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MONOMIAL AND MONOLITHIC CHARACTERS OF FINITE SOLVABLE GROUPS

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Abstract

Let G be a finite solvable group and let p be a prime divisor of $|G|$. We prove that if every monomial monolithic character degree of G is divisible by p, then G has a normal p-complement and, if p is relatively prime to every monomial monolithic character degree of G, then G has a normal Sylow p-subgroup. We also classify all finite solvable groups having a unique imprimitive monolithic character.

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1. Introduction

One of the notable questions in the character theory of finite groups is to determine the group structure from the number of its irreducible characters. The earliest result concerning the number of nonlinear irreducible characters was provided by Seitz. He classified all finite groups having a unique nonlinear irreducible character [12]. Later, Berkovich gave a classification for the finite groups with two nonlinear irreducible characters in [2, Theorem 31.6]. Recently, all finite solvable groups having at most two imprimitive irreducible characters have been classified in [5].

Before presenting our results, we introduce some definitions and notation. Let $\mathrm{Irr}(G)$ be the set of irreducible characters of a finite group G and denote the set of irreducible character degrees of G by $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. An irreducible character of G is said to be monomial if it is induced from a linear character of a proper subgroup of G. In the light of this definition, the monomial characters of G may be regarded as nonlinear. When all nonlinear irreducible characters of G are monomial, G is called an M-group. A character $\chi \in \mathrm{Irr}(G)$ is called primitive if $\chi \neq \lambda^G$ for any character λ of a proper subgroup of G. For an irreducible character χ of G, if the factor group $G/\ker\chi$ has a unique minimal normal subgroup, then the character χ is called monolithic.

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There have been several studies devoted to monomial and monolithic characters of finite groups. There are recent studies on the monolithic characters of real groups [4] and irreducible monomial characters of solvable groups [10]. However, there are not many studies of the monolithic characters of finite groups. To the best of our knowledge, the first studies of monolithic characters of finite groups were by Berkovich [2, 1]. In [2, Ch. 30], Berkovich made important contributions by generalising the celebrated theorems of Thompson and Ito–Michler (see [8, Corollaries 12.2 and 12.34]). Since these theorems are directly related to our work here, it is helpful to recall their statements. Let p be a prime dividing the order of a group G. Thompson's theorem asserts that if all nonlinear character degrees in $\mathrm{cd}(G)$ are divisible by p, then G has a normal p-complement, and Ito–Michler's theorem states that all character degrees in $\mathrm{cd}(G)$ are not divisible by p if and only if G has an abelian normal Sylow p-subgroup.

Motivated by these results, we consider monomial and monolithic characters of a finite solvable group. More precisely, we prove the following theorem.

THEOREM 1.1. *Let G be a finite solvable group and let p be a prime divisor of $|G|$.*

(i) *If p divides $\chi(1)$ for every monomial monolithic character χ of G, then G has a normal p-complement.*
(ii) *If p does not divide $\chi(1)$ for every monomial monolithic character χ of G, then G has a normal Sylow p-subgroup.*

It is worth remarking that results (i) and (ii) do not remain true if G is not a solvable group. For example, consider the symmetric group S_5 in which irreducible characters are monolithic. From [3], the only monomial irreducible character of S_5 has degree 6 while the group S_5 has no normal p-complement or normal Sylow p-subgroup for $p = 2$ or 3. On the other hand, the converse of (ii) turns out to be false. As a counterexample, we can take the semidirect product $\mathrm{C}_9 \rtimes \mathrm{C}_6$, where C_6 acts faithfully on C_9. Even though this group is an M-group having a normal Sylow 3-subgroup, its unique faithful character is monolithic of degree 6. Also, a group G satisfying the hypotheses of (ii) does not need to have an abelian Sylow p-subgroup, unlike Ito's theorem. It can be easily seen that the special linear group $\mathrm{SL}(2, 3)$ has a unique monomial and monolithic character of degree 3. So, the conclusion of (ii) remains valid for $\mathrm{SL}(2, 3)$, but the Sylow 2-subgroup of this group is the quaternion group Q_8.

Now, let us state our second result, which is concerned with the classification of solvable groups with their imprimitive and monolithic characters. This result is a generalised version of Theorem A in [5].

THEOREM 1.2. *A finite solvable group G has a unique imprimitive monolithic character if and only if one of the following statements holds:*

(i) $G \cong \mathrm{ES}(2^{2m+1}) \times A$, *where A is an abelian group of odd order;*

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(ii) *$G/\mathrm{Z}(G)$ is a Frobenius group of order $p^a(p^a-1)$ for some prime p and also $G' \cap \mathrm{Z}(G) = 1$, $G' \cong \mathrm{E}(p^a)$ and $G \cong N \times P$, where $P \in \mathrm{Syl}_p(\mathrm{Z}(G))$;*

(iii) *$G/\mathrm{Z}(G) \cong \mathrm{A}_4$ and also $G' \cap \mathrm{Z}(G) = G'' > 1$ and $G \cong N \rtimes Q$, where Q is an abelian Sylow 3-subgroup of G and N is a nilpotent group whose Sylow p-subgroups are abelian for every prime p different from two.*

Throughout this paper, we will use the notation C_n, $\mathrm{E}(n)$, Q_n and $\mathrm{ES}(n)$ respectively to denote a cyclic group, an elementary abelian group, a generalised quaternion group and an extraspecial group of order n.

2. The proofs of the main results

In this section, we will prove Theorems 1.1 and 1.2. In the proofs, some preliminary results are required. For the convenience of the reader, these are given with their complete statements before the proofs of the related theorems.

THEOREM 2.1 [12]. *A finite group G has a unique nonlinear irreducible character if and only if one of the following holds:*

(i) *G is an extraspecial 2-group;*

(ii) *G is a Frobenius group of order $p^a(p^a-1)$ with an elementary abelian Frobenius kernel of order p^a and a cyclic Frobenius complement of order p^a-1, where p is a prime.*

THEOREM 2.2 [1, Proposition 3]. *A finite group G has a unique nonlinear monolithic character if and only if one of the following assertions holds:*

(i) *$G \cong \mathrm{ES}(2^{2m+1}) \times A$, where A is an abelian group of odd order;*

(ii) *$G/\mathrm{Z}(G)$ is a Frobenius group of order $p^a(p^a-1)$ for some prime p and also $G' \cap \mathrm{Z}(G) = 1$, $G' \cong \mathrm{E}(p^a)$ and $G \cong N \times P$, where $P \in \mathrm{Syl}_p(\mathrm{Z}(G))$.*

We call the groups described in Theorem 2.2 Berkovich groups. Observe that these groups are solvable and the Fitting subgroup $\mathrm{F}(G)$ of the group in the case (ii) is the abelian group $\mathrm{Z}(G) \times G'$ since $\mathrm{Z}(G) < \mathrm{F}(G)$ for a solvable group G.

LEMMA 2.3 [6, Lemma 1]. *Let $G = NH$, where H is a subgroup of G and N is a normal subgroup of G with $N \cap H = 1$. Then each linear character of N which is invariant under G extends to a character of G.*

If $G = NH$ for $H \leq G$ and $N \trianglelefteq G$ with $N \cap H = 1$, we say that the group G splits over N. Also, we use the notation $\mathrm{Irr}_{mm}(G)$ to denote the set of monomial monolithic characters of G. Now, we are ready to present the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. We will prove the theorem by way of contradiction. To do this, we assume that the assertions of (i) and (ii) given in the theorem are not true and let G be a minimal counterexample to all these statements. Suppose that M and N are distinct minimal normal subgroups of G. Since $\mathrm{Irr}_{mm}(G/M)$ and

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$\mathrm{Irr}_{mm}(G/N)$ are subsets of $\mathrm{Irr}_{mm}(G)$, both factor groups G/M and G/N have a normal p-complement by induction. Since G is isomorphic to a subgroup of $G/M \times G/N$, it follows that G has a normal p-complement, which is a contradiction. Obviously, if both G/M and G/N have a normal Sylow p-subgroup, then we again reach a contradiction.

From now on, we assume that G has a unique minimal normal subgroup M for the assertions (i) and (ii). Since G is solvable, M is an elementary abelian q-group for some prime q. First of all let us suppose that p divides all monomial monolithic character degrees of G. Since G/M has a normal p-complement by induction, we see that $p = q$. It follows that $A := MH$ is a normal subgroup of G, where H is a Hall p'-subgroup of G. By using Theorem 1 of [11], we see that $G = A\mathrm{N}_G(H) = MH\mathrm{N}_G(H) = M\mathrm{N}_G(H)$. Since M is abelian, it is well known that $M \cap \mathrm{N}_G(H) \trianglelefteq G$, which is possible only when $M \leq \mathrm{N}_G(H)$ or $M \cap \mathrm{N}_G(H) = 1$. If $M \leq \mathrm{N}_G(H)$, then $H \trianglelefteq G$, which is a contradiction. Therefore, we can assume that $G = M \rtimes \mathrm{N}_G(H)$, that is, G splits over the unique minimal normal subgroup M. On the other hand, there exists a Sylow p-subgroup P of G satisfying $M \leq P$. Since $M \trianglelefteq P$, we observe that P acts by conjugation on M and so P acts on $\mathrm{Irr}(M)$. Also, we know that $\mathrm{C}_P(x) = P$ for $x \in M \cap \mathrm{Z}(P) > 1$. This implies that there exists a nontrivial linear character λ of M with inertia subgroup $\mathrm{I}_P(\lambda) = P$ and hence we conclude that $P \leq \mathrm{I}_G(\lambda)$ for some nontrivial linear character λ of M. Since G splits over M and $M \leq \mathrm{I}_G(\lambda)$, the group $\mathrm{I}_G(\lambda)$ splits over M and also the character λ is invariant in $\mathrm{I}_G(\lambda)$. It follows from Lemma 2.3 that $\varphi_M = \lambda$ for a linear character φ of $\mathrm{I}_G(\lambda)$. By the Clifford correspondence, $\varphi^G = \chi \in \mathrm{Irr}(G)$. Notice that χ is a monomial character. If $\ker\chi > 1$, then $M \leq \ker\chi$. Therefore, $M \leq \ker\varphi^G \leq \ker\varphi$ and hence $M \leq \ker\varphi_M = \ker\lambda$. This leads to a contradiction as λ is a nontrivial character of M. Thus, $\ker\chi = 1$, that is, χ is a monomial monolithic character of G. By the hypothesis of (i), $p \mid \chi(1)$. By considering $P \leq \mathrm{I}_G(\lambda)$, we reach a contradiction that $p \nmid |G : \mathrm{I}_G(\lambda)| = \varphi^G(1) = \chi(1)$. This completes the proof of (i).

Now, we assume that p does not divide all monomial monolithic character degrees of G. Since G/M has a normal Sylow p-subgroup by induction, we see that $p \neq q$. Thus, $K := MP$ becomes a normal subgroup of G for $P \in \mathrm{Syl}_p(G)$. It follows by a Frattini argument that $G = K\mathrm{N}_G(P) = MP\mathrm{N}_G(P) = M\mathrm{N}_G(P)$. Also, we deduce that $M \cap \mathrm{N}_G(P) = 1$, that is, G splits over the minimal normal subgroup M. Let $1_M \neq \lambda \in \mathrm{Irr}(M)$ and $T := \mathrm{I}_G(\lambda)$. It can be easily seen that T also splits over M and λ is invariant in T. By Lemma 2.3, we conclude that there exists a linear character θ of T such that $\theta_M = \lambda$. Since T is the inertia group of λ, by the Clifford correspondence we deduce that $\theta^G = \chi$ for an irreducible character χ of G. If $\ker\chi > 1$, then $M \leq \ker\theta_M = \ker\lambda$, which is a contradiction. It follows that χ is a monomial and monolithic character of G and so $p \nmid \chi(1)$ from the hypothesis of (ii). On the other hand, since M is an abelian and normal subgroup of K, we know by Ito's theorem that $\xi(1)$ divides $|K : M| = |P|$ for every irreducible character ξ of K. Since $p \nmid \chi(1)$, we conclude from Clifford's theorem that every irreducible constituent of χ_K must be linear and so $K = M \times P$ is an abelian group. This yields a contradiction $P \trianglelefteq G$ and completes the proof of the theorem. □

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It is a well-known result that any normal subgroup N of a finite group G is an intersection of the kernels of some irreducible characters of G. The following lemma is the slightly more general (but elementary) result for determining the normal subgroup structure of G.

LEMMA 2.4. *Every proper normal subgroup of G is the intersection of kernels of some monolithic characters of G.*

PROOF. Let $N \trianglelefteq G$ be the maximal counterexample among normal subgroups of G. First suppose that G/N has two distinct minimal normal subgroups M_1/N and M_2/N. Since $M_1/N \cap M_2/N = N$ and both the normal subgroups M_1 and M_2 are the intersections of the kernels of some monolithic characters of G, we arrive at a contradiction that N is the intersection of the kernels of some monolithic characters of G. Thus, we can assume that G/N has a unique minimal normal subgroup. This implies that G/N has a monolithic character χ such that $N = \ker\chi$, which contradicts our assumption. The conclusion follows. □

In the proof of Theorem 1.2, we will make use of the following two lemmas.

LEMMA 2.5 [1, Lemma 2]. *The intersection of the kernels of nonlinear monolithic characters of a finite group G is a subgroup $\mathrm{Z}(G)$. Also, the intersection of the kernels of all monolithic characters of G is trivial.*

Recall that a symplectic vector space V is a finite-dimensional vector space over a field F together with a nondegenerate, alternating and bilinear form w on V.

LEMMA 2.6 [7, Satz II.9.23]. *Let C be a cyclic group and V a finite symplectic vector space of order n^2. Suppose that C acts on V preserving the symplectic form. If V is an irreducible and faithful C-module, then $|C|$ divides $n + 1$.*

An irreducible character having a maximal kernel among the kernels of nonlinear irreducible characters of a solvable group G must be imprimitive and monolithic by [8, Lemma 12.3]. Also, if G has a nonlinear primitive character θ and G/N is nilpotent or supersolvable, then $\theta_N \in \mathrm{Irr}(N)$ by [8, Theorem 6.22]. In particular, $\theta_{G'} \in \mathrm{Irr}(G')$ and hence G' cannot be abelian. Now, we give the proof of Theorem 1.2.

PROOF OF THEOREM 1.2. Let χ be the unique imprimitive monolithic character of G. If G has a unique nonlinear monolithic character, then G is one of the Berkovich groups in Theorem 2.2. By the hypothesis of the theorem, we may assume that G has at least one nonlinear primitive monolithic character θ. Let $\ker\theta$ be maximal among the kernels of nonlinear primitive monolithic characters of G. By [5, Lemma 2.2], $\ker\theta < \ker\chi$. It can be seen immediately that $G/\ker\chi$ is either an extraspecial 2-group or a Frobenius group as in Theorem 2.1. First suppose that $G/\ker\chi$ is an extraspecial 2-group and set $\mathrm{Z}(G/\ker\theta) =: \mathrm{Z}(\overline{G})$. By [8, Corollary 6.13], $\mathrm{Z}(\overline{G}) > 1$. Since $\overline{G}/\mathrm{Z}(\overline{G})$ has exactly one nonlinear monolithic character χ, we see that $\overline{G}/\mathrm{Z}(\overline{G})$ is the nilpotent group in Theorem 2.2(i). This leads to a contradiction that $G/\ker\theta$ is a nilpotent group.

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We may now assume that $G/\ker\chi$ is a Frobenius group of order $p^a(p^a-1)$ for some prime p and integer a. Suppose that every nonlinear primitive monolithic character of G is faithful. We will show that $\ker\chi = \mathrm{Z}(G)$. Since we know that $\mathrm{Z}(G)$ is the unique maximal abelian normal subgroup of G from [8, Corollary 6.13], it is obvious that $\mathrm{Z}(G) = \mathrm{Z}(\ker\chi)$. Since $G/\mathrm{Z}(G)$ is the Berkovich group in Theorem 2.2(ii), we see that $\ker\chi/\mathrm{Z}(G)$ is abelian and so $\ker\chi$ is nilpotent. Since $p \nmid \chi(1) = p^a - 1$, we deduce from Theorem 1.1 that the Sylow p-subgroup of G is normal. Since G has a unique minimal normal subgroup M, $\mathrm{F}(G)$ is a p-group and hence $\ker\chi < \mathrm{F}(G) = G'\ker\chi$. Since $M < G' \leq \mathrm{F}(G)$ and $\mathrm{F}(G)/M$ is abelian, we obtain $\mathrm{F}'(G) = G'' = M$. Notice that $|M| = p$ because $\mathrm{Z}(G)$ is cyclic. Observing that $1 \neq \mathrm{Z}(G') \leq \ker\chi \cap G'$ and $M \leq \ker\chi \cap G'$, we obtain from Lemma 2.4 that $\mathrm{Z}(G') = M$. This shows that G' is an extraspecial p-group. Since $\theta_{G'} \in \mathrm{Irr}(G')$, we have $\theta(1)^2 = |G' : M|$. Since $G/\mathrm{F}(G)$ is abelian, we also get $\theta_{\mathrm{F}(G)} \in \mathrm{Irr}(\mathrm{F}(G))$. From [8, Theorem 2.31], $\theta(1)^2 = |\mathrm{F}(G) : \mathrm{Z}(G)|$ and hence $|\mathrm{F}(G)/\mathrm{Z}(G)| = |G'/M|$. Since $\mathrm{F}(G)/\mathrm{Z}(G) = \ker\chi/\mathrm{Z}(G) \times G'\mathrm{Z}(G)/\mathrm{Z}(G)$ and $G'\mathrm{Z}(G)/\mathrm{Z}(G) \cong G'/M$, from Lemma 2.4, we obtain $\ker\chi = \mathrm{Z}(G)$. On the other hand, from [9, Corollary 16.3], the cyclic group $G/\mathrm{F}(G)$ acts on $\mathrm{F}(G)/\mathrm{Z}(G)$ preserving the symplectic form and $\mathrm{F}(G)/\mathrm{Z}(G)$ is an irreducible and faithful $G/\mathrm{F}(G)$-module. From Lemma 2.6, $p^a - 1 = |G : \mathrm{F}(G)| \leq \sqrt{p^a} + 1$. It follows that $p^a = 3$ or 4. Since $\theta(1)^2 = |\mathrm{F}(G) : \mathrm{Z}(G)| = p^a$, we conclude $p^a = 4$ and hence $G/\mathrm{Z}(G) \cong \mathrm{A}_4$, as desired.

Now, we may assume that G has at least one nonfaithful monolithic character θ and that $\ker\theta$ is maximal among the kernels of nonlinear primitive monolithic characters of G. Since all nonlinear primitive monolithic characters of $G/\ker\theta$ are faithful and $1 < \ker\theta < \ker\chi$ by the first paragraph of the proof, $G/\ker\chi \cong \mathrm{A}_4$ and $\theta(1) = 2$. Thus, the Sylow 2-subgroup P of G is normal and nonabelian by Theorem 1.1. Since $1 \neq P' \leq \Phi(P)$ and $P' \trianglelefteq G$, we have $P' \leq \Phi(G)$. Notice that $\theta_P \in \mathrm{Irr}(P)$ for every nonlinear primitive character θ of G because G/P is abelian. Thus, $P' \leq \Phi(G) \not\leq \ker\theta$ for every nonlinear primitive monolithic character θ of G. Therefore, $G/\Phi(G)$ has exactly one nonlinear monolithic character χ. It follows from Theorem 2.2(ii) that $\mathrm{F}(G/\Phi(G)) = \mathrm{F}(G)/\Phi(G) = G'\ker\chi/\Phi(G)$ and this yields $G'\ker\chi = \mathrm{F}(G)$.

Suppose that there exists a primitive monolithic character φ of G with $\ker\varphi < \ker\theta$. Let $\widetilde{M} := M/\ker\varphi$ be the unique minimal normal subgroup of $\widetilde{G} := G/\ker\varphi$. Since $\mathrm{Z}(\widetilde{G})$ is cyclic and the unique maximal abelian subgroup among normal subgroups of $\widetilde{G}$, there exists $\widetilde{E} \trianglelefteq \widetilde{G}$ satisfying $\mathrm{F}(\widetilde{G}) = \widetilde{E}\,\mathrm{Z}(\widetilde{G})$ and $\widetilde{E} \cap \mathrm{Z}(\widetilde{G}) = \widetilde{M}$ from [9, Theorems 1.4(vii) and 1.9]. Therefore, $\mathrm{F}(\widetilde{G})/\widetilde{M} = \widetilde{E}/\widetilde{M} \times \mathrm{Z}(\widetilde{G})/\widetilde{M}$ becomes an abelian group. This shows that $\widetilde{M} = \mathrm{F}'(\widetilde{G}) \leq \ker\theta \leq \mathrm{F}(\widetilde{G})$, but this is a contradiction because $\theta_{\mathrm{F}(\widetilde{G})} \in \mathrm{Irr}(\mathrm{F}(\widetilde{G}))$. Thus, we conclude that every nonlinear primitive monolithic character of G has the maximal kernel among the kernels of nonlinear primitive monolithic characters. Hence, these primitive monolithic characters cannot be faithful.

Next, we shall show by induction on $|G|$ that $\ker\chi = \mathrm{Z}(G)$. If $\ker\lambda > 1$ for every nonlinear $\lambda \in \mathrm{Irr}(G)$, then $\mathrm{Z}(G/\ker\lambda) = \ker\chi/\ker\lambda = \mathrm{Z}(\lambda)/\ker\lambda$ by induction because $\ker\lambda < \ker\chi$. It follows from [8, Corollary 2.28] that $\mathrm{Z}(G) = \ker\chi$, as desired. Now, we suppose that G has a faithful irreducible character μ and then we will determine

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$\mu(1) = 2$. First let us consider the case where $\mathrm{F}(G) \neq P$ for the normal and nonabelian Sylow 2-subgroup P of G. Recall that G/P is abelian, which implies that $F := \mathrm{F}(G) = P \times A$, where A is an abelian group of odd order. Since F/A is nonabelian and $A \trianglelefteq G$, we have $A \leq \ker\chi$. By induction, $\mathrm{Z}(G/A) = \ker\chi/A$. Let K/A be maximal among abelian normal subgroups of F/A. It is well known that K/A is self centralising, that is, $\mathrm{Z}(F/A) = \mathrm{Z}(G/A) < K/A$, and hence $|F : K| = 2$. Thus, applying [8, Lemma 12.12] for the group F/A, we find $2 = |(F/A)'|$. Since $F' \leq P$, we see that $(F/A)' \cong F'$ of order two. Moreover, $G/\ker\theta$ has a unique minimal normal subgroup. Therefore, $F/\ker\theta$ is a 2-group and so $A \leq \ker\theta$ for every nonlinear primitive monolithic character θ of G. Also, by Lemma 2.5, $A \leq \mathrm{Z}(G)$ and hence $F'A \leq \mathrm{Z}(G)$. On the other hand, since $F' \leq \mathrm{Z}(F)$ and $\theta_F \in \mathrm{Irr}(F)$, we have $4 = \theta(1)^2 = |F : \mathrm{Z}(F)|$ from [8, Theorem 2.31]. Thus, every nonlinear irreducible character degree of F is equal to 2. Since $|G : F| = 3$, we see by [8, Corollary 11.29] that every nonlinear irreducible character degree of G may be 2, 3 or 6. Since P is nonabelian, by Clifford's theorem and Frobenius reciprocity, 2 divides the faithful character degrees of G. This shows that the faithful character degree $\mu(1) = 2$ or 6. Since G has an irreducible character χ of degree 3 and μ is faithful, μ must be primitive when $\mu(1) = 2$ and, by [8, Corollary 6.13], $\mathrm{Z}(F) \leq \mathrm{Z}(G) \leq \ker\chi < F$. Thus, $\mathrm{Z}(G) = \ker\chi$, as desired, because $4 = |F : \mathrm{Z}(F)|$. Suppose instead that $\mu(1) = 6$. Since $F' \leq G' \cap \ker\chi$ and $G'' \leq G' \cap \ker\chi$, we have $F' = G'' = G' \cap \ker\chi$ by Lemma 2.4. Also, since $G'/G'' \cong G'\ker\chi/\ker\chi = F/\ker\chi$ is of order four, $G' \cong \mathrm{Q}_8$. Since $|G : F| = 3$, we can pick an element x of G such that $|x| = 3$ and $x \notin A$. Then we have a normal subgroup $L := A \times (G' \rtimes \langle x\rangle)$ of G because $A \leq \mathrm{Z}(G)$. Notice that $L < G$ since G has an irreducible character μ of degree 6 and that $G' \rtimes \langle x\rangle \cong \mathrm{SL}(2, 3)$. If δ is an irreducible constituent of μ_L, then, by [8, Corollary 11.29], $\mu(1)/\delta(1)$ divides $|G : L|$, which is a power of two. Therefore, considering $\mathrm{cd}(L) = \{1, 2, 3\}$, we get $\delta(1) = 3$. Since the kernels of odd degree irreducible characters of L contain the group L'' of order two, we have a contradiction $L'' \leq \ker\mu_L = 1$. This contradiction shows that G cannot have a faithful irreducible character of degree 6.

Now, we restrict our attention to the case where $\mathrm{F}(G) = P$ for the normal nonabelian Sylow 2-subgroup P of G. Assume that there exists a minimal normal subgroup M of G such that $M \leq \ker\theta$ but $M \nleq F'$. By induction, we have $\mathrm{Z}(G/M) = \ker\chi/M = \mathrm{Z}(F/M)$. Applying [8, Theorem 12.12] for the group F/M, we get $|(F/M)'| = 2$ since $|F/M : K/M| = 2$ for the self-centralising normal subgroup K/M of F/M. Hence, $|F'| = 2$ because $F' \cap M = 1$. From [8, Theorem 2.31], $\theta(1)^2 = 4 = |F : \mathrm{Z}(F)|$ for every nonlinear primitive monolithic character θ of G. By a similar argument to that used in the case $\mathrm{F}(G) \neq P$, we can show that if the faithful character degree $\mu(1)$ is 2, then $\ker\chi = \mathrm{Z}(G)$. Using the same argument again, we get a normal subgroup $R := G' \rtimes \langle x\rangle$, where $G' \cong \mathrm{Q}_8$ and $x \in G$ of order three. If $\mu(1) = 6$, then by [8, Corollary 11.29] we reach a contradiction that $\mathrm{Z}(R) = 1$ as $\mu_R = 2\delta$ for $\delta \in \mathrm{Irr}(R)$ of degree 3.

From now on suppose, by way of contradiction, that if $M \leq \ker\theta$ for any nonlinear primitive monolithic character θ and minimal normal subgroup M of G, then $M \leq F'$. Recall that G has no faithful primitive monolithic character as we have seen before. Let N be the intersection of the kernels of nonlinear monolithic characters of G. If

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$N > 1$, then $N \cap F' \leq N \cap G' = 1$ from Lemma 2.5, which contradicts our assumption. Hence, N must be trivial. It follows that G has a minimal normal subgroup S and a nonlinear primitive monolithic character $\tilde{\theta}$ such that $S \leq \ker\theta$ but $S \nleq \ker\tilde{\theta}$. Since primitive characters are homogeneous and $S \cap \ker\tilde{\theta} = 1$, we see that $\tilde{\theta}_S = 2\alpha$ for $1_S \neq \alpha \in \mathrm{Irr}(S)$. Since the elementary abelian group S has a faithful irreducible character, we get $|S| = 2$. Again by induction and [8, Theorem 12.12], we obtain $|(F/S)'| = 2$. From our assumption $S \leq F'$, we already know that $|F'| = 4$. Since G' must be nonabelian, we see that $G'' = F' < G'$ from Lemma 2.4. Since $G'/G'' \cong G'\ker\chi/\ker\chi$ has order four, we obtain $|G'| = 16$ and so $|\mathrm{Z}(G')| = 2$. Therefore, $G' \cong \mathrm{D}_8, \mathrm{Q}_{16}$ or SD_{16} whose automorphism groups are 2-groups. Consider $G' \rtimes \mathrm{C}_3 =: V \trianglelefteq G$. Then $\mathrm{N}_V(G')/\mathrm{C}_V(G') = V/\mathrm{C}_V(G')$ is a subgroup of the 2-group $\mathrm{Aut}(G')$ and so $\mathrm{C}_3 \leq \mathrm{C}_V(G')$. This shows that $G' \times \mathrm{C}_3 = V \trianglelefteq G$, giving $\mathrm{C}_3 \trianglelefteq G$, which is a contradiction since the Fitting subgroup F is a 2-group. From this contradiction, it follows that $G/\mathrm{Z}(G) \cong \mathrm{A}_4$.

Observe that since G' must be nonabelian, $1 < G'' = G' \cap \ker\chi$ from Lemma 2.4. On the other hand, since G has a unique monomial monolithic character χ of degree 3, we conclude by Theorem 1.1 that $G \cong N \rtimes Q$, where $Q \in \mathrm{Syl}_3(G)$ and N is the nilpotent Hall $3'$-subgroup of G. By [2, Proposition 30.18(c)], the Sylow p-subgroups of N are abelian for every prime p different from two. We observe that $N \nleq \ker\chi$, which implies that $G/N \cong Q$ is an abelian group, as desired in (iii) of the theorem.

For the converse of the theorem, it is easy to check that G has a unique imprimitive monolithic character if (i) or (ii) of the theorem holds. Suppose that $G/\mathrm{Z}(G) \cong \mathrm{A}_4$. By [8, Corollaries 2.28 and 2.30], it is immediate that $\mathrm{Z}(G) = \mathrm{Z}(\varphi)$ for every nonlinear irreducible character φ of G and $\mathrm{cd}(G) = \{1, 2, 3\}$. Let $\ker\chi = \mathrm{Z}(G)$ for a nonlinear irreducible character χ of G. Thus, χ is imprimitive and monolithic of degree 3. If χ is the unique nonlinear monolithic character of G, then G is a group as in (i) and (ii) of the theorem. Therefore, assume that G has another nonlinear monolithic character φ and that φ is imprimitive too. We know that $\ker\varphi < \mathrm{Z}(G) = \ker\chi$. If $\varphi(1) = 2$, then there exist a proper subgroup $H/\ker\varphi$ of $G/\ker\varphi$ and $\lambda \in \mathrm{Lin}(H/\ker\varphi)$ such that $\lambda^G = \varphi$. Since φ is a faithful character of $G/\ker\varphi$ and $|G : H| = 2$, the factor group $H/\ker\varphi$ must be an abelian normal subgroup of $G/\ker\varphi$. It follows from Ito's theorem that every nonlinear character degree of $G/\ker\varphi$ is equal to 2, which contradicts $\chi(1) = 3$. Thus, we may assume that $\varphi(1) = 3$. Since $G/\ker\varphi$ has a unique minimal normal subgroup and $\varphi(1) = 3$, by Clifford's theorem the Fitting subgroup $\mathrm{F}(G/\ker\varphi) = \mathrm{F}(G)/\ker\varphi$ becomes an abelian Sylow 2-subgroup of $G/\ker\varphi$. Since $G/\ker\varphi$ has the unique minimal normal subgroup of order two and $\ker\varphi < \mathrm{Z}(G)$, we get a contradiction that $2 \nmid |(G/\ker\varphi)' \cap \mathrm{Z}(G)/\ker\varphi|$ by applying [8, Theorem 5.6]. Thus, if $G/\mathrm{Z}(G) \cong \mathrm{A}_4$, then G has exactly one imprimitive monolithic character. This completes the proof. □

As an immediate consequence of Theorem 1.2, we have the following result.

COROLLARY 2.7. *Let G be a finite solvable group having a unique imprimitive monolithic character. If G is not a Berkovich group, then $cd(G) = cd_m(G) = \{1, 2, 3\}$, where $cd_m(G)$ is the set of degrees of monolithic characters of G.*

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PROOF. By the last paragraph of the proof of Theorem 1.2, it is easy to see that $\mathrm{cd}(G) = \mathrm{cd}_m(G) = \{1, 2, 3\}$ since $G/\mathrm{Z}(G) \cong \mathrm{A}_4$. In fact, G has a unique monolithic character of degree 3 and the kernel of this character is equal to $\mathrm{Z}(G)$. □

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