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Base invariance of the manipulability index

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SUMMARY

The manipulability index suggested by Yoshikava is an important tool for the design of mechanisms and their control. It represents a quantitative measure of the functionality and the ability for realizing some tasks or groups of tasks. This index is some kind of performance measure and should be taken into consideration in the design phase of a mechanism and also in the design of control algorithms.

In this paper two important properties of the manipulability index are investigated. The first part of the present work demonstrates that manipulability of a mechanism is independent of task space coordinates. In the second part, a proof of the independency of the manipulability index on the first DOF is given.

This invariance is important for simplification of the mechanism's Jacobian matrix and gives excellent insight into the dependences of configuration space coordinates on this index. Moreover, it proves that the manipulability index is determined only by relative positions of the mechanism itself and by the mechanism's geometry.

Finally, the properties of the manipulability index are illustrated by some examples for fundamental open kinematical chain structures.

KEYWORDS: Robots; Manipulability index; Invariance.

1. INTRODUCTION

Matrix analysis is a valuable tool to study a physical problem. However, some limitations result from the fact that the physical system does not know which coordinate system in a task space is used. To explain where the physical system – the mechanism – is in the environment and what it does, the configuration space vector and the task space vector are introduced and must be related. The configuration space vector $\vec{q}$ has components $q_i, i = 1 \dots m$ which usually are the relative joint displacements and relative joint angles, respectively. The coordinate system of the task space is defined by the class of tasks to be handled and can be described by n variables $x_j, j = 1 \dots n$ and $n \leq m$. The relation between q_i and x_j is given by the – in general highly nonlinear-equation

$$\vec{x} = \vec{f}(\vec{q}). \tag{1}$$

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Velocity $\dot{\vec{x}}$ in task space is related to the joint velocity $\dot{\vec{q}}$ by

$$\dot{\vec{x}} = J(\vec{q})\,\dot{\vec{q}} \tag{2}$$

where $\dot{\vec{x}} = \frac{d\vec{x}}{dt}$, $\dot{\vec{q}} = \frac{d\vec{q}}{dt} \in \Re^m$ with $m \geq n$. In most cases $\dot{\vec{x}} \in \Re^n$, the n-dimensional Euclidean space. $J(\vec{q})$ denotes the Jacobian matrix with dimension $(n \times m)$.

1.1. Manipulability index

Yoshikava[1–3] was the first who introduced the manipulability index in its basic mathematical form. The scalar quantity is defined for non-redundant mechanisms $(m = n)$ as

$$w = |\det(J)| \tag{3}$$

and for redundant mechanisms $(m > n)$ by

$$w = \sqrt{\det(J\,J^T)}. \tag{4}$$

It describes the mechanism's ability to perform a certain task. If $w = 0$ then the mechanism is in a singular position. Therefore, w should be as large as possible. The mechanisms treated in this work have an open serial kinematic chain structure. For mechanisms, which have both rotational and prismatic joints, the entries of J have different physical dimension. Hence,[4,5] (3) has to be replaced by

$$w = |\det(J\,M)| \tag{5}$$

and (4) by

$$w = \sqrt{\det(J\,M(J\,M)^T)} \tag{6}$$

where M is a suitably chosen matrix which guarantees consistency of physical dimensions.

2. MANIPULABILITY INDEX AND TASK SPACE COORDINATES

In this section we want to show the influence of a change of coordinates on the manipulability index of open kinematic chain structures. The main point is that the transformation matrix for a new base frame is also a Jacobian matrix. The manipulability index of Yoshikava has the following property: If two open chain structures C_1 and C_2 with manipulability indices w_1 and w_2 are given and $w_1 > w_2$ holds, then the structure C_1 is better adopted to perform the given task. Hence, it is not the absolute size of w but the relative size of w_1 and w_2 which are important for comparison of chain structures.

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Take a system, which is described geometrically by the vector function (1). Then, velocities in task space $\vec{x}$ and configuration space $\vec{q}$ are related by (2),[6*]

$$\dot{\vec{x}} = \frac{\partial \vec{f}}{\partial \vec{q}^T}\frac{d\vec{q}}{dt} = \frac{\partial \vec{f}}{\partial \vec{q}^T}\dot{\vec{q}} = J\dot{\vec{q}}. \tag{7}$$

Introduce now a new coordinate frame (with elements $\vec{y}$) in task space according

$$\vec{y} = S\vec{x} \tag{8}$$

where S is a constant regular matrix. Multiplication of (1) by S yields

$$\vec{y} = S\vec{f}(\vec{q}). \tag{9}$$

Differentiation with respect to time gives

$$\frac{d\vec{y}}{dt} = \dot{S}\vec{f}(\vec{q}) + S\dot{\vec{f}}(\vec{q}) = S\dot{\vec{f}}(\vec{q}(t)) \tag{10}$$

because the matrix S is constant. Use of the Jacobian results in

$$\dot{\vec{y}} = SJ(\vec{q})\,\dot{\vec{q}}. \tag{11}$$

Comparison of (11) with (1) leads to the question: "Is $SJ(\vec{q})$ also a Jacobian matrix?" The answer is "yes" as can be seen as follows. First, rewrite the new geometric relation (9) as

$$\vec{y} = S\vec{f}(\vec{q}) = \vec{f}_S(\vec{q}) \tag{12}$$

where $\vec{f}_S(\vec{q})$ gives the relation between the configuration space coordinates $\vec{q}$ and the new task space coordinates $\vec{y}$. Differentiation with respect to time gives

$$\dot{\vec{y}} = \frac{\partial \vec{f}_S}{\partial \vec{q}^T}\frac{d\vec{q}}{dt} = \frac{\partial \vec{f}_S}{\partial \vec{q}^T}\dot{\vec{q}} \tag{13}$$

with

$$\frac{\partial \vec{f}_S}{\partial \vec{q}^T} = \frac{\partial}{\partial \vec{q}^T}(S\vec{f}) = S\frac{\partial \vec{f}}{\partial \vec{q}^T} = SJ(\vec{q}) \tag{14}$$

because S is constant and hence, independent of $\vec{q}$.

Let $J_S(\vec{q})$ denote the Jacobian of $\vec{f}_S(\vec{q})$. Consequently, $SJ(\vec{q})$ is also a Jacobian and the two Jacobians are related by

$$S\frac{\partial \vec{f}}{\partial \vec{q}^T} = \frac{\partial \vec{f}_S}{\partial \vec{q}^T} \Leftrightarrow SJ(\vec{q}) = J_S(\vec{q}). \tag{15}$$

[*] We use throughout the following notation: If $\vec{a} \in \Re^n$ and $\vec{b} \in \Re^m$ are vectors, then $\frac{d\vec{a}}{d\vec{b}^T}$ denotes the matrix with n rows and m columns

$$\frac{d\vec{a}}{d\vec{b}^T} = \left[\frac{\partial \vec{a}}{\partial b_1}\frac{\partial \vec{a}}{\partial b_2}\cdots\frac{\partial \vec{a}}{\partial b_m}\right] = \begin{bmatrix} \frac{\partial a_1}{\partial b_1} & \cdots & \frac{\partial a_1}{\partial b_m} \\ \vdots & & \vdots \\ \frac{\partial a_n}{\partial b_1} & \cdots & \frac{\partial a_n}{\partial b_m} \end{bmatrix}$$

From this results that computation of the manipulability index can be reasonably simplified by appropriate choice of the coordinate frame in task space.

For a non-redundant open kinematic chain structures

$$w^* = |\det(J_S)| = |\det(SJ)| = |\det S \det J| = |\det S| \cdot |\det J| = C \cdot w \tag{16}$$

with

$$C = |\det(S)|. \tag{17}$$

For redundant open kinematic chain structures

$$w^* = \sqrt{\det\left(J_S J_S^T\right)} = \sqrt{\det(SJ(SJ)^T)} \tag{18}$$

$$(SJ)^T = J^T S^T \tag{19}$$

$$\begin{aligned} w^* &= \sqrt{\det(SJJ^TS^T)} = \sqrt{\det(S)\det(JJ^T)\det(S^T)} \\ &= \sqrt{\det(SS^T)} \cdot \sqrt{\det(JJ^T)} \\ &= C\sqrt{\det(JJ^T)} = Cw \end{aligned} \tag{20}$$

with

$$C = \sqrt{\det(SS^T)} = |\det(S)|. \tag{21}$$

Hence, in both cases the manipulability indices w and w^* are related by

$$w^* = Cw. \tag{22}$$

The common factor C is defined by the transformation matrix S as

$$C = |\det(S)|. \tag{23}$$

The preceding investigations can be summarized, as follows:

Proposition 1: Let a chain mechanism with task space coordinates $\vec{x}$ and configuration $\vec{q}$ with geometric transformation (1) and Jacobian (2) be given. A coordinate transformation (9) in task space with a constant and regular matrix S results in the geometric relation (12). The two Jacobians are related by (15). Moreover, manipulability indices are related by (22).

From this results a fact which is important for the evaluation of the suitability of mechanisms for certain task:

Proposition 2: For two mechanisms with manipulability indices w_i and w_i^* $(i = 1, 2)$

$$\frac{w_1^*}{w_2^*} = \frac{w_1}{w_2} \tag{24}$$

holds. Therefore, their quality for certain tasks can be judged by using any suitable coordinate frame in task space.

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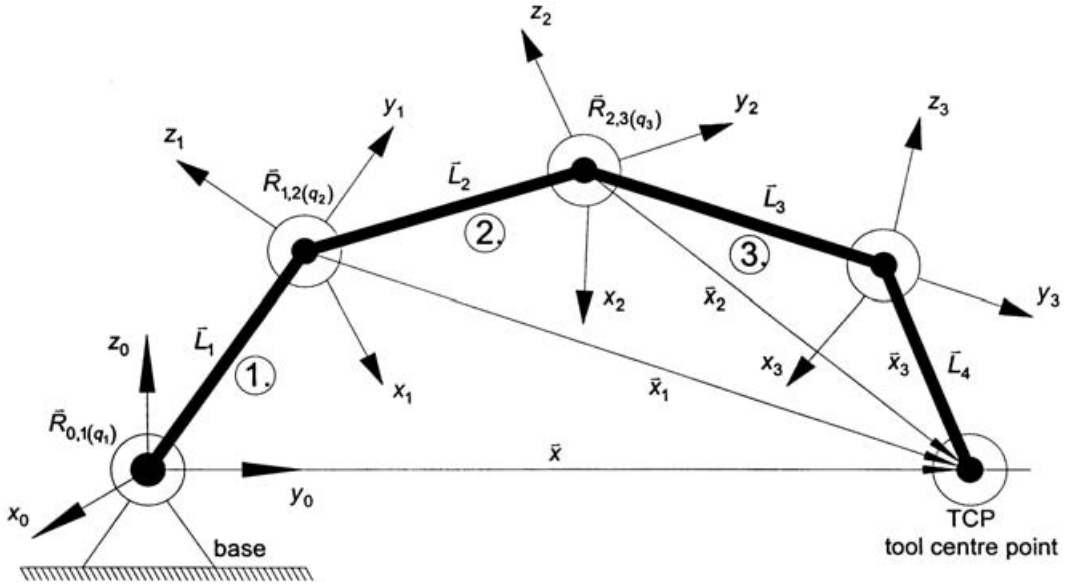


Fig. 1. Open loop chain mechanism.

3. MANIPULABILITY INDEX IS INDEPENDENT OF THE RELATIVE POSITION OF THE FIRST LINK

For manipulators with the reference frame attached at the first joint, the manipulability index w is independent of rotation or translation, respectively, of the first joint; it is a function of rotations or translations of the other joints only. This is due to the fact that rotation or translation, respectively, of the first joint is equivalent to rotation or translation respectively of the reference frame.[7] Therefore the position of the mechanism in the task space has no influence on the manipulability index. Its relative value depends only on the chosen kinematical structure of the mechanism and the inner relative positions of the mechanism elements. To show this, we take a close look on expression (3) for non-redundant mechanisms ($n = m$) and (4) for redundant mechanisms ($m > n$), respectively.

The elements of the Jacobian matrix J in equation (2) are dependent on the relative positions q_i and on the geometry of the links. When considering only positions of the links, the transformation between the configuration space coordinates and the task space coordinates is given by (1). Putting the local coordinate frame properly into each link, Fig. 1, equation (1) can be written in more detailed form as

$$\vec{x} = R_{0,1}(\vec{x}_1 + \vec{l}_1) \tag{25a}$$

$$\vec{x}_1 = R_{1,2}(\vec{x}_2 + \vec{l}_2) \tag{25b}$$

$$\vdots$$

$$\vec{x}_{m-1} = R_{m-1,m}(\vec{x}_m + \vec{l}_m) \tag{25c}$$

$$\vec{x}_m = \vec{l}_{m+1} \tag{25d}$$

with $R_{i-1,i}$ transformation matrix between the coordinate frames "i" and "$i-1$" with $R_{i-1,i} = R_{i-1,i}(q_i)$ if the "i" joint is rotational and independent of q_k for all $k \neq i$ and $R_{i-1,i} = I$ if the joint "i" is a translational joint.

$\vec{x}_i$ vector from the origin of local coordinate frame "i" to the end-effector, $\vec{l}_i$ link geometry vector given in local coordinate frame "i", $\vec{l}_i = \vec{e}_i$ is constant for rotational joint "i", $\vec{l}_i = q_i\vec{e}_i$ if the joint "i" is translational. The unit vector $\vec{e}_i$ is the motion direction vector.

The description of mechanism is separated into its geometric and joint properties. The geometric properties are contained in the vector $\vec{l}_i$ for rotational joints and in the vector $\vec{e}_i$ for translational joint. It is important to note that each link parameter is presented in its own coordinate frame. The properties of the joints are given in transformation matrices "$R_{i-1,i}$" for rotational joints and in vector $\vec{l}_i$ for translational joints. Moreover, the vector $\vec{x}_i$ is given as

$$\begin{aligned}\vec{x}_i &= R_{i,i+1}(\vec{x}_{i+1} + \vec{l}_{i+1}) \\ &= R_{i,i+1}R_{i+1,i+2}\ldots R_{m-1,m}(\vec{l}_{m+1} + \vec{l}_m) \\ &\quad + R_{i,i+1}\ldots R_{m-2,m-1}\vec{l}_{m-1} + \ldots + R_{i,i+1}\vec{l}_{i+1} \\ &= R_{i,m}(\vec{l}_{m+1} + \vec{l}_m) + R_{i,m-1}\vec{l}_{m-1} + \ldots + R_{i,i+1}\vec{l}_{i+1} \\ &= \vec{x}_i(q_{i+1}, q_{i+2}, \ldots, q_m)\end{aligned} \tag{26}$$

with $\vec{l}_i = \vec{e}_i =$ constant if the i-th joint is rotational and with

$$R_{i,k} = R_{i,i+1}R_{i+1,i+2}\ldots R_{k-1,k}, \quad 0 \leq i < k \leq m \tag{27}$$

and especially

$$R_{0,i} = R_{0,1}R_{1,2}\cdots R_{i-1,i}. \tag{28}$$

The columns of the Jacobian matrix can be calculated from (25a):

$$\begin{aligned}J = \frac{\partial\vec{x}}{\partial\vec{q}^T} &= \left[\frac{\partial\vec{x}}{\partial q_1}, \frac{\partial\vec{x}}{\partial q_2}, \cdots \frac{\partial\vec{x}}{\partial q_m}\right] \\ &= \left[\frac{\partial}{\partial q_1}(R_{0,1}(\vec{x}_1 + \vec{l}_1)), \frac{\partial}{\partial q_2}(R_{0,1}(\vec{x}_1 + \vec{l}_1)),\right. \\ &\quad \left.\cdots \frac{\partial}{\partial q_m}(R_{0,1}(\vec{x}_1 + \vec{l}_1))\right]\end{aligned} \tag{29}$$

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According to (26) the vector $\vec{x}_i$ depends only on configuration space coordinates with index from "$i+1$" to 'm". Consequently

$$\frac{\partial}{\partial q_1}(R_{0,1}(\vec{x}_1+\vec{l}_1)) = \begin{cases} \frac{\partial R_{0,1}}{\partial q_1}(\vec{x}_1+\vec{l}_1)\ldots \text{ rotational joint} \\ \vec{e}_1 \ldots \text{ translational joint.} \end{cases} \tag{30}$$

Case 1:
Consider that the first DOF is a rotation around an arbitrary axes $\vec{a}$, which goes through the origin of the fixed reference frame x_{OR}, y_{OR} and z_{OR}. In this case the matrix $R_{0,1}$ can be written in the form:

$$R_{0,1} = B R_a B^{-1} \tag{31}$$

$$R_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_1 & -\sin q_1 \\ 0 & \sin q_1 & \cos q_1 \end{bmatrix} \tag{32}$$

R_a is chosen to be the first of the elementary orthogonal rotational matrices R_x, R_y and R_z. This choice is free, but determines the matrix B. For our choice matrix B is a matrix composed of three vectors,

$$B = \lfloor \vec{a} \quad \vec{b} \quad \vec{c} \rfloor \tag{33}$$

where vector $\vec{a}$ is determined by the direction of the rotation axes of the first DOF and has length $\|\vec{a}\| = 1$. Vector $\vec{b}$ is orthogonal to the vector $\vec{a}$ ($\vec{a}^T\vec{b} = 0$) with $\|\vec{b}\| = 1$. Vector $\vec{c}$ is chosen such that a Cartesian coordinate system results, what yields $\vec{c} = \vec{a} \times \vec{b}$, where $\vec{a} \times \vec{b}$ denotes the usual vector product. Matrix $R_{0,1}$ is regular. Hence, there exists a matrix W such that

$$\frac{\partial R_{0,1}}{\partial q_1} = R_{0,1} W. \tag{34}$$

Matrix B in (33) is given in the fixed reference coordinate frame and is therefore independent of q_1. Consequently differentiation of (31) gives

$$\frac{\partial R_{0,1}}{\partial q_1} = B \frac{\partial R_a}{\partial q_1} B^{-1} = B R_a W_a B^{-1}. \tag{35}$$

Direct computation shows that for the rotation matrix R_a given by (32), W_a is the skew symmetric constant matrix[8]

$$W_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{36}$$

Hence,

$$\frac{\partial R_a}{\partial q_1} = R_a W_a \tag{37}$$

(31) can be rewritten as

$$R_a = B^{-1} R_{0,1} B \tag{38}$$

and (35) gives

$$\frac{\partial R_{0,1}}{\partial q_1} = B B^{-1} R_{0,1} B W_a B^{-1} = R_{0,1} W \tag{39}$$

where

$$W = B W_a B^{-1} \tag{40}$$

is a constant matrix. Equation (39) holds for an arbitrary orientation of the first DOF in the fixed coordinate frame.

For columns of the Jacobian matrix (except for the first column) in (29)

$$\frac{\partial}{\partial q_i}(R_{0,1}(\vec{x}_1+\vec{l}_1)) = R_{0,1}\left(\frac{\partial \vec{x}_1}{\partial q_1}\right) \qquad i = 2, \ldots, m \tag{41}$$

holds. Substituting the resulting equations (39) and (41) into the Jacobian matrix (29) gives for a rotational first joint

$$\begin{aligned} J = \frac{\partial \vec{x}}{\partial \vec{q}^T} &= \left[\frac{\partial R_{0,1}}{\partial q_1}(\vec{x}_1+\vec{l}_1),\, R_{0,1}\left(\frac{\partial \vec{x}_1}{\partial q_2}\right), \ldots R_{0,1}\left(\frac{\partial \vec{x}_1}{\partial q_m}\right)\right] \\ &= \left[R_{0,1} W(\vec{x}_1+\vec{l}_1),\, R_{0,1}\left(\frac{\partial \vec{x}_1}{\partial q_2}\right), \ldots R_{0,1}\left(\frac{\partial \vec{x}_1}{\partial q_m}\right)\right] \\ &= R_{0,1}\left[W(\vec{x}_1+\vec{l}_1),\, \left(\frac{\partial \vec{x}_1}{\partial q_2}\right), \ldots \left(\frac{\partial \vec{x}_1}{\partial q_m}\right)\right]. \end{aligned} \tag{42}$$

Introducing

$$\tilde{J} = \left[W(\vec{x}_1+\vec{l}_1),\, \left(\frac{\partial \vec{x}_1}{\partial q_2}\right), \ldots \left(\frac{\partial \vec{x}_1}{\partial q_m}\right)\right] \tag{43}$$

where $\tilde{J} = \tilde{J}(q_2, \ldots, q_m)$ is independent of q_1, the Jacobian can be written as

$$J = R_{0,1}\tilde{J} = R_{0,1}\tilde{J}(q_2, \ldots, q_m) \tag{44}$$

$R_{0,1}$ is a rotation matrix with $\det(R_{0,1}) = 1$ and therefore, the manipulability index w according (3) and (4), respectively, is independent of q_1.

Case 2:
If the first joint is prismatic – pure translation – then matrix $R_{0,1}$ is the identity matrix I and $\vec{l}_1 = q_1\vec{e}_1$ and

$$\frac{\partial \vec{x}}{\partial q_1} = \frac{\partial \vec{l}_1}{\partial q_1} = \vec{e}_1. \tag{45}$$

All other columns are again given by (41) with $R_{0,1} = I$.

Vector $\vec{e}_1$ is the direction unit vector of the translational axes and is constant in the local coordinate frame "1". Therefore, the Jacobian matrix can be written as

$$J = \frac{\partial \vec{x}}{\partial \vec{q}^T} = \left[\vec{e}_1,\, \frac{\partial \vec{x}_1}{\partial q_2}, \cdots \frac{\partial \vec{x}_1}{\partial q_m}\right] \tag{46}$$

and is independent of q_1 also for translational first DOF. Consequently, w is also independent of q_1.

This completes the proof.

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For mechanisms with rotational first DOF, the following relations can be helpful.

For non-redundant mechanisms we have because of $\det(R_{0,1}) = 1$,

$$w = |\det(J)| = |\det(R_{0,1}\tilde{J})| = |\det(\tilde{J})|. \tag{47}$$

For redundant mechanisms with rotational first DOF we have:

$$w = \sqrt{\det(J J^T)} = \sqrt{\det(R_{0,1}\tilde{J}(R_{0,1}\tilde{J})^T)}$$
$$= \sqrt{\det(R_{0,1})\det(\tilde{J}\tilde{J}^T)\det\left(R_{0,1}^T\right)}$$

and hence,

$$w = \sqrt{\det(J J^T)} = \sqrt{\det(\tilde{J}\tilde{J}^T)}. \tag{48}$$

Theorem: For both, redundant and non-redundant open chain mechanisms, the manipulability index w of Yoshikava is independent of the first DOF no matter whether this DOF which connects the mechanism with the base is rotational or prismatic.

4. EXAMPLES

To illustrate the previous theoretic investigations some examples are given. The first two examples are plane mechanisms with two and three rotational joints respectively. The next two mechanisms are also plane mechanisms but each one has a prismatic joint. The last mechanism has four rotational joints in $\Re^3$. For each of these five examples, the Jacobian matrix, the manipulability index, the matrix $\tilde{J}$ and the manipulability index from the matrix $\tilde{J}$ are calculated.

Example 1: Plane non-redundant mechanism with two rotational DOF's.

For this case, Fig. 2, the matrix (31) and the skew symmetric matrix (36) are

$$R_{0,1} = \begin{bmatrix} \cos\varphi_1 & -\sin\varphi \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix} \quad W = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The Jacobian matrix from (29) is

$$J = \begin{bmatrix} -l_1\sin\varphi_1 - l_2\sin(\varphi_1+\varphi_2) & -l_2\sin(\varphi_1+\varphi_2) \\ l_1\cos\varphi_1 + l_2\cos(\varphi_1+\varphi_2) & l_2\cos(\varphi_1+\varphi_2) \end{bmatrix}.$$

The manipulability index according (3) is

$$w = |\det(J)| = |l_1 l_2 \sin\varphi_2|.$$

The matrix $\tilde{J}$ from (42–43) reads as

$$J = R_{0,1}\tilde{J} = \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix} \cdot \begin{bmatrix} -l_2\sin\varphi_2 & -l_2\sin\varphi_2 \\ l_1 + l_2\cos\varphi_2 & l_2\cos\varphi_2 \end{bmatrix}$$

$$\tilde{J} = \begin{bmatrix} -l_2\sin\varphi_2 & -l_2\sin\varphi_2 \\ l_1 + l_2\cos\varphi_2 & l_2\cos\varphi_2 \end{bmatrix}.$$

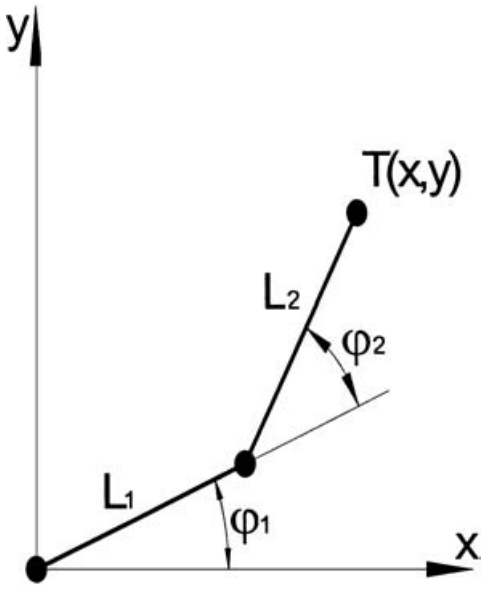


Fig. 2. Mechanism with two DOF.

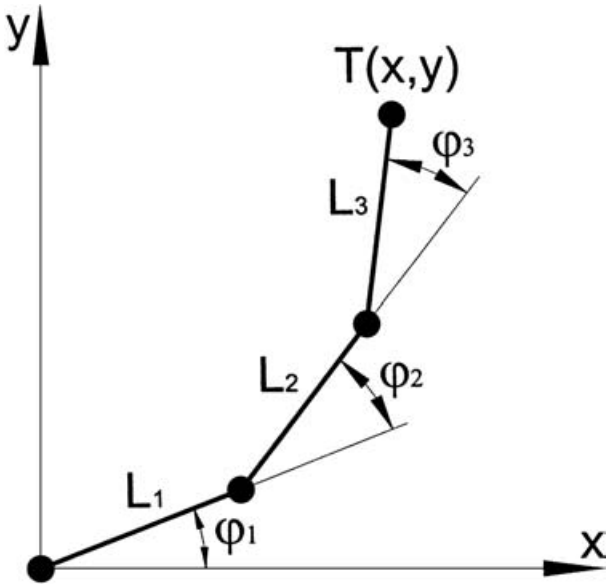


Fig. 3. Plane mechanism with three DOF (redundant mechanism for the point T(x,y)).

The manipulability index from (47) is easier to calculate and again

$$w_{(\tilde{J})} = |\det(\tilde{J})| = |l_1 l_2 \sin\varphi_2|.$$

The Jacobian matrix from (15) transformed into the last local coordinate frame on the mechanism is

$$J_{ef} = \begin{bmatrix} l_1\sin\varphi_2 & 0 \\ l_2 + l_1\cos\varphi_2 & l_2 \end{bmatrix}.$$

Example 2: Redundant plane mechanism with three rotational DOFs.

For the mechanism of Fig. 3, the matrix $R_{0,1}$ (31) and W of (34) are

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$$R_{0,1} = \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix} \quad W = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

respectively the Jacobian matrix, (29) is

$$J = \begin{bmatrix} -l_1\sin\varphi_1 - l_2\sin(\varphi_1+\varphi_2) - l_3\sin(\varphi_1+\varphi_2+\varphi_3) & -l_2\sin(\varphi_2+\varphi_3) - l_3\sin(\varphi_1+\varphi_2+\varphi_3) & -l_3\sin(\varphi_1+\varphi_2+\varphi_3) \\ l_1\cos\varphi_1 + l_2\cos(\varphi_1+\varphi_2) + l_3\cos(\varphi_1+\varphi_2+\varphi_3) & l_2\cos(\varphi_2+\varphi_3) + l_3\cos(\varphi_1+\varphi_2+\varphi_3) & l_3\cos(\varphi_1+\varphi_2+\varphi_3) \end{bmatrix}.$$

The manipulability index according (4) is

$$\begin{aligned} w &= \sqrt{\det(JJ^T)} \\ &= \Big(\frac{l_1^2l_2^2}{2} + l_1^2l_3^2 + l_2^2l_3^2 + l_1l_2l_3^2\cos\varphi_2 - \frac{1}{2}l_1^2l_2^2\cos(2\varphi_2) \\ &\quad + l_1^2l_2l_3\cos\varphi_3 - l_2^2l_3^2\cos(2\varphi_3) - l_1^2l_3^2\cos(2\varphi_2+\varphi_3) \\ &\quad - l_1^2l_2l_3\cos(2\varphi_2+\varphi_3) - l_1l_2l_3^2\cos(\varphi_2+2\varphi_3)\Big)^{\frac{1}{2}}. \end{aligned}$$

The matrix $\tilde{J}$ from (43–44) is

$$J = R_{0,1}\tilde{J} = \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix}.$$

$$\begin{bmatrix} -l_2\sin\varphi_2 - l_3\sin(\varphi_2+\varphi_3) & -l_2\sin\varphi_2 - l_3\sin(\varphi_2+\varphi_3) & -l_3\sin(\varphi_2+\varphi_3) \\ l_1 + l_2\cos\varphi_2 + l_3\cos(\varphi_2+\varphi_3) & l_2\cos\varphi_2 + l_3\cos(\varphi_2+\varphi_3) & l_3\cos(\varphi_2+\varphi_3) \end{bmatrix}$$

$$\tilde{J} = \begin{bmatrix} -l_2\sin\varphi_2 - l_3\sin(\varphi_2+\varphi_3) & -l_2\sin\varphi_2 - l_3\sin(\varphi_2+\varphi_3) & -l_3\sin(\varphi_2+\varphi_3) \\ l_1 + l_2\cos\varphi_2 + l_3\cos(\varphi_2+\varphi_3) & l_2\cos\varphi_2 + l_3\cos(\varphi_2+\varphi_3) & l_3\cos(\varphi_2+\varphi_3) \end{bmatrix}.$$

The manipulability index according (48) is

$$\begin{aligned} w_{(\tilde{J})} &= \sqrt{\det(\tilde{J}\tilde{J}^T)} \\ &= \Big(\frac{l_1^2l_2^2}{2} + l_1^2l_3^2 + l_2^2l_3^2 + l_1l_2l_3^2\cos\varphi_2 - \frac{1}{2}l_1^2l_2^2\cos(2\varphi_2) \\ &\quad + l_1^2l_2l_3\cos\varphi_3 - l_2^2l_3^2\cos(2\varphi_3) - l_1^2l_3^2\cos(2\varphi_2+\varphi_3) \\ &\quad - l_1^2l_2l_3\cos(2\varphi_2+\varphi_3) - l_1l_2l_3^2\cos(\varphi_2+2\varphi_3)\Big)^{\frac{1}{2}}. \end{aligned}$$

The Jacobian matrix (15) transformed into the last local coordinate frame of the mechanism is

$$J_{ef} = \begin{bmatrix} l_2\sin\varphi_3 + l_1\sin(\varphi_2+\varphi_3) & l_2\sin\varphi_3 & 0 \\ l_3 + l_2\cos\varphi_3 + l_1\cos(\varphi_2+\varphi_3) & l_3 + l_2\cos\varphi_3 & l_3 \end{bmatrix}.$$

Example 3: Plane mechanism with one prismatic DOF and one rotational DOF.

Here, Fig. 4, $R_{0,1} = I$ and the Jacobian matrix according (29) is

$$J = \begin{bmatrix} 1 & -l\sin\varphi \\ 0 & l\cos\varphi \end{bmatrix}.$$

The manipulability index according (3) is

$$w = |\det(J)| = |l\cos\varphi|.$$

Because of $R_{0,1} = I$, the relation

$$J = I\tilde{J} \Leftrightarrow J = \tilde{J}$$

hold together with $w_{(\tilde{J})} = |\det(\tilde{J})| = |l\cos\varphi|$.

The Jacobian matrix (15) transformed into the last local coordinate frame on the mechanism is

$$J_{ef} = \begin{bmatrix} \cos\varphi & 0 \\ \sin\varphi & l \end{bmatrix}.$$

Example 4: Plane mechanism with one rotational DOF and one prismatic DOF.

For the plane mechanism of Fig. 5, the matrices (31) and (34) are

$$R_{0,1} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \quad W = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

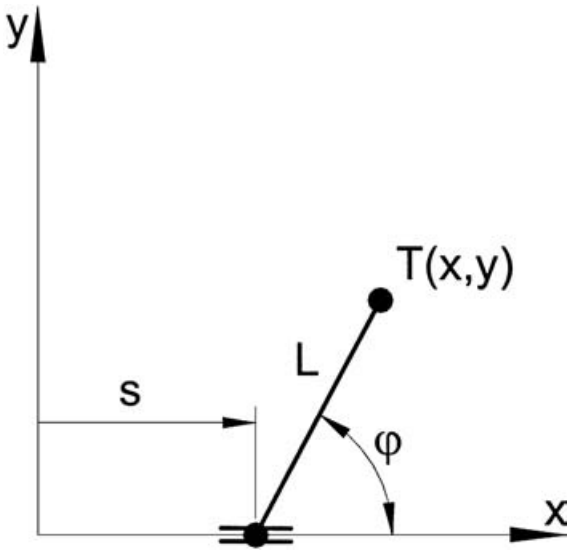


Fig. 4. Plane mechanism with one prismatic and one rotational joint.

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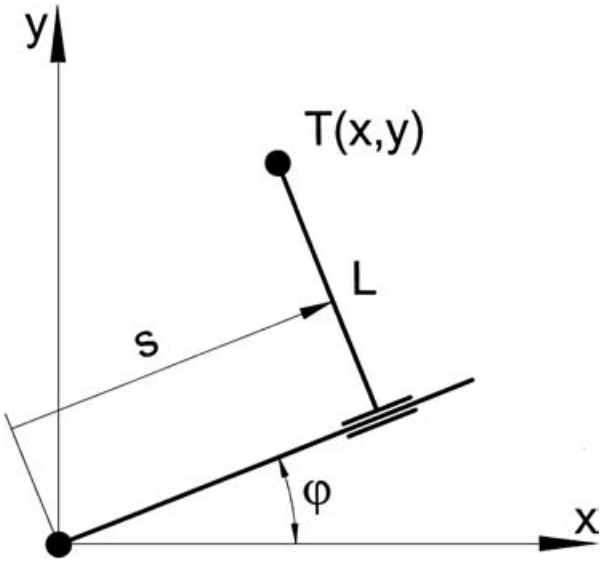


Fig. 5. Plane mechanism with rotational and prismatic joint.

The Jacobian matrix according (29) is

$$J = \begin{bmatrix} \cos\varphi & -s\sin\varphi - l\cos\varphi \\ \sin\varphi & s\cos\varphi - l\sin\varphi \end{bmatrix}$$

and yields a manipulability index, (3)

$$w = |\det(J)| = |s|.$$

The matrix $\tilde{J}$ (43–44)

$$J = R_{0,1}\tilde{J} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \cdot \begin{bmatrix} 1 & -l \\ 0 & s \end{bmatrix}$$

$$\Leftrightarrow \tilde{J} = \begin{bmatrix} 1 & -l \\ 0 & s \end{bmatrix}$$

what results in the manipulability index, (47)

$$w_{(\tilde{J})} = |\det(\tilde{J})| = |s|.$$

Example 5: 4DOF's mechanism in $\Re^3$.

For the mechanism of Fig. 6 the matrices $R_{0,1}$ (31) and W (34) are

$$R_{0,1} = \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 & 0 \\ \sin\varphi_1 & \cos\varphi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

respectively the Jacobian matrix (29) is

$$J = \begin{bmatrix} j_{11} & j_{12} & j_{13} & j_{14} \\ j_{21} & j_{22} & j_{23} & j_{24} \\ j_{31} & j_{32} & j_{33} & j_{34} \end{bmatrix}$$

$$j_{11} = \cos\varphi_1(l_2\cos\varphi_2 + l_3\cos(\varphi_2+\varphi_3) + l_4\cos(\varphi_2+\varphi_3+\varphi_4))$$

$$j_{12} = -\sin\varphi_1(l_2\sin\varphi_2 + l_3\sin(\varphi_2+\varphi_3) + l_4\sin(\varphi_2+\varphi_3+\varphi_4))$$

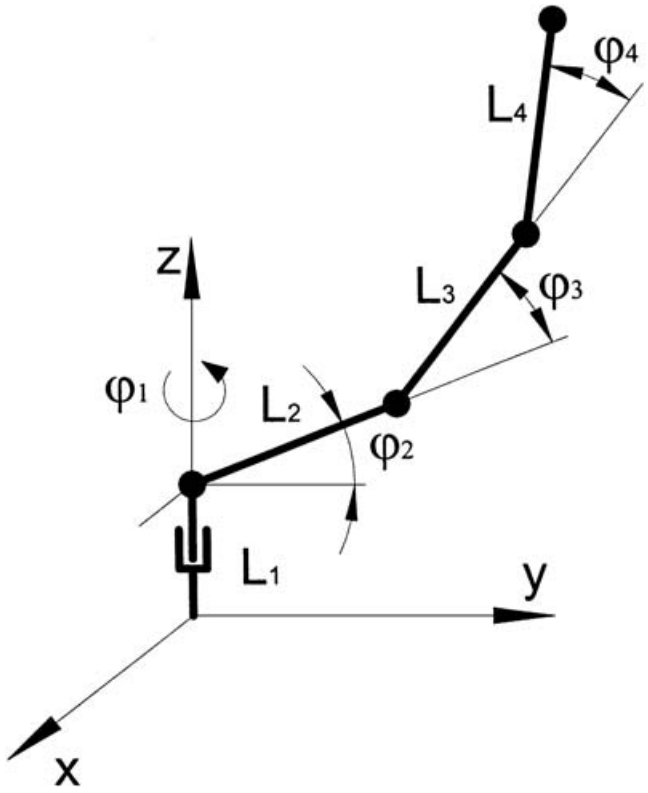


Fig. 6. 4DOFs mechanism in $\Re^3$.

$$j_{13} = -\sin\varphi_1(l_3\sin(\varphi_2+\varphi_3) + l_4\sin(\varphi_2+\varphi_3+\varphi_4))$$

$$j_{14} = -l_4\sin\varphi_1\sin(\varphi_2+\varphi_3+\varphi_4)$$

$$j_{21} = (l_2\cos\varphi_2 + l_3\cos(\varphi_2+\varphi_3) + l_4\cos(\varphi_2+\varphi_3+\varphi_4))\sin\varphi_1$$

$$j_{22} = \cos\varphi_1(l_2\sin\varphi_2 + l_3\sin(\varphi_2+\varphi_3) + l_4\sin(\varphi_2+\varphi_3+\varphi_4))$$

$$j_{23} = \cos\varphi_1(l_3\sin(\varphi_2+\varphi_3) + l_4\sin(\varphi_2+\varphi_3+\varphi_4))$$

$$j_{24} = l_4\cos\varphi_1\sin(\varphi_2+\varphi_3+\varphi_4)$$

$$j_{31} = 0$$

$$j_{32} = -l_2\cos\varphi_2 - l_3\cos(\varphi_2+\varphi_3) - l_4\cos(\varphi_2+\varphi_3+\varphi_4)$$

$$j_{33} = -l_3\cos(\varphi_2+\varphi_3) - l_4\cos(\varphi_2+\varphi_3+\varphi_4)$$

$$j_{34} = -l_4\cos(\varphi_2+\varphi_3+\varphi_4).$$

The manipulability index according (4) is:

$$w = \sqrt{\det(JJ^T)} = \Big(-\frac{1}{2}(l_2\cos\varphi_2 + l_3\cos(\varphi_2+\varphi_3) + l_4\cos(\varphi_2+\varphi_3+\varphi_4))^2\big(-l_2^2l_3^2 - 2l_2^2l_4^2 - 2l_3^2l_4^2 - 2l_2l_3l_4^2\cos\varphi_3 + l_2^2l_3^2\cos(2\varphi_3) - 2l_2^2l_3l_4\cos\varphi_4 + 2l_3^2l_4^2\cos(2\varphi_4) + 2l_2^2l_4^2\cos(2(\varphi_3+\varphi_4)) + 2l_2^2l_3l_4\cos(2\varphi_3+\varphi_4) + 2l_2l_3l_4^2\cos(\varphi_3+2\varphi_4)\big)\Big)^{\frac{1}{2}}.$$

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The matrix $\tilde{J}$ from (43–44) is

$$J = R_{0,1}\tilde{J} = \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 & 0 \\ \sin\varphi_1 & \cos\varphi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \tilde{j}_{11} & \tilde{j}_{12} & \tilde{j}_{13} & \tilde{j}_{14} \\ \tilde{j}_{21} & \tilde{j}_{22} & \tilde{j}_{23} & \tilde{j}_{24} \\ \tilde{j}_{31} & \tilde{j}_{32} & \tilde{j}_{33} & \tilde{j}_{34} \end{bmatrix}$$

$$\begin{aligned}
\tilde{j}_{11} &= l_2\cos\varphi_2 + l_3\cos(\varphi_2+\varphi_3) + l_3\cos(\varphi_2+\varphi_3+\varphi_4) \\
\tilde{j}_{12} &= \tilde{j}_{13} = \tilde{j}_{14} = \tilde{j}_{21} = \tilde{j}_{31} = 0 \\
\tilde{j}_{22} &= l_2\sin\varphi_2 + l_3\sin(\varphi_2+\varphi_3) + l_4\sin(\varphi_2+\varphi_3+\varphi_4) \\
\tilde{j}_{23} &= l_3\sin(\varphi_2+\varphi_3) + l_4\sin(\varphi_2+\varphi_3+\varphi_4) \\
\tilde{j}_{24} &= l_4\sin(\varphi_2+\varphi_3+\varphi_4) \\
\tilde{j}_{32} &= -l_2\cos\varphi_2 - l_3\cos(\varphi_2+\varphi_3) - l_4\cos(\varphi_2+\varphi_3+\varphi_4) \\
\tilde{j}_{33} &= -l_3\cos(\varphi_2+\varphi_3) - l_4\cos(\varphi_2+\varphi_3+\varphi_4) \\
\tilde{j}_{34} &= -l_4\cos(\varphi_2+\varphi_3+\varphi_4).
\end{aligned}$$

The manipulability index according to (48) is easier to calculate and has the same value as the manipulability index of the original Jacobian matrix:

$$w = \sqrt{\det(J\,J^T)} = \sqrt{\det(\tilde{J}\,\tilde{J}^T)} = w_{(\tilde{J})}.$$

The Jacobian matrix (15) transformed into the last local coordinate frame on the mechanism is

$$J_{ef} = \begin{bmatrix} l_2C_2+l_3C_{23}+l_4C_{234} & 0 & 0 & 0 \\ 0 & -l_3S_4-l_2S_{23} & -l_3S_4 & 0 \\ 0 & -l_4-l_3C_4-l_2C_{23} & -l_4-l_3C_4 & -l_4 \end{bmatrix}$$

with $C_i = \cos\varphi_i$, $C_{ij} = \cos(\varphi_i+\varphi_j)$, $S_i = \sin\varphi_i$ and $S_{ij} = \sin(\varphi_i+\varphi_j)$.

It is to be noted that – contrary to the original Jacobian J – the Jacobian J_{ef} is a sparse matrix and hence, much easier to handle.

5. CONCLUSIONS

Invariance properties of the manipulability index are very important in the design phase of a mechanism and in the design and implementation of the control. The manipulability index is a quantitative measure of the ability of the mechanism for performing a prescribed task. Especially, for redundant mechanisms, it contains important information about the mechanism's properties, the best position, and can be used in the control algorithm synthesis.

Two important properties of this index have been discussed and proved. First, it was demonstrated that the relative size of the manipulability index is invariant with respect to the coordinate frame used in task space. The Jacobian matrix can be transformed into an arbitrary chosen coordinate system, in particular into the end-effector local coordinate system. In this local coordinate system, as shown in examples 1–5, the elements of the Jacobian matrix provide good insight into dependences and relations and are simple mathematical expressions. The manipulability differs at most by a constant factor from the index calculated from the Jacobian matrix in the origin base.

Secondly, it is shown that the manipulability index does not depend on the first DOF, i.e. on the one that connects the mechanism with the base coordinate system. This fact is important because it shows that this index is dependent just on inner DOFs of the mechanism and in no way of the position in the task space.

This fact opens new opportunities in developing control strategies and show new possibilities in the optimal design of open chain mechanisms.

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