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Selectionable distributions for a random set

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Abstract

Artstein has given a characterization of the distributions induced by selections of a random compact subset of a complete separable metric space. In this paper, Artstein's results are extended to spaces which may be neither metric nor separable.

0. *Introduction*

A *random set* is a random variable whose values are subsets of some fixed space X. Theorems about random sets and their measurable selections are usually framed and proved under the strong hypothesis that X is a separable metric space. Such a strong condition on X is often quite unnecessary for the formulation of a problem (see Ross [**9**]), and is artificial in many applications, for example, some arising in mathematical economics.

Zvi Artstein [**2**] has given a very pretty characterization of the distributions associated with measurable selections of a random compact subset of a complete separable metric space X (Theorem 1, below). The result has the flavour of Hall's 'marriage theorem' (see Halmos [**6**]), and in fact employs and extends a generalization, due to Bollobás and Varopoulos [**3**], of that theorem.

In this paper, Artstein's theorem is proved in a much more general setting, in which X need be neither metric nor separable. The idea is to represent the measures involved by hyperfinite measures (in the sense of nonstandard analysis). The original form of the marriage theorem is then applied to obtain the result.

1. *Statement of results*

Let X be a Hausdorff regular space, and let $\mathscr{F}$ be the space of closed subsets of X. There is a natural topology on $\mathscr{F}$ (see [**7**]), defined as follows. For $E \subseteq X$, write $\mathscr{F}^E$ for $\{A \in \mathscr{F} : A \cap E = \varnothing\}$, and $\mathscr{F}_E$ for $\{A \in \mathscr{F} : A \cap E \neq \varnothing\}$. A subbasis for the topology on $\mathscr{F}$ consists of all sets of the form $\mathscr{F}_u$ and $\mathscr{F}^{(u^c)}$, where u is an open subset of X.

A *random closed subset* of X is a probability space $(S, \mathscr{B}_S, P)$, together with a map $F : S \to \mathscr{F}$, measurable with respect to the Borel σ-algebra $\mathscr{B}_\mathscr{F}$ on $\mathscr{F}$. This random set induces a natural image measure DF on $(\mathscr{F}, \mathscr{B}_\mathscr{F})$, which is called the *distribution* of F.

Suppose $F \neq \varnothing$ almost surely, and $f : S \to X$ is measurable with $f(s) \in F(s)$ for almost all s; then f is called a *selection* of F.

If σ is a probability measure on $\mathscr{F}$, and ρ is a probability measure on X, then ρ is

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called *selectionable* with respect to σ provided there is a random set F and a selection f of F with $DF = \sigma$ and $Df = \rho$.

The following is Artstein's theorem.

THEOREM 1. *Suppose X is a complete, separable metric space, ρ is a probability on X, and σ is a probability on $\mathscr{F}$ for which $\sigma(\mathscr{K}) = 1$, where $\mathscr{K} = \{E \in \mathscr{F} : E$ is compact$\}$. Then ρ is selectionable with respect to σ if and only if*

$$(*) \qquad \rho(E) \leqslant \sigma(\mathscr{F}_E) \quad \textit{for every closed } E \subseteq X.$$

In addition, Artstein shows that the domain probability space for the random variables F and f can be chosen independently of ρ and σ, and that the requirement that $\sigma(\mathscr{K}) = 1$ may be dropped if X is locally compact.

It happens that every Borel probability measure on a Polish space is Radon [5], and if X is Polish then so is $\mathscr{K}$ (see [7]). (Recall that a Borel probability space is *Radon* provided it is inner regular with respect to compact sets.) The following theorem, which is the main result of this paper, is thus stronger than Theorem 1.

THEOREM 2. *Let X be any Hausdorff regular space, ρ a Radon probability on X, and σ a Radon probability on $\mathscr{F}$. Then ρ is selectionable with respect to σ if and only if condition (*) holds.*

2. *Preliminaries*

The proof uses the hyperfinite measure construction from nonstandard analysis. The reader is referred to Cutland [**4**] for terminology and details. We assume that the model is β-saturated, where $\beta \geqslant 2^{\operatorname{card}(X)}$.

A *uniform hyperfine measure space* is a triple $(S, L(S), L(\mu))$ where S is a hyperfinite set, $\mu : {}^*\mathscr{P}(S) \to {}^*[0, 1]$ is the internal counting measure giving each point a mass of $|S|^{-1}$, $L(\mu)$ is the (external) measure obtained from μ by means of the hyperfinite measure construction, and $L(S)$ is the $L(\mu)$-completion of the smallest σ-algebra containing all internal subsets of S.

The first lemma asserts that a fixed uniform hyperfinite probability space can represent all (sufficiently small) Radon spaces.

LEMMA 1. *Let $(S, L(S), L(\mu))$ be a uniform hyperfinite probability space, let $(Y, \mathscr{B}_Y, P)$ be a Radon space, and suppose $\beta \geqslant 2^{\operatorname{card}(Y)}$. There is an internal function $\Theta_S : S \to {}^*Y$ such that*

(1) *$NS(S) = (\operatorname{st} \circ \Theta_S)^{-1}(Y)$ is in $L(S)$ and has $L(\mu)$-measure one;*

(2) *$\operatorname{st}_S = \operatorname{st} \circ \Theta_S$ is a measurable, measure-preserving function from $NS(S)$ onto Y;*

(3) *if $K \subseteq Y$ is compact then $\Theta_S^{-1}({}^*K) \subseteq \operatorname{st}_S^{-1}(K)$; and*

(4) *if $U \subseteq Y$ is open then $\Theta_S^{-1}({}^*U) \supseteq \operatorname{st}_S^{-1}(U)$.*

Proof. This is due to Anderson [**1**]; see Cutland [**4**], theorem 4·8.

The next lemma resolves a technical difficulty in applying the marriage theorem to hyperfinite measures. The basic idea was first used in Ross [**8**].

LEMMA 2. *Suppose that $(S, L(S), L(\mu))$ and $(S', L(S'), L(\mu'))$ are uniform hyperfinite probability spaces, where S and S' have the same internal cardinality H. Let $W \subseteq S \times S'$ be an internal relation, let E be the domain of W, and suppose that for every internal $A \subseteq E$, $L(\mu)(A) \leqslant L(\mu')(W(A))$, where $W(A) = \{s' \in S' : (s, s') \in W$ for some $s \in S\}$. Fix*

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$r > 0$; there is then an internal $Z \subseteq E$, and there is a one-to-one function $g: Z \to S'$, such that $L(\mu)(Z) \geqslant L(\mu)(E) - r$, and such that $(s, g(s)) \in W$ for every $s \in Z$.

Proof. Let M be the greatest hyperinteger less than rH, and let $J = \{0, 1, \ldots, M\}$. Put $W' = W \cup (E \times J)$. For $A \subseteq E$ internal,

$$^\circ(|A|/H) = L(\mu)(A) \leqslant L(\mu')(W(A)) < L(\mu')(W(A)) + r = {}^\circ(|W'(A)|/H)$$

so $|A| < |W'(A)|$. By the *marriage theorem there is an internal one-to-one function $g: E \to (S' \cup J)$ with $(s, g(s)) \in W'$ for all $s \in E$. Let $Z = g^{-1}(S')$. It is easy to see that Z and g satisfy the conclusion of the lemma.

3. *Proof of Theorem 2*

Write st for the standard part map on $^*X$, and ST for the standard part map on $^*\mathscr{F}$. Let H be an infinite integer. Since σ and ρ are Radon, there are uniform hyperfinite probability spaces $(S, L(S), L(\mu_\sigma))$ and $(T, L(T), L(\mu_\rho))$, and internal functions $\Theta_S: S \to {}^*\mathscr{F}$ and $\Theta_T: T \to {}^*X$, which represent $(\mathscr{F}, \mathscr{B}_\mathscr{F}, \sigma)$ and $(X, \mathscr{B}_X, \rho)$ as in Lemma 1.

For $E \subseteq X$ write $\mathrm{CL}(E)$ for the closure of E in X. If u and v are subsets of X, write $u)(v$ if $\mathrm{CL}(u) \cap \mathrm{CL}(v) = \varnothing$.

The first step in the proof is to show that the following conditions (on S' and g) are finitely satisfiable:

(1) $S' \in {}^*\mathscr{P}(S)$;

(2) $\mu_\sigma(S \backslash S') < 2r$ for each (standard) $r > 0$;

(3) $g: S' \to T$ is one-to-one; and

(4) $\Theta_T(g(s)) \in {}^*u$ implies $\Theta_S(s) \notin {}^*v$, for each pair u, v of open subsets of X with $u)(v$, and for all $S \in S'$.

Fix any $r > 0$, and open sets $u_1, \ldots, u_n, v_1, \ldots, v_n$ with $u_i)(v_i$ for each $i \leqslant n$. Let $T' \subseteq NS(T)$ be internal with $L(\mu_\rho)(T') > 1 - r$. For $t \in T'$, put $u(t) = \bigcap \{u_i : \Theta_T(t) \in {}^*u_i\}$ and $v(t) = \bigcup \{v_i : \Theta_T(t) \in {}^*u_i\}$. To ensure that these intersections and unions exist, assume (without loss of generality) that $u_1 = X$ and $v_1 = \varnothing$. For A an internal subset of T', put $u(A) = \bigcup_{t \in A} u(t)$ and $v(A) = \bigcap_{t \in A} v(t)$. Note that all these unions and intersections are finite, so $u(A)$ and $v(A)$ are open; moreover, $u(A))(v(A)$.

Define an internal binary relation W on $T' \times S$ by $(t, s) \in W$ if and only if $\Theta_S(s) \notin {}^*v(t)$. Fix an internal subset A of T'. Since $\Theta_T(A) \subseteq {}^*u(A)$ we have $\mathrm{st}_T(A) \subseteq \mathrm{CL}(u(A))$; it follows that

$$L(\mu_\rho)(A) \leqslant \rho(\mathrm{CL}(u(A))) \leqslant \sigma(\mathscr{F}_{\mathrm{CL}(u(A))}) = L(\mu_\sigma)(\mathrm{ST}_S^{-1}(\mathscr{F}_{\mathrm{CL}(u(A))})).$$

If $s \in NS(S)$ and $s \notin W(A)$ then $\Theta_S(s) \subseteq {}^*v(A)$, so $\Theta_S(s) \notin {}^*\mathscr{F}_{\mathrm{CL}(v(A))^c}$, which means that $\mathrm{ST}_S(s) \notin \mathscr{F}_{\mathrm{CL}(v(A))^c}$, i.e. $\mathrm{ST}_S(s) \subseteq \mathrm{CL}(v(A))$; since $u(A))(v(A)$, $\mathrm{ST}_S(s) \notin \mathscr{F}_{\mathrm{CL}(u(A))}$. Thus $\mathrm{ST}_S^{-1}(\mathscr{F}_{\mathrm{CL}(u(A))}) \subseteq W(A)$, so $L(\mu_\rho)(A) \leqslant L(\mu_\sigma)(W(A))$.

To apply Lemma 2 it remains to show that T' is the domain of W. Given $t \in T'$, put $A = \{t\}$ and let $s \in S$ with $\mathrm{ST}_S(s) = \mathrm{CL}(u(t))$ (which is possible since $u(t) \neq \varnothing$ and ST_S is surjective). By the preceding argument, $s \in \mathrm{ST}_S^{-1}(\mathscr{F}_{\mathrm{CL}(u(t))}) \subseteq W(\{t\})$, so $(t, s) \in W$, and T' is the domain of W.

By Lemma 2 there is an internal $T'' \subseteq T'$ with $L(\mu_\rho)(T'') > 1 - 2r$, and there is an internal one-to-one function $h: T'' \to S$ with $(t, h(t)) \in W$ for each $t \in T''$. Let S' be the

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range of h, and put $g = h^{-1}$ on S'. It is easy to verify (1)–(4) for this g and S', with the given r and sets u_i and v_i.

By saturation of the nonstandard model, there exist an internal g and S' satisfying (1)–(4) for all r, u, v.

Put $S'' = NS(S') \cap g^{-1}(NS(T))$. Clearly $L(\mu_\sigma)(S'') = 1$. Define $f: S \to X$ and $F: S \to \mathscr{F}$ so that $f = \mathrm{st}_T \circ g$ and $F = \mathrm{ST}_S$ on S''. By the construction, f and F are measurable with the appropriate distributions. It remains to show that $f \in F$ almost surely.

Let $s \in S''$, and suppose (for a contradiction) that $f(s) \notin F(s)$. Since X is regular, there are open sets u, u', and v such that $f(s) \in u \subseteq \mathrm{CL}(u) \subseteq u'$, $F(s) \subseteq v$, and $u' \cap v = \varnothing$. Note that $u)(v$. By definition of f, $\mathrm{st}(\Theta_T(g(s))) = f(s) \in u$, so $\Theta_T(g(s)) \in {}^*u$, and condition (4) ensures that $\Theta_S(s) \notin {}^*v$. However $\mathrm{ST}(\Theta_S(s)) = F(s) \subseteq v$, a contradiction. The theorem is proved.

4. *Extensions*

In the proof of Theorem 2, regularity of X is used only once, to show that $f(s)$ and $F(s)$ can be separated by open sets $u)(v$ when $f(s) \notin F(s)$, which is used to prove that $f(s) \in F(s)$ almost surely.

Suppose then that X is any Hausdorff space (not necessarily regular), and – as in Theorem 2 – that σ is concentrated on $\mathscr{K}$. The space $\mathscr{K}$ is Hausdorff in the subspace topology; moreover, if $E \in \mathscr{K}$ and $x \in X \backslash E$ then x and E can be separated by disjoint open subsets u, v of X. If in addition X is locally compact, the sets can be chosen so that $u)(v$. The proof of Theorem 2 will give the following result.

THEOREM 3. *Let X be any locally compact Hausdorff space, ρ a Radon probability on X, and σ a Radon probability on $\mathscr{K}$. Then ρ is selectionable with respect to σ if and only if (*) holds.*

Note that the same base space for the random set and selection, namely S, will work in Theorems 2 and 3 for all sufficiently small spaces X; in particular, the choice of S does not depend on σ. This is similar to Artstein's Theorem 2·2.

Moreover, once σ is given, the same F (namely, $F = \mathrm{ST}_S$) will work for all choices of ρ satisfying (*); this is useful for the integration of random sets: see [**10**].

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