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Relative Zariski Open Objects

by

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Abstract

In [TV], Bertrand Toën and Michel Vaquié define a scheme theory for a closed monoidal category $(\mathcal{C},\otimes,1)$. One of the key ingredients of this theory is the definition of a Zariski topology on the category of commutative monoidal objects in $\mathcal{C}$. The purpose of this article is to prove that under some hypotheses, Zariski open subobjects of affine schemes can be classified almost as in the usual case of rings $(\mathbb{Z}\text{-mod},\otimes,\mathbb{Z})$. The main result states that for any commutative monoidal object A in $\mathcal{C}$, the locale of Zariski open subobjects of the affine scheme $\mathrm{Spec}(A)$ is associated to a topological space whose points are prime ideals of A and whose open subsets are defined by the same formula as in rings. As a consequence, we can compare the notions of scheme over $\mathbb{F}_1$ in [D] and in [TV].

Key Words: Category, enriched category, sieves, relative geometry.

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Introduction

In [TV], Bertrand Toën and Michel Vaquié define a scheme theory for a closed monoidal category $(\mathcal{C},\otimes,1)$. This theory recovers the classical notion of scheme when $(\mathcal{C},\otimes,1) = (\mathbb{Z}\text{-mod},\otimes,\mathbb{Z})$. Their first step is to define affine schemes in a formal way. The category $Aff_{\mathcal{C}}$ of affine schemes relative to $\mathcal{C}$ is defined to be the category $\mathrm{Comm}(\mathcal{C})^{op}$ (Definition 2.7 in [TV]), i.e. the dual category of the category of commutative monoidal objects in $\mathcal{C}$. One of the key ingredients of the theory is the definition of a Zariski topology on the category of commutative monoidal objects of $\mathcal{C}$. This topology is used to glue affine schemes, seen as sheaves through the Yoneda functor, in order to obtain the general notion of scheme over $\mathcal{C}$.

The aim of the current article is to study the Zariski topology above in more detail. The category $Aff_{\mathcal{C}}$ will then be identified through Yoneda functor with the category of representable sheaves for the Zariski topology and $\mathrm{Spec}(A)$ will refer to the representable sheaf on $\mathrm{Comm}(\mathcal{C})^{op}$: $Hom_{Comm(\mathcal{C})}(A,-)$. We work in a "relative context" which is a symmetric monoidal category provided with some additional hypotheses described in the first paragraph of the preliminaries. Here are some examples of contexts :

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- The category ($\mathbb{Z}$-mod,$\otimes_{\mathbb{Z}}$,$\mathbb{Z}$), with which we recover the usual notion of scheme.

- The category ($\mathbb{N}$-mod,$\otimes_{\mathbb{N}}$,$\mathbb{N}$)[1] with which we get a notion of $\mathbb{N}$-scheme that may be understood as a notion of positive scheme (and that might be linked with tropical geometry).

- The category (Set,$\times$,$\mathbb{F}_1$), where $\mathbb{F}_1$ is *the field with one element*.

- The category (Log,$\otimes^{log}$,($\mathbb{Z}$,$\mathbb{F}_1$)), where an object of Log is a triple (M,E,u) such that $M \in \mathbb{Z}$-mod, $E \in Set$, $u \in Hom_{Set}(M,E)$ and the tensor product is $\mathbb{Z}$-mod on the first component and the direct product on the second. This context recovers the notion of prelog-scheme. Additional work is necessary to recover the notion of log scheme.

We also give an example in which some hypotheses do not hold. This is the context of differential schemes. The base category is the category of differential modules with the usual tensor product. Its unit is $\mathbb{Z}$ with a zero differential. In this context, there are two choices of self enrichment for the base category . The choice which recovers the notion of differential ideal corresponds to a context which is not strong. In this case, the Hom between two modules is the set of their differential morphisms (the constants of the classical Hom) and the unit does not generate $\mathbb{Z}$-mod^{diff}, but there is another generator, $\mathbb{Z}[X] \backsimeq \mathbb{Z}[X,X',X^{(2)},....]$. Nevertheless, it seems that it should be possible to prove theorems similar to those proved in this article.

The definition of Zariski open object in [TV] differs from the usual definition which could not be generalized naively to the relative context. In fact, a theorem in [SGA4] (part 4, theorem 17.9.1) asserts that the usual notion of Zariski open object is equivalent to the notion of *finitely presented flat epimorphism* which is well defined in the relative context. The usual notions of image, generator, ideal (Definition 1.11), prime ideal (Definition 2.10) or localization of a monoid have generalisations to this relative setting. For instance for a monoid A, localization by an element f of its underlying set $A_0 := Hom_{\mathcal{C}}(1,A)$, denoted by A_f, makes sense. Furthermore, they are the first examples of relative Zariski open objects and they play a fundamental role in the main theorems. Eventually, these theorems provide a link between two ways of defining the category of relative schemes. These two definitions are equivalent for $\mathbb{Z}$-schemes; more precisely the category of schemes (seen as ringed spaces) is equivalent to the category of sheaves on affine schemes

[1] The tensor product $\otimes_{\mathbb{N}}$ is defined like $\otimes_{\mathbb{Z}}$ as a quotient of the direct product by the same kind of relations

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(opposite to the category of commutative unital rings) (for the Zariski topology) provided with an affine covering ([TV] definition 2.15 p 19).

Here, we forge the inverse path. We start from the definitions in [TV] and construct, for a commutative monoidal object in $\mathcal{C}$, a topological space which can be provided with a natural structure sheaf. Indeed, as it is a sheaf, we just have to define it on a basis of open objects of the topology. In theorem 3.7 of this paper, we prove that this basis of open objects are in facts ideal q_f generated by one element f. Then A_f is the image of q_f, and this defines a unique sheaf. The topological space we get, together with the structure sheaf, should then be called a monoidal space, in analogy to ringed spaces. Here the topological space associated to $A \in \mathrm{Comm}(\mathcal{C})$ in the theorems plays the role of the underlying topological space of the spectrum of the commutative monoidal object A.

In fact, what is hidden behind the definition of a relative Zariski open immersion as a *finitely presented flat epimorphism* is a notion of open sub-object for the module categories, i.e. $\mathrm{Spec}(B)$ is a relative Zariski open sub-object of $\mathrm{Spec}(A)$ in $\mathrm{Comm}(\mathcal{C})$ if B-mod is an "open" sub-category of A-mod in some sense. This notion of "open" corresponds to being a localization of A-mod ([BQ]), i.e. a full subcategory which is reflexive (there is an exact left adjoint to the forgetful functor).

This notion leads us to sheaf theory. As we work with the category $\mathcal{C}$, in the article, we need the $\mathcal{C}$-enriched sheaf theory of the article [BQ]: $\mathcal{C}$-enriched localizations are obtained by defining the Zariski open immersion as $\mathcal{C}$-flat morphisms instead of just flat ones. So, now, the $\mathcal{C}$-category of presheaves is A-mod. It is the category of presheaves on the category with one object $*$ and $End(*) := A$. And the "open subcategories" are just the categories of sheaves. To each such subcategory corresponds a $\mathcal{C}$-Grothendieck topology[2] which is in particular a set of sieves, which are ideals of A in our case. We will call these topologies Gabriel Filters.

Working then a little more on the stability property of Gabriel filters, and showing that they are characterized by their subsets of prime ideals, we recover a notion of Zariski topology analogous to the one for $\mathbb{Z}$-schemes or $\mathbb{F}_1$-schemes in [D]. In the two following theorems, we call the topology obtained the Zariski topology on $\mathrm{Spec}(A)$.

Recall that $\mathrm{Spec}(A)$ hasn't been defined as the set of prime ideals of A. In this article, we use $Lp(A)$ to denote the topological space whose points are the prime ideals of A and whose topology is the one described previously (definition 3.12). Such notation is not made to last a long time, but it is necessary to choose one at this stage of the work.

[2]We give the definition fitting our case in 2.2, the general (enriched) definition is in [BQ]

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Our first theorem makes the link between the topological space $Lp(A)$ and the already existing structure of locale (a topological space without points; it is discussed in [TV] at the end of section 2.4, bottom of page 24), denoted by Zar(Spec(A)) on the category of Relative Zariski open subobjects of Spec(A) (objects are Zariski open sub-objects Spec(B) $\rightarrow$Spec(A) and morphisms are open immersions).

Theorem 1 (See 3.13) *Let A be a commutative monoidal object in a relative context $\mathcal{C}$. The poset of Zariski open subobjects $Zar(X)$ of $X :=Spec(A)$ is the locale associated to the topological space $Lp(A)$ whose points are the prime ideals of A and whose closed sets are given, for an ideal q of A, by $V(q) := \{p \ st \ q \subset \ p\}$.*

The second theorem describes an open base of the topology (Proved before in the text). This theorem can be seen as a corollary of the first, as the basis of open sub-objects is in fact clear in $Lp(A)$.

Theorem 2 (See 3.7) *Let A be a commutative monoidal object in a (strong) relative context $\mathcal{C}$. Then the Zariski open objects $(Spec(A_f))_{f\in A_0}$ form a basis of Zariski open sub-objects in Zar(Spec(A)).*

The object $Lp(A)$ that we get in the end has a structure sheaf of commutative monoidal objects in $\mathcal{C}$ (the definition of [D] section 2.1 can be generalized straightforward), and it's an object we would like to call also Spec(A), because it's the analogue of the ringed space associated to a ring. This will certainly become evident in the proof of the equivalence of the two points of view in the relative case, i.e the equivalence between the categorical construction of schemes used here and the ringed space like construction (as in [D] for $(Set,\times,\mathbb{F}_1)$) where schemes would be monoidal spaces (topological spaces together with a sheaf of commutative monoidal objects in $\mathcal{C}$) locally isomorphic to spectra of commutative monoidal objects in $\mathcal{C}$.

In this paper, we will apply our constructions and results in the relative context $(Set,\times,\mathbb{F}_1)$ in order to compare the notions of $\mathbb{F}_1$-scheme of [TV] and [D] and to prove that the associated topological spaces are homeomorphic.

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Preliminaries

Through the entire paper $(\mathcal{C},\otimes,1)$ is a closed symmetric monoidal category which is complete, cocomplete, locally finitely presentable, regular in the sense of Barr and has a compact(i.e. finitely presented) unit. Let $\mathcal{C}_0$ denote the full subcategory of finitely presented objects. Recall that $\mathcal{C}$ is regular in the sense of Barr if the pullback of a regular epimorphism (i.e. a cokernel) is also a regular epimorphism and $\mathcal{C}$ is locally finitely presentable if the Yoneda functor $i : \mathcal{C} \to pr(\mathcal{C}_0)$ is fully-faithful. We assume furthermore that $\mathcal{C}_0$ is stable under tensor product and that $Hom_{\mathcal{C}}(1,-)$ preserves regular epimorphisms (we say that $\mathcal{C}$ respects images). A category $\mathcal{C}$ verifying these hypotheses will be called a *relative context*.

The functor i has a left adjoint K defined, for $F \in Pr(\mathcal{C}_0)$ by $K(F) := Colim_{(k,f)\in\mathcal{C}_0/F} k$ where $\mathcal{C}_0/F$ is the category whose objects are couples (k,f), $k \in \mathcal{C}_0$, $f \in F(k)$ and morphisms from (k,f) to (k',f') are morphisms $h : k \to k'$ such that $f' \circ h = f$.

The hypotheses have three first important consequences. The functor i reflects epimorphisms, filtered colimits are exact in $\mathcal{C}$ and the collection of subobjects of a given object X, i.e. of isomorphism classes of couples (Y,j) where $Y \xrightarrow{j} X$ and j is a monomorphism, is a set.

The functors $Hom_{\mathcal{C}}(1,-)$, denoted by $(-)_0 : X \to X_0$ and $Hom_{\mathcal{C}}(k,-)$, denoted by $(-)_k : X \to X_k$, k in $\mathcal{C}_0$, from $\mathcal{C}$ to Set will be called respectively the *underlying set functor* and the *weak underlying set functor*. For $X \in \mathcal{C}$, the object X_0 is the underlying set of X. If there is no confusion, we may forget the word *weak* for the other functors. An element of X is an element of one of its weak underlying sets X_k or X_0. If f is such an element, we will write $f \in X$.

A relative context is called *strong* when the functor $(-)_0$ reflects isomorphisms i.e. when the unit 1 generates $\mathcal{C}$. For $X,Y \in \mathcal{C}$, let r_X, l_X, $s_{(X,Y)}$ and $a_{X,Y,Z}$ denote respectively the natural isomorphisms $X \otimes 1 \to X$, $1 \otimes X \to X$, $X \otimes Y \to Y \otimes X$, and $(X \otimes Y) \otimes Z \to X \otimes (Y \otimes Y)$. The Mac Lane coherence theorem (cf [McL]) asserts that any monoidal category is equivalent to one in which the morphisms r,l and a are identities, so they can be considered as identities in all demonstrations concerning properties of $\mathcal{C}$ that are stable under monoidal equivalences.

In such a category, there exists a notion of commutative monoidal object and of A-module for a given commutative monoidal object A (see [McL] or [TV]). For a commutative monoidal object A and an A-module M, let m_A, i_A, μ_M denote respectively the multiplication $A \otimes A \to A$, the unit morphism $1 \to A$ and the multiplication $M \otimes A \to M$. It is known that module categories over a commutative

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monoidal object A are also symmetric monoidal categories for the tensor product[3] $\otimes_A$ upon A. The notations for structural morphisms for the category $\mathcal{C}$ will then be the same as for any category of modules over a commutative monoidal object A. Comm($\mathcal{C}$) will refer to the category of commutative monoidal objects in $\mathcal{C}$ and for $A \in$Comm($\mathcal{C}$), A-mod will refer to the category of A-modules.

In fact, all the hypotheses on our basis category will be true in any of its module categories. If A be a commutative monoidal object in $\mathcal{C}$, then the category A-mod is a relative context. Let us give some details of this fact. It's well known that A-mod is closed, complete and cocomplete. Moreover, since the forgetful functor A-mod$\rightarrow \mathcal{C}$ commutes with small limits and colimits, it is easy to prove in A-mod regularity in the sense of Barr, the compactness of the unit A and the fact that A-mod respects images. Most difficult is to prove that the full sub-category of finitely presentable objects of A-mod, denoted A-mod$_0$ is a generating subcategory. It can be proved for example using Ind-objects.

In a relative context, the category of commutative monoidal objects has also some interesting properties. In particular, it can be proved that it is locally finitely presentable. As the module categories are also relative contexts, these properties are true in any category of algebras in $\mathcal{C}$. Recall that, for a commutative monoidal object A, the category of algebras over A, denoted A-alg is Comm(A-mod). It is also equivalent to $A/$Comm($\mathcal{C}$).

Let us now recall some well known properties of monoidal categories. A pushout in A-alg is a tensor product in the sense that for commutative monoidal objects $B, C \in A$-alg, $B \otimes_A C \backsimeq B \coprod_A C$. There are two adjunctions:

$$\mathcal{C} \underset{i}{\overset{(-\otimes A)}{\rightleftarrows}} A\text{-}mod \qquad \mathcal{C} \underset{i}{\overset{L}{\rightleftarrows}} Comm(\mathcal{C})$$

where the forgetful functor i is a right adjoint and the *free associated commutative monoidal object* functor L is defined by $L(X) := \coprod_{n\in\mathbb{N}} X^{\otimes n}/S_n$[4] (where S_n refers to the symmetric group, regarded as a category with one object). In these adjunctions, $\mathcal{C}$ can be replaced by B-mod for $B \in$Comm($\mathcal{C}$). Let φ (resp φ_B) and ψ (resp ψ_B) denote these adjunctions for the category $\mathcal{C}$ (resp B-mod). For $X \in \mathcal{C}$ and $M \in A$-mod, $\varphi : Hom_{\mathcal{C}}(X, M) \rightarrow Hom_{A\text{-}mod}(X \otimes A, M)$ is easy to describe :

$$\varphi : f \rightarrow \mu_M \circ Id_A \otimes f \qquad \varphi^{-1} : g \rightarrow g \circ (Id_X \otimes i_A) \circ r_X^{-1}.$$

[3]Given by $M \otimes_A N := coker(\ M \otimes A \otimes N \underset{\mu_N}{\overset{\mu_M}{\rightrightarrows}} M \otimes N\ )$ where the colimit is taken in $(\mathcal{C}, \otimes, 1)$

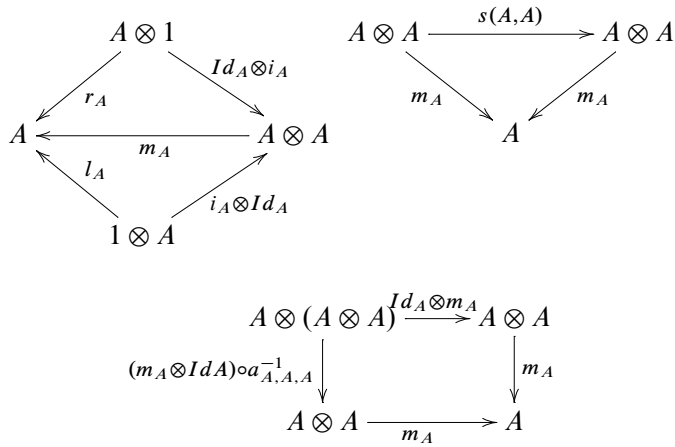
[4]Where $X^{\otimes n}/S_n := Colim_{S_n} F$, $F(*) = X^{\otimes n}$ and $S_n \rightarrow End(X^{\otimes n})$ is the evident permutation map.

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1. Algebraic Structures

Commutative monoids in $\mathcal{C}$ and commutative algebras in $\mathcal{C}$ In such a general setting, the objects that are of interest to us for defining affine schemes are commutative monoidal objects in $\mathcal{C}$. This notion is in fact analogous to that of rings in $\mathbb{Z}$-mod (commutative monoidal objects in the category of commutative groups $\mathbb{Z}$-mod) and that of commutative monoids in Sets (Deitmar). Thus, to simplify vocabulary, we will call these objects *commutative monoids* in $\mathcal{C}$.

Definition 1.1 A commutative monoid in $\mathcal{C}$ is defined as a commutative monoidal object in $\mathcal{C}$. Recall that it consists of an object A, a morphism $m_A : A \otimes A \to A$ and a morphism $i_A : 1 \to A$ such that the following diagrams commute:



The usual notion of commutative monoid in the category of sets will then be referred to as *commutative monoid in sets*. A *commutative monoid* in $\mathcal{C}$ will just be called a *commutative monoid* when the monoidal category is clear.

Definition 1.2 Let A be a commutative monoid in $\mathcal{C}$. A commutative A-algebra is a commutative monoid in the monoidal category $(A\text{-mod}, \otimes_A, A)$.

Images and generators We define and describe now two fundamental notions in a relative context $\mathcal{C}$. These notions will be particularly important in the relative context A-mod for a commutative monoid A.

Definition 1.3 (Images) Let $u : X \to Y$ be a morphism in $\mathcal{C}$.

▷ Let $\mathcal{C}_Y$ denote the sub-category of $\mathcal{C}/Y$ consisting of all objects Z such that the morphism $Z \to Y$ is a monomorphism and whose morphisms are

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monomorphisms in $\mathcal{C}/Y$. A subobject of Y is an isomorphism class of objects in $\mathcal{C}_Y$. The category of subobjects of Y will be denoted $sub(Y)$.

▷ A subobject Z of Y *contains the image of* u if u factors through the inclusion $Z \hookrightarrow Y$. We denote by $sub_u Y$ the sub-category of $sub(Y)$ whose objects are subobjects of Y *containing the image of* u.

▷ The image of u, denoted $Im(u)$ is $Im(u) := Lim_{sub_u Y} Z$.

▷ Let $|u|$ denote the object $|u| := Coker(\ X \times_Y X \rightrightarrows X\ )$.

The existence of the limit is due to the smallness of the category $sub_u Y$.

Lemma 1.4 *Let $u : X \to Y$ be a morphism in $\mathcal{C}$.*

◇ *If u is a monomorphism, then there is an isomorphism $X \backsimeq Im(u)$.*

◇ *Let $v : Y \to Z$ be in $\mathcal{C}$. Then there is an isomorphism $Im(v \circ u) \backsimeq Im(v \circ u')$ where u' is the monomorphism $u' : Im(u) \to Y$.*

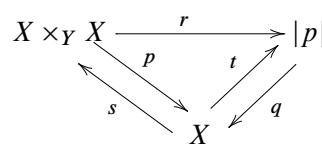
◇ *There is an isomorphism $Im(u) \backsimeq |u|$.*

Proof: If u is a monomorphism, the isomorphism class associated to the object X is initial in $sub_u Y$, thus $X \backsimeq Im(u)$. The second isomorphism comes from the equality $v \circ u = v \circ u' \circ u''$ where u'' is the morphism $u'' : X \to Im(u)$. And by the universal property of the image, $Im(v \circ u' \circ u'') = Im(v \circ u')$.

Let us prove the third isomorphism. There is a unique natural morphism from $|u|$ to any subobject of Y *containing the image of* u and thus from $|u|$ to $Im(u)$. We need to prove that $|u|$ is a subobject of Y to achieve the proof. Let us prove first that $X \times_Y |u| \backsimeq X$. As a relative context is regular in the sense of Barr,

$$|u| \times_Y X \backsimeq Coker(\ X \times_Y X \times_Y X \rightrightarrows X \times_Y X\ ) \backsimeq |p|$$

where the natural morphism $p : X \times_Y X \to X$ is split by a morphism s, i.e. $p \circ s = Id$. Let r,q and t denote respectively the morphism $X \times_Y X \to |p|$, the morphism $|p| \to X$ and the morphism $r \circ s$. There is a diagram



in which $q \circ t = q \circ r \circ s = p \circ s = Id$. Reciprocally, $t \circ q = Id$ is a clear consequence of the universal property of the cokernel $|p|$.

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Finally

$$|u| \times_Y |u| \backsimeq Coker(X \times_Y X \times_Y |u| \rightrightarrows X \times_Y |u|) \backsimeq |u|$$

so $|u|$ represents a subobject of Y. □

Corollary 1.5 *Let $u : X \to Y$ be a morphism in $\mathcal{C}$. The morphism $u_0 : X_0 \to Im(u)_0$ is surjective.*

Proof: As $\mathcal{C}$ respects images, the functor $(-)_0$ commutes with regular epimorphim and thus sends them to epimorphisms in the category of sets, i.e. to surjective morphisms. □

Lemma 1.6 *Let $u : A \to B$ be a morphism in $Comm(\mathcal{C})$. Its image in $\mathcal{C}$ is isomorphic (in $\mathcal{C}$) to its image in $Comm(\mathcal{C})$.*

Proof: Let $Im(u)$ be its image in $\mathcal{C}$. We provide it with a structure of commutative monoid. Let u' denote the morphism $A \to Im(u)$. The unit morphism of $Im(u)$ is $u' \circ i_A$. Let us now define a multiplication morphism. Recall that the forgetful functor from A-mod to $\mathcal{C}$ commutes with small colimits. Thus $Im(u)$ has a natural structure of A-module. As $\mathcal{C}$ is closed, the functors $X \otimes -$, $X \in \mathcal{C}$ commute with small colimits and preserve epimorphisms. Thus

$$Im(u) \otimes Im(u) \backsimeq Coker(A \times_B A \otimes Im(u) \rightrightarrows A \otimes Im(u))$$
$$Im(u) \otimes A \backsimeq Coker(A \times_B A \otimes A \rightrightarrows A \otimes A).$$

Moreover, the morphism $Id \otimes u' : A \times_B A \otimes A \to A \times_B A \otimes Im(u)$ is an epimorphism. There is a commutative diagram

$$\begin{array}{ccc} A \times_B A \otimes A & \rightrightarrows & A \otimes A \\ \downarrow{\scriptstyle Id \otimes u'} & & \downarrow{\scriptstyle Id_A \otimes u'} \\ A \times_B A \otimes Im(u) & \rightrightarrows & A \otimes Im(u) \xrightarrow{\mu_{Im(u)}} Im(u) \end{array}$$

where $Id_A \otimes u'$ equalizes the two top morphisms. Thus $\mu_{Im(u)}$ equalizes the two morphisms from $A \times_B A \otimes A$ to $A \otimes Im(u)$ and as $Id \otimes u'$ is an epimorphism, $\mu_{Im(u)}$ equalizes the two morphisms of the coker $Im(u) \otimes Im(u)$. Finally $\mu_{Im(u)}$ has a lift $m_{Im(u)} : Im(u) \otimes Im(u) \to Im(u)$. The commutativity of the structure diagrams of this monoid is clear. □

Definition 1.7 (Generators) Let X be in $\mathcal{C}$ and $F = (f_i)_{i \in I}$ be a family of elements of X, i.e. each f_i is an element of some $Hom_{\mathcal{C}}(k, X)$ with $k \in \mathcal{C}_0$. Let $sub_F X$ denote the full subcategory of $sub(X)$ whose objects are subobjects Y of X containing the family F. The object generated by F is

$$X_F := Lim_{sub_F X} Y.$$

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Definition 1.8 Let X be in $\mathcal{C}$ and $F = (f_i)_{i \in I}$ be a family of elements of X. The presheaf associated to F, denoted F', is defined by

$$F' : k \to F(k) := \{f \in X_k \; st \; \exists \, h : k \to k' \; and \; g \in F \cap C_{k'} \; st \; f = g \circ h\}.$$

It is a presheaf by construction. Moreover, the family $\coprod_k F'(k)$, still denoted F', generates the same object, i.e. $X_{F'} \backsimeq X_F$. In fact, this object is the image of the morphism of presheaves $\coprod_F k \to X$.

Lemma 1.9 *Let X be in $\mathcal{C}$, $F = (f_i)_{i \in I}$ be a family of elements of X and F' be its associated presheaf. Then the object X_F is isomorphic in $\mathcal{C}$ to the image of the natural morphism*

$$p : K(F') := Colim_{\mathcal{C}_0^{F'}} k \to X$$

where K is the left adjoint of the Yoneda functor $i : \mathcal{C} \to Pr(\mathcal{C}_0)$.

Proof: Considering the adjunction between K and i, the category $sub_p X$ is clearly isomorphic to $sub_F X$. □

Let us now list the properties of an object generated by a family.

Lemma 1.10 *Let X be in $\mathcal{C}$ and F be a family of elements of X.*

1. *If Y, Y' are two objects of $sub(X)$ generated by F, they are isomorphic.*
2. *If $(-)_0$ reflect isomorphisms, X_0 is a generating family of X.*
3. *Let $u : X \to Y$ be in $\mathcal{C}$. Then the family $u_*(F) := \{u \circ f, f \in F\}$ generates $Im(u)$.*
4. *The family F generates X if and only if it is an epimorphic family of morphisms.*
5. *A morphism is an epimorphism if and only if it preserves generating families.*
6. *Let $\mathcal{G}$ be the diagram of finite subsets of F whose morphisms are inclusions. Then there is an isomorphism $X_F \backsimeq Colim_{G \in \mathcal{G}} X_G$.*
7. *If X is generated by F and finitely presented in $\mathcal{C}$ ($Hom_{\mathcal{C}}(X, -)$ commutes with filtered colimits), it is generated by a finite family included in F.*
8. *Let A, B be in $\mathcal{C}$. Then the family $i(A) \times i(B)$ generates $A \otimes B$.*
9. *Let $u : A \to B$ and C be in $\mathcal{C}$. Then the image of $u \otimes Id_C$ is generated by $i(Im(u)) \times i(C)$.*

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Proof: (1) is clear. For (2), let us assume that $(-)_0$ reflect isomorphisms. Let Y be the subobject of X generated by X_0. There is an isomorphism $X_0 \backsimeq Y_0$, thus $X \backsimeq Y$.

(3) The elements of $u_*(F)$ are clearly elements of $Im(u)$. Let $Im(u)_F$ be the subobject of $Im(v)$ generated by $u_*(F)$. The natural morphism $F \to Im(u)_F$ lifts to a morphism $X \to Im(u)_F$ which factors u. Thus there is a monomorphism $Im(u) \to Im(u)_F$. As $Im(u)$ is a sub-object of $Im(u)_F$ in $Sub_{u_*(F)}(Im(u))$, there is an isomorphism $Im(u) \backsimeq Im(u)_F$.

(4) A generating family is clearly epimorphic. Reciprocally, if F is epimorphic, let F' be its associated presheaf. There is an isomorphism $Hom_{\mathcal{C}}(X,Y) \backsimeq Hom_{\mathcal{C}}(K(F'),Y)$ for all $Y \in \mathcal{C}$. Thus, by Yoneda, $X \backsimeq K(F') \backsimeq X_F$.

(5) is clear, as epimorphisms preserves epimorphic families by composition.

(6) We prove first that $Colim_{G\in\mathcal{G}} X_G$ is a subobject of X. The Yoneda functor $i : \mathcal{C} \to Pr(\mathcal{C}_0)$ commutes with filtered colimits. Thus we compute this colimit levelwise in $Pr(\mathcal{C}_0)$. More precisely, there are isomorphisms $colim_{G\in\mathcal{G}}(X_G)_k \backsimeq Colim_{G\in\mathcal{G}}((X_G)_k)$. For all $k \in \mathcal{C}_0$, the sets $(X_G)_k$ are subsets of X_k, therefore their filtered colimit is a subset of X_k. This implies that $Colim_{G\in\mathcal{G}} X_G$ is a subobject of X in $\mathcal{C}$. Finally, as $Colim_{G\in\mathcal{G}} X_G$ clearly contains F, $X \backsimeq Colim_{G\in\mathcal{G}} X_G$.

(7) The object X is finitely presented and isomorphic to $Colim_{G\in\mathcal{G}} X_G$. Thus there exists a finite set G such that this isomorphism factors through X_J. The morphism $X_J \to X$ is then a monomorphism and a split epimorphism, hence an isomorphism.

For (8), we write $A \otimes B \backsimeq K(i(A)) \otimes K(i(B))$, and set $F := i(A) \times i(B)$, so that

$$A \otimes B \backsimeq colim_{(k,f),(k',f')\in \mathcal{C}_0^{i(A)} \times \mathcal{C}_0^{i(B)}} k \otimes k' \backsimeq K(F).$$

(9) The morphism $u \otimes Id_C$ factors through $Im(u) \otimes C \to B \otimes C$ and $Im(u) \otimes C$ is generated by $i(Im(u)) \times i(C)$. The morphism $i(A) \times i(C) \to i(Im(u)) \times i(C)$ is surjective thus the image of $u \otimes Id_C$ is isomorphic to the image of $Im(u) \otimes C \to B \otimes C$, which is generated by $i(Im(u)) \times i(C)$. □

Ideals and multiplication

Definition 1.11 (Ideals) Let A be a commutative monoid and M be an A-module. A sub-A-module of M is a subobject of M in the category A-mod. An ideal of A is a sub-A-module of A.

We adopt the following convention for the rest of the paper. For any ideal q, an element of the isomorphism class will be chosen and also called q. The corresponding monomorphism will be denoted $j_q : q \to A$.

Lemma 1.12 *Let A be in Comm($\mathcal{C}$), M be an A-module and q be an ideal of A.*

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◇ *The object A_0 is a monoid in $(Set,\times,*)$ and the object M_0 is a A_0-module in $(Set,\times,*)$.*

◇ *The object q_0 is a subset of A_0.*

◇ *The morphism φ defines an isomorphism of monoids in $(Set,\times,*)$ and $A_0 \backsimeq End_{A\text{-mod}}(A)$.*

◇ *If there exists $f \in q_0$ which is invertible in A_0, then $q \backsimeq A$.*

Proof: The proof of the three first items is clear. Let us prove the fourth. The morphism $\varphi(f)$ is an automorphism of A in A-mod which factors through j_q. The morphism j_q is thus both a monomorphism and a split epimorphism, hence an isomorphism. □

Definition 1.13 Let $A \xrightarrow{u} B$ be a morphism of commutative monoids and q be an ideal of A. The image of the ideal q in B, denoted qB, is the image of the morphism $q \otimes_A B \to B$ in B-mod.

Lemma 1.14 *Let $A \xrightarrow{u} B$ be a flat morphism in Comm(℃) (i.e. $- \otimes_A B$ is an exact functor,[TV] page 3), and let q be an ideal of A. Then $q \otimes_A B \backsimeq qB$.*

Proof: The functor $- \otimes_A B$ preserves monomorphisms thus $q \otimes_A B$ is an ideal of B. □

Definition 1.15 (Multiplication) Let A be in Comm(℃), f be an element of A and M an A-module. Then multiplication by f in M, denoted τ_f is defined by the commutative diagram

$$\begin{array}{ccc} M \otimes_A A \otimes k & \xrightarrow{Id_M \otimes f} & M \otimes_A A \\ {\scriptstyle r_M \otimes Id_k}\downarrow & & \downarrow{\scriptstyle r_M} \\ M \otimes k & \xrightarrow[\tau_f]{} & M. \end{array}$$

In particular, when $M = A$, $\tau_f = \varphi(f)$ where φ is the adjunction defined in the preliminaries.

Lemma 1.16 *Let A be in Comm(℃), M,N be two A-modules and u be in $Hom_{℃}(M,N)$. The morphism u is in A-mod if and only if it commutes with morphisms τ_f for any element f of a generating subset of A, i.e. satisfies $\tau_f \circ (u \otimes Id_k) = u \circ \tau_f$.*

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Proof: It is easy to check, keeping in mind that generating families of elements are epimorphic families and the functors $Z \otimes -$, $Z \in \mathcal{C}$ preserve epimorphisms. □

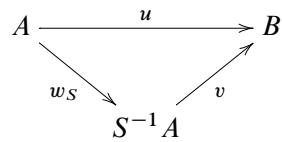
Lemma 1.17 *Let A be in Comm($\mathcal{C}$) and q be an ideal of A. An element f of A is an element of q if and only if $\varphi(f)$ factors through j_q.*

Proof: It comes from $\varphi(j_q \circ f) = j_q \circ \varphi(f)$. □

Localization

Definition 1.18 Let A be in Comm($\mathcal{C}$) and S be a multiplicative part of the pointed monoid $A_0 := Hom_{\mathcal{C}}(1, A)$. A localization of A along S is a couple $(S^{-1}A, w_S)$, where $w_S : A \to S^{-1}A \in$Comm($\mathcal{C}$), satisfying the following properties:

1. The image of the elements of S are invertible, i.e. $(w_S)_0(S) \subset (S^{-1}A)_0^*$.
2. Any morphism verifying i factors through w_S. More precisely, $\forall\ A \xrightarrow{u} B \in Comm(\mathcal{C})$, *if* $u_0(S) \subset B_0^*$ then $\exists!\ v$ such that the diagram



commutes.

If $S =< \{f\} >$ is the multiplicative part generated by the one element f, A_f will refer to $S^{-1}A$ and w to w_S.

Remark 1.19 Lemma 1.21 and proposition 1.20 prove the existence of $S^{-1}A$, and the morphism $A \to S^{-1}A$ is an epimorphism by construction. It is well known that a morphism of monoids is an epimorphism if and only if the corresponding forgetful functor on the module categories is fully faithful. In particular, the category $S^{-1}A$-mod is a full subcategory of A-mod.

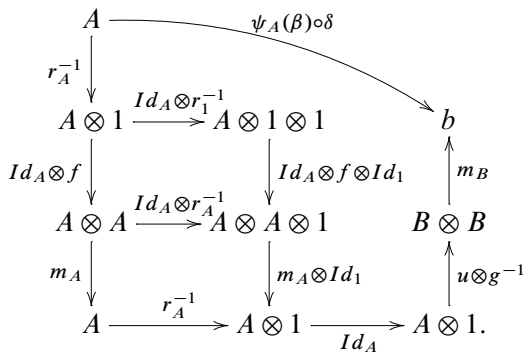
Proposition 1.20 *Let A be a commutative monoid and f be in A_0. Let j_1 and j_0 denote respectively the inclusion morphisms of $A = A^{\otimes_A 1}/S^1$ and $A = A^{\otimes_A 0}/S^0$ in $L_A(A) = \coprod_{n\in\mathbb{N}} A^{\otimes_A n}/S^n$. Define δ as $j_1 \circ m_A \circ Id \otimes f \circ r_A^{-1}$. Then*

$$A_f \backsimeq coker(\ L_A(A) \underset{\psi_A(j_0)}{\overset{\psi_A(\delta)}{\rightrightarrows}} L_A(A)\ ).$$

The colimit is taken in Comm($\mathcal{C}$).

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Proof: Let us prove first that this colimit satisfies (2) in definition 1.18. Let $A \xrightarrow{u} B$ be in $A/\mathrm{Comm}(\mathcal{C})$ such that $g := u_0(f) = u \circ f$ is invertible in B_0. Let g^{-1} denote its inverse. Thus $m_b \circ (g \otimes g^{-1}) \otimes r_1^{-1} = i_B$. Consider the morphism $\psi_A(\beta) : L_A(A) \to B$ in A-alg adjoint to $\beta = m_b \circ (u \otimes g^{-1}) \circ r_A^{-1} : A \to B$. Then $\psi_A(\beta) \circ j_0$ is the unit morphism of B in A-alg, i.e. is u. We get $\psi_A(\beta) \circ j_1 = \beta$. Thus $\psi_A(\beta) \circ \delta = \beta \circ m_A \circ (Id \otimes f) \circ r_A^{-1}$. Moreover the following diagram is commutative:



Then using the compatibility between $\otimes$ and $\circ$, the commutative monoid morphism's structure of u and the associativity of multiplication, the computation of $\beta \circ m_A \circ (Id \otimes f) \circ r_A^{-1}$ gives u. Thus $\psi_A(\beta) \circ \delta = \psi_A(\beta) \circ j_0$, hence $\psi_A(\beta) \circ \psi_A(\delta) = \psi_A(\beta) \circ \psi_A(j_0)$. Finally, by the universal property of colimits, there exists a unique morphism from this colimit to B.

Let us prove now that the colimit satisfies (1) in definition 1.18. We now denote this colimit by A_f, and the morphism $A \to A_f$ by ω. There exists a morphism $\bar{p}$ from $L_A(A)$ to A_f such that $\bar{p} \circ \delta = \omega$. Let $p : A \to A_f$ denote the adjoint of $\bar{p}$. Then $p \circ m_A \circ (Id \otimes f) \circ r_A^{-1} = p$. Moreover, considering the identity $m_{A_f} \circ (\omega \otimes (p \circ i_A)) \circ r_A^{-1} = p$, it is clear that the morphism p plays the same role as β, and $p \circ i_A$ will play the role of g^{-1}. Thus, an analogous computation gives

$$\omega \circ i_A = p \circ m_A \circ (Id \otimes f) \circ r_A^{-1} \circ i_A = m_{A_f} \circ ((\omega \circ f) \otimes p \circ i_A) \otimes r_1^{-1}.$$

Thus the product of $\omega \circ f$ and $p \circ i_A$ is the unit element of $(A_f)_0$. □

Lemma 1.21 *A category, still denoted S, is associated to S. Its elements are elements of S and for two elements f,g of S, the morphisms from f to g are the elements h of S satisfying $f * h = g$. There is an isomorphism $S^{-1}A \backsimeq Colim_S A_f$. Furthermore, as S is multiplicative, this colimit is filtered.*

Proof: By the universal property of objects $S^{-1}A$ and A_f. □

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Proposition 1.22 *Let A be a commutative monoid and f be in A_0.*

The object $B := colim(\ A \xrightarrow{\varphi(f)} A \xrightarrow{\varphi(f)} A \xrightarrow{\varphi(f)} \dots\ )$ can be provided with a structure of a commutative monoid and is isomorphic in A-alg to A_f.

Proof: First, B can be provided with the structure of commutative monoid. The unit morphism $1 \to B$ is induced by the morphism from 1 to the first term A in the colimit. Let $\mathcal{N}^{+1}$ denotes the category generated by the graph whose vertices are integers with a vertex from i to $i+1$ for any i. Let F denote the functor from $\mathcal{N}^{+1}$ to A-mod which sends any integer to A and any generator morphism $i \mapsto i+1$ to φ_f. Then, clearly, $B \backsimeq colim(F)$ and $B \otimes_A B \backsimeq Colim(F \otimes_A F) \backsimeq ColimF \backsimeq B$. This isomorphism defines the multiplication morphism of B. One can check easily that these morphisms provide B with the structure of monoid in $\mathcal{C}$.

The next step is to prove that $g := i_B \circ f$ is invertible in $Hom_{A\text{-}mod}(A,B) \backsimeq B_0$. There exists a morphism from the second term A in the colimit to B, denoted arbitrarily by g^{-1}. One verifies that $g^{-1} \circ \varphi(f) = i_B$. As $\varphi(f) = m_A \circ (Id_A \otimes_A f) \circ r_A^{-1}$, we get

$$i_B = g^{-1} \circ \varphi(f) = g^{-1} \circ m_A \circ (Id_A \otimes_A f) \circ r_A^{-1}.$$

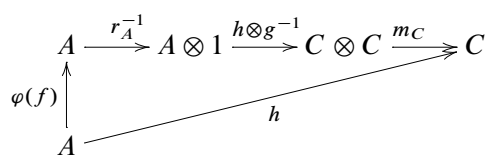
By A-linearity of g^{-1}, this equals

$$\mu_B \circ (g^{-1} \otimes_A Id_A) \circ (Id_A \otimes_A f) \circ r_A^{-1}$$

where $\mu_B = m_B \circ (Id_B \otimes_A i_B)$. Thus i_B equals

$$\mu_B \circ (g^{-1} \otimes_A f) \circ r_A^{-1} = m_B \circ (g^{-1} \otimes_A g) \circ r_A^{-1}.$$

Last but not least, let us prove that B satisfies property (2) of the definition of A_f. Let v denote the unit morphism of B in A-alg. Let $h : A \to C$ be a morphism in Comm($\mathcal{C}$) such that $g := h_0(f)$ is invertible in C_0. Let g^{-1} denote the inverse of $g := h \circ f$ in C_0. Then the following diagram commutes:



Indeed $m_C \circ (h \otimes g^{-1}) \circ r_A^{-1}$ is just $\varphi(g^{-1})$ and $h \circ \varphi(f)$ is $\varphi(g)$. So this diagram gives their product in $Hom_{A\text{-}mod}(A,C)$ which is the unit morphism of C in A-alg, h. Using the same method with inverses g^{-n} of g^n, we construct a family of morphisms from A to C which makes the diagram of the colimit B commute. Thus there is an

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induced morphism in A-mod, $s : B \to C$ such that $s \circ v = h$.
The morphism s is in fact a morphism of A-alg. Indeed, the commutativity of the diagram

$$\begin{array}{ccc} B & \xrightarrow{s} & C \\ {\scriptstyle m_B}\uparrow & & \uparrow{\scriptstyle m_C} \\ B \otimes_A B & \xrightarrow[s \otimes_A s]{} & C \otimes_A C \end{array}$$

is equivalent, by the universal property of colimits, to the commutativity, for each n, of the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi(g^{-(n+m)})} & C \\ {\scriptstyle m_A}\uparrow & & \uparrow{\scriptstyle m_C} \\ A \otimes_A A & \xrightarrow[\varphi(g^{-n}) \otimes_A \varphi(g^{-m})]{} & C \otimes_A C \end{array}$$

which is clear. □

Corollary 1.23 *Let q be an A-module, f be in A_0, and S be a multiplicative part of A_0. Then*

1. $q_f := q \otimes_A A_f \backsimeq colim(\, q \xrightarrow{\tau(f)} \ldots\,)$.
2. $q \otimes_A S^{-1}A \backsimeq colim_S q_f$.

Proof: The result is implied by lemma 1.21, proposition 1.22 and the fact that $q \otimes_A -$ commutes with colimits. □

Corollary 1.24 *Let S be a multiplicative part of A_0. The category $S^{-1}A$-mod of modules over $S^{-1}A$ is exactly the subcategory of A-mod of all modules M such that for any element f of S, τ_f is an invertible endomorphism of M.*

Proof: Clearly, if q is in $S^{-1}A$-mod, it satisfies the property. Let q be an A-module satisfying the property. Then, by the previous corollary, $q_f \backsimeq q \ \forall f \in S$. Hence $q \otimes_A S^{-1}A \backsimeq q$ and thus q is in $S^{-1}A$-mod. □

The localizations of A are important because they are the basic examples of Zariski open subobjects of Spec(A). Thus, the next step is to prove it.

Zariski open objects

Definition 1.25 (cf [TV] page 3) Let $A \to B$ in a morphism in Comm$(\mathcal{C})$. Then:

1. $A \to B$ is $\mathcal{C}$-flat if the functor $- \otimes_A B$ commutes with finite enriched limits (cf definition 4.2 in appendix).
2. $A \to B$ is finitely presented, if B is finitely presented in A-alg.

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3. $A \to B$ is a formal Zariski open morphism, if it is both a $\mathcal{C}$-flat morphism and an epimorphism.

4. $A \to B$ is a Zariski open morphism, if it is a finitely presented formal Zariski open morphism. In this case, the subsheaf $\mathrm{Spec}(B)$ of $\mathrm{Spec}(A)$ will be called a Zariski open subobject of $\mathrm{Spec}(A)$.

Remark 1.26 This definition is not exactly the definition of [TV]. Indeed, $\mathcal{C}$-flat means that the functor is not only exact but also commutes with finitely presented exponentiation (see proposition 4.3 in the appendix). This more precise definition is necessary to prove the theorem and is also a generalisation of usual cases. In fact, there is an equivalence between flat and $\mathcal{C}$-flat, if finitely presented objects in A-mod are quotients of free objects of finite type. In this case, exponential objects depend only on finite limits. One example is the case of abelian groups $(\mathbb{Z}\text{-mod}, \otimes, \mathbb{Z})$ in which for any ring A and finitely presented A-module M, there exists a resolution $A^q \to A^p \to M$.

Lemma 1.27 *Let A be in Comm($\mathcal{C}$).*

1. *If $X \in \mathcal{C}$ (resp A-mod) is finitely presented then so is $L_A(X) \in$Comm($\mathcal{C}$) (resp A-alg).*

2. *Finitely presented objects are stable under finite colimits.*

Moreover, in A-mod, filtered colimits commute with finite enriched limits (Corollary 4.4 in appendix). In particular, localizations are Zariski open objects. Indeed, the following corollary results.

Corollary 1.28 *(of 1.20, 1.19, 1.23 and 1.27)*
Let A be in Comm($\mathcal{C}$), f in A_0, and S a multiplicative part of A_0. Then $A \to S^{-1}A$ is a formal Zariski open morphism and $A \to A_f$ is a Zariski open morphism, i.e. Spec$(S^{-1}A)$ is a formal Zariski open subobject of Spec(A) and Spec(A_f) is a Zariski open subobject of Spec(A).

2. Gabriel Filters

Gabriel filters and Borceux-Quinteiro's theorem. The key idea of the classification is to introduce the notion of Gabriel filter to make a link between Zariski open objects and ideals. A Gabriel filter on a monoid A will be a Grothendieck topology on the category $\mathcal{B}A$, defined here in an enriched sense.

Definition 2.1 Let A be in Comm($\mathcal{C}$).

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1. Let $\mathcal{B}A$ denote the $\mathcal{C}$-category consisting of one object $*$ and an object of morphisms $\underline{Hom}_{\mathcal{B}A}(*,*) := A \in \mathcal{C}$.

2. Let $\mathrm{Pr}(\mathcal{B}A)$ denote the $\mathcal{C}$-category of $\mathcal{C}$-presheaves i.e. of $\mathcal{C}$-functors from $\mathcal{B}A$ to $\mathcal{C}$.

3. The Yoneda $\mathcal{C}$-functor consists of the $\mathcal{C}$-functor $* \to h_*$ which is defined on objects by $h_*(*) = A$ and on morphisms by the adjoint $h_* : A \to \underline{End}_{\mathcal{C}}(A)$ of m_A.

In fact, any $\mathcal{C}$-presheaf on $\mathcal{B}A$ sends $*$ to an A-module and the functor

$$F : \mathrm{Pr}(\mathcal{B}A) \to A\text{-mod} \quad (G : \mathcal{B}A \to \mathcal{C}) \mapsto G(*)$$

defines a $\mathcal{C}$-equivalence from $\mathrm{Pr}(\mathcal{B}A)$ to A-mod.

Definition 2.2 (cf [BQ] for general definition) A Grothendieck $\mathcal{C}$-topology on $\mathcal{B}A$ consists of a set J of ideals of A such that:

1. $A \in J$.

2. $\forall q \in J$, $\forall k \in \mathcal{C}_0$ and $\forall f \in A_k$, $f^{-1}(q) = (q^k \times_{A^k} A) \in J$.

3. Let q,r be two ideals of A such that $q \in J$. If $\forall f : k \to q$, $k \in \mathcal{C}_0$ the ideal $f^{-1}(r)$ is in J, then so is r.

Here $q^k = \underline{Hom}(k,q)$ and $A^k = \underline{Hom}(k,A)$ are exponential objects. This topology will be called a Gabriel filter. In (2) the morphism from A to $\underline{Hom}_{\mathcal{C}}(k,A)$ is the adjoint of the morphism $\varphi(f) : (A \otimes k) \to A$.

Lemma 2.3 *Let G be a Gabriel filter, q an ideal in G, k in $\mathcal{C}_0$ and f in A_k. The ideal $f^{-1}q$ is the sup (or union), in the poset of ideals of A containing q, of the ideals p such that the multiplication $\tau_f : p \to p$ factors through $q \hookrightarrow p$. This family of ideals is denoted* $\mathrm{Cont}_f q$.

Proof: Weak underlying set functors commute with limits, and the commutative diagram

$$\begin{array}{ccc} A_{k'} & \xrightarrow{\tau_f^*} & A_{k' \otimes k} \\ \uparrow & & \uparrow \\ f^{-1}q & \longrightarrow & q_{k' \otimes k}. \end{array}$$

proves that $f^{-1}q \in \mathrm{Cont}_f q$. Moreover, if q' is in $cont_f q$, it is easy to check, by the universal property of $f^{-1}q$, that $q' \hookrightarrow f^{-1}q$. □

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Definition 2.4 (Borceux-Quinteiro [BQ]) A localization of A-mod consists of a full reflexive (saturated) sub-$\mathcal{C}$-category of A-mod such that the left adjoint of the inclusion $\mathcal{C}$-functor is left $\mathcal{C}$-exact.

Theorem 2.5 (Borceux-Quinteiro [BQ]) *For A in Comm($\mathcal{C}$), there is a bijection between the set of Gabriel filters of A and the set of localizations of A-mod.*

The poset of Gabriel filters of A will be denoted $\mathrm{Gab}\,A$ and the poset of localizations of A-mod will be denoted $\mathrm{Loc}\,A$.

Lemma 2.6 *[BQ] Let $\mathcal{L}$ be a localisation of A-mod. The Gabriel filter associated to $\mathcal{L}$, denoted $G_{\mathcal{L}}$ consists of the set of ideals q of A satisfying for all M in $\mathcal{L}$*

$$\underline{Hom}_{A\text{-mod}}(A, M) \backsimeq \underline{Hom}_{A\text{-mod}}(q, M).$$

Conversely, let G be a Gabriel filter. The localisation associated to G, denoted $\mathcal{L}_G$, is the full sub-$\mathcal{C}$-category of A-mod whose objects are A-modules M satisfying for all q in G

$$\underline{Hom}_{A\text{-mod}}(A, M) \backsimeq \underline{Hom}_{A\text{-mod}}(q, M).$$

Remark 2.7 The bijection $\mathrm{Gab}\,A \backsimeq \mathrm{Loc}\,A$ is a contravariant isomorphism of posets. This implies that $\mathrm{Loc}\,A$ and $\mathrm{Gab}\,A$ have small unions. The poset $\mathrm{Gab}\,A$ has small intersections. A small intersection in $\mathrm{Gab}\,A$ is an intersection in the sense of sets.

The poset $\mathrm{Loc}\,A$ has small intersections. If $\mathcal{L}_i$ is a small family of localisations with left adjoint functors l_i, its intersection consists of the sub-$\mathcal{C}$-category of A-mod whose set of objects is $\cap_i Ob(\mathcal{L}_i)$ and whose left adjoint functor is $Colim(l_i)$.

Proposition 2.8 *Let $Y := Spec(B)$ be a formal Zariski open subobject of $X := Spec(A) \in Aff_{\mathcal{C}}$. The category B-mod is a localization of A-mod. Its associated Gabriel filter is the set of ideals q such that $q \otimes_A B \backsimeq B$.*

Proof: The category B-mod is clearly a localization of A-mod. The characterization of 2.6 induces by adjunction, for all $M \in B$-mod, an isomorphism

$$\underline{Hom}_{B\text{-}mod}(B, M) \backsimeq \underline{Hom}_{B\text{-}mod}(q \otimes_A B, M).$$

Thus, by Yoneda $q \otimes_A B \backsimeq B$. □

The object $\mathrm{Spec}(S^{-1}A)$ is a formal Zariski open subobject of $\mathrm{Spec}(A)$. Thus, there is an associated localization of $\underline{A\text{-mod}}$. The following proposition describes its associated Gabriel filter.

Proposition 2.9 *Let A be a Commutative monoid, $f \in A_0$ and S a multiplicative part of A_0. Then the Gabriel filter G_f associated to A_f is the set of ideals of A containing a power of f, and the Gabriel filter G_S associated to $S^{-1}A$ is $\cup_{f \in S} G_f$*

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Proof: Let G denote the set of ideals of A containing a power of f. Let q be an element of G. There exists n such that $f^n \in q_0$. Thus q_f contains an invertible element of A_f and by 1.12 is isomorphic to A_f. Therefore, $G \subset G_f$.

Reciprocally, let q be in G_f. Then $i_{q_f} : q_f \backsimeq q \otimes_A A_f \to A_f$ is an isomorphism and the diagram

$$\begin{array}{ccc} A_0 & \longrightarrow & (A_f)_0 \backsimeq (A_0)_f \\ {\scriptstyle -\circ i_q}\uparrow & & \uparrow \\ q_0 & \longrightarrow & (q_f)_0 \backsimeq (q_0)_{-\circ \tau_f} \end{array}$$

is commutative. As this diagram is in Sets, $\exists n \in \mathcal{N}$ such that $f^n \in q_0$. Thus $q \in G$. Finally $G = G_f$. For $S^{-1}A$, clearly $\cup_{f \in S} G_f \subset G_S$ and the other inclusion is obtained with an analogous commutative diagram in Set. □

Primitive quasi-compact and locally primitive Gabriel filters

Definition 2.10 Let A be a commutative monoid.

- ▷ The product $q.q'$ of two ideals q,q' of A is the image of the morphism $q \otimes q' \to A$.
- ▷ An ideal p of A is prime if for all q,q' such that $q.q' \subset p$, either $q \subset p$ or $q' \subset p$.
- ▷ A set G of ideals of A is filtered, if for all $q \subset q'$, $q \in G \Rightarrow q' \in G$.
- ▷ A set of ideals of A is commutative, if it is closed under product of ideals.
- ▷ A set of ideals G is *quasi-compact*, if for any ideal q in G which is isomorphic to a filtered colimit $colim_{i \in \mathcal{I}} q_i$, there exists an i such that q_i is in G.

Lemma 2.11 *Let A be a commutative monoid. Then:*

- ◇ *A commutative Gabriel filter is a monoid in Sets.*
- ◇ *Any Gabriel filter is filtered.*
- ◇ *Any Gabriel filter is commutative.*
- ◇ *A finite union or a finite intersection of quasi-compact sets of ideals is quasi-compact.*
- ◇ *A Gabriel filter G is quasi-compact if and only if for any ideal q in G there exists an ideal $q' \subset q$, g' in G, which is finitely generated, .*

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◇ *Let A be a commutative monoid, q,q' two ideals of A and F a generating family of q. Then there is an isomorphism $q.q' \backsimeq \cup_{f\in F} Im(\tau_f)$ where $\tau_f : q' \to q'$.*

Proof: The first property is clear. The second property comes from axiom (2) in the definition of Gabriel filter. The third comes from axiom (3). The fourth property follows from the fact that finite unions and intersections of Gabriel filters are unions and intersections in the sense of sets.

Let us now prove the fifth property. If G is a quasi-compact Gabriel filter, any ideal q of G is the filtered colimit of its finitely generated sub-ideals (1.10). Thus there is a finitely generated sub-ideal of q in G. Reciprocally, if $q \backsimeq colim_{\mathcal{I}}(q_i)$ is a filtered colimit in G along $\mathcal{I}$ and q' is a finitely generated sub-ideal of q in G, then the ideals q' is of finite type and the morphism from q' to the colimit factors through an ideal q_i. The morphism $q \to q_i$ is clearly a monomorphism, thus $q_i \in G$.

We now prove the sixth property. There is an isomorphism $q \backsimeq Im(K(F) \to q)$. Moreover the image of $K(F)$ under this morphism is a colimit, hence it commutes with tensor product. Furthermore $q' \backsimeq K(i(q')) \backsimeq Im(K(i(q')) \to q'))$ where i is the Yoneda functor from $\mathcal{C}$ to $Pr(\mathcal{C}_0)$. By 1.10, the image of $K(F) \otimes q' \to q \otimes q'$ is $q \otimes q'$. Thus

$$q \otimes q' \backsimeq Im(Colim_{(k,f),(k',f')\in \mathcal{C}_0^F \times \mathcal{C}_0^{i(q')}} k \otimes k' \to q \otimes q').$$

In particular, $F \times i(q')$ generates $q.q'$. Moreover for all $k \in \mathcal{C}_0$

$$k \otimes q' \backsimeq Colim_{(k',f')\in \mathcal{C}_0^{i(q')}} k \otimes k'.$$

Thus $Im(\tau_f)$ is generated by $\{f\} \times i(q')$. Therefore, any generator of $q.q'$ is included in an object $Im(\tau_f)$ where $f \in F$ and finally $\cup_{f\in F} Im(\tau_f) \backsimeq q.q'$. □

The elements of an ideal play a particularly important role. We describe them more precisely.

Definition 2.12 Let A be a commutative monoid. The ideal of A generated by an element f is denoted (f). An isomorphism between two elements $f \in A_k$, $g \in A_{k'}$ is an isomorphism from k to k' equalizing the two elements.

Lemma 2.13 *Let A be a commutative monoid. The collection of isomorphism classes of elements of A is a commutative monoid in sets, whose unit is i_A. The multiplication is denoted by $f \oslash g$. Moreover $(f \oslash g) = (f).(g)$. In particular $(f^{\oslash n}) = (f)^n$.*

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Proof: The product given by $f,g \to f \oslash g := f \circ (g \otimes Id_k)$ provides the set of isomorphism classes of elements with the structure of commutative monoid. The formula $(f \oslash g) = (f).(g)$ is a particular case of the last property proved in the previous lemma. □

We will now give a more convenient equivalent definition for these commutative Gabriel filters, in particular those of finite type.

Proposition 2.14 *Let A be a commutative monoid. A Gabriel filter of A consists of a subset G of the set of ideals of A, which is non-empty, filtered, and satisfies property (3) of the definition of Gabriel filter.*

A quasi-compact Gabriel filter of A consists of a subset of the set of ideals of A, which is non-empty, filtered, commutative and quasi-compact.

Proof: Any Gabriel filter is non-empty and filtered. Reciprocally, we need to check that properties (1) and (2) hold in G. G is filtered and non empty and thus contains A. Let q be in G and $f \in A_k$. Then the ideal $f^{-1}q$ contains q and thus is in G.

Let us now prove the characterisation of quasi-compact Gabriel filters. Let G be a set of ideals satisfying these four properties. We have already proved that properties (1) and (2) of the definition of Gabriel filter hold in G. Let us prove now that (3) holds. Let q be in G and q' be such that $\forall f \in_k q$, $f^{-1}q'$ is in G. Let I be the family of finite subets of elements of q, i.e. of $\prod_{k\in\mathcal{C}_0} Hom_{\mathcal{C}}(k,q)$. Let q_X denote the ideal generated by a subset X of $\prod_{k\in\mathcal{C}_0} Hom_{\mathcal{C}}(k,q)$. By 1.10, $q \backsimeq colim_{i\in I} q_i$. As G is finitely presented, there exists X such that q_X is in G. The diagram

$$\begin{array}{ccc} A\otimes k & \xrightarrow{\tau_f} & A \\ \uparrow & & \uparrow{\scriptstyle i_{q'}} \\ f^{-1}(q')\otimes k & \xrightarrow[\tau_f]{} & q' \end{array}$$

is commutative. Thus, $q_X . \cap_{f\in X} f^{-1}q' \backsimeq \cup_{f\in X} Im_{\cap_{f\in X} f^{-1}q'}(\tau_f) \subset q'$. As G is closed under finite intersections and finite products, q' is in G. □

Proposition 2.15 *Let $u : A \to B$ be in $Comm(\mathcal{C})$. If u is a formal Zariski open morphism then the Gabriel filter associated to B is quasi-compact.*

Proof: Let q be in G and $\mathcal{J}$ be a filtered diagram such that $q \backsimeq colim_{i\in\mathcal{J}} q_i$. There is an isomorphism $q \otimes_A B \backsimeq B$. Hence $B \backsimeq colim_{i\in\mathcal{J}}(q_i \otimes_A B)$. As B is finitely presented in A-mod, there exists i such that this morphism factors through $q_i \otimes_A B$. As u is flat, $q_i \otimes_A B \backsimeq q_i.B$ is an ideal of B. Finally, by lemma 1.12, $q_i \otimes_A B \backsimeq B$. Hence $q_i \in G$. □

Definition 2.16 Let A be a commutative monoid and q an ideal of A.

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- ▷ The Gabriel filter G_q is the smallest Gabriel filter of A containing q in the poset $\mathrm{Gab}\,A$. A Gabriel filter generated by one ideal in this sense will be called a primitive Gabriel filter.
- ▷ The Gabriel filter G^q is defined by $G^q = \cap_{f\in q} G_f$ in the poset $\mathrm{Gab}\,A$. A Gabriel filter generated by one ideal in this sense will be called a locally primitive Gabriel filter.

Remark 2.17 The set of prime ideals included in G^q is the set of prime ideals containing q.

Lemma 2.18 *Let A be a commutative monoid and q a finitely generated ideal of A. The Gabriel filter G_q consists of the set of all ideals of A containing a power of q.*

Proof: Let G be the set of ideals containing a power of q. By 2.14, it is a quasi-compact Gabriel filter. Thus $G_q \subset G$. Moreover its elements are obtained from q by products and inclusions. Therefore $G \subset G_q$. Finally $G = G_q$. □

Corollary 2.19 *Let A be a commutative monoid and q an ideal of A*

- ◇ *If F is a generating family of q, then $G^q = \cap_{f\in F} G_f$.*
- ◇ *If $q = \cup_{i\in I} q_i$, then $G^q = \cap_{i\in I} G^{q_i}$.*

Proof: The proofs of these two properties are equivalent. So let us prove the first. There is clearly an inclusion $G^q \subset \cap_{f\in F} G_f$. Let p be an ideal in $\cap_{f\in F} G_f$. Let g be an element of q. Recall that (g) refers to the ideal generated by g. By 1.10, there is an isomorphism

$$q \backsimeq colim_{F'\subset F,\, fini}(q_{F'}).$$

As (g) is an ideal of finite type , the monomorphism $(g) \to q$ factors through an object $q_{F'}$ in the colimit. There is an inclusion $\cap_{f\in F} G_f \subset G^{q_{F'}}$. Thus p is in $G^{q_{F'}}$. As F' is finite, $G^{q_{F'}} = G_{q_{F'}}$. It follows that p contains a power of $q_{F'}$. As g is an element of $q_{F'}$, p contains a power of g. This is true for all $g \in q$. Thus $G^q = \cap_{f\in F} G_f$. □

Lemma 2.20 *Let G and G' be two locally primitive (resp primitive) Gabriel filters generated respectively by q and q'. Their union in the poset of locally primitive (resp primitive) Gabriel filters is the locally primitive (resp primitive) Gabriel filter generated by $q.q'$. In particular, the prime ideals contained in the union are exactly the prime ideals contained in one of the two Gabriel filters.*

Proof: In the primitive case, $G_{q.q'}$ contains G and G' and for any ideal p such that G_p contains G and G', $G_{q.q'}$ is clearly included in G_p.

In the locally primitive case, we write $G^q \cup G^{q'} = \cap_{f\in q, f'\in q'} G_f \cup G_{f'} = \cap_{f\in q, f'\in q'} G_{f\oslash f'} = G^{q.q'}$. □

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3. Classification of Zariski Open Objects

Zariski open subobjects We first discuss affine Zariski Open subobjects, and then give the general definition.

Lemma 3.1 *Let A be a commutative monoid and let $Spec(B)$ and $Spec(B')$ be two Zariski open subobjects of $Spec(A)$. Then $Spec(B \otimes_A B')$ is a Zariski open subobject of $Spec(A)$ and $B \otimes_A B' - mod = B\text{-mod} \cap B'\text{-mod}$. The intersection is taken in the poset* Loc A.

Proof: The localizations B-mod $\cap$ B'-mod and $B \otimes_A B'$-mod have clearly the same objects. Hence they are equivalent $\mathcal{C}$-categories. The two left adjoint functors are thus adjoint to the same forgetful functor and thus are isomorphic. □

We give now the first interesting result to make a link between Gabriel filters and Zariski open subobjects.

Theorem 3.2 (Finite covering theorem) *Let A be a commutative monoid and $(Spec(B_i))_{i\in I}$ a finite family of Zariski open subobjects of $Spec(A)$. This family is a covering of the Zariski open subobject $Spec(B)$ for the topology defined on $Comm(\mathcal{C})$ ([TV] definition 2.10.2) if and only if $B\text{-mod} = \cup_{i\in I}(B_i\text{-mod})$, where the union is taken in the poset* Loc A.

Proof: We define a functor $l : A\text{-mod} \to A\text{-mod}$

$$l(M) := ker(\ \prod_{i,j} M \otimes_A B_i \otimes_A B_j \rightrightarrows \prod_i M \otimes_A B_i\ ).$$

This functor commutes with finite enriched limits (all the monoids in the kernel are $\mathcal{C}$-flat). It also satisfies $l(M) \otimes_A B_i \backsimeq l(M \otimes_A B_i)$ for all $M \in A$-mod and all i, by flatness of B_i.

Let us prove first that l is isomorphic to identity on the category B_i-mod for all i. Let M be in B_i-mod. There is a unique monomorphism $M \to l(M)$ equalizing the two arrows of the kernel, induced by the natural morphisms $M \to M \otimes_A B_i \otimes_A B_j$. Furthermore

$$l(M) \to \prod_{j\in I} M \otimes_A B_j \to M \otimes_A B_i \backsimeq M.$$

Thus the composition on M is the identity and the composition on $l(M)$ is the unique endomorphism of $l(M)$ equalizing the two arrows of the kernel. Hence it is identity. Finally $M \backsimeq l(M)$.

Let us now verify that $l^2 = l$. For all $M \in A$-mod

$$l^2(M) \backsimeq Ker(\ \prod_{i,j} l(M) \otimes_A B_i \otimes_A B_j \rightrightarrows \prod_i l(M) \otimes_A B_i\ ).$$

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And for all i

$$l(M) \otimes_A B_i \backsimeq l(M \otimes_A B_i) \backsimeq M \otimes_A B_i.$$

Thus $l^2(M)$ is isomorphic to $l(M)$.

We prove that the category $\mathcal{L}$ defined as the essential image of l is a localization. First, we prove that l is left adjoint to the forgetful functor. The unit of the adjunction is

$$M \to l(M).$$

The counit of the adjunction, for $N \backsimeq l(M)$ in the essential image of l, is the isomorphism

$$l(N) \backsimeq N.$$

The triangular identities for the unit and the counit are obtained from the triangular identities for the unit and counit of the adjunctions between forgetful functors and functors $- \otimes_A B_i$.

Finally, $\mathcal{L}$ is a localization with left adjoint functor l. It contains all the B_i-mod and thus their union in $\mathrm{Loc}\, A$. It is contained in B-mod, by definition.

Let us now prove the finite covering theorem. If $B\text{-mod} = \cup_{i \in I}(B_i\text{-mod})$, then $\cup_i (B_i\text{-mod}) \subset \mathcal{L} \subset B\text{-mod}$ and $\cup_i (B_i\text{-mod}) = B\text{-mod}$. Thus $\mathcal{L} = B\text{-mod}$. For all $M \in B\text{-mod}$, there is an isomorphism $M \backsimeq l(M)$, which proves that the functor $B\text{-mod} \to \prod_i B_i\text{-mod}$ reflects isomorphisms.

Reciprocally, if the family of functors $- \otimes_A B_i$ reflects isomorphisms, then for $M \in B$-mod,

$$l(M) \otimes_A B_i \backsimeq l(M \otimes_A B_i) \backsimeq M \otimes_A B_i.$$

Thus $l(M) \backsimeq M$ and $B\text{-mod} = \mathcal{L} = \cup_i (B_i\text{-mod})$. □

Proposition 3.3 *Let A be a commutative monoid and $Y := Spec(B)$ an affine Zariski open subobject of $X := Spec(A)$. Then the Gabriel filter associated to B is primitive and generated by an ideal of finite type.*

Proof: We write $G_B = \cup_{q \in G_B^{ft}} G_q$ in $\mathrm{Gab}\, A$, where G_B^{ft} is the set of ideals of finite type in G_B. Thus $\mathcal{L}_B = \cap_{q \in G_B^{ft}} \mathcal{L}_q$ in $\mathrm{Loc}\, A$. In particular $- \otimes_A B \backsimeq Colim(l_q)$ where the functors l_q are the left adjoints for the localizations $\mathcal{L}_q$. As B is a finitely presented A-algebra, the isomorphism $B \backsimeq colim(l_q(A))$ factors through a $l_q(A)$. Therefore $l_q(A) \in B$-mod. As all functors l_q are isomorphic to identity on B-mod, we obtain for all $q' \in G_B$ that $l_{q'} \circ l_q(A) \backsimeq l_q(A)$. Hence $B \otimes_A l_q(A) \backsimeq colim_{q'}(l_{q'} \circ l_q(A)) \backsimeq l_q(A)$. As there is a functorial isomorphism $(B \otimes_A -) \circ l_q \backsimeq (B \otimes_A -)$, $B \backsimeq l_q(A)$.

Last but not least, let us prove that $\mathcal{L}_q$ is a localization of B-mod. The functor l_q is left $\mathcal{C}$-adjoint. Thus it commutes with $\mathcal{C}$-colimits. In particular, for all $X \in \mathcal{C}$

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and $Y \in A$-mod, $X \otimes Y \in A$-mod and $l_q(X \otimes Y) = X \otimes l_q(Y)$. Moreover, if X, Y are two A-modules,

$$X \otimes_A Y := coker(\ X \otimes A \otimes Y \rightrightarrows X \otimes Y\ ).$$

Thus

$$l_q(X \otimes_A Y) \backsimeq X \otimes_A l_q(Y).$$

And the functor l_q makes an A-module M into a $l_q(A)$-module (hence a B-module) $l_q(M)$. Finally $G_B = G_q$, i.e. G_B is primitive and generated by an ideal of finite type. □

Definition 3.4 Let A be a commutative monoid in $\mathcal{C}$ and F a Zariski open subobject of Spec(A). There exists a family of affine Zariski open subobjects $(\mathrm{Spec}(B_i))_{i\in I}$ of Spec(A) such that F is the image of $\coprod_{i\in I}\mathrm{Spec}(B_i) \to \mathrm{Spec}(A)$ ([TV] definition 2.12). The Gabriel filter associated to F is $G_F := \cap_{i\in I} G_{B_i}$.

Lemma 3.5 *This definition does not depend on the choice of the covering of F.*

Proof: If $(\mathrm{Spec}(B_i))_{i\in I}$ and $(\mathrm{Spec}(C_j))_{j\in J}$ are two coverings, define $D_{i,j} := B_i \coprod_A C_j$. The family $(\mathrm{Spec}(D_{i,j}))_{(i,j)\in I\times J}$ is also a covering and each Spec(B_i) (resp Spec(C_j)) is covered by $(\mathrm{Spec}(D_{i,j}))_{j\in J}$ resp $((\mathrm{Spec}(D_{i,j}))_{i\in I})$ and finally $\cap_{i\in I} G_{B_i} = \cap_{i,j\in I\times J} G_{D_{i,j}} = \cap_{j\in J} G_{C_j}$. □

Theorem 3.6 *Let A be a commutative monoid in $\mathcal{C}$ and F a Zariski open subobject of Spec(A). Then the Gabriel filter associated to F is locally primitive.*

Proof: The scheme F is covered by a family $\mathrm{Spec}(B_i)_{i\in I}$. Let q_i be the ideal of finite type generating the Gabriel filter G_{q_i} associated to B_i. As the q_i are of finite type, $G_F = \cap_{i\in I} G_{q_i} = \cap_{i\in I} G^{q_i}$. Thus $G_F = G^{\cup_{i\in I} q_i}$. □

Zariski open subobjects in strong relative contexts We assume in this section that the relative context $\mathcal{C}$ is strong, i.e. that the functor $(-)_0$ reflects isomorphisms. By 1.10, this implies that for any object X, the set X_0 is a generating family.

Theorem 3.7 *Let A be a commutative monoid in $\mathcal{C}$. Then any Zariski affine open subobject $Y :=$Spec(B) of $X :=$Spec(A) has a finite covering by objects Spec(A_f).*

Proof: The Gabriel filter G_B associated to B is primitive and generated by an ideal q of finite type. Therefore, as $\mathcal{C}$ is strong, there exists a generating finite family of elements of q_0 in q. We write $G_B = G_q = \cap_{f\in F} G_f$. As F is finite, 3.2 implies that $(\mathrm{Spec}(A_f))_{f\in F}$ is a covering of Spec(B). □

Theorem 3.8 (Covering theorem) *Let A be a commutative monoid and $(Spec(B_i))_{i\in I}$ a family of Zariski open subobjects of Spec(A). Then this family is a covering of the Zariski open subobject Spec(B) for the topology defined on*

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Comm($\mathcal{C}$) *([TV], definition 2.10) if and only if* B*-mod* $= \cup_{i\in I} B_i$*-mod in the poset* Loc A.

Proof: Assume that B-mod $= \cup_{i\in I} B_i$-mod. We have $G_q := G_B = \cap_{i\in I} G_{q_i}$ where $G_{q_i} := G_{B_i}$. The ideal q is the union of the q_i. Thus by 1.10, q is isomorphic to the filtered colimit $colim_{J\ fini\subset I} \cup_{j\in J} q_j$. As it is of finite type, there exists J such that this isomorphism factors through $q \to \cup_{j\in J} q_j$. The morphism $\cup_{j\in J} q_j \to q$ is then a monomorphism and a split epimorphism. Hence it is an isomorphism. By 3.2, the family $(\mathrm{Spec}(B_j))_{j\in J}$ is a covering of Spec(B). Thus so is the family $(\mathrm{Spec}(B_i))_{i\in I}$.

Reciprocally, if the family $(\mathrm{Spec}(B_i))_{i\in I}$ is a covering of Spec(B), it has a finite covering sub-family $(\mathrm{Spec}(B_j))_{j\in J}$ and by 3.2, B-mod $= \cup_{j\in J} B_j$-mod. As $\cup_{j\in J} B_j$-mod $\subset \cup_{i\in I} B_i$-mod $\subset B$-mod, we have also that $\cup_{i\in I} B_i$-mod $= B$-mod. □

Theorem 3.9 (Classification of Zariski open subobjects) *Let A be a commutative monoid. Then there is a (contravariant) bijection between the poset of Zariski open subobject of $X :=$Spec(A), denoted Zar(X), and the poset of locally primitive Gabriel filters of A, denoted $lp(A)$. Moreover a Zariski open subobject of Spec(A) is quasi-compact if and only if its associated Gabriel filter is quasi-compact (and primitive).*

Proof: By 3.6, there is a morphism in sets from Zar(X) to $lp(A)$. We prove first that this morphism is a contravariant morphism of posets. Let $F \subset F'$ be two Zariski open subobjects of X, and let $(\mathrm{Spec}(C_j))_{j\in J}$,$(\mathrm{Spec}(B_i))_{i\in I}$ be respectively coverings of F' and F. Let q, (resp q_j,q_i) be the ideal generating the Gabriel filter associated to F (resp C_j, B_i). The scheme F is covered by the family $(\mathrm{Spec}(B_i \otimes_A C_j))_{i,j\in I\times J}$. Thus $G_F = \cap_{j\in J} \cap_{i\in I} G^{q_i\cdot q_j} = \cap_{j\in J} G^{q\cdot q_j} \supset \cap_{j\in J} G_{q_j} = G_{F'}$.

Let us prove surjectivity. Let G^q be a locally primitive Gabriel filter. As $\mathcal{C}$ is strong, $G^q = \cap_{f\in q_0} G_f$. The Gabriel filter associated to the Zariski open subobject of X defined as the image of $\coprod_{f\in q_0} \mathrm{Spec}(A_f) \to A$ is G^q.

Let us prove injectivity. Let F, F' be two Zariski open subobjects of X with the same associated Gabriel filter. Let $(\mathrm{Spec}(A_f)_{f\in J}$ and $(\mathrm{Spec}(A_f))_{f\in J'}$ be respectively coverings of F and F'. If q and q' are the ideals generated by J and J', then $G^q = G_F = G_{F'} = G^{q'}$. For all $f \in q_0$, there exists n such that $f^n \in q'$. As $A_{f^n} \backsimeq A_f$, any affine Zariski open subobject A_f from the covering family of F is a Zariski open subobject of F'. Therefore F is a Zariksi open subobject of F'. As the roles of F and F' are symmetric, $F \backsimeq F'$.

Finally, if G^q is a quasi-compact locally primitive Gabriel filter, it is primitive as $G^q = G_q$. Moreover we can write it as a finite intersection of filters G_f and its associated Zariski open subobject is covered by a finite family Spec(A_f). Thus is

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quasi-compact. □

Remark 3.10 The poset $lp(A)$ is a colocale. It is the colocale of the closed subsets of the the Zariski topological space associated to Spec(A).

Lemma 3.11 *Let A be in Comm$(\mathcal{C})$.*

- ▷ *A locally primitive Gabriel filter of A is irreducible if and only if it has a unique prime representive.*
- ▷ *Two irreducible locally primitive Gabriel filter are equals if and only if they have the same subsets of prime ideals.*
- ▷ *Two locally primitive Gabriel filters are equals if and only if they have the same subsets of prime ideals.*

Proof: Let G^q be an irreducible locally primitive Gabriel filter. The radical $\sqrt{q}$ of q (the intersection of the prime ideals containing q) represents the same locally primitive Gabriel filter and is prime. Indeed $q \subset \sqrt{q}$ and q contains a power of any element of $\sqrt{q}$. If $\sqrt{q} \backsimeq q'.q''$, then q' and q'' contain $\sqrt{q}$ and as $G^{\sqrt{q}}$ is irreducible, $\sqrt{q}$ contains a power of any element of one of these two ideals, for example q'. Then, by definition of the radical of an ideal, $\sqrt{q}$ contains q' and thus $\sqrt{q} = q'$. This proves the existence of a prime representative. The uniqueness is provided by 2.17.

Two irreducible locally primitive Gabriel filters which have the same prime ideals, are generated by the same prime ideal. Thus are equal.

Let $G^q, G^{q'}$ be two locally primitive Gabriel filters. We write them as unions of irreducible ones $G^q = \cup_{i\in I} G^{p_i}$ and $G^{q'} = \cup_{j\in J} G^{p_j}$. For all i, there exists j such that $p_j \subset p_i$. Hence $G^{p_i} \subset G^{p_j}$. Therefore $G^q \subset G^{q'}$. As the roles are symmetric, there is equality. □

Definition 3.12 The Zariski topological space associated to $X :=$Spec(A), denoted $Lp(A)$, is the topological space whose points are all prime ideals of A and whose closed subsets are the sets of prime ideals contained in a locally primitive Gabriel filter.

Theorem 3.13 *Let A be a commutative monoid. Then the Zariski topological space $Lp(A)$ associated to A is sober and its associated locale is Zar(Spec(A)).*

Proof: By 3.11 and 3.9. □

Examples of contexts

Proposition 3.14 *The context ($\mathbb{Z}$-mod,$\otimes_{\mathbb{Z}}$,$\mathbb{Z}$). Let A be a ring. Then there is an homeomorphism between the Zariski topological space $Lp(A)$ and the underlying Zariski topological space of Spec(A).*

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Proof: By construction. □

Proposition 3.15 *The context $(Set,\times,\mathbb{F}_1)$. Let A be a commutative monoid in sets. Then there is an homeomorphism between the Zariski topological space $Lp(A)$ and the underlying Zariski topological space of $Spec(A)$ (defined in [D] section 2.1 by Deitmar[5]).*

Proof: The Zariski topological space constructed in [D] is homeomorphic to $Lp(A)$. □

Appendix - Enriched Category Theory

See [K], [DK] and [H] for more details. Let $\mathcal{C}$ be a relative context. We will consider enriched categories over $\mathcal{C}$, referred to as $\mathcal{C}$-categories. Any usual notion will have its enriched equivalent, referred to as the $\mathcal{C}$-notion, for example $\mathcal{C}$-limit, $\mathcal{C}$-colimit... There is an "underlying category" functor. To an enriched category is associated the category whose objects are the same and morphisms between objects proceed from the "underlying set functor" $Hom_{\mathcal{C}}(1,-)$.

Definition 4.1 A $\mathcal{C}$-category $\mathcal{B}$ consists of a collection of objects $ob(\mathcal{B})$, for any two objects (X,Y), $\mathcal{B}$ of an object $\underline{Hom}_{\mathcal{B}}(X,Y) \in \mathcal{C}$, and of a composition law:

$$M_{X,Y,Z} : \underline{Hom}_{\mathcal{B}}(Y,Z) \otimes \underline{Hom}_{\mathcal{B}}(X,Y) \to \underline{Hom}_{\mathcal{B}}(X,Z)$$

such that the appropriate commutative diagrams commute. A $\mathcal{C}$-functor $F : \mathcal{B} \to B'$ consists of an object $F(X)$ in B' for each object X in B, and a morphism

$$\underline{Hom}_{B}(X,Y) \xrightarrow{F_{X,Y}} \underline{Hom}_{B'}(X,Y)$$

for each couple of objects (X,Y) such that the appropriate diagrams commute. An equivalence of $\mathcal{C}$-categories consists of a $\mathcal{C}$-functor which is $\mathcal{C}$-fully faithful and its underlying functor is essentially surjective.

Definition 4.2

1. Let $\mathcal{B}$ be a $\mathcal{C}$-category, $\mathcal{K}$ a small $\mathcal{C}$-category, and $F : \mathcal{K} \to \mathcal{C}$ and $G : \mathcal{K} \to \mathcal{B}$ two $\mathcal{C}$-functors. If the functor $\mathcal{B}^{op} \to \mathcal{C}$:

$$B \to \underline{Hom}_{\mathcal{C}^{\mathcal{K}}}(F, \underline{Hom}_{\mathcal{B}}(B, G-))$$

is represented by an object $lim_F(G)$, then G has a $\mathcal{C}$-limit indexed by F.

2. Let $\mathcal{B}$ be a $\mathcal{C}$-category, $\mathcal{K}$ a small $\mathcal{C}$-category, and $F : \mathcal{K}^{op} \to \mathcal{C}$ and $G : \mathcal{K} \to \mathcal{B}$ two C-functors. If the functor $\mathcal{B} \to \mathcal{C}$:

$$B \to \underline{Hom}_{\mathcal{C}^{\mathcal{K}^{op}}}(F, \underline{Hom}_{\mathcal{B}}(G-, B))$$

is represented by an object $colim_F(G)$, then G has a C-colimit indexed by F.

[5]It involves monoidal spaces i.e. topological spaces together with sheaves of monoids in set, analogous to ringed spaces.

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3. A $\mathcal{C}$-category is finite if and only if its collection of objects is a finite set and for each couple of objects (X,Y), the object $\underline{Hom}(X,Y)$ is finitely presented in $\mathcal{C}$.

4. A $\mathcal{C}$-limit is finite if and only if it is indexed by a functor $F : \mathcal{K} \to \mathcal{C}_0$ from a finite $\mathcal{C}$-category $\mathcal{K}$ to the generating subcategory of $\mathcal{C}$ consisting of finitely presented objects.

For a tensored and cotensored (exponential objects exists) $\mathcal{C}$-category , the enriched limit of G along F is in fact the end of G^F and the enriched colimit is the coend of $F \otimes G$. Relative contexts are tensored and cotensored self enriched categories.

Proposition 4.3 *(cf [K], 3.10) A $\mathcal{C}$-Category $\mathcal{B}$ is $\mathcal{C}$-complete if and only if it is complete and cotensored. Furthermore, $\mathcal{B}$ has finite $\mathcal{C}$-limits if and only if it has finite limits and if all exponential objects X^Y, Y finitely presented in $\mathcal{C}$, are representable in $\mathcal{C}$. In particular, a functor F is left $\mathcal{C}$-exact if and only if it is left exact and commutes with all exponential objects X^Y (i.e. $F(X^Y) \backsimeq F(X)^Y$), Y finitely presented in $\mathcal{C}$.*

Corollary 4.4 *Let $\mathcal{C}$ be a tensored and cotensored symmetric monoidal category which is locally finitely presentable. Assume that one of the following hypotheses holds in $\mathcal{C}$:*

- *The generating subcategory is stable under tensor product.*

- *The unit of $\mathcal{C}$ generates $\mathcal{C}$, i.e. $Hom(1,-)$ reflect isomorphisms.*

Let A be in $Comm(\mathcal{C})$. Then filtered colimits are $\mathcal{C}$-exact in A-mod.

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