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ARTICLE

Syllogistic logic with "Most"

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Abstract
We add Most X are Y to the syllogistic logic of All X are Y and Some X are Y. We prove soundness, completeness, and decidability in polynomial time. Our logic has infinitely many rules, and we prove that this is unavoidable.

Keywords: Syllogistic logic; majority quantifier; completeness theorem

1. Introduction

The classical syllogistic is the logical system whose sentences are of the form All X are Y, Some X are Y, and No X are Y. These sentences are evaluated in a model by assigning a set $[\![X]\!]$ to the variable X and then using the evident truth definition. This logical system lies at the root of the Western logical tradition. For this reason, modern logicians have occasionally looked back on it with an eye to its theoretical properties or to extending it in various ways.

This paper presents an extension of the syllogistic which includes sentences of the form Most X are Y. Variables are interpreted by subsets of a given finite set, with the understanding that Most X are Y means that strictly more than half of the X's are Y's. We present a proof system which is *strongly complete* relative to the semantics: for every finite set Γ of sentences and every sentence φ, $\Gamma \vdash \varphi$ in our system if and only if $\Gamma \models \varphi$. (This last assertion means that every model of all sentences in Γ is also a model of φ.)

To get a feeling for the logical issues, we present a few valid and invalid assertions. Note first that

$$\{\text{Most } X \text{ are } Y,\ \text{Most } X \text{ are } Z\} \models \text{Some } Y \text{ are } Z. \tag{1}$$

For if $[\![Y]\!] \cap [\![Z]\!] = \emptyset$, then it cannot be the case that $[\![X]\!] \cap [\![Y]\!]$ and $[\![X]\!] \cap [\![Z]\!]$ each have more than half of the elements of $[\![X]\!]$. For a second example, we might ask whether

$$\{\text{Most } X \text{ are } Y,\ \text{Most } Y \text{ are } Z,\ \text{Most } Z \text{ are } W\} \models \text{Some } X \text{ are } W.$$

The answer here is negative: we may take $[\![X]\!] = \{1, 2, 3\}$, $[\![Y]\!] = \{2, 3, 4\}$, $[\![Z]\!] = \{3, 4, 5\}$, and $[\![W]\!] = \{4, 5, 6\}$. Another positive assertion:

$$\{\text{All } Y \text{ are } X,\ \text{All } X \text{ are } Z,\ \text{Most } Z \text{ are } Y\} \models \text{Most } X \text{ are } Y.$$

This turns out to be a sound rule of inference in our system. Continuing, we may ask whether or not

$$\{\text{All } X \text{ are } Z,\ \text{All } Y \text{ are } Z,\ \text{Most } Z \text{ are } Y,\ \text{Most } Y \text{ are } X\} \models \text{Most } X \text{ are } Y\,?$$

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Again, the conclusion does not follow. One can take $[\![X]\!] = \{1, 2, 3, 4, 5, 6, 7\}$, $[\![Y]\!] = \{5, 6, 7, 8, 9\}$, and $[\![Z]\!] = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. This last example is from Moss (2008).

For a final point in this direction, here is a challenge for the reader. Let Γ contain the sentences below:

$$\begin{array}{llll} \mathsf{Most}\ U\ \mathsf{are}\ A^* & \mathsf{Most}\ V\ \mathsf{are}\ B^* & \mathsf{Most}\ W\ \mathsf{are}\ A^* & \mathsf{Most}\ A^*\ \mathsf{are}\ U \\ \mathsf{All}\ A^*\ \mathsf{are}\ W & \mathsf{All}\ U\ \mathsf{are}\ V & \mathsf{All}\ V\ \mathsf{are}\ W & \mathsf{All}\ D\ \mathsf{are}\ A^* \\ \mathsf{All}\ D\ \mathsf{are}\ B^* & \mathsf{All}\ A^*\ \mathsf{are}\ E & \mathsf{All}\ B^*\ \mathsf{are}\ E & \mathsf{Most}\ E\ \mathsf{are}\ U \end{array} \tag{2}$$

We ask two questions:

(Q1) Does $\Gamma \models \mathsf{Most}\ U\ \mathsf{are}\ E$, or not?
(Q2) Does $\Gamma \models \mathsf{Some}\ A^*\ \mathsf{are}\ B^*$, or not?

The answers may be found in Section 6.

The main work of this paper presents a sound and complete proof system for this semantics. The proof system is found in Section 3 and the completeness itself is in Section 4. We also give an algorithm in Section 5 to tell whether or not $\Gamma \models \varphi$ for arbitrary finite Γ and φ in our fragment. The algorithm is based on our proof system and the completeness proof. We present a few examples in Section 6. The last section discusses a fine point on our logical system: it has infinitely many rules, and this is unavoidable.

Prior work on this topic To the best of our knowledge, the problem of axiomatizing the syllogistic logic of Most originates with Moss (2008). That paper obtained some very simple results in the area, such as a completeness result for syllogistic reasoning using Some and Most (but not All), and also explicit statements of some of the very simplest of the infinite rule scheme that we employ in this paper, the scheme of ($\triangleright$) rules. The full formulation of these ($\triangleright$) rules in our logic is new, as is the completeness result.

Lai et al. (2016) is also about Most A are B. But that paper goes in a different direction. It has the logic of sentences Most A are B and $\neg$(Most A are B), but it does not include the rest of the syllogistic sentences like All A are B and Some A are B. We leave as an open problem the combination of results from these two papers.

There are a number of papers dealing with the *numerical syllogistic*. This is the logical language whose sentences are of the form At least C X are Y and At most C X are Y, where C is a numeral. For example, Pratt-Hartmann (2009) shows that there is no finite syllogistic logic for this fragment.

2. Syntax and semantics

For the syntax of our language, we start with a collection of *nouns*. (These are also called *unary atoms* or *variables* in this area, and we shall use these terms interchangeably.) We use uppercase Roman letters like $A, B, \ldots, X, Y, Z$ for nouns. We are only interested in *sentences* of one of the following three forms: All X are Y, Some X are Y, and Most X are Y.

We mentioned sentences No X are Y in our opening sentence. But we do not treat No in what follows; it is open to extend what we do to the larger syllogistic fragment with No.

For the semantics, we use *models* $\mathcal{M}$ consisting of a finite set M together with *interpretations* $[\![X]\!] \subseteq M$ of each noun X. We then interpret our sentences in a model as follows:

$$\begin{aligned} \mathcal{M} &\models \mathsf{All}\ X\ \mathsf{are}\ Y \quad \text{iff} \quad [\![X]\!] \subseteq [\![Y]\!] \\ \mathcal{M} &\models \mathsf{Some}\ X\ \mathsf{are}\ Y \quad \text{iff} \quad [\![X]\!] \cap [\![Y]\!] \neq \emptyset \\ \mathcal{M} &\models \mathsf{Most}\ X\ \mathsf{are}\ Y \quad \text{iff} \quad \mathit{card}([\![X]\!] \cap [\![Y]\!]) > \tfrac{1}{2}\mathit{card}([\![X]\!]) \end{aligned}$$

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$$\frac{}{\mathsf{All}\,X\,\mathsf{are}\,X}\,axiom \qquad \frac{\mathsf{All}\,X\,\mathsf{are}\,Y \quad \mathsf{All}\,Y\,\mathsf{are}\,Z}{\mathsf{All}\,X\,\mathsf{are}\,Z}\,barbara$$

$$\frac{\mathsf{Some}\,X\,\mathsf{are}\,Y}{\mathsf{Some}\,Y\,\mathsf{are}\,X}\,conv \qquad \frac{\mathsf{Some}\,X\,\mathsf{are}\,Y}{\mathsf{Some}\,X\,\mathsf{are}\,X}\,some \qquad \frac{\mathsf{Some}\,X\,\mathsf{are}\,Y \quad \mathsf{All}\,Y\,\mathsf{are}\,Z}{\mathsf{Some}\,X\,\mathsf{are}\,Z}\,darii$$

$$\frac{\mathsf{Most}\,X\,\mathsf{are}\,Y}{\mathsf{Some}\,X\,\mathsf{are}\,Y}\,m_1 \qquad \frac{\mathsf{Some}\,X\,\mathsf{are}\,X}{\mathsf{Most}\,X\,\mathsf{are}\,X}\,m_2 \qquad \frac{\mathsf{Most}\,X\,\mathsf{are}\,Y \quad \mathsf{All}\,Y\,\mathsf{are}\,Z}{\mathsf{Most}\,X\,\mathsf{are}\,Z}\,m_3$$

$$\frac{\mathsf{Most}\,X\,\mathsf{are}\,Z \quad \mathsf{All}\,X\,\mathsf{are}\,Y \quad \mathsf{All}\,Y\,\mathsf{are}\,X}{\mathsf{Most}\,Y\,\mathsf{are}\,Z}\,m_4$$

$$\frac{\mathsf{All}\,Y\,\mathsf{are}\,X \quad \mathsf{All}\,X\,\mathsf{are}\,Z \quad \mathsf{Most}\,Z\,\mathsf{are}\,Y}{\mathsf{Most}\,X\,\mathsf{are}\,Y}\,m_5$$

$$\frac{X_1 \triangleright_{A,B} Y_1 \quad Y_1 \triangleright_{B,A} X_2 \quad \cdots \quad X_n \triangleright_{A,B} Y_n \quad Y_n \triangleright_{B,A} X_1}{\mathsf{Some}\,A\,\mathsf{are}\,B}\,\triangleright$$

Figure 1. Rules of the logical system for All, Some, and Most. The last line is an infinite rule scheme, and the syntax is explained in Section 3.

Observe that if $[\![X]\!]$ is empty, then automatically $\mathcal{M} \not\models \mathsf{Most}\ X\ \mathsf{are}\ Y$.

We sometimes use φ and ψ as variables ranging over sentences in the language of this paper. The symbol Γ always denotes a finite set of sentences.

We say that $\mathcal{M} \models \Gamma$ if $\mathcal{M} \models \psi$ for all $\psi \in \Gamma$. $\Gamma \models \varphi$ means that for all (finite) models $\mathcal{M}$, if $\mathcal{M} \models \Gamma$, then $\mathcal{M} \models \varphi$.

The central point of this paper is to provide a proof system which defines a relation $\Gamma \vdash \varphi$ in terms of proof trees, and to prove the soundness and completeness of the system: $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$.

3. Proof system

The logical system is based on the rules in Figure 1. The rules of All and Some are familiar from basic logic, and the interesting rules of the system are the ones involving Most. In fact, the first line is complete for All, and the first two lines are complete for All and Some, see further Moss (2008).

We write $\Gamma \vdash \varphi$ to mean that there is a tree $\mathcal{T}$ labeled with sentences from our language such that (a) all of the leaves of $\mathcal{T}$ are labeled with sentences which belong to Γ (or are *axioms* of the form All X are X); (b) each node which is not a leaf matches one of the rules in the system; and (c) the root is labeled φ.

As an example, {Some X are X, All X are Y} $\vdash$ Most X are Y via the tree below:

$$\frac{\dfrac{\mathsf{Some}\ X\ \mathsf{are}\ X}{\mathsf{Most}\ X\ \mathsf{are}\ X}\,m_2 \qquad \mathsf{All}\ X\ \mathsf{are}\ Y}{\mathsf{Most}\ X\ \mathsf{are}\ Y}\,m_3$$

For more on syllogistic logics in general, see Pratt-Hartmann and Moss (2009).

The system is sound: if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$. The proof is a routine induction on proof trees in the system. We comment on the soundness of the rules concerning Most:

- (m_1) For (m_1), if Most X are Y in a model $\mathcal{M}$, then in that model, $card([\![X]\!] \cap [\![Y]\!]) > 0$, and so Some X are Y holds.
- (m_2) For (m_2), if $card([\![X]\!]) > 0$, then $card([\![X]\!] \cap [\![X]\!]) = card([\![X]\!]) > \frac{1}{2} card([\![X]\!])$.
- (m_3) For the rule (m_3), suppose that in $\mathcal{M}$, we have $card([\![X]\!] \cap [\![Y]\!]) > \frac{1}{2} card([\![X]\!])$ and that $[\![Y]\!] \subseteq [\![Z]\!]$. Then $card([\![X]\!] \cap [\![Z]\!]) \geq card([\![X]\!] \cap [\![Y]\!]) > \frac{1}{2} card([\![X]\!])$, and so we have Most X are Z.

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(m_4) Turning to (m_4), if $card([\![X]\!] \cap [\![Z]\!]) > \frac{1}{2} card([\![X]\!])$, and also $[\![X]\!] \subseteq [\![Y]\!] \subseteq [\![X]\!]$, then $[\![X]\!] = [\![Y]\!]$, and so $card([\![Y]\!] \cap [\![Z]\!]) > \frac{1}{2} card([\![Y]\!])$.

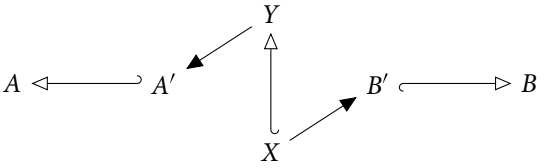
(m_5) For (m_5), assume that $[\![Y]\!] \subseteq [\![X]\!] \subseteq [\![Z]\!]$ and that $card([\![Z]\!] \cap [\![Y]\!]) > \frac{1}{2} card([\![Z]\!])$. Then $Z \cap Y = Y = X \cap Y$, and so $card([\![X]\!] \cap [\![Y]\!]) > \frac{1}{2} card([\![Z]\!]) \geq \frac{1}{2} card([\![X]\!])$.

For the infinite scheme of ($\triangleright$) rules, we need a preliminary result. In the figures below, and for later in this paper, we present facts about the interpretations of various variables inside a given model using special diagrams. For the most part, the notation is self-explanatory given our statements in Proposition 1 below. We merely alert the reader to the two types of arrows:

$\longrightarrow\!\!\triangleright$ for All sentences and $\longrightarrow$ for Most sentences .

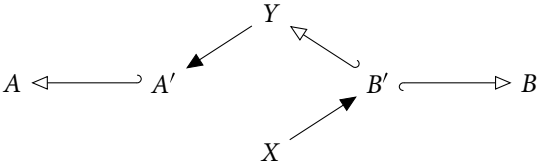
Proposition 1. *Let $\mathcal{M}$ be a (finite) model which either satisfies all of the sentences below:*

Most X are B',
All A' are A,
Most Y are A',
All B' are B,
All X are Y,



or else all sentences below:

Most X are B',
All A' are A,
Most Y are A',
All B' are B,
All B' are Y.



Also, assume that in addition, $[\![A]\!] \cap [\![B]\!] = \emptyset$. Let

$$X_A = \text{card}([\![X]\!] \cap [\![A]\!]),$$

and similarly for X_B, Y_A, and Y_B. Then

$$\min(Y_A, Y_B) > \min(X_A, X_B).$$

Proof. In $\mathcal{M}$, Most X are B', All A' are A, Most Y are A', and All B' are B. Note that $Y_A > \frac{1}{2} card(Y)$. So since $[\![A]\!] \cap [\![B]\!] = \emptyset$, $Y_A > Y_B$. Similarly, $X_B > X_A$.

We have two cases, depending on whether All X are Y is true in $\mathcal{M}$, or All B' are Y.

In the first case, $Y_A > Y_B \geq X_B > X_A$.

In the second case, we show that $X_A < Y_B, Y_A$ by the following calculation:

$$\begin{aligned} X_A &= card(X \cap A) \\ &\leq card(X \setminus B) \quad \text{since } [\![A]\!] \cap [\![B]\!] = \emptyset \\ &\leq card(X \setminus B') \quad \text{since All } B' \text{ are } B \\ &< card(X \cap B') \quad \text{since Most } X \text{ are } B' \\ &\leq card(B') \end{aligned}$$

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$$= card(Y \cap B') \text{ since All } B' \text{ are } Y$$
$$\leq card(Y \cap B)$$
$$= Y_B$$

□

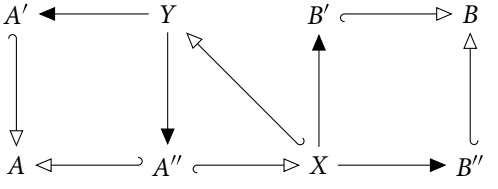
Recall that this entire discussion is part of a specification of the rules of the logical system in this paper. In order to state the rules in a concise way, we introduce some notation based on what we saw in Proposition 1. We write $X \triangleright_{A,B} Y$ for either of the two sets of five assertions in Proposition 1. This notation is found in the last rule in Figure 1; actually, this is a rule scheme with infinitely many instances. When we write $X \triangleright_{A,B} Y$, we fix the variables X, Y, A, and B, but the additional variables A' and B' are arbitrary. When we have more than one assertion with a $\triangleright$, we permit different additional variables to be used. To be concrete, the first ($\triangleright$) rule would be

$$\frac{X \triangleright_{A,B} Y \quad Y \triangleright_{B,A} X}{\text{Some } A \text{ are } B} \tag{3}$$

This is shorthand for four rules. One of them is: From

$$\begin{array}{l}\text{Most } X \text{ are } B',\ \text{All } A' \text{ are } A,\ \text{Most } Y \text{ are } A',\ \text{All } B' \text{ are } B,\ \text{All } X \text{ are } Y,\\ \text{Most } Y \text{ are } A'',\ \text{All } A'' \text{ are } A,\ \text{Most } X \text{ are } B'',\ \text{All } B'' \text{ are } B,\ \text{All } A'' \text{ are } X\end{array} \tag{4}$$

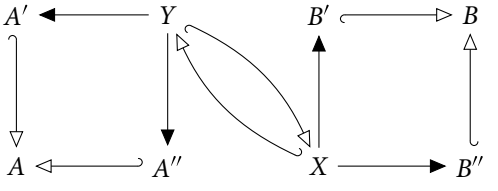
infer Some A are B. We have written $X \triangleright_{A,B} Y$ on the top line and $Y \triangleright_{B,A} X$ on the bottom line. Here is a diagrammatic form of this rule:



We mentioned that the ($\triangleright$) rule shown in (3) is shorthand for *four* rules. The other three differ in the last assertions on the two lines above, corresponding to the differing possibilities in Proposition 1. One of these other rules is: From

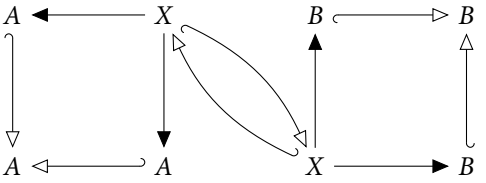
$$\begin{array}{l}\text{Most } X \text{ are } B',\ \text{All } A' \text{ are } A,\ \text{Most } Y \text{ are } A',\ \text{All } B' \text{ are } B,\ \text{All } X \text{ are } Y,\\ \text{Most } Y \text{ are } A'',\ \text{All } A'' \text{ are } A,\ \text{Most } X \text{ are } B'',\ \text{All } B'' \text{ are } B,\ \underline{\text{All } Y \text{ are } X}\end{array} \tag{5}$$

infer Some A are B. (We have underlined the premise that changed.) Diagrammatically, the premises are



As with all rules in logic, we may identify variables. For example, taking Y to be X, also A' and A'' to be A, and finally B' and B'' to be B, we get

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Dropping repeated premises and the premises All X are X, All A are A, and All B are B, we obtain a simpler form of this rule:

$$\frac{\text{Most } X \text{ are } A \quad \text{Most } X \text{ are } B}{\text{Some } A \text{ are } B}$$

This was the rule that we began with, back in (1).

Lemma 2. *Every (▷) rule is sound.*

Proof. The result follows by repeated application of Proposition 1, yielding a contradiction of the form

$$\min(X_{1,A}, X_{1,B}) < \min(Y_{1,A}, Y_{1,B}) < \min(X_{2,A}, X_{2,B}) < \cdots < \min(X_{1,A}, X_{1,B})$$

for the case that $[\![A]\!] \cap [\![B]\!] = \emptyset$.

For concreteness, we shall only go into details concerning one of the rules, the one described just before the statement of this lemma. Let $\mathcal{M}$ be a (finite) model satisfying all 10 sentences in (4). Assume towards a contradiction that $[\![A]\!] \cap [\![B]\!] = \emptyset$. We appeal to Proposition 1 and to the notation there. One use of Proposition 1 shows that $\min(Y_A, Y_B) > \min(X_A, X_B)$. A second use shows $\min(X_A, X_B) > \min(Y_A, Y_B)$. This is a contradiction. □

4. Completeness

Theorem 3. *Let Γ be a finite set of sentences in our fragment. If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.*

This is the main result in this paper. The rest of this section is devoted to the proof.

Notation 4. *We write $\Gamma \vdash_0 \varphi$ if there is a proof tree for $\Gamma \vdash \varphi$* without using any (▷) rules.

If $\Gamma \vdash_0$ All X are Y, *we write $X \hookrightarrow Y$. If $\Gamma \vdash_0$* Most X are Y, *we write $X \to Y$. If $\Gamma \vdash_0$* Some X are Y, *we write $X \downarrow Y$.*

The logical system $\vdash_0$ will be important in Section 5 because the smaller system now has only finitely many rules.

Notice that our notation suppresses the underlying set Γ of assumptions. It should also be noted that the shortened notations $\hookrightarrow$, $\to$, and $\downarrow$ are intended to be used only for the statements having to do with the provability notion $\vdash_0$. When discussing a particular model $\mathcal{M}$ of Γ, we always write, for example, $\mathcal{M} \models$ Most X are Y. We would never write "$X \to Y$ in $\mathcal{M}$."

Proposition 5. *If* All X are Y *belongs to Γ, then $X \hookrightarrow Y$; and similarly for* Some *and* Most. *The relation $\hookrightarrow$ is a preorder, and $\downarrow$ is symmetric. If $X \downarrow Y \hookrightarrow Z$, then $X \downarrow Z$. If $X \downarrow Y$, then $X \downarrow X$. If $X \to Y \hookrightarrow Z$, then $X \to Z$. If $X \to Y$ and $Y \hookrightarrow Z \hookrightarrow Y$, then $X \hookrightarrow Z$. If $X \hookrightarrow Y \hookrightarrow Z$ and $Z \to X$, then $Y \to X$.*

We use Proposition 5 in the rest of this paper, usually without mention.

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The proof of Theorem 3 is by cases as to φ. We usually write the atoms in φ as A^* and B^*, with the asterisks just as marks to focus our attention. To keep the cases separate, we treat them in separate subsections. The bulk of the work turns out to be for the case φ is of the form Some A^* are B^*.

4.1 *The proof when φ is* All A^* are B^*

This is the easiest case. Let Γ_{all} be the set of All sentences in Γ. We claim that in this case $\Gamma_{\text{all}} \models \varphi$. To see this, let $\mathcal{M} \models \Gamma_{\text{all}}$. Like all models in this paper, $\mathcal{M}$ is finite. Add k fresh points to the interpretation $[\![X]\!]$ of every noun, where k is chosen large enough so that the new interpretations now overlap in most elements. Then the expanded model $\mathcal{M}^+$ satisfies all most sentences, and it still satisfies Γ_{all}. Thus $\mathcal{M}^+ \models \Gamma$. So $\mathcal{M}^+ \models \varphi$. Since φ is an All-sentence, the original model $\mathcal{M}$ also satisfies it.

This establishes our claim, and now our result follows from the completeness of the logic of All; see Moss (2008).

4.2 *The proof when φ is* Some A^* are A^*

This is another easy case. Let

$$\begin{aligned}\Delta = \Gamma_{\text{all}} &\cup \{\text{Some } X \text{ are } Y : \text{Some } X \text{ are } Y \in \Gamma\} \\ &\cup \{\text{Some } X \text{ are } Y : \text{Most } X \text{ are } Y \in \Gamma\}\end{aligned} \tag{6}$$

Notice that Δ is a set of sentences which use All and Some (but not Most). We claim that $\Gamma \models \varphi$ iff $\Delta \models \varphi$. One direction is obvious. For the other, assume that $\Gamma \models \varphi$, and let $\mathcal{M} \models \Delta$. Let S be a fresh set whose size is larger than the size of the universe of $\mathcal{M}$, and add the elements of S to $[\![U]\!]$ for all U such that $[\![U]\!] \neq \emptyset$ in $\mathcal{M}$. Let $\mathcal{N}$ be the new model. We claim that $\mathcal{N} \models \Gamma$. The two models $\mathcal{M}$ and $\mathcal{N}$ satisfy all All and Some sentences in Δ, and hence in Γ. Consider a sentence in Γ of the form Most V are W. Then Δ contains Some V are W. Thus $[\![V]\!]$ and $[\![W]\!]$ are not empty in $\mathcal{M}$. Since S is larger in size than the universe of $\mathcal{M}$, $\mathcal{N}$ satisfies Most V are W. This proves that $\mathcal{N} \models \Gamma$. Since we assumed that $\Gamma \models \varphi$, we see that $\mathcal{N} \models \varphi$. Thus, in $\mathcal{N}$, $[\![A^*]\!] \neq \emptyset$. By construction of $\mathcal{N}$, we see that in $\mathcal{M}$, we also have $[\![A^*]\!] \neq \emptyset$. (That is, the passage from $\mathcal{M}$ to $\mathcal{N}$ had the property that a variable has an empty interpretation in $\mathcal{M}$ if and only if it has an empty interpretation in $\mathcal{N}$.) Thus $\mathcal{M} \models \varphi$, as desired.

As in Section 4.1, we are now done by the completeness of the syllogistic logic of All and Some.

4.3 *The proof when φ is* Most A^* are B^*

We assume in this section that $\Gamma \not\vdash$ Most A^* are B^*. We build a finite model $\mathcal{M} \models \Gamma$ where $card([\![A^*]\!] \cap [\![B^*]\!]) \leq \frac{1}{2} card([\![A^*]\!])$. We have three subcases.

The first subcase: $A^* \hookrightarrow B^*$**.** We claim that in this subcase, $\neg(A^* \downarrow A^*)$. Here is the reason: if $\Gamma \vdash$ Some A^* are A^*, then by (m_2) and (m_3), $\Gamma \vdash$ Most A^* are B^*. This would contradict the basic assumption in Section 4.3. We define a model $\mathcal{M}$ using $M = \{*\}$, and

$$[\![X]\!] = \begin{cases} \{*\} \text{ if } X \downarrow X \\ \emptyset \quad \text{otherwise} \end{cases}$$

We check that $\mathcal{M} \models \Gamma$. Consider a sentence $X \to Y$ in Γ. Then $X \downarrow X$ and $Y \downarrow Y$, and indeed Most X are Y holds in the model. The same holds for sentences $X \downarrow Y$. For the sentences $X \hookrightarrow Y$, note that if $[\![X]\!] \neq \emptyset$, then $X \downarrow X$. And so by the logic, $Y \downarrow Y$, and thus $[\![X]\!] = \{*\} = [\![Y]\!]$. This concludes the verification that $\mathcal{M} \models \Gamma$. Clearly $[\![A^*]\!] = \emptyset$, and so $\mathcal{M} \not\models$ Most A^* are B^*.

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The second subcase: $\neg(A^* \hookrightarrow B^*)$ and $B^* \hookrightarrow A^*$. Divide the variables into three classes:

$$\begin{aligned}\mathcal{A} &= \{X : A^* \hookrightarrow X\}\\ \mathcal{B} &= \{X : X \hookrightarrow B^*\}\\ \mathcal{C} &= \text{all others}\end{aligned}$$

Define a model $\mathcal{M}$ using $M = \{1, 2, 3, 4\}$ and

$$[\![X]\!] = \begin{cases}\{1,2,3,4\} & \text{if } X \in \mathcal{A}\\ \{1,2\} & \text{if } X \in \mathcal{B}\\ \{1,2,3\} & \text{if } X \in \mathcal{C}\end{cases} \tag{7}$$

Every sentence of the form Some X are Y is true in $\mathcal{M}$. So is every sentence of the form All X are Y, except for the ones with $X \in \mathcal{C}$ and $Y \in \mathcal{B}$, and those with $X \in \mathcal{A}$ and $Y \in \mathcal{B} \cup \mathcal{C}$. But if $Y \in \mathcal{B}$, and if Γ contains All X are Y, then $X \in \mathcal{B}$ as well. If $X \in \mathcal{A}$, and if Γ contains All X are Y, then $Y \in \mathcal{A}$ also. So the All sentences in Γ all hold in $\mathcal{M}$.

Every sentence of the form Most X are Y is true in $\mathcal{M}$, except for the ones with $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. But if $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, then we cannot have $X \to Y$: if we did have this, then using (m_5) we would have $A^* \to B^*$, contradicting our assumption in this section that $\neg(A^* \to B^*)$.

We conclude that $\mathcal{M} \models \Gamma$. Finally, Most A^* are B^* is false in $\mathcal{M}$.

The final subcase: $\neg(A^* \hookrightarrow B^*)$ and $\neg(B^* \hookrightarrow A^*)$. We divide the variables into six classes:

Class	Definition: variables X such that	Interpretation
$\mathcal{A}$	$\neg(X \hookrightarrow A^*),\ A^* \hookrightarrow X$	$\{0, 1, 2, 3, 4, 5, 6\}$
$\mathcal{B}$	$X \hookrightarrow A^*,\ A^* \hookrightarrow X$	$\{1, 2, 3, 4, 5, 6\}$
$\mathcal{C}$	$X \hookrightarrow A^*,\ \neg(A^* \hookrightarrow X),\ \neg(X \hookrightarrow B^*)$	$\{1, 2, 3, 4\}$
$\mathcal{D}$	$\neg(X \hookrightarrow A^*),\ X \hookrightarrow B^*$	$\{0, 1, 2, 3\}$
$\mathcal{E}$	$X \hookrightarrow A^*,\ X \hookrightarrow B^*$	$\{1, 2, 3\}$
$\mathcal{F}$	All others	$\{0, 1, 2, 3, 4\}$

We have listed the classes, the definitions, and the interpretation of each variable in the various classes. This defines a model which we call $\mathcal{M}$. $\mathcal{M}$ satisfies all Some sentences. It satisfies all All sentences except those of the following forms; we list them together with reasons why no sentences in each form can belong to Γ. Our list is in "backwards lexicographic" order (roughly).

All A are B with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then $A \hookrightarrow B \hookrightarrow A^*$, contradicting $A \in \mathcal{A}$.
All D are B, with $D \in \mathcal{D}$ and $B \in \mathcal{B}$. In this case, $D \hookrightarrow B \hookrightarrow A^*$ implies that $D \hookrightarrow A^*$, contradicting $D \in \mathcal{D}$.
All F are B, with $F \in \mathcal{F}$ and $B \in \mathcal{B}$. In this case, $F \hookrightarrow B \hookrightarrow A^*$ implies that $F \hookrightarrow A^*$. So $F \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{E}$, contradicting $F \in \mathcal{F}$.
All X are C, with $X \in \mathcal{A} \cup \mathcal{F}$ and $C \in \mathcal{C}$. We have $X \hookrightarrow A^*$, and so $X \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{E}$.
All B are C with $B \in \mathcal{B}$ and $C \in \mathcal{C}$. Then $A^* \hookrightarrow B \hookrightarrow C$, contradicting $C \in \mathcal{C}$.
All D are C with $D \in \mathcal{D}$ and $C \in \mathcal{C}$. Then $D \hookrightarrow C \hookrightarrow A^*$, contradicting $D \in \mathcal{D}$.
All X are D with $X \in \mathcal{A} \cup \mathcal{B}$ and $D \in \mathcal{D}$. Then $A^* \hookrightarrow X \hookrightarrow D \hookrightarrow B^*$, contradicting this subcase.
All C are D with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Then $C \hookrightarrow D \hookrightarrow B^*$, contradicting $C \in \mathcal{C}$.
All F are D with $F \in \mathcal{F}$ and $D \in \mathcal{D}$. Then $F \hookrightarrow B^*$, and so $F \in \mathcal{D} \cup \mathcal{E}$.
All X are E with $X \in \mathcal{A} \cup \cdots \cup \mathcal{F}$ and $E \in \mathcal{E}$. Then $X \hookrightarrow A^*$ and $X \hookrightarrow B^*$, and so $X \in \mathcal{E}$.
All X are F with $X \in \mathcal{A} \cup \mathcal{B}$ and $F \in \mathcal{F}$. Then $A^* \hookrightarrow X \hookrightarrow F$, and so $F \in \mathcal{A} \cup \mathcal{B}$.

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Turning to the Most sentences, our model $\mathcal{M}$ satisfies all of them except for two kinds. We list these below along with an explanation of why no sentence of either kind can belong to Γ.

Most B are X, with $B \in \mathcal{B}$ and $X \in \mathcal{D} \cup \mathcal{E}$. Then $A^* \to B^*$, using (m_4). This contradicts the fact that $\Gamma \nvdash \varphi$ (see the opening of Section 4.3).

Most A are E, with $A \in \mathcal{A}$ and $E \in \mathcal{E}$. We have $E \hookrightarrow A^* \hookrightarrow A$ and $A \to E$. So by (m_5), $A^* \to E$. But since $E \hookrightarrow B^*$, we also have $A^* \to B^*$, again contradicting $\Gamma \nvdash \varphi$.

This concludes the proof that $\mathcal{M} \models \Gamma$. Recall the assumptions in this subcase: $\neg(A^* \hookrightarrow B^*)$ and $\neg(B^* \hookrightarrow A^*)$. Therefore $A^* \in \mathcal{B}$ and $B^* \in \mathcal{D}$. Thus $\mathcal{M} \not\models$ Most A^* are B^*.

4.4 The next case: φ is Some A^* are B^*, *and also either* $A^* \hookrightarrow B^*$ *or* $B^* \hookrightarrow A^*$

If $A^* \hookrightarrow B^*$ (or $B^* \hookrightarrow A^*$), then we get a model of Γ where φ is false simply by getting a model where Some A^* are A^* (or Some B^* are B^*, respectively) is false. So we are done by the work in Section 4.2.

4.5 The last case: φ is Some A^* are B^*, $\neg(A^* \hookrightarrow B^*)$, *and* $\neg(B^* \hookrightarrow A^*)$

We are still proving Theorem 3. We are left with the case that $\Gamma \nvdash$ Some A^* are B^*, and with the extra assumptions that $\neg(A^* \hookrightarrow B^*)$ and also that $\neg(B^* \hookrightarrow A^*)$. We build a finite model $\mathcal{M} \models \Gamma$ where $[\![A^*]\!] \cap [\![B^*]\!] = \emptyset$.

The construction is long, and the reader may wish to consult Section 6.3 for an example that illustrates our work in Sections 4.5–4.8.

We divide the unary atoms (nouns) in $\Gamma \cup \{\varphi\}$ into five classes:

$$\begin{aligned}
\mathcal{A} &= \{X : X \hookrightarrow A^* \text{ but } \neg(X \hookrightarrow B^*)\} \\
\mathcal{B} &= \{X : X \hookrightarrow B^* \text{ but } \neg(X \hookrightarrow A^*)\} \\
\mathcal{D} &= \{X : X \hookrightarrow A^* \text{ and } X \hookrightarrow B^*\} \\
\mathcal{C} &= \{X \notin \mathcal{A} \cup \mathcal{B} \cup \mathcal{D} : \text{for some } Y, X \hookrightarrow Y \to A^* \text{ or } X \hookrightarrow Y \to B^*\} \\
\mathcal{E} &= \text{all other nouns}
\end{aligned}$$

(The reason we list them in this order is that a major role is played by class $\mathcal{C}$, but it is easier to make the definition of $\mathcal{C}$ using $\mathcal{D}$, as we have done.) Before going on, we should note that $\mathcal{B} \cap \mathcal{A} = \emptyset$, and also $\mathcal{C} \cap (\mathcal{A} \cup \mathcal{B}) = \emptyset$.

Notation 6. *Henceforth, we use A as a variable for the elements of $\mathcal{A}$, and similarly for B, C, D, and E. Also, we continue to use X as an arbitrary element of $\mathcal{A} \cup \cdots \cup \mathcal{E}$.*

Proposition 7. *The following hold:*

1. *$A^* \in \mathcal{A}$ and $B^* \in \mathcal{B}$.*
2. *$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are pairwise disjoint.*
3. *If $E \in \mathcal{E}$, then $\neg(E \hookrightarrow X)$ for all $X \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$.*
4. *For $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $C \in \mathcal{C}$, $\neg(C \hookrightarrow A)$ and $\neg(C \hookrightarrow B)$.*
5. *If $E \in \mathcal{E}$, then $\neg(E \to X)$ for all $X \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{D}$.*

Proof. Part (1) is a reminder of the heading of Section 4.5: $\neg(A^* \hookrightarrow B^*)$ and also $\neg(B^* \hookrightarrow A^*)$.

Part (2) is an easy consequence of the definitions. Part (3) comes down to two similar facts: if $X \hookrightarrow A^*$ or $X \hookrightarrow B^*$, then $X \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{D}$. And if $C \in \mathcal{C}$ and $X \hookrightarrow C$, then $X \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$. For (4), we use again that if $X \hookrightarrow A^*$ or $X \hookrightarrow B^*$, then $X \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{D}$.

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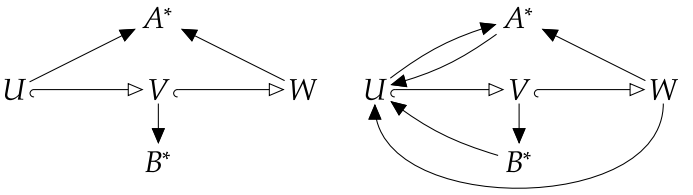


Figure 2. Motivation of the construction.

For the last part, we argue the contrapositive. Assume that $E \to X$ where $X \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{D}$. Then $E \to A^*$ or $E \to B^*$. So if $E \notin \mathcal{A} \cup \mathcal{B} \cup \mathcal{D}$, we surely would have $E \in \mathcal{C}$. Hence $E \notin \mathcal{E}$. □

We now begin the model construction.

The idea, part I. The high-level description of our semantics is that each interpretation $[\![X]\!]$ will be a disjoint union of two sets (that depend on the variable X):

(i) A set $\{1, \ldots, n_{X_a}\}$ which will be $[\![X]\!] \cap [\![A^*]\!]$.

(ii) A set $\{1, \ldots, n_{X_b}\}$ which will be $[\![X]\!] \cap [\![B^*]\!]$.

To ensure $[\![A^*]\!] \cap [\![B^*]\!] = \emptyset$, we stipulate $n_{A_b} = 0$ and $n_{B_a} = 0$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We use n_A as shorthand for n_{A_a} and n_B for n_{B_b}. Note that in order that a sentence of the form All U are V be true in the model, we only need $n_{U_a} \leq n_{V_a}$ and $n_{U_b} \leq n_{V_b}$. To arrange that a sentence of the form Most U are V is true, we will choose the n_{X_a} (and likewise n_{X_b}) in such a way that the following holds for all U, V: if $n_{U_a} \neq 0$ and $n_{V_a} \neq 0$, then $\min\{n_{U_a}, n_{V_a}\} > \frac{n_{U_a}}{2}$. This will ensure that many Most sentences hold in our model; in particular all sentences of the form Most A are A', Most B are B', Most A are C, Most B are C, Most C are C' and Most X are E.

The situation is more delicate for sentences ψ in Γ of the form Most C are A (and sentences Most C are B), where $C \in \mathcal{C}$ and $A \in \mathcal{A}$. We wish to ensure that ψ holds. Observe that ψ is true iff $\min(n_{C_a}, n_{A_a}) > \frac{1}{2}(n_{C_a} + n_{C_b})$; thus, in particular, $n_{C_a} > n_{C_b}$. Likewise, for Most C are B we need $\min(n_{C_b}, n_{B_b}) > \frac{1}{2}(n_{C_a} + n_{C_b})$.

Let us consider Figure 2 (left) for motivating the construction ahead. We want to get a model of this system with $[\![A^*]\!] \cap [\![B^*]\!] = \emptyset$. The smallest such model is:

$$[\![U]\!] = 1 + 0 \qquad [\![A^*]\!] = 3 + 0$$
$$[\![V]\!] = 1 + 2 \qquad [\![B^*]\!] = 0 + 2$$
$$[\![W]\!] = 3 + 2$$

Here and below, in giving the definition of a set, $+$ denotes disjoint union, and each number m denotes $\{1, \ldots, m\}$. However, suppose that we extend the assumptions as shown in Figure 2 (right). The model so far does not have Most A^* are U or Most W are U. But notice: in order to have Most U are A^* and also keep $[\![A^*]\!] \cap [\![B^*]\!] = \emptyset$, we must have $n_{U_a} > n_{U_b}$. Similar remarks apply to V and W. This gives a system of constraints like:

					height
				W_a	3
				$\vee$	
		V_b	$\leq$	W_b	2
		$\vee$			
U_a	$\leq$	V_a			1
$\vee$					
U_b					0

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But the sets in the smallest model also satisfy these, so more is needed. We also must arrange that $n_{A_a} > \frac{n_{C_a}+n_{C_b}}{2}$ for $C = U$ and $C = W$, and $n_{B_b} > \frac{n_{C_a}+n_{C_b}}{2}$ for $C = V$. Further, to make Most W are U true, the easiest way would be to have $n_{U_b} > \frac{1}{2} n_{W_a}$.

We find the sizes of the sets above by using a function of the expressions U_b, U_a, ..., W_a. We use a function f of the height h in the well-founded relation $<$. Specifically, we shall use $f(h) = \sum_{l=0}^{K} 2^{K-l}$ where h is the height of the set and $K = 3$ is the maximum height. This gives rise to

$$\begin{aligned} [\![U]\!] &= 12 + 8 & [\![A^*]\!] &= 15 + 0 \\ [\![V]\!] &= 12 + 14 & [\![B^*]\!] &= 0 + 14 \\ [\![W]\!] &= 15 + 14 \end{aligned}$$

The definition of the size function $f(h)$ automatically guarantees that all most sentences hold within each class $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and from $\mathcal{A}$ and $\mathcal{B}$ to $\mathcal{C}$. Moreover, the definition of $f(h)$ allows us to simplify constraints of the form $n_{A_a} > \frac{n_{X_a}+n_{X_b}}{2}$ with $n_{X_a} > n_{X_b}$ arising from most sentences. The definition of $f(h)$ implies that these constraints are equivalent to $n_{A_a} > n_{X_b}$ with $n_{X_a} > n_{X_b}$ (see Lemma 13). All of this is to motivate what comes next.

Definition 8. *We define a structure* $(\mathcal{G}, \blacktriangleleft, \leq)$ *from* Γ, A^*, *and* B^*. *Let*

$$\mathcal{G} = \mathcal{A} \cup \mathcal{B} \cup \{C_a : C \in \mathcal{C}\} \cup \{C_b : C \in \mathcal{C}\}.$$

$\mathcal{G}$ *includes* $\mathcal{A} \cup \mathcal{B}$, *and also two copies of* $\mathcal{C}$. *Those copies are tagged with the letters a and b. When we need to refer to arbitrary elements of* $\mathcal{G}$, *we use the letter g. We then define* $\blacktriangleleft$ *and* $\leq$ *by*

$$\frac{C \to A}{C_b \blacktriangleleft C_a} \qquad \frac{C \to A}{C_b \blacktriangleleft A} \qquad \frac{C \to B}{C_a \blacktriangleleft C_b} \qquad \frac{C \to B}{C_a \blacktriangleleft B}$$

$$\frac{C \hookrightarrow C'}{C_a \leq C'_a} \qquad \frac{C \hookrightarrow C'}{C_b \leq C'_b} \qquad \frac{A \hookrightarrow A'}{A \leq A'} \qquad \frac{B \hookrightarrow B'}{B \leq B'} \qquad \frac{A \hookrightarrow C}{A \leq C_a} \qquad \frac{B \hookrightarrow C}{B \leq C_b}$$

We call $(\mathcal{G}, \blacktriangleleft, \leq)$ *the* structure associated with Γ, A^*, and B^*.

Proposition 9. $\leq$ *is a preorder on* $\mathcal{G}$.

The idea, part II There is still a way to go to satisfy the sentences Most C are A and Most C are B. But before we get to this, we need to see how $\leq$ and $\blacktriangleleft$ give us a well-founded relation. This is the content of Lemma 11 below, and Lemma 10 is a preliminary to that result.

We write $<$ for the strict part of this preorder, so $g < g'$ means $g \leq g'$ but $\neg(g' \leq g)$.
Here is the only use of the $(\triangleright)$ rules of the logic:

Lemma 10. *There are no cycles in* $\blacktriangleleft \cup \leq$ *which involve a* $\blacktriangleleft$ *relation. That is, if S is the relation* $\blacktriangleleft \cup \leq$, *then there is no sequence of length* ≥ 2 *of the form*

$$g_1 \, S \, g_2 \, \cdots g_i \blacktriangleleft g_{i+1} \, \cdots g_{k-1} \, S \, g_k = g_1.$$

We say that $(\mathcal{G}, \blacktriangleleft, \leq)$ *is* acyclic.

Proof. Assume towards a contradiction that we had a sequence as above. By rotating the sequence around, we might as well assume that $i = 1$. Moreover, since $\leq$ is transitive, we may assume that no two successive instances of S are $\leq$. Without loss of generality, g_1 is of the form C^1_a. Thus g_2 is either C^1_b (for the same C) or else g_2 is some B.

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We first claim that there must be another instance of $\blacktriangleleft$ on our sequence. For if not, then $g_2 = g_1$. However, we have just seen that g_2 cannot equal g_1. The next instance of $\blacktriangleleft$ must be of one of the two forms $C_b^2 \blacktriangleleft C_a^2$, or else $C_b^2 \blacktriangleleft A$. Our cycle thus begins

$$C_a^1 \blacktriangleleft g_2 \leq C_b^2 \blacktriangleleft g_4 \cdots \tag{8}$$

The points on the "pointy side" of the $\blacktriangleleft$'s in (8) alternate between those of the form C_a and those of the form C_b. And so the chain overall may be written as:

$$C_a^1 \blacktriangleleft g_2 \leq C_b^2 \blacktriangleleft g_4 \leq C_a^3 \blacktriangleleft \cdots \leq C_b^r \blacktriangleleft g_{2r} \leq C_a^{r+1} = C_a^1.$$

Now in the first section of the cycle we have $C_a^1 \blacktriangleleft g_2 \leq C_b^2 \blacktriangleleft g_4$. If g_2 is C_b^1, then we have some $B \in \mathcal{B}$ and $A \in \mathcal{A}$, so that all of the following are provable from Γ:

$$\mathsf{Most}\ C^1\ \mathsf{are}\ B,\ \mathsf{All}\ C^1\ \mathsf{are}\ C^2,\ \mathsf{Most}\ C^2\ \mathsf{are}\ A,$$

and also $\mathsf{All}\ B\ \mathsf{are}\ B^*$ and $\mathsf{All}\ A\ \mathsf{are}\ A^*$. (These last two are from the definitions of $\mathcal{A}$ and $\mathcal{B}$.) In short, we get $C^1 \triangleright_{A^*,B^*} C^2$. If g_2 is of the form B, then we again get $C^1 \triangleright_{A^*,B^*} C^2$.

In a similar way, the section of the cycle $C_b^2 \blacktriangleleft g_4 \leq C_b^4 \blacktriangleleft g_6$, tells us that $C^2 \triangleright_{B,*,A^*} C^3$.

Continuing, we get $C^3 \triangleright_{A^*,B^*} C^4, \ldots, C^r \triangleright_{B,*,A^*} C^{r+1} = C^1$.

Then by one of the ($\triangleright$) rules, we see that $\Gamma \vdash \mathsf{Some}\ A^*\ \mathsf{are}\ B^*$. But this contradicts the overall assumption made at the very beginning of this section that $\Gamma \nvdash \mathsf{Some}\ A^*\ \mathsf{are}\ B^*$. □

We write R for the union $\blacktriangleleft \cup <$. (Note that S used the relation $\leq$ while R uses its strict form $<$.)

Lemma 11. *The relation R is well founded on $\mathcal{G}$.*

Proof. Suppose $g_0, \ldots, g_n, \ldots$ is an infinite sequence with the property that $g_{n+1}\ R\ g_n$ for all n. Since $\mathcal{G}$ is a finite set, there are $j < k$ such that $g_k = g_j$. Thus

$$g_j = g_k\ R\ g_{k-1}\ R \cdots R\ g_{j+1}\ R\ g_j.$$

In this cycle, the instances of R cannot all be from $<$; otherwise, we would have $g_j < g_k$ and $g_j = g_k$, a contradiction. So at least one pair must be related by $\blacktriangleleft$. But then we have a cycle containing an instance of $\blacktriangleleft$, contradicting Lemma 10. □

Since R is well founded, there is a unique *rank function* $|\cdot| : \mathcal{G} \to N$ such that for all $g \in \mathcal{G}$,

$$|g| = \max\big(\{1 + |h| : h \blacktriangleleft g\} \cup \{|h| : h < g\}\big). \tag{9}$$

Here we stipulate that $\max(\emptyset) = 0$.

Lemma 12. *Concerning $|\cdot| : \mathcal{G} \to N$:*

1. *If $A \hookrightarrow A'$, then $|A| \leq |A'|$.*
2. *If $A \hookrightarrow A'$ and $A' \hookrightarrow A$, then $|A| = |A'|$.*
3. *If $C \hookrightarrow C'$ and $C' \hookrightarrow C$, then $|C_a| = |C'_a|$.*
4. *If $C \hookrightarrow C'$, then $|C_a| \leq |C'_a|$.*
5. *If $A \hookrightarrow C$, then $|A| < |C_a|$.*

Proof. For part 1, assume that $A \hookrightarrow A'$. Every predecessor of A under either $\blacktriangleleft$ or $<$ is a predecessor of A' for the same relation. This immediately implies that $|A| \leq |A'|$. And then part (2) follows from part 1.

We turn to (3) and (4). Before that, we have an observation: the R-predecessors in $\mathcal{G}$ of C'_a are (1) the elements C_a, where $C \hookrightarrow C'$ but not conversely; and (2) the elements C'_b, where $C' \to A$ for some A. Now for (3), if $C \hookrightarrow C' \hookrightarrow C$, then C and C' have all of the same predecessors

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under $\blacktriangleleft$ and $\leq$. (This uses Proposition 5 and ultimately (m_4).) And then (3) follows. As for (4), assume that $C \hookrightarrow C'$. If we also had $C' \hookrightarrow C$, then we could apply (3). Otherwise, $C_a < C'_a$. Every $\blacktriangleleft$-predecessor of C is a $\blacktriangleleft$-predecessor of C'. So $|C_a| \leq |C'_a|$.

Here is the argument for (5). If $A \hookrightarrow C$, then $A \leq C_a$ by the definition of $\leq$. We cannot have $C_a \hookrightarrow A$, by the definition of $\mathcal{C}$. So $A < C_a$. This implies that $|A| < |C_a|$. □

Remark We stated Lemma 12 in terms of variables $A \in \mathcal{A}$ and $C \in \mathcal{C}$. But the symmetry of the definitions implies that there are parallel results for $B \in \mathcal{B}$ as well. We tacitly include these in the statement of Lemma 12.

4.6 A lemma on falling sums

Let K be any number. We define a function f_K with domain $\{0, \ldots, K\}$ by

$$f_K(i) = \sum_{l=0}^{i} 2^{K-l}. \tag{10}$$

Note that $1 \leq f_K(i) \leq 2^{K+1} - 1$.

We use these functions f_K in order to ensure that the Most sentences in Γ are true in the model which we build. The key point in the verification hinges on the following result.

Lemma 13. *For all i and j between 0 and K (inclusive), and all $k > i$, $f_K(k) > \frac{1}{2}(f_K(i) + f_K(j))$.*

Proof. It is easy to check that f is strictly increasing. Fix $0 \leq i < j, k \leq K$. Note that $i < K$ so that $K - i \geq 1$. We drop the subscript K on f, and then we calculate:

$$\begin{aligned} f(i) + f(j) &\leq f(i) + f(K) = \left(\textstyle\sum_{l=0}^{i} 2^{K-l}\right) + \sum_{l=0}^{K} 2^{l} = \left(\textstyle\sum_{l=0}^{i} 2^{K-l}\right) + 2^{K+1} - 1 \\ &< \left(\textstyle\sum_{l=0}^{i} 2^{K-l}\right) + 2^{K+1} = 2\left(\left(\textstyle\sum_{l=0}^{i} 2^{K-l-1}\right) + 2^{K}\right) = 2\textstyle\sum_{l=0}^{i+1} 2^{K-l} = 2f(i+1) \\ &\leq 2f(k) \end{aligned}$$

□

4.7 Notation for sets in our model construction

We need some notation for sets. Given numbers a and b, we let

$$a + b = (\{1, \ldots, a\} \times \{1\}) \cup (\{1, \ldots, b\} \times \{2\}) \tag{11}$$

For example, $1 + 3$ is a short for $\{(1, 1), (1, 2), (2, 2), (3, 2)\}$. Observe that $a + b \subseteq a' + b'$ if and only if both $a \leq a'$ and $b \leq b'$.

4.8 The model and the verification

At this point we return to the proof of Theorem 3. We have a set Γ, and we want to build a model of it where $[\![A^*]\!] \cap [\![B^*]\!] = \emptyset$. Definition 8 gives the structure $(\mathcal{G}, \blacktriangleleft, \leq)$ associated with Γ, A^*, and B^*. And (9) gives a function $g \mapsto |g|$ whose properties were studied in Lemma 12. We also remind the reader of the functions f_K defined in (10) above. In what follows, we take

$$K = \max_{g \in \mathcal{G}} |g|.$$

Then with this value of K, we define numbers n_g for $g \in \mathcal{G}$ by

$$n_g = f_K(|g|). \tag{12}$$

So $n_g = \sum_{l=0}^{|g|} 2^{K-l}$. We also let $N = \max\{n_g : g \in \mathcal{G}\}$.

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We now present our model, $\mathcal{M}$, using all of the notation above. The universe M is $N + N$; this is a set with $2N = 2^{K+2} - 2$ elements. The rest of the structure is given as follows:

$$\begin{aligned} \text{For } A \in \mathcal{A},\ [\![A]\!] &= n_A + 0 \\ \text{For } B \in \mathcal{B},\ [\![B]\!] &= 0 + n_B \\ \text{For } C \in \mathcal{C},\ [\![C]\!] &= n_{C_a} + n_{C_b} \\ \text{For } D \in \mathcal{D},\ [\![D]\!] &= 0 + 0 \\ \text{For } E \in \mathcal{E},\ [\![E]\!] &= N + N \end{aligned} \tag{13}$$

Note that $[\![D]\!] = \emptyset$, while $[\![E]\!] = M$. This defines our model $\mathcal{M}$.

We turn to the verification that the model has the properties needed for our theorem: $\mathcal{M} \models \Gamma$, but $\mathcal{M} \not\models$ Some A^* are B^*. The reader might like to look back at our description of the overall idea in Section 4.5 and the example in Section 6.

First, another basic fact:

Proposition 14. *If $D \in \mathcal{D}$, then for all X, $\neg(D \downarrow X)$.*

Proof. If $D \downarrow X$, then $D \downarrow D$. And from this it follows that $A^* \downarrow B^*$. This contradicts our overall assumption that $\Gamma \not\vdash$ Some A^* are B^*. □

Lemma 15. *If $X \hookrightarrow Y$, then $[\![X]\!] \subseteq [\![Y]\!]$.*

Proof. For $X, Y \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, this result comes from Lemma 12 and the definitions of the model. We also use the fact that no $C \in \mathcal{C}$ is related by $\hookrightarrow$ to any $A \in \mathcal{A}$ or to any $B \in \mathcal{B}$.

If $X \in \mathcal{D}$, then $[\![X]\!] = \emptyset$. If $Y \in \mathcal{D}$, then $X \in \mathcal{D}$ also, by the definition of $\mathcal{D}$.

If $X \in \mathcal{E}$, then Y cannot belong to $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ by Proposition 7, part (3). For $Y \in \mathcal{E}$, our result comes from the fact that for all X, $[\![X]\!] \subseteq [\![E]\!]$. □

Lemma 16. *If $X \to Y$, then* $\text{card}([\![X]\!] \cap [\![Y]\!]) > \frac{1}{2}\text{card}([\![X]\!])$.

Proof. Every model satisfies the sentences Most X are X, provided $[\![X]\!] \neq \emptyset$. Our model has $[\![X]\!] \neq \emptyset$ for $X \notin \mathcal{D}$. Obviously, $\mathcal{M}$ satisfies all sentences Most X are E where $X \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{E}$ and $E \in \mathcal{E}$. For all $A, A' \in \mathcal{A}$, $B, B' \in \mathcal{B}$, $C, C' \in \mathcal{C}$, and $E \in \mathcal{E}$, our $\mathcal{M}$ satisfies all sentences of all of the forms Most A are A', Most A are C, Most B are B', Most B are C, Most C are C', and Most E are C. The reason for all of these has to do with the choice of the parameters n_g. For all $g \in \mathcal{G}$ we have $2^K \leq n_g < 2^{K+1}$, and hence, for all $g, g' \in \mathcal{G}$ it holds that $\min\{n_g, n_{g'}\} > \frac{n_g}{2}$. In other words, we have $card([\![X]\!] \cap [\![Y]\!]) > \frac{1}{2} card([\![X]\!])$ in all the cases which we mentioned, even without the assumption that $X \to Y$.

From Proposition 14, we have $\neg(D \to X)$ and $\neg(X \to D)$ for all X and also $\neg(E \to X)$ for $X \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{D}$. Finally, we cannot have $A \to B$, since this would easily entail $A^* \downarrow B^*$.

Much of the work in our construction was devoted to ensuring that when $C \to A$, the sentence Most C are A is true in $\mathcal{M}$. (The same holds for B replacing A, mutatis mutandis.) In this case $C_b \blacktriangleleft C_a, A$. By Lemma 13,

$$n_A > \tfrac{1}{2}(n_{C_a} + n_{C_b}) \tag{14}$$

And since $n_{C_a} > n_{C_b}$,

$$n_{C_a} > \tfrac{1}{2}(n_{C_a} + n_{C_b}) \tag{15}$$

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Thus

$$\begin{aligned} card([\![C]\!] \cap [\![A]\!]) &= \min(n_A, n_{C_a}) \text{ by (13)} \\ &> \tfrac{1}{2}(n_{C_a} + n_{C_b}) \text{ by (14) and (15)} \\ &= \tfrac{1}{2} card([\![C]\!]) \end{aligned}$$

This completes the proof. □

Lemma 17. *If $X \downarrow Y$, then $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$.*

Proof. $\mathcal{M}$ satisfies all sentences Some X are Y except those of two kinds: first, the sentences Some X are Y where either X or Y belongs to $\mathcal{D}$; and second those where $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ (or vice versa). By Proposition 14, sentences of the first kind cannot follow from Γ. And as for the sentences of the second kind, notice that for $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $\neg(A \downarrow B)$, lest $A^* \downarrow B^*$. □

Proposition 5 and Lemmas 15, 16, and 17 imply that $\mathcal{M} \models \Gamma$. The next result shows that $\mathcal{M} \not\models$ Some A^* are B^*.

Lemma 18. $[\![A^*]\!] \cap [\![B^*]\!] = \emptyset$.

Proof. Our construction has arranged that $[\![A]\!] \cap [\![B]\!] = \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Recall that we are assuming that $A^* \in \mathcal{A}$ and $B^* \in \mathcal{B}$; see Proposition 7, part (1). Our result follows. □

Remark 19. We are done with the case of Theorem 3 mentioned in the heading of Section 4.5. We assumed that $\Gamma \nvdash \varphi$ and then built a model of Γ falsifying φ. But the assumption that $\Gamma \nvdash \varphi$ only was used in Lemma 10 and Proposition 14. We did not need this assumption to build the associated structure $(\mathcal{G}, \blacktriangleleft, \leq)$. And while Lemma 10 was used heavily in what we did, as we noted in the proof of that result, if $(\mathcal{G}, \blacktriangleleft, \leq)$ were not acyclic, then we could use a $(\triangleright)$ rule to show that $\Gamma \vdash \varphi$. Similarly, if the statement of Proposition 14 failed, then again we would have $\Gamma \vdash \varphi$.

Our work thus shows the following. Let Γ be any set of sentences, and consider the sentence Some A^* are B^*. Define $\hookrightarrow$, $\rightarrow$, $\downarrow$, $\mathcal{G}$, $\blacktriangleleft$, and $\leq$ from Γ, A^*, and B^* as we did earlier in the paper. Assume that $\neg(A^* \hookrightarrow B^*)$ and $\neg(B^* \hookrightarrow A^*)$. Then the following are equivalent:

1. For all $D \in \mathcal{D}$ and all X, $\neg(D \downarrow X)$; and in addition, $(\mathcal{G}, \blacktriangleleft, \leq)$ is acyclic.
2. $\Gamma \not\models$ Some A^* are B^*.

This concludes the proof of Theorem 3. We now know that our proof system is sound and complete. The next section addresses the decidability.

5. Algorithmic analysis and complexity

We check that the problem of whether or not $\Gamma \models \varphi$ (equivalently, $\Gamma \vdash \varphi$) can be decided in time polynomial in the number of symbols in the finite set Γ.

Proposition 20. (Pratt-Hartmann and Moss (2009), Theorem 4.11). *The problem of determining whether $\Delta \models \psi$ for $\Delta \cup \{\psi\}$ a finite set of sentences in the logic of* All *and* Some *is in NLogSpace.*

Proof. We sketch the proof for the convenience of the reader. Fix Δ, and write ψ as All X are Y. Let $\mathcal{G}^{all} = \mathcal{G}^{all}_{\Delta}$ be the *all-graph of* Δ: the points are the variables occurring in Δ, and with $U \rightarrow V$ iff Δ contains the sentence All U are V. We claim that $\Delta \models \psi$ iff there is a directed path $\mathcal{G}^{all}$ from

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X to Y. (We allow the empty path in this result and in the rest of this paper.) To see this, note that each path in $\mathcal{G}^{all}$ easily gives a proof of ψ from Δ using the two rules (*axiom*) and (*barbara*). Since the logic is sound, we have $\Delta \models \psi$. More interestingly, if there is no path in $\mathcal{G}^{all}$ from X to Y, then we get a model $\mathcal{M} \models \Delta$ with $\mathcal{M} \not\models \psi$ by

$$[\![U]\!] = \{V : \text{there is a path from } V \text{ to } U\}.$$

We omit the easy proof that $\mathcal{M}$ has the required properties.

Now we return to our overall result. Let ψ now be Some X are Y. We claim that $\Delta \models \psi$ iff there is a Some-sentence in Δ, say Some K are L, such that each of the two elements of $\{X, Y\}$ is reachable in $\mathcal{G}^{all}$ from some element of $\{K, L\}$. It is clear that our condition is sufficient to have $\Delta \vdash \psi$; by soundness, gives $\Delta \models \psi$. To show the contrapositive, assume that there is no sentence Some K are $L \in \Delta$ as above. We construct $\mathcal{M} \models \Delta$ which falsifies ψ by taking as points the sentences in Δ of the form Some K are L, and with

$$[\![U]\!] = \{(\text{Some } K \text{ are } L) \in \Delta : \text{there is a path in } \mathcal{G}^{all} \text{ from } K \text{ to } U, \text{ or from } L \text{ to } U\}.$$

Again, we omit the details that this works. The important fact is that the problem of deciding whether or not $\Delta \vdash \psi$ is reducible to questions about reachability in $\mathcal{G}^{all}_{\Delta}$. Since those questions can be answered in NLogSpace, we are done. □

We state our result as a corollary to Theorem 3.

Corollary 21. *The problem of determining whether* $\Gamma \models \psi$ *for* $\Gamma \cup \{\psi\}$ *a finite set of sentences in* All, Some, *and* Most *is in PTime.*

Proof. First, for φ of the form All A^* are B^*, the result follows from the work we did in Section 4.1. Let Γ_{all} be the set of All sentences in Γ. To begin, decide whether $\Gamma_{\text{all}} \models \varphi$. The point is that $\Gamma \models \varphi$ iff $\Gamma_{\text{all}} \models \varphi$. Then we are done by Proposition 20.

Second, we consider φ of the form Some A^* are A^*. Define Δ by (6). Decide whether or not $\Delta \vdash \varphi$ in the logic of All and Some. Again, by Proposition 20, this can be done in PTime.

Third, we consider the case when φ is of the form Most A^* are B^*. Recall our work in Sections 4.5–4.8. First, we see whether $A^* \hookrightarrow B^*$ or not by looking at $\mathcal{G}^{all}$, the all-graph of Γ (see the proof of Proposition 20). (a) If $A^* \hookrightarrow B^*$, we ask whether or not $A^* \downarrow A^*$. The work in the previous paragraph shows how to determine this. If so, then $\Gamma \models \varphi$; and if not, then the one-point model in Section 4.3 shows that $\Gamma \not\models \varphi$.

(b) If $\neg(A^* \hookrightarrow B^*)$ but $B^* \hookrightarrow A^*$, then we build the four-point model $\mathcal{M}$ in (7). We then evaluate all the sentences in Γ to see if they are true in this $\mathcal{M}$. If so, then we have a model of Γ where φ fails, and so $\Gamma \not\models \varphi$. If some $\psi \in \Gamma$ fails in $\mathcal{M}$, then the argument concerning the model shows that $\Gamma \vdash \varphi$.

(c) If $\neg(A^* \hookrightarrow B^*)$ and $\neg(B^* \hookrightarrow A^*)$, then we do the same thing with the seven-point model mentioned in the text.

Finally, we consider the case when φ is of the form Some A^* are B^*. We may assume that $\neg(A^* \hookrightarrow B^*)$ and $\neg(B^* \hookrightarrow A^*)$, or else we can reduce to the case of sentences of the form Some A^* are A^*.

Recall from Notation 4 that we also defined $\Gamma \vdash_0 \varphi$ if φ can be proved from Γ without using any ($\triangleright$) rules. The problem of deciding whether $\Gamma \vdash_0 \varphi$ can be solved in time $O(n^3)$, where n is the total number of atoms in $\Gamma \cup \{\varphi\}$. (The exponent 3 comes from the fact that each of the rules has at most 3 atoms.) For more details on this, see Lemma 2.1 of Pratt-Hartmann and Moss (2009). It follows that in PTime we can define the relations $\hookrightarrow$, $\rightarrow$, and $\downarrow$, and then construct the associated structure $(\mathcal{G}, \blacktriangleleft, \leq)$. Ask: is it true that for all $D \in \mathcal{D}$, $\neg(D \downarrow D)$? And, is $(\mathcal{G}, \blacktriangleleft, \leq)$ acyclic? These questions can be answered in PTime. If the answers to either of these is "No," then $\Gamma \vdash \varphi$. If both are "Yes," then $\Gamma \not\models \varphi$. These points were justified in Remark 19. □

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The proof also shows the following:

Proposition 22. *Let* Γ *be a finite set of sentences in the language of this paper. If* Γ *has a model where* Some A^* are B^* *is false, then there is such a model of size* $\leq 2^{M+2} - 2$*, where* M *is the number of* most *sentences in* Γ.

Proof. Here is a sketch. We use the model from Section 4.8. The largest possible rank $|U_a|$ of any element of $\mathcal{G}$ is M. The largest size of the universe is $2f_M(M) = 2(2^{M+1} - 1) = 2^{M+2} - 2$. □

6. Examples

We illustrate the methods of this paper by answering the two questions posed at the end of the Introduction and by modifying the first question.

6.1 Solution to (Q1)

We give the solution to (Q1) including the main definitions and the construction of the model. Consider Γ from (2) in the Introduction. (Q1) asks whether or not $\Gamma \models$ Most V are U.

For this, we follow the work in Corollary 21. Since the conclusion is a Most-sentence, we must follow the details in Section 4.3. Notice that $A^* = U$ and $B^* = V$. First, we calculate the relation $\hookrightarrow$ explicitly. The relation $\hookrightarrow$ consists of the reflexive instances $X \hookrightarrow X$ and also

$$\begin{array}{lllll} A^* \hookrightarrow E & A^* \hookrightarrow W & B^* \hookrightarrow E & D \hookrightarrow A^* & D \hookrightarrow B^* \\ D \hookrightarrow E & D \hookrightarrow W & U \hookrightarrow V & U \hookrightarrow W & V \hookrightarrow W \end{array} \tag{16}$$

Note that $\neg(V \hookrightarrow U)$ but $U \hookrightarrow V$, so we are in the second subcase of Section 4.3. We then split the variables into three subsets: $\mathcal{A} = \{V,\ W\}$, $\mathcal{B} = \{U\}$, and $\mathcal{C} = \{A^*,\ B^*,\ D,\ E\}$. We construct the four-point model in (7). Explicitly, this is

$$\begin{aligned} [\![V]\!] = [\![W]\!] &= \{1, 2, 3, 4\} \\ [\![U]\!] &= \{1, 2\} \\ [\![A^*]\!] = [\![B^*]\!] = [\![D]\!] = [\![E]\!] &= \{1, 2, 3\} \end{aligned} \tag{17}$$

We check that this model satisfies the sentences in Γ and falsifies Most V are U. Thus we have determined that $\Gamma \not\vdash$ Most V are U.

6.2 Strengthening Γ *in (Q1)*

We again consider Γ from (2). Does $\Gamma \cup \{$Most W are $U\} \vdash$ Most V are U? As in our discussion just above, we construct a four-point model $\mathcal{M}$ from the data in (16). We have seen $\mathcal{M}$ in (17) above. However, when we go to verify that all of the sentences in Γ hold in $\mathcal{M}$, we see that Most W are U does *not* hold. The argument in Corollary 21 already tells us that $\Gamma \cup \{$Most W are $U\} \vdash$ Most V are U, answering our question. Indeed, we can derive Most V are U using (m_5). The point of this example is to illustrate the technique of finding proof trees from the *failure* of a small model to satisfy various sentences.

6.3 Solution to (Q2)

We next present an example of all the work in Sections 4.5–4.8, including the main definitions and the construction of the model. Again consider Γ from (2) in the Introduction. We asked whether or not $\Gamma \models$ Some A^* are B^*. We follow our previous details to see. We may re-express Γ using the notation from Proposition 1. This is shown on the left in Figure 3.

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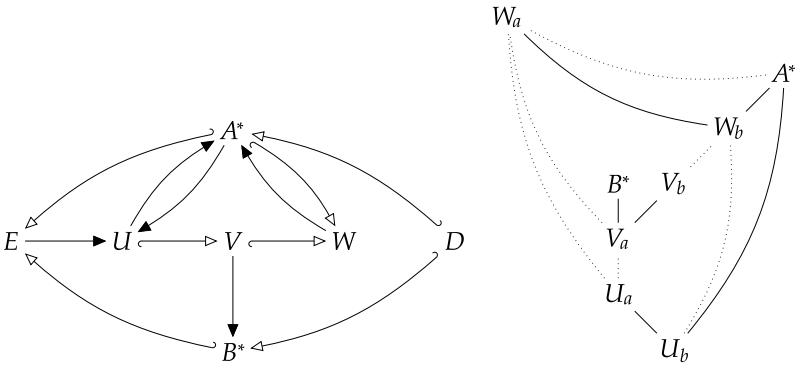


Figure 3. The set Γ in Section 6.3 on the left, along with a picture of its associated relation R on the right.

Our variables are A^*, B^*, U, V, W, D, and E. With this Γ, A^*, and B^*, the variables are divided into subsets as follows:

$$\mathcal{A} = \{A^*\} \qquad \mathcal{B} = \{B^*\} \qquad \mathcal{C} = \{U, V, W\} \qquad \mathcal{D} = \{D\} \qquad \mathcal{E} = \{E\}$$

Let us exhibit the three relations on our variables in Notation 4. The relation $\hookrightarrow$ is easy to read off of Figure 3; see also (16). The "most" arrows $X \to Y$ are those of the form $X \to X$ for all X other than D, and also

$$\begin{array}{lllll} A^* \to E & A^* \to U & A^* \to V & A^* \to W & B^* \to E \\ E \to U & E \to V & E \to W & & \\ U \to A^* & U \to V & U \to W & V \to B^* & V \to W \quad W \to A^* \end{array}$$

For this example, the relation $\downarrow$ is the union of $\to$ with its converse.

We next exemplify the structure $(\mathcal{G}, \blacktriangleleft, \leq)$ associated with Γ, A^*, and B^*; see Definition 8. The set $\mathcal{G}$ is $\{A^*, B^*, U_a, U_b, V_a, V_b, W_a, W_b\}$, and the relations $\leq$ and $\blacktriangleleft$ are

$$\begin{array}{llllll} U_a \leq V_a & U_b \leq V_b & V_a \leq W_a & V_b \leq W_b & A^* \leq W_a \\ U_a \leq W_a & U_b \leq W_b & U_b \blacktriangleleft U_a & U_b \blacktriangleleft A^* & \\ V_a \blacktriangleleft V_b & V_a \blacktriangleleft B^* & W_b \blacktriangleleft W_a & W_b \blacktriangleleft A^* & \end{array} \tag{18}$$

and all the reflexive values $g \leq g$. All of the $\leq$ statements in (18) above in fact hold strictly. So the relation R in Lemma 11 has exactly the pairs listed in (18). R is pictured on the right in Figure 3 as a Hasse diagram, showing $g\ R\ h$ by putting g below h. We depict $\leq$-edges as dotted lines and $\blacktriangleleft$-edges as solid lines. We have shown this relation R completely; it is not transitive, and there are no missing edges. From the picture, we can see that $(\mathcal{G}, \blacktriangleleft, \leq)$ is acyclic. This already tells us the answer to the opening question: it is not true that $\Gamma \models \mathsf{Some}\ A^*\ \mathsf{are}\ B^*$. And by following the work in the completeness proof, we can exhibit a counter-model.

The rank function $|\cdot| : \mathcal{G} \to N$ is shown below:

$$\begin{array}{llll} |U_b| = 0 & |V_a| = 1 & |B^*| = 2 & |A^*| = 3 \\ |U_a| = 1 & |V_b| = 2 & |W_b| = 2 & |W_a| = 3 \end{array}$$

The number K is the maximum rank, 3. And $N = 2^{K+1} - 1 = 15$. The function f_3 from Section 4.6 is $f_3(0) = 8, f_3(1) = 12, f_3(2) = 14$, and $f_3(3) = 15$. The numbers $n_g = f_6(|g|)$ are

$$\begin{array}{llll} n_{U_b} = 8 & n_{V_a} = 12 & n_{B^*} = 14 & n_{A^*} = 15 \\ n_{U_a} = 12 & n_{V_b} = 14 & n_{W_b} = 14 & n_{W_a} = 15 \end{array}$$

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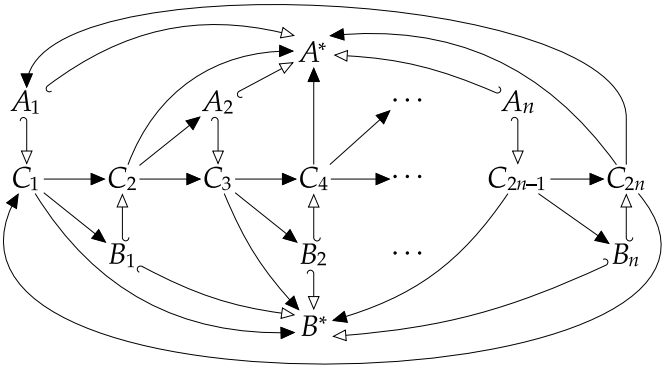


Figure 4. The set Γ in Theorem 23

The model $\mathcal{M}$ is given in terms of the notation for disjoint sets in Section 4.7:

$$\begin{aligned} [\![A^*]\!] &= 15 + 0 & [\![B^*]\!] &= 0 + 14 \\ [\![U]\!] &= 12 + 8 & [\![V]\!] &= 12 + 14 \\ [\![W]\!] &= 15 + 14 & [\![D]\!] &= 0 + 0 \\ [\![E]\!] &= 15 + 15 \end{aligned}$$

Notice that $\mathcal{M} \models \Gamma$, but $[\![A^*]\!] \cap [\![B^*]\!] = \emptyset$, as desired.

7. No finite axiomatization

Our set of rules in Figure 1 is infinite, due to the scheme ($\triangleright$). It is natural to ask whether there is any finite set of rules which is sound and complete.

Theorem 23. *There is no finite set $\mathcal{R}$ of rules which is sound and complete for the logic of this paper.*

Proof. The method here comes from similar results concerning syllogistic logics: see Pratt-Hartmann (2009) and Pratt-Hartmann and Moss (2009).

Let $\mathcal{R}$ be any finite set of rules which is sound. We show that $\mathcal{R}$ is not complete. Let n be larger than the number of antecedents in any rule in $\mathcal{R}$. Let Γ be given diagrammatically in Figure 4. We also have left a few arrows implicit: every arrow is also a bidirectional some arrow, and the graph is reflexive for most, all, and some arrows. We also left off some arrows: we have all some arrows $A_i \updownarrow A_j$, $A_i \updownarrow A^*$, $B_i \updownarrow B_j$, $B_i \updownarrow B^*$, and $C_j \updownarrow X$, where X is any variable.

The diagram contains an instance of ($\triangleright$), and so $\Gamma \models$ Some A^* are B^*. Actually, it is worthwhile to examine the structure $(\mathcal{G}, \blacktriangleleft, \leq)$ associated with Γ, A^*, and B^*. We have $\mathcal{A} = \{A_1 \ldots, A_n\}$, $\mathcal{B} = \{B_1 \ldots, B_n\}$, $\mathcal{C} = \{C_1, \ldots, C_{2n}\}$, and $\mathcal{D} = \emptyset = \mathcal{E}$. There is one and only one cycle in $\blacktriangleleft \cup \leq$ involving a pair related by $\blacktriangleleft$:

$$A_1 \leq C_{1,a} \blacktriangleleft B_1 \leq C_{2,b} \blacktriangleleft A_2 \leq C_{3,a} \blacktriangleleft B_2 \leq \cdots \leq C_{2n,b} \blacktriangleleft A_1 \tag{19}$$

Claim 1. *For every sentence φ except for* Some A^* are B^* *and* Some B^* are A^*, *if $\Gamma \vdash \varphi$ is the logic of this paper, then $\varphi \in \Gamma$.*

Proof. By induction on the derivations from Γ. We argue by cases as to the rule at the root of the derivation. If the rule is (*axiom*), (*barbara*), (*conv*), (m_1), (m_2), or (m_3), the induction is trivial. For (*darii*), note that the only sentences in Γ of the form All A^* are X have $X = A^*$, and similarly for B^*. For (m_4) and (m_5), note that these can only be used in a trivial way, where the All X are Y

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sentences have $X = Y$. We are left with ($\triangleright$). The only Some sentences not in Γ are Some A_j are B_k, Some A_j are B^*, Some A^* are B_k, Some A^* are B^*, and their converses. There are two ways to argue. First, one could show directly that no ($\triangleright$) rule has sentences in Γ as antecedents and one of these as a conclusion. Alternatively (and probably more easily), one could show that for all of these sentences Some X are Y, if we form the structure $(\mathcal{G}, \blacktriangleleft, \leq)$ associated with Γ, X, and Y, there is no cycle with a $\blacktriangleleft$. □

Let

$$\Delta_i = \Gamma \setminus \{\text{All } A_i \text{ are } A^*\}. \tag{20}$$

Claim 2. *If $\Delta_i \models \varphi$, then $\varphi \in \Gamma$.*

Proof. First, suppose that φ is different from Some A^* are B^* and Some B^* are A^*. If $\Delta_i \models \varphi$, then $\Gamma \models \varphi$. So by completeness, $\Gamma \vdash \varphi$. By our previous claim, $\varphi \in \Gamma$.

We need to only show that Δ_i has a model where $[\![A^*]\!] \cap [\![B^*]\!] = \emptyset$; hence neither Some A^* are B^* nor Some B^* are A^* is a consequence of Δ_i. Consider the structure $(\mathcal{G}, \blacktriangleleft, \leq)$ associated with Δ_i, A^*, and B^*. Δ_i does not contain All A_i are A^*. With Δ_i, $A_i \notin \mathcal{A}$, but rather $A_i \in \mathcal{C}$. Then the cycle in (19) is broken: the link $C_{2i-2,b} \blacktriangleleft A_i$ is missing, and indeed there is no way to go from $C_{2i-2,b}$ to $C_{2i-1,a}$. The upshot is that $(\mathcal{G}, \blacktriangleleft, \leq)$ is acyclic, and so we can build a model of Δ_i in which $[\![A^*]\!] \cap [\![B^*]\!] = \emptyset$. □

We next show that Γ is closed under one-step provability in $\mathcal{R}$. Suppose that $\Gamma \vdash \varphi$ by applying one rule in $\mathcal{R}$ one time. The antecedent of the rule application must have $< n$ sentences, by choice of n. So it must be missing one of the n sentences All A_i are A^*. Thus the antecedent must be a subset of Δ_i for some i. Thus, for some i, $\Delta_i \vdash \varphi$ using the rules of $\mathcal{R}$. By soundness, $\Delta_i \models \varphi$. By the last claim, $\varphi \in \Gamma$.

From this, an induction on derivations shows that Γ is closed under provability using the rules in $\mathcal{R}$. In particular, $\Gamma \nvdash$ Some A^* are B^* using $\mathcal{R}$. So $\mathcal{R}$ is not complete, just as we want to show. □

Conclusion

This paper presented a sound and complete axiomatization of the logic of All X are Y, Some X are Y, and Most X are Y, interpreted on finite models by taking Most X are Y to mean that a strict majority of the X are Y. We have also shown that the decision problem for this language is decidable in polynomial time. One next step in this line of work would be to add Boolean sentential connectives, as was done in Lai et al. (2016) for the language with Most X are Y only. Another step would be to allow the nouns to be complemented.

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