \to \mathbb{R}$ is a majorant-computable function and U, V are effectively open sets that define f. Then,

$$x \in \mathrm{dom}(f) \leftrightarrow (\exists n \in \omega)\, \exists y \exists z \left(U(x,y) \wedge V(x,z) \wedge |y - z| < \frac{1}{2} \right).$$

It is clear that $\mathrm{dom}(f)$ is the effective intersection of effectively open sets. □

Below we use the result from Proposition 2.1 which states that there exists a computable sequence of computable trees $\{T_m\}_{m\in\omega}$ such that the set $I = \{m | T_m \text{ has an infinite path}\}$ is Σ^1_1-complete.

Lemma 6.6. There exists an effective procedure which generates a computable sequence $\{A^m_n\}_{m,n\in\omega}$ of effectively open subsets of $\mathcal{B}$ such that $\bigcap_{n\in\omega} A^m_n \neq \varnothing \leftrightarrow m \in I$.

Proof. Let $T_m \subseteq \mathcal{N}^{<\omega}$ such that $m \in I$. Define

$$\{A^m_n\}_{m,n\in\omega} = \{f \in \mathcal{B} | < f(0), \ldots, f(n) > \in T_m\}.$$

It is easy to see that $f \in \bigcap_{n\in\omega} A^m_n$ iff f is an infinite path in T_m iff $m \in I$. □

Proof. (Theorem 6.6) First let us note that the standard Tarski–Kuratovski algorithm (Rogers 1967) shows that $Ix(\{f | f \neq \bot_{\mathbb{R}}\}) \in \Sigma^1_1$. Our goal is to show that $I \leqslant_m Ix(\{f | f \neq \bot_{\mathbb{R}}\})$. For that, we construct a computable sequence $\mathcal{F} = \{f_m\}_{m\in\omega}$ of majorant-computable functions such that $m \in I \leftrightarrow f_m \neq \bot_{\mathbb{R}}$. Using Lemma 6.6 and Lemma 6.4 we construct a computable sequence $\{A^m_n\}_{m,n\in\omega}$ of effectively open subsets of $\mathcal{B}$ and a computable sequence $\{\widetilde{E}^m_n\}_{m,n\in\omega}$ of effectively open subsets of $(0,1)$ such that $\bigcap_{n\in\omega} A^m_n \neq \varnothing \leftrightarrow m \in I$

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and $\bigcap_{n\in\omega} A_n^m = \varnothing \leftrightarrow \bigcap_{n\in\omega} \widetilde{E}_n^m = \varnothing$. Using Lemma 6.5 we can effectively construct a majorant-computable function f_m such that $\mathrm{dom}(f_m) = \bigcap_{n\in\omega} \widetilde{E}_n^m$. So, there exists a computable function $\chi : \omega \to \omega$ such that $f_m = \gamma(\chi(m))$. The function χ is a reduction, since $m \in I \leftrightarrow \gamma(\chi(m)) \neq \bot_{\mathbb{R}}$.

□

Theorem 6.7. For majorant-computable functions $f : \mathbb{R} \to \mathbb{R}$, the problem of equality is Π^1_1-complete.

Proof. This follows immediately. □

7. Computable closed sets over effectively enumerable topological spaces

The notions of computable closed and computably compact sets of Euclidean spaces have been proposed in Brattka and Weihrauch (1999). We generalise these notions for effectively enumerable topological spaces.

Definition 7.1. Let $\mathcal{X} = (X, \tau, \alpha)$ be an effectively enumerable topological space. A set $A \subseteq X$ is called *computable closed* if

1. A is co-effectively closed, i.e., $X \setminus A = \bigcup_{i\in V} \alpha(i)$, where V is computably enumerable;
2. $U = \{n | \alpha(n) \cap A \neq \varnothing\}$ is computably enumerable.

If A and V satisfy the first condition of Definition 7.1 we say that the constraint $*(V; A)$ holds, furthermore, if, additionally, U satisfy the second condition of Definition 7.1 we say that the constraint $*(V, U; A)$ holds. For an effectively enumerable space X we denote the set of co-effectively closed sets as $\mathcal{CO}_X$ and the set of computable closed sets as $\mathcal{C}_X$. It is clear that $\mathcal{CO}_X$ and $\mathcal{C}_X$ are constrained classes.

Proposition 7.1.

1. X and $\varnothing$ are computable closed sets.
2. A point x in a T_0-space $\mathcal{X}$ is computable and co-effective if and only if $\{x\}$ is a computable closed subset of $\mathcal{X}$.

Proof. The claims follow from the definitions. □

To illustrate our approach below we state several results on computable metric spaces. For the definition of computable metric space we refer to Moschovakis (1964); Weihrauch (1993, 2000).

Proposition 7.2 (Brattka 2001). Let $\mathcal{M} = (M, \mathcal{B}, d)$ be a complete computable metric space. Then every computable closed set $A \subseteq M$ contains a computable point $x \in M$.

Proof. Assume (M, τ, α) is the effectively enumerable topological space corresponding to M such that $\alpha(n) = B(b_{m_n}, r_n)$, i.e., $\alpha(n)$ is the open ball with the center $b_{m_n} \in \mathcal{B}$ and the radius $r_n \in \mathbb{Q}^+$. Now we effectively construct a sequence $\{\alpha(n_s)\}_{s\in\omega}$ by steps.
Step 0. Find the first n_0 such that $\alpha(n_0) \cap A \neq \varnothing$.
Step s + 1. Chose $\alpha(n_{s+1})$ satisfying the following conditions:

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a. $\alpha(n_{s+1}) \cap A \neq \varnothing$,
b. the radius of the open ball $\alpha(n_{s+1})$ is at least two times less than the radius of $\alpha(n_s)$,
c. $d(b_{m_{n_{s+1}}}, b_{m_{n_s}}) + r_{n_{s+1}} < r_{n_s}$.

By construction, $\mathrm{cl}(\alpha(n_{s+1})) \subseteq \alpha(n_s)$. From the Nested Sphere Theorem it follows that there exists x such that $\{x\} = \bigcap_{s\in\omega} \mathrm{cl}(\alpha(n_s))$. Therefore, for all s, $x \in \alpha(n_s) \cap A$. Since $x \in \alpha(n) \leftrightarrow \exists s(d(b_{m_n}, b_{m_{n_s}}) + r_{n_s} < r_n)$, x is a computable point. □

Proposition 7.3. There exist a computable metric space $\mathcal{M} = (M, \mathcal{B}, d)$ and a computable closed non-empty subset $A \subseteq M$ such that $A \cap M_c = \varnothing$.

Proof. Let $\mathcal{B} \rightleftharpoons \{\sum_{i=0}^{N} \frac{a_i}{3^{i+1}} | N \in \omega, a_i \in \{0,1,2\}, \mathrm{card}(\{i | a_i = 1\}) > 1\}$ and $\mathcal{C}$ be the Cantor set in $[0,1]$, i.e., $\mathcal{C} = \{\sum_{i=0}^{\infty} \frac{b_i}{3^{i+1}} | b_i \in \{0,2\}\}$. We define $M \rightleftharpoons \big([0,1] \setminus \mathbb{R}_c\big) \cup \mathcal{B}$, $A \rightleftharpoons \big(\mathcal{C} \setminus \mathbb{R}_c\big)$ and d is induced by the standard metric on $[0,1]$. It is worth noting that $B \cap C = \varnothing$, $A = \mathcal{C} \cap M$ and $M \setminus A = \big((0,1) \setminus \mathcal{C}\big) \cap M$. Therefore, A is co-effectively closed. Since, for $x \in \mathcal{B}$ and $r \in \mathbb{Q}^+$, $B(x,r) \cap A = B(x,r) \cap \mathcal{C} \cap M = (B(x,r) \cap \mathcal{C}) \setminus \mathbb{R}_c$, $B(x,r) \cap A \neq \varnothing$ if and only if $B(x,r) \cap \mathcal{C} \neq \varnothing$. So, A is a computable closed set such that $A \cap M_c = \varnothing$. □

Proposition 7.4. Let $\mathcal{X} = (X, \tau, \alpha)$ be an effectively enumerable topological space. A sequence $\{A_n\}_{n\in\omega}$ of computable closed subsets of X is computable if

i. there exists a computable function $g : \omega \times \omega \to \omega$ such that

$$X \setminus A_n = \bigcup_{i\in\omega} \alpha(g(n,i));$$

ii. there exists a computable function $\gamma : \omega \to \omega$ such that

$$\{k | \alpha(k) \cap A_n \neq \varnothing\} = W_{\gamma(n)}.$$

Proof. The claim follows from the definitions. □

Theorem 7.1. Let $\mathcal{M} = (M, B, d)$ be a complete computable metric space. There is no computable sequence that contains all computable closed sets of M.

Proof. First it is worth noting that, by Proposition 7.2, for every computable sequence $\{A_n\}_{n\in\omega}$ of closed sets we can effectively construct a computable sequence $\{x\}_{n\in\omega}$ of points such that, for all $n \in \omega$, $x_n \in M_c$ and $x_n \in A_n$. Now assume contrary, i.e., $\{A_n\}_{n\in\omega}$ contains all computable closed sets of M. Then, $\{x\}_{n\in\omega}$ contains all computable points of M. This contradicts the non-existence of a computable numbering of M_c (Brattka 2001). □

Definition 7.2. Let $\mathcal{X} = (X, \tau, \alpha)$ be an effectively enumerable topological space. A compact set $K \subseteq X$ is called *computably compact* if

$$L_K = \{k \in \omega \mid K \subseteq \bigcup_{i\in D_k} \alpha(i)\} \text{ is computably enumerable.}$$

For an effectively enumerable space X we denote the set of computably compact sets as $\mathcal{CC}_X$.

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Proposition 7.5. Let $\mathcal{X} = (X, \tau, \alpha)$ be a strongly effectively enumerable topological space. Then every computably compact set $K \subseteq X$ is co-effectively closed.

Proof.
From Definitions 3.3 and 7.2, it is easy to see that

$$X \setminus K = \bigcup_{n \in L_K} \left(\bigcap_{m \in D_n} (X \setminus \mathrm{cl}(\alpha(\mathrm{m}))) \right)$$

is effectively open. □

Now we consider the computable closed and the computably compact sets as co-effectively closed subsets of Euclidean spaces and calculate the complexity of their index sets. Similar results for the Cantor and Baire spaces can be found in Brodhead and Cenzer (2008). For proof simplicity we consider $n = 1$. Suppose that the set $\mathcal{CO}_{\mathbb{R}}$ is indexed by the numbering $\alpha^{co}_{\mathbb{R}} : \omega \to \mathcal{CO}_{\mathbb{R}}$ defined as follows: $\alpha^{co}_{\mathbb{R}}(n) = \mathbb{R} \setminus \alpha^{e}_{\mathbb{R}}(n)$. It is clear that $\alpha^{co}_{\mathbb{R}}$ is a principal computable numbering of $\mathcal{CO}_{\mathbb{R}}$.

Theorem 7.2. The index set for the computable closed sets of a Euclidean space is Σ^0_3–complete.

Proof. First, we show that $Ix(\mathcal{C}_{\mathbb{R}}) \in \Sigma^0_3$. It is easy to see that a closed set A is computable if and only if it is co-effectively closed and its distance function d_A is computable (Brattka and Weihrauch 1999). In other words A is a computable closed set if and only if it is co-effectively closed and there exists $k \in \omega$ such that $\gamma(k) = f$, where f is a total computable function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

1. $f \geqslant 0$;
2. $f \leqslant d_A$ on $\mathbb{R} \setminus A$;
3. $f \geqslant d_A$ on $\mathbb{R} \setminus A$;
4. $f = 0$ on A.

Let us denote $\alpha^{co}_{\mathbb{R}}(e) = A_e$. We have $e \in Ix(\mathcal{C}_{\mathbb{R}})$ iff

$$(\exists k) \left(\gamma(k) \in \mathrm{Tot}_{\mathbb{R}} \wedge \gamma(k) \geqslant 0 \wedge \left((\forall x)\, x \notin A_e \to \gamma(k)(x) \leqslant d_{A_e}(x) \right) \wedge \right.$$
$$\left. \left((\forall x)\, x \notin A_e \to \gamma(k)(x) \geqslant d_{A_e}(x) \right) \wedge ((\forall x)\, \gamma(k)(x) > 0 \to x \notin A_e) \right).$$

From Theorem 6.5 it follows that the first conjunct has the form Π^0_2. Let us note that

$$(\exists x)\gamma(k)(x) < 0 \leftrightarrow (\exists q_1 \in \mathbb{Q})(\exists q_2 > q_1 \in \mathbb{Q})(\forall x \in [q_1, q_2])\, V(x, 0),$$

where V is the epigraph of $\gamma(k)$. By Propositions 3.1–3.3, this has the form Σ^0_1. Therefore the second conjunct has the form Π^0_1. Now we calculate the complexity of the rest of conjuncts. First, we note that $\mathbb{R} \setminus A_e = \bigcup_{n \in \omega} I_n$, where I_n is an open interval with rational endpoints. It is easy to see that $I_n = \bigcup_{m \in \omega} K_{m,n}$, where $K_{m,n}$ is a compact interval with rational endpoints and $\{K_{m,n}\}_{m,n \in \omega}$ is a computable sequence of computable compact intervals. Therefore, a formula $(\forall x)(x \notin A_e \to \Theta(x))$ is equivalent to the formula $\bigwedge_{m,n \in \omega} (\forall x \in K_{m,n})\, \Theta(x)$.

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So, Condition (2), i.e., the third conjunct, is equivalent to the formula

$$\bigwedge_{m,n\in\omega}\bigwedge_{a,b\in\mathbb{Q}}(\forall x\in K_{m,n})([a,b]\subseteq B(x,\gamma(k)(x))\to[a,b]\subseteq\mathbb{R}\setminus A_e),$$

where $B(x,y)$ is an open ball with the center x and the radius y. We rewrite it as

$$\bigwedge_{m,n\in\omega}\bigwedge_{a,b\in\mathbb{Q}}(\forall x\in K_{m,n})([a,b]\not\subseteq B(x,\gamma(k)(x))\vee\Psi_{a,b}),$$

where $\Psi_{a,b}$ is a Π^0_1-sentence. From Theorem 3.2 it follows that the first and the second disjuncts are Π^0_1. Therefore, Condition (2) is Π^0_1.

Condition (3), i.e., the fourth conjunct, is equivalent to the following formula:

$$\bigwedge_{m,n\in\omega}(\forall x\in K_{m,n})\neg\Psi(x,e,k),$$

where $\Psi(x,e,k)\rightleftharpoons\forall y(d(y,x)\leqslant\gamma(k)(x)\to y\notin A_e)$ which is rewritable as

$$(\exists a\in\mathbb{Q})(\exists b\in\mathbb{Q})(\exists c\in\mathbb{Q})(x\in(a,b)\wedge c>\sup_{z\in[a,b]}\gamma(k)(z)\wedge$$
$$(\forall y\in[a-c,b+c])\,y\notin A_e).$$

From Propositions 3.1–3.3 it follows that the second conjunct and the third conjunct in the formula above are equivalent to an effective disjunction of quantifier free sentences. Therefore, Condition (3) is Π^0_1. Condition (4), i.e., the fifth conjunct, is equivalent to the formula

$$\bigwedge_{K_m\subseteq\{y|\gamma(k)(y)>0\}}(\forall x\in K_m)\,x\notin A_e.$$

By analogy, Conditions (4) is Π^0_2. So, the relation $e\in Ix(\mathcal{C}_{\mathbb{R}})$ is Σ^0_3. It is worth noting that Σ^0_3-completeness of $Ix(\mathcal{C}_{\mathbb{R}})$ follows from the fact that the Σ^0_3-complete set $R=\{n|W_n \text{ is computable}\}$ is m-reducible to $Ix(\mathcal{C}_{\mathbb{R}})$. Indeed, if we define $\alpha^{co}_{\mathbb{R}}(\beta(n))=A_n=\omega\setminus W_n$ then $n\in R\leftrightarrow\beta(n)\in Ix(\mathcal{C}_{\mathbb{R}})$. Therefore, $Ix(\mathcal{C}_{\mathbb{R}})$ is Σ^0_3–complete. □

Lemma 7.1. Assume $K\subseteq\mathbb{R}$ is co-effectively closed. Then K is computably compact if and only it is bounded.

Proof. The proof is standard and omitted. □

Theorem 7.3.

a. The index set for the computably compact sets of a Euclidean space is Σ^0_3–complete.
b. The index set for the class $\mathcal{K}_{\mathbb{R}}$ of computably compact and computable closed sets of a Euclidean space is Σ^0_3–complete.

Proof.

a. First, we show that $Ix(\mathcal{CC}_{\mathbb{R}})\in\Sigma^0_3$. Let us denote $\alpha^{co}_{\mathbb{R}}(n)=A_n$.
By Lemma 7.1, we have $n\in Ix(\mathcal{CC}_{\mathbb{R}})$ iff A_n is bounded. To say that A_n is bounded we write that

$$(\exists c\in\mathbb{Q}^+)(\forall p\in\mathbb{Q})(\forall q\in\mathbb{Q})(p>q\vee[p,q]\setminus(-c,c)\subseteq\alpha^e_{\mathbb{R}}(n)).$$

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This is Σ^0_3. It is worth noting that Σ^0_3–completeness of $Ix(\mathcal{CC}_{\mathbb{R}})$ follows from the fact that the Σ^0_3-complete set Cofin $= \{n|W_n \text{ is cofinite}\}$ is m-reducible to $Ix(\mathcal{CC}_{\mathbb{R}})$.
Let us define a reduction function g such that

$$\alpha^{co}_{\mathbb{R}}(g(n)) = \bigcup_{k\in W_n}(k-1,k+1)\bigcup(-\infty,0).$$

Then, $n \in \text{Cofin} \leftrightarrow \alpha^{co}_{\mathbb{R}}(g(n)) \in Ix(\mathcal{CC}_{\mathbb{R}})$. Indeed, if $n \notin \text{Cofin}$ then $X \setminus \alpha^{e}_{\mathbb{R}}(g(n))$ is unbounded. So, $\alpha^{co}_{\mathbb{R}}(g(n)) \notin Ix(\mathcal{CC}_{\mathbb{R}})$. If $n \in \text{Cofin}$ then $X \setminus \alpha^{e}_{\mathbb{R}}(g(n))$ is finite. So, $\alpha^{co}_{\mathbb{R}}(g(n)) \in Ix(\mathcal{CC}_{\mathbb{R}})$. Therefore, $Ix(\mathcal{CC}_{\mathbb{R}})$ is Σ^0_3–complete.

b. By Theorem 7.2 and (a) it is easy to see that $Ix(\mathcal{K}_{\mathbb{R}}) \in \Sigma^0_3$. It is worth noting that Σ^0_3–completeness of $Ix(\mathcal{K}_{\mathbb{R}})$ follows from the fact that the Σ^0_3-complete set $R = \{n|W_n \text{ is computable}\}$ is m-reducible to $Ix(\mathcal{K}_{\mathbb{R}})$.
Let us define a reduction function g such that

$$\alpha^{e}_{\mathbb{R}}(g(n)) = B_n = \{0\} \cup \left\{\tfrac{1}{2}\right\} \cup \left\{\tfrac{1}{2^i} \mid i \in A_n\right\} \subseteq [0,1],$$

where $A_n = \omega \setminus W_n$. By construction, for every n, B_n is bounded and co-effectively closed. So B_n is computably compact. It is worth noting that B_n is computable closed if and only if W_n is computable. Therefore, $Ix(\mathcal{K}_{\mathbb{R}})$ is Σ^0_3–complete.

□

Corollary 7.1. The classes of computably compact and computable closed sets of a Euclidean space are incomparable.

Proof. The claim follows from the construction of B_n in the proof of Theorem 7.3. □

Theorem 7.4. For any non-empty $K \subseteq \mathcal{CC}_X$, where X is a Euclidean space, the index set $Ix(K)$ is Π^0_2-hard.

Proof. Let us fix $A \in K$. Without loss of generality we assume that $A \subseteq [0,1]$. We define a computable sequence of co-effectively closed sets as follows:

$$A_0 \leftrightharpoons A; A_n \leftrightharpoons A \bigcup_{k\geqslant n}[2k, 2k+1].$$

It is easy to see that for every $n > 0$, the set A_n is not compact, i.e., $A_n \notin K$ and $A = \bigcap_{n\in\omega} A_n$. Now we show that $\{n|W_n \text{ is infinite}\} \leqslant_m Ix(K)$. Define

$$B_n = \bigcap_{m\in W_n} A_m.$$

Since $\{B_n\}_{n\in\omega}$ is a computable sequence of co-effectively closed sets, there exists a computable function $F : \omega \to \omega$ such that $B_n = \alpha^{co}_{\mathbb{R}}(F(n))$. We have the following equivalences:

$$W_n \text{ is infinite} \leftrightarrow B_n \in K \leftrightarrow F(n) \in Ix(K).$$

Therefore, $Ix(K)$ is Π^0_2–hard. □

Corollary 7.2. Let X be a Euclidean space. Then, for any $K \subseteq \mathcal{CC}_X$, $Ix(K) \notin \Delta^0_2$.

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8. Conclusion and related work

Recently, several approaches to measure complexity of undecidable problems in computable analysis have been intensively developed. Our work is closely related to the approach originated from the research on computing index sets of closed subsets of classical Polish spaces and computably continuous functions (Cenzer and Remmel 1998, 1999).

Our main results extend this approach as follows. In the framework of effectively enumerable topological spaces we have constructed principal computable numberings of the computable elements, partial majorant-computable functions and co-effective closed subsets. With respect to these principal computable numberings we have proven analogues of Rice's theorem and shown that partial function equality is Π^1_1-complete. We also have shown that the index sets for the computable closed sets and for the computably compact sets of a Euclidean space are Σ^0_3–complete.

Another well-established approach to measure complexity of closed, open sets and continuous functions is based on the classical Borel hierarchy and its effective version (see among others Becher *et al.* 2014, Brattka and Gherardi 2009, Montalban and Nies 2013 and Selivanov and Schröder 2014). The relationship between these two approaches is still unclear and we leave its investigation for future work.

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References

Ash, C.J. and Knight, J.F. (2000). *Computable Structures and the Hyperarithmetical Hierarchy*. Studies in Logic and the Foundations of Mathematics, **144**, Elsevier.

Becher, V., Heiber, P. and Slaman, T.A. (2014). Normal numbers and the Borel hierarchy. *Fundamenta Mathematicae* **22**, 63–77.

Berger, U. (1993). Total sets and objects in domain theory. *Annals of Pure and Applied Logic* **60** (2) 91–117.

Brattka, V. (2001). Computable versions of Baire's category theorem. In: MFCS'99. *Springer Lecture Notes in Computer Science* **2136** 224–235.

Brattka, V. and Gherardi, G. (2009). Borel complexity of topological operations on computable metric spaces. *Journal of Logic and Computation* **19** (1) 45–76.

Brattka, V. and Gherardi, G. (2011). Weihrauch degrees, omniscience principles and weak computability. *Journal of Symbolic Logic* **76** (1) 143–176.

Brattka, V. and Weihrauch, K. (1999). Computability on subsets of euclidean space I: Closed and compact sets. *Theoretical Computer Science* **219** (1-2) 65–93.

Brodhead, P. and Cenzer, D.A. (2008). Effectively closed sets and enumerations. *Archive for Mathematical Logic* **46** (7-8) 565–582.

Calvert, W., Fokina, E., Goncharov, S.S., Knight, J.F., Kudinov, O.V., Morozov, A.S. and Puzarenko, V. (2007). Index sets for classes of high rank structures. *Journal of Symbolic Logic* **72** (4) 1418–1432.

Calvert, W., Harizanov, V.S., Knight, J.F. and Miller, S. (2006). Index sets of computable structures. *Journal Algebra and Logic* **45** (5) 306–325.

Ceitin, G.S. (1971). Mean value theorems in constructive analysis. *American Mathematical Society Translations: Series* 2 **98** 11–40.

https://doi.org/10.1017/S0960129516000141 Published online by Cambridge University Press

Cenzer, D.A. and Remmel, J.B. (1998). Index sets for Π^0_1 classes. *Annals of Pure and Applied Logic* **93** (1-3) 3–61.

Cenzer, D.A. and Remmel, J.B. (1999). Index sets in computable analysis. *Theoretical Computer Science* **219** (1-2) 111–150.

Delzell, C.N. (1982). A finiteness theorem for open semi-algebraic sets, with application to Hilbert's 17th problem. In: *Ordered Fields and Real Algebraic Geometry*, Contemp. Math., **8**, AMS 79–97.

Downey, R.G., Hirschfeldt, D.R. and Khoussainov, B. (2003). Uniformity in computable structure theory. *Journal of Algebra and Logic* **42** (5) 318–332.

Downey, R.G. and Montalban, A. (2008). The isomorphism problem for torsion-free Abelian groups is analytic complete. *Journal of Algebra* **320** 2291–2300.

Ershov, Yu.L. (1973). Theorie der Numerierungen I. *Zeitschrift fur mathematische Logik Grundlagen der Mathematik* **19** 289–388.

Ershov, Yu.L. (1977). Model $\mathbb{C}$ of partial continuous functionals. In: *Logic Colloquium*, **76**, North-Holland 455–467.

Ershov, Yu.L. (1999). Theory of numberings. In: Griffor, E.R. (ed.) *Handbook of Computability Theory,* Elsevier Science B.V., 473–503.

Frolov, A., Harizanov, V., Kalimullin, I., Kudinov, O. and Miller, R. (2012). Spectra of high$_n$ and nonlow$_n$ degrees. *Electronic Notes in Theoretical Computer Science* **22** (4) 755–777.

Gierz, G., Heinrich Hofmann, K., Keime, K., Lawson, J.D. and Mislove, M.W. (2003). *Continuous Lattices and Domain*. Encyclopedia of Mathemtics and its Applications **93**, Cambridge University Press.

Grubba, T. and Weihrauch, K. (2007). On computable metrization. *Electronic Notes in Theoretical Computer Science* **167**, 345–364.

Grubba, T. and Weihrauch, K. (2009). Elementary computable topology. *Journal of UCS* **15** (6) 1381–1422.

Korovina, M.V. (2003a). Gandy's theorem for abstract structures without the equality test. In: Vardi, M.Y. and Voronkov, A. (eds.) LPAR'03. *Springer Lecture Notes in Computer Science* **2850** 290–301.

Korovina, M.V. (2003b). Computational aspects of Σ-definability over the real numbers without the equality test. In: Baaz, M. and Makowsky, J.A. (eds.) CSL'03. *Springer Lecture Notes in Computer Science,* **2803** 330–344.

Korovina, M.V. and Kudinov, O.V. (2015). Positive predicate structures for continuous data. *Journal of Mathematical Structures in Computer Science* **25** (8), 1669–1684.

Korovina, M.V. and Kudinov, O.V. (2005). Towards computability of higher type continuous data. In: Proceedings of the CiE'05. *Springer-Verlag Lecture Notes in Computer Science* **3526** 235–241.

Korovina, M.V. and Kudinov, O.V. (2008). Towards computability over effectively enumerable topological spaces. *Electronic Notes in Theoretical Computer Science* **221** 115–125.

Korovina, M.V. and Kudinov, O.V. (2009). The Uniformity Principle for Sigma-definability. *Journal of Logic and Computation* **19** (1) 159–174.

Martin-Löf, P. (1970). *Notes on Constructive Mathematics*. Almqvist & Wiksell.

Montalban, A. and Nies, A. (2013). Borel structures: A brief survey. *Lecture Notes in Logic* **41** 124–134.

Morozov, A.S. and Korovina, M.V. (2008). On Σ-definability without equality over the real numbers. *Mathematical Logic Quarterly* **54** (5) 535–544.

Moschovakis, Y.N. (1964). Recursive metric spaces. *Fundamenta Mathematicae* **55** 215–238.

Rogers, H. (1967). *Theory of Recursive Functions and Effective Computability,* McGraw-Hill.

Selivanov., V. and Schröder, M. (2014) Hyperprojective hierarchy of qcb_0-Space. *Journal Computability* **4** (1) 1–17, North-Holland.

https://doi.org/10.1017/S0960129516000141 Published online by Cambridge University Press

Shoenfield, J.R. (1971). *Degrees of Unsolvability*, North-Holland Publ.

Soare, R.I. (1987). *Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets*, Springer Science and Business Media.

Spreen, D. (1984). On r.e. inseparability of cpo index sets. *Logic and Machines: Decision Problems and Complexity, Lecture Notes in Computer Science* **171** 103–117.

Spreen, D. (1995). On some decision problems in programming. *Information and Computation* **122** (1) 120–139.

Spreen, D. (1998). On effective topological spaces. *Journal of Symbolic Logic* **63** (1) 185–221.

Weihrauch, K. (1993) Computability on computable metric Spaces. *Theoretical Computer Science* **113** (1) 191–210.

Weihrauch, K. (2000). *Computable Analysis*, Springer Verlag.

Weihrauch, K. and Deil, Th. (1980) Berechenbarkeit auf cpo-s. *Schriften zur Angew. Math. u. Informatik* **63**. RWTH Aachen.

Welch, L.V. (1984). A hierarchy of families of recursively enumerable degrees. *Journal of Symbolic Logic* **49** (4) 1160–1170.

Yates, C.E.M. (1965). Three theorems of the degrees of recursively enumerable sets. *Duke Mathematical Journal* **32** (3) 461–468.

Yates, C.E.M. (1969). On the degrees of index sets II. *Transactions of the American Mathematical Society* **135** 249–266.

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