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Mass transference principle for limsup sets generated by rectangles

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Abstract

We generalise the mass transference principle established by Beresnevich and Velani to limsup sets generated by rectangles. More precisely, let $\{x_n\}_{n\geqslant 1}$ be a sequence of points in the unit cube $[0,1]^d$ with $d \geqslant 1$ and $\{r_n\}_{n\geqslant 1}$ a sequence of positive numbers tending to zero. Under the assumption of full Lebesgue measure theoretical statement of the set

$$\big\{x \in [0,1]^d : x \in B(x_n, r_n), \text{ for infinitely many } n \in \mathbb{N}\big\},$$

we determine the lower bound of the Hausdorff dimension and Hausdorff measure of the set

$$\big\{x \in [0,1]^d : x \in B^{\mathbf{a}}(x_n, r_n), \text{ for infinitely many } n \in \mathbb{N}\big\},$$

where $\mathbf{a} = (a_1, \ldots, a_d)$ with $1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_d$ and $B^{\mathbf{a}}(x, r)$ denotes a rectangle with center x and side-length $(r^{a_1}, r^{a_2}, \ldots, r^{a_d})$. When $a_1 = a_2 = \cdots = a_d$, the result is included in the setting considered by Beresnevich and Velani.

1. *Introduction*

We begin with the classic Diophantine approximation to illustrate our settings. Diophantine approximation concerns about the approximating irrationals by rationals. From a measure theoretical point of view, one concerns about the size of the following limsup set

$$W(\psi) = \big\{x \in [0,1] : |x - p/q| < \psi(q), \text{ for infinitely many } (p,q) \in \mathbb{N}^2\big\}$$

in the sense of, especially, Lebesgue measure and Hausdorff measure, where ψ is a positive function.

For the Lebesgue measure of the set $W(\psi)$, Khintchine's pioneering work can be viewed as a foundation of metric Diophantine approximation. Since then, many great contributions have been achieved concerning the Lebesgue measure of the set $W(\psi)$ and its generalisations. We refer the reader to the references [**3**, **11**–**13**, **17**, **18**] for examples.

When ψ tends to zero very fast, the limsup set $W(\psi)$ will be of Lebesgue measure zero. Then Hausdorff measure is introduced to discriminate these Lebesgue null sets. Jarník [**14**] is the one who first applied the Hausdorff measure to measure the size of $W(\psi)$. In the last three decades, the Hausdorff measure of the set $W(\psi)$ and its generalisations have been

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well studied (see for examples [**2**, **4**–**8**]). Among these works, a cornerstone is the "mass transference principle" due to Beresnevich and Velani [**1**], which states that, in principle, a Lebesgue measure theoretical statement for a limsup set can induce a Hausdorff measure theoretical statement. To make this clear, we cite here their theorem: Let $\{B(x_n, r_n)\}_{i\in\mathbb{N}}$ be a sequence of balls in $\mathbb{R}^d$ with $r_n \to 0$ as $n \to \infty$ and f a dimension function such that $x^{-d}f(x)$ is monotonic. Denote by $\dim_{\mathrm{H}}$ the Hausdorff dimension, by $\mathcal{L}$ the d-dimensional Lebesgue measure, by $\mathcal{H}^f$ the f-Hausdorff measure and use *i.o.* to stand for *infinitely often*.

THEOREM 1·1 (Mass transference principle [**1**]). *Assume that*

$$\mathcal{L}\left(\left\{x \in [0,1]^d : |x - x_n| < f(r_n)^{1/d},\ i.o.\ n \in \mathbb{N}\right\}\right) = 1.$$

Then, for any ball $B \subset [0,1]^d$,

$$\mathcal{H}^f\left(\left\{x \in B : |x - x_n| < r_n,\ i.o.\ n \in \mathbb{N}\right\}\right) = \infty.$$

Theorem 1·1 discloses a profound transference principle from Lebesgue measure theoretic statements to Hausdorff measure theoretic statements for limsup sets. It unifies "almost all" known results about the Hausdorff measure of $W(\psi)$ and its generalisations.

Loosely speaking, the setting of the mass transference principle is: first, we are given a full Lebesgue measure theoretic statement of some limsup set generated by "balls" or more generally some neighbourhoods of a sequence of sets, and then one considers the Hausdorff measure of the limsup sets with the radius of each generating ball shrinking in all directions in a same way. So how about the case when the generating sets shrink in different manner in different directions. For example, how about the size of the following set:

$$\left\{x = (y_1, \cdots, y_d) \in [0,1]^d : \|qy_i\| < q^{-\tau_i}, 1 \leqslant i \leqslant d,\ \text{i.o. } q \in \mathbb{N}\right\}, \tag{1·1}$$

where $\|z\| = \min_{p\in\mathbb{Z}} |z - p|$ denotes the distance from z to the nearest integer and $\tau_i \geqslant 0$ for all $1 \leqslant i \leqslant d$. (This was studied by Rynne [**16**].) Instead of the "balls", the limsup set given in (1·1) is generated by rectangles.

This motivates us to wonder whether there is a similar mass transference principle for the limsup sets generated by rectangles? This is the main aim of this paper. Similarly, we also assume that we already have a full Lebesgue measure theoretical statement of a limsup set in hand, and then consider the Hausdorff measure theoretical results of the shrunk limsup set. Now we give our settings.

Let $\{x_n\}_{n\geqslant 1}$ be a sequence of points in the unit cube $[0,1]^d$ with $d \geqslant 1$ and $\{r_n\}_{n\geqslant 1}$ be a sequence of positive numbers tending to zero. Use $\mathbf{a}$ to denote a d-dimensional vector $(a_1, \ldots, a_d)$. Define

$$W_{\mathbf{1}} := \left\{x \in [0,1]^d : x \in B(x_n, r_n),\ \text{i.o. } n \in \mathbb{N}\right\},$$

and for any $\mathbf{a} = (a_1, \ldots, a_d)$ with $1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_d$, define

$$W_{\mathbf{a}} := \left\{x \in [0,1]^d : x \in B^{\mathbf{a}}(x_n, r_n),\ \text{i.o. } n \in \mathbb{N}\right\},$$

where we use $B^{\mathbf{a}}(x, r)$ to denote a rectangle with center x and side-length $(r^{a_1}, r^{a_2}, \ldots, r^{a_d})$.

The first result concerns the Hausdorff dimension of the set $W_{\mathbf{a}}$.

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THEOREM 1·2. *Assume that* $\mathcal{L}(W_{\mathbf{1}}) = 1$. *Let* $\mathbf{a} = (a_1, \ldots, a_d)$ *with* $1 \leqslant a_1 \leqslant \cdots \leqslant a_d$. *Then we have*

$$\dim_{\mathrm{H}} W_{\mathbf{a}} \geqslant \min\Big\{\frac{d + ja_j - \sum_{i=1}^{j} a_i}{a_j}, 1 \leqslant j \leqslant d\Big\} := s.$$

Concerning the Hausdorff measure of $W_{\mathbf{a}}$ at the critical value s, we have

THEOREM 1·3. *Assume that* $\mathcal{L}(W_{\mathbf{1}}) = 1$. *Let* $\mathbf{a} = (a_1, \ldots, a_d)$ *with* $1 \leqslant a_1 \leqslant \cdots \leqslant a_d$ *and* $a_d > 1$. *Then* $\mathcal{H}^s(W_{\mathbf{a}}) = \infty$.

Remark 1. The condition $a_d > 1$ is a natural restriction. Because if $a_1 = \cdots = a_d = 1$, we get $s = \min\big\{(d + ja_j - \sum_{i=1}^{j} a_i)/a_j, 1 \leqslant j \leqslant d\big\} = d$ and $\mathcal{L}(W_{\mathbf{a}}) = 1$. Then we have $0 < \mathcal{H}^d(W_{\mathbf{a}}) < \infty$ by the fact that the measures $\mathcal{L}$ and $\mathcal{H}^d$ differ only by a constant multiple.

It is clear that Theorem 1·2 can be induced from Theorem 1·3. However, since the ideas used to prove these two results are just a combining of the method in [**1**] to show the mass transference principle with the method in [**13**] to study the shrinking target problems in torus, we show Theorem 1·2 in detail but only give an outline of the proof of Theorem 1·3.

2. *Preliminaries*

In this section, we recall the $K_{G,B}$ lemma presented in [**1**] which is also the main tool to show Theorems 1·2 and 1·3. Then we give some estimates of some quantities for later use.

Throughout this paper, we use $a \ll b$ to mean that there exists a constant $c > 0$ such that $a \leqslant cb$ and use $a \approx b$ to denote $a \ll b$ and $b \ll a$. In the sequel, all the constants implied in '$\ll$' (or in '$\approx$' respectively) can be taken to be a unified constant $c_{\ll}$ (or $c_{\approx}$ respectively) depending only on d, the dimension of the underlying space $[0, 1]^d$.

Recall that

$$W_{\mathbf{1}} = \big\{x \in [0, 1]^d : |x - x_n| < r_n, \text{ i.o. } n \in \mathbb{N}\big\}$$

and that

$$s = \min\Big\{\frac{d + ja_j - \sum_{i=1}^{j} a_i}{a_j}, 1 \leqslant j \leqslant d\Big\}.$$

LEMMA 2·1 ($K_{G,B}$-lemma [**1**]). *Assume that* $\mathcal{L}(W_{\mathbf{1}}) = 1$. *Then for any integer* $G \geqslant 1$ *and any cube or ball* B, *there exists a collection* $K_{G,B}$ *of balls* $B(x_n, r_n)$ *fulfilling the following conditions:*

(i) *separation condition*: *for any two distinct balls* $B(x_n, r_n)$ *and* $B(x_m, r_m)$ *in* $K_{G,B}$, *one has*

$$B(x_n, 8dr_n) \cap B(x_m, 8dr_m) = \varnothing;$$

(ii) *packing condition*:

$$\bigcup_{B(x_n, r_n) \in K_{G,B}} B(x_n, 8dr_n) \subset B, \qquad \sum_{B(x_n, r_n) \in K_{G,B}} \mathcal{L}(B(x_n, r_n)) \approx \mathcal{L}(B).$$

where the constant implied in $\approx$ *depends only on* d.

The next two estimates will be used later in estimating the measure of a ball $B(x, r)$.

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LEMMA 2·2. *Let $1 \leqslant j \leqslant d$ be an integer and r a positive number with $r_{n_k}^{a_{j+1}} < r \leqslant r_{n_k}^{a_j}$ for some $r_{n_k} > 0$. Set*

$$u_r = \left(\prod_{i=1}^{j} \frac{r}{r_{n_k}^{a_i}}\right) \cdot r_{n_k}^d. \tag{2·1}$$

Then we have $u_r \leqslant r^s$.

Proof. Taking log and divided by $\log r$ on both sides of (2·1), we arrive at

$$\frac{\log u_r}{\log r} = j + \frac{d \log r_{n_k} - \sum_{i=1}^{j} a_i \log r_{n_k}}{\log r}.$$

Since the right-hand side is monotone with respect to r, the minimal value is attained at one of the endpoints of the range $(r_{n_k}^{a_{j+1}}, r_{n_k}^{a_j}]$ of r. Thus, it follows that

$$\begin{aligned}\frac{\log u_r}{\log r} &\geqslant \min\left\{j + \frac{d - \sum_{i=1}^{j} a_i}{a_j}, j + \frac{d - \sum_{i=1}^{j} a_i}{a_{j+1}}\right\} \\ &= \min\left\{\frac{d + ja_j - \sum_{i=1}^{j} a_i}{a_j}, \frac{d + (j+1)a_{j+1} - \sum_{i=1}^{j+1} a_i}{a_{j+1}}\right\} \\ &\geqslant s.\end{aligned}$$

LEMMA 2·3. *Let r be a positive number with $r \leqslant r_{n_k}^{a_d}$. Set*

$$u_r = \frac{r^d}{\prod_{j=1}^{d} r_{n_k}^{a_j}} \cdot r_{n_k}^d. \tag{2·2}$$

Then we have $u_r \leqslant r^s$.

Proof. Taking log and divided by $\log r$ on both sides of (2·2), we arrive at

$$\frac{\log u_r}{\log r} = d + \left(d - \sum_{i=1}^{d} a_i\right) \frac{\log r_{n_k}}{\log r}.$$

Noticing that $\log r_{n_k} / \log r \leqslant 1/a_d$ and $d - \sum_{i=1}^{d} a_i$ is non-positive, we get

$$\frac{\log u_r}{\log r} \geqslant d + \frac{d - \sum_{i=1}^{d} a_i}{a_d} = \frac{d + da_d - \sum_{i=1}^{d} a_j}{a_d} \geqslant s.$$

3. *Proof of Theorem* 1·2

The idea to determine the Hausdorff dimension of $W_{\mathbf{a}}$ is quite classical: firstly, we construct a Cantor subset $\mathbb{F}_\infty$ of $W_{\mathbf{a}}$; secondly, define a suitable mass distribution μ supported on $\mathbb{F}_\infty$; thirdly, estimate the Hölder exponent of the measure μ; and finally, we conclude the result by applying the following mass distribution principle.

PROPOSITION 3·1 (Mass Distribution Principle [**9**]). *Let μ be a probability measure supported on a set F. Suppose there are positive constants c and r_o such that*

$$\mu(B(x, r)) \leqslant cr^s$$

for any ball $B(x, r)$ with radius $r \leqslant r_o$ and center $x \in F$. Then $\mathcal{H}^s(F) \geqslant 1/c$ and so $\dim_{\mathrm{H}} F \geqslant s$.

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3·1. *Construction of a Cantor subset of* $W_{\mathbf{a}}$

The 1st *level of the Cantor set*

Let $B_0 = [0, 1]^d$. Applying the $K_{G,B}$-Lemma to B_0, we get a collection K_{G,B_0} of balls $B(x_n, r_n)$. Then the first level of the Cantor set is defined as

$$\mathbb{F}_1 = \bigcup_{B(x_n,r_n)\in K_{G,B_0}} B^{\mathbf{a}}(x_n, r_n).$$

In other words, shrink each ball $B(x_n, r_n)$ in K_{G,B_0} to a rectangle $B^{\mathbf{a}}(x_n, r_n)$.

The other levels of the Cantor subset is defined recursively by the following way. Assume that the kth level $\mathbb{F}_k$ has been well defined which is a collection of rectangles with the form $B^{\mathbf{a}}(x_{n_k}, r_{n_k})$. For ease of notation, $\mathbb{F}_k$ will also refer to the collection of rectangles which constitute the kth level as well as the union of those rectangles. However, from the context, there will be no ambiguity should arise.

The $(k+1)$*th level of the Cantor set*

We give at first an outline about the construction of the Cantor set from an element $C_k \in \mathbb{F}_k$ to a local level $\mathbb{F}_{k+1}(C_k)$ of the $(k+1)$th level. Instead of using $K_{G,B}$-Lemma to the rectangles $C_k \in \mathbb{F}_k$ directly, three steps are used to carry out $\mathbb{F}_{k+1}(C_k)$:

(i) *partition*: divide the rectangle C_k into cubes $\{B_k^{(i)}\}$ with the side-length being the smallest one of the rectangle C_k;
(ii) *packing*: use $K_{G,B}$-Lemma to pack every cube $B_k^{(i)}$ by balls $B(x_n, r_n)$;
(iii) *scaling*: shrink the balls $B(x_n, r_n)$ obtained from the $K_{G,B}$-Lemma to rectangles $B^{\mathbf{a}}(x_n, r_n)$ which are what we need.

Compared with the method for proving the mass transference principle developed by Beresnevich and Velani, a newly added process is the step about *partition*. Since in [**1**], they are confronted with balls instead of rectangles, the *partition* process is unnecessary.

Now we present the construction in detail. Let $C_k = B^{\mathbf{a}}(x_{n_k}, r_{n_k})$ be an element in $\mathbb{F}_k$. At first, divide the rectangle $B^{\mathbf{a}}(x_{n_k}, r_{n_k})$ into a collection of cubes with side-length $r_{n_k}^{a_d}$. Denote by $\{B_k^{(i)}\}_{1\leqslant i\leqslant t_k(C_k)}$ the collection of these cubes. We choose a well separated sub-collection $\mathfrak{G}(C_k)$ of these cubes in the sense that for any $B_k^{(i)}$ and $B_k^{(j)}$ in $\mathfrak{G}(C_k)$, we have

$$3B_k^{(i)} \cap 3B_k^{(j)} = \varnothing.$$

Then clearly the number $t_k(C_k)$ of cubes in $\mathfrak{G}(C_k)$ is

$$t_k(C_k) \approx \left(\prod_{j=1}^{d} r_{n_k}^{a_j}\right) / \left(r_{n_k}^{a_d}\right)^d = \prod_{j=1}^{d} \left(r_{n_k}^{a_j} / r_{n_k}^{a_d}\right).$$

For each $B_k^{(i)}$ with $1 \leqslant i \leqslant t_k(C_k)$, apply the $K_{G,B}$-lemma to $B_k^{(i)}$, then we get a collection $K_{G,B_k^{(i)}}$ of balls $B(x_n, r_n)$. Define

$$\mathbb{F}_{k+1}(C_k) = \bigcup_{i=1}^{t_k(C_k)} \bigcup_{B(x_n,r_n)\in K_{G,B_k^{(i)}}} B^{\mathbf{a}}(x_n, r_n).$$

We call $C_k \in \mathbb{F}_k$ the mother rectangle of the rectangles in $\mathbb{F}_{k+1}(C_k)$.

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Then the $(k+1)$th level of the Cantor set is defined as

$$\mathbb{F}_{k+1} = \bigcup_{C_k \in \mathbb{F}_k} \mathbb{F}_{k+1}(C_k).$$

Remark 2. (1) When applying the $K_{G,B}$-lemma to the cube $B_k^{(i)}$, we can require that G is sufficiently large that the radius of the balls in $K_{G,B_k^{(i)}}$ are as small as we want compared with the radius of balls occurring in all the previous levels.

(2) By the separation condition of $\{B_k^{(i)}\}$ and the $K_{G,B}$-lemma, we notice that the rectangles in $\mathbb{F}_{k+1}(C_k)$ are well separated in the sense that for any two distinct rectangles $B^{\mathbf{a}}(x_n, r_n)$, $B^{\mathbf{a}}(x_m, r_m)$ in $\mathbb{F}_{k+1}(C_k)$, we have

$$B(x_n, 3r_n) \cap B(x_m, 3r_m) = \varnothing.$$

Finally, we define the desired Cantor set as

$$\mathbb{F}_\infty = \bigcap_{k \geqslant 1} \bigcup_{B^{\mathbf{a}}(x_n, r_n) \in \mathbb{F}_k} B^{\mathbf{a}}(x_n, r_n).$$

By the construction of the Cantor set $\mathbb{F}_\infty$ given above, it is clear that

PROPOSITION 3·2. $\mathbb{F}_\infty \subset W_{\mathbf{a}}$.

3·2. *Mass distribution on* $\mathbb{F}_\infty$

We define a probability measure supported on $\mathbb{F}_\infty$ as follows. At first let $\mu(B_0) = 1$. For each rectangle $C_1 = B^{\mathbf{a}}(x_{n_1}, r_{n_1}) \in \mathbb{F}_1$, define

$$\mu\left(B^{\mathbf{a}}(x_{n_1}, r_{n_1})\right) = \frac{r_{n_1}^d}{\sum_{B(x_m, r_m) \in K_{G,B_0}} r_m^d}.$$

Assume that the measure of the rectangles in the kth level $\mathbb{F}_k$ of the Cantor set has been defined. Now we define the measure of the rectangles in the $(k+1)$th level $\mathbb{F}_{k+1}$. For each $C_{k+1} = B^{\mathbf{a}}(x_{n_{k+1}}, r_{n_{k+1}}) \in \mathbb{F}_{k+1}$, let $C_k = B^{\mathbf{a}}(x_{n_k}, r_{n_k}) \in \mathbb{F}_k$ be the mother rectangle in $\mathbb{F}_k$ of C_{k+1}. Then define

$$\mu\left(B^{\mathbf{a}}(x_{n_{k+1}}, r_{n_{k+1}})\right) = \frac{r_{n_{k+1}}^d}{\sum_{i=1}^{t_k(C_k)} \sum_{B(x_n, r_n) \in K_{G,B_k^{(i)}}} r_n^d} \times \mu(C_k). \tag{3·1}$$

Clearly the measure constructed in the above way fulfills the Kolmogorov's consistency theorem, so it can be expanded uniquely into a probability measure supported on $\mathbb{F}_\infty$.

We end this subsection with a lemma concerning the quantity of the denominator in (3·1).

LEMMA 3·3. *Let* $C_k = B^{\mathbf{a}}(x_{n_k}, r_{n_k}) \in \mathbb{F}_k$ *and* $\{B_k^{(i)}\}_{1 \leqslant i \leqslant t_k(C_k)}$ *be the cubes in* $\mathfrak{G}(C_k)$. *Then*

$$\sum_{i=1}^{t_k(C_k)} \sum_{B(x_n, r_n) \in K_{G,B_k^{(i)}}} r_n^d \approx \sum_{i=1}^{t_k(C_k)} \mathcal{L}(B_k^{(i)}) \approx \mathcal{L}(C_k), \tag{3·2}$$

where the constants implied in $\approx$ *depends only on* d. *Moreover for each cube* $B_k^{(i)}$ *with* $1 \leqslant i \leqslant t_k(C_k)$,

$$\mu(B_k^{(i)}) \approx \frac{1}{t_k(C_k)} \mu(C_k).$$

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Proof. The first assertion follows from the second item of the $K_{G,B}$-lemma, while the second one is direct. Since μ is a measure supported on $\mathbb{F}_\infty$,

$$\begin{aligned}\mu(B_k^{(i)}) &= \sum_{B^{\mathbf{a}}(x_n,r_n)\in\mathbb{F}_{k+1},\, B^{\mathbf{a}}(x_n,r_n)\subset B_k^{(i)}} \mu\left(B^{\mathbf{a}}(x_n,r_n)\right) = \sum_{B(x_n,r_n)\in K_{G,B_k^{(i)}}} \mu\left(B^{\mathbf{a}}(x_n,r_n)\right)\\ &\overset{(3\cdot2)}{\approx} \sum_{B(x_n,r_n)\in K_{G,B_k^{(i)}}} \frac{r_n^d}{\mathcal{L}(C_k)}\cdot\mu(C_k) \overset{K_{G,B}\text{-Lemma}}{\approx} \frac{\mathcal{L}(B_k^{(i)})}{\mathcal{L}(C_k)}\cdot\mu(C_k)\\ &\approx \frac{1}{t_k(C_k)}\cdot\mu(C_k).\end{aligned}$$

3·3. *Hölder exponent of μ*

In this subsection, we will estimate the μ-measure of an arbitrary ball $B(x,r)$ and then apply the mass distribution principle (Proposition 3·1) to conclude the desired result.

At first, we estimate the measure of a rectangle $C_k = B^{\mathbf{a}}(x_{n_k}, r_{n_k}) \in \mathbb{F}_k$. Let $C_{k-1} \in \mathbb{F}_{k-1}$ be the mother rectangle in $\mathbb{F}_{k-1}$ containing B_k. Then by (3·1) on the definition of the measure of rectangle and Lemma 3·3, it follows that

$$\mu\left(B^{\mathbf{a}}(x_{n_k}, r_{n_k})\right) \approx \frac{r_{n_k}^d}{\mathcal{L}(C_{k-1})} \times \mu(C_{k-1}).$$

As remarked in Remark 2 (1) on the choice of G, it can be expected that $\mu(C_{k-1})/\mathcal{L}(C_{k-1}) \leqslant r_{n_k}^{-\epsilon}$ for any $\epsilon > 0$ and $k \geqslant 1$. Thus we have

$$\mu\left(B^{\mathbf{a}}(x_{n_k}, r_{n_k})\right) \ll r_{n_k}^{d-\epsilon}.$$

Recall that

$$s = \min\left\{\frac{d + ja_j - \sum_{1\leqslant i\leqslant j} a_i}{a_j} : 1 \leqslant j \leqslant d\right\} \leqslant \frac{d}{a_1},$$

and that the diameter of the rectangle $B^{\mathbf{a}}(x_{n_k}, r_{n_k})$ is equivalent to $r_{n_k}^{a_1}$. We have

$$\mu\left(B^{\mathbf{a}}(x_{n_k}, r_{n_k})\right) \ll \left(r_{n_k}^{a_1}\right)^{\frac{d-\epsilon}{a_1}} \leqslant \left(r_{n_k}^{a_1}\right)^{s-\epsilon}. \tag{3·3}$$

This convinces us that the measure μ defined above may fulfill the conditions needed in the mass distribution principle.

Now we turn to the μ-measure of an arbitrary ball $B(x,r)$. Assume that r is small enough. If $B(x,r)$ can intersect at most one rectangle $C_k \in \mathbb{F}_k$ for each $k \geqslant 1$, then

$$\mu(B(x,r)) \leqslant \mu(C_k) \longrightarrow 0, \text{ as } k \longrightarrow \infty.$$

Naturally, $\mu(B(x,r)) = 0 \leqslant r^s$.

Let k be the integer such that $B(x,r)$ can intersect only one rectangle $C_k = B^{\mathbf{a}}(x_{n_k}, r_{n_k}) \in \mathbb{F}_k$, while $B(x,r)$ intersects at least two rectangles in $\mathbb{F}_{k+1}$. Let $C_k = B^{\mathbf{a}}(x_{n_k}, r_{n_k})$ be the unique rectangle in $\mathbb{F}_k$ for which the ball $B(x,r)$ intersects. Then

$$\mu\left(B(x,r)\right) = \mu\left(B(x,r) \bigcap C_k\right).$$

We divide three cases to estimate the measure of the ball $B(x,r)$ according to the range of r compared with the sidelength of C_k.

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Case 1. When $r > r_{n_k}^{a_1}$, i.e. r is larger than the diameter of C_k. In this case, by (3·3), it follows that

$$\mu(B(x,r)) \leqslant \mu(C_k) \ll \left(r_{n_k}^{a_1}\right)^{s-\epsilon} \leqslant r^{s-\epsilon}.$$

Case 2. $r_{n_k}^{a_d} < r \leqslant r_{n_k}^{a_1}$, i.e. r is larger than the side-lengh of the cubes $\{B_k^{(i)}\}$ partitioning C_k. In this case, $B(x,r) \bigcap C_k$ may contain many $B_k^{(i)}$. We estimate at first the Lebesgue measure of $B(x,r) \bigcap C_k$. It is clear that

$$\mathcal{L}\left(B(x,r) \bigcap C_k\right) \ll \prod_{i: r_{n_k}^{a_i} \geqslant r} r \cdot \prod_{i: r_{n_k}^{a_i} < r} r_{n_k}^{a_i}.$$

Thus the number of $B_k^{(i)}$ for which $B(x,r) \bigcap C_k$ can intersect is

$$\ll t := \left(\prod_{i: r_{n_k}^{a_i} \geqslant r} r \cdot \prod_{i: r_{n_k}^{a_i} < r} r_{n_k}^{a_i}\right) / (r_{n_k}^{a_d})^d.$$

As a result,

$$\begin{aligned}\mu(B(x,r)) &\leqslant \sum_{B_k^{(i)}: B_k^{(i)} \cap B(x,r) \neq \varnothing} \mu(B_k^{(i)}) \ll \frac{t}{t_k(C_k)} \mu(C_k) \\ &\ll \left(\prod_{i: r_{n_k}^{a_i} \geqslant r} \frac{r}{r_{n_k}^{a_i}}\right) \cdot r_{n_k}^{d-\epsilon}.\end{aligned}$$

Assume that $1 \leqslant j \leqslant d$ be the integer such that $r_{n_k}^{a_{j+1}} < r \leqslant r_{n_k}^{a_j}$, then

$$\mu(B(x,r)) \ll \left(\prod_{i=1}^{j} \frac{r}{r_{n_k}^{a_i}}\right) \cdot r_{n_k}^{d-\epsilon} \leqslant \left(\prod_{i=1}^{j} \frac{r}{r_{n_k}^{a_i}}\right) \cdot r_{n_k}^{d} \cdot r^{-\epsilon},$$

where the last inequality follows from $r \leqslant r_{n_k}^{a_1} \leqslant r_{n_k}$. Then by Lemma 2·2, it follows that

$$\mu\left(B(x,r)\right) \ll r^{s-\epsilon}.$$

Case 3. $r \leqslant r_{n_k}^{a_d}$. Recall that $B(x,r)$ intersects at least two rectangles in $\mathbb{F}_{k+1}$ and these rectangles are all contained in C_k. Denote by $\mathfrak{T}$ the collection of rectangles $B^{\mathbf{a}}(x_{n_{k+1}}, r_{n_{k+1}}) \in \mathbb{F}_{k+1}(C_k)$ intersecting $B(x,r)$. Since $\sharp\mathfrak{T} \geqslant 2$ and by Remark 2 (2), these rectangles are well separated, then we must have

$$B(x_{n_{k+1}}, r_{n_{k+1}}) \subset B(x, 3r)$$

for each $B^{\mathbf{a}}(x_{n_{k+1}}, r_{n_{k+1}}) \in \mathfrak{T}$. Moreover, the balls

$$\{B(x_{n_{k+1}}, r_{n_{k+1}}) \colon B^{\mathbf{a}}(x_{n_{k+1}}, r_{n_{k+1}}) \in \mathfrak{T}\}$$

are disjoint. As a result,

$$\sum_{B^{\mathbf{a}}(x_{n_{k+1}}, r_{n_{k+1}}) \in \mathfrak{T}} r_{n_{k+1}}^d \approx \sum_{B^{\mathbf{a}}(x_{n_{k+1}}, r_{n_{k+1}}) \in \mathfrak{T}} \mathcal{L}(B(x_{n_{k+1}}, r_{n_{k+1}})) \ll r^d.$$

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Therefore

$$\mu\left(B(x,r)\right) \ll \sum_{B^{\mathbf{a}}(x_{n_{k+1}},r_{n_{k+1}})\in\mathfrak{T}} \frac{r_{n_{k+1}}^d}{\mathcal{L}(C_k)} \times \mu(C_k) \ll \frac{r^d}{\mathcal{L}(C_k)} \times r_{n_k}^{d-\epsilon}$$
$$\leqslant \left(\frac{r^d}{\prod_{j=1}^d r_{n_k}^{a_j}} \times r_{n_k}^d\right) \cdot r^{-\epsilon}.$$

Then by Lemma 2·3, it follows that

$$\mu\left(B(x,r)\right) \ll r^{s-\epsilon}.$$

This completes the proof by applying the mass distribution principle (Proposition 3·1).

4. *Proof of Theorem* 1·3: *Hausdorff measure*

In this section, we investigate the Hausdorff measure of $W_{\mathbf{a}}$ at the critical value s. We will use almost the same idea developed in [**1**] and combine the *partition* process to show that $\mathcal{H}^s(W_{\mathbf{a}}) = \infty$ for any $\mathbf{a} = (a_1, \ldots, a_d)$ with $1 \leqslant a_1 \leqslant \cdots \leqslant a_d$ and $a_d > 1$.

We refine the Cantor set $\mathbb{F}_\infty$ defined in Section 3 to a new one $\mathbb{G}_\infty$ for which each level contains sufficiently many sublevels. At first, fix a large positive number $\eta > 0$. The final goal is to show that $\mathcal{H}^s(W_{\mathbf{a}}) \geqslant \eta$.

The first level $\mathbb{G}_1$: recall $B_0 = [0,1]^d$. Let $\ell_1 = \lfloor \eta \rfloor$. Set the first level of the Cantor set $\mathbb{F}_\infty$ constructed in Section 3 as the first sublevel of $\mathbb{G}_1$, i.e.

$$\mathbb{G}_1(B_0, 1) = \bigcup_{B(x_n,r_n)\in K_{G,B_0}} B^{\mathbf{a}}(x_n, r_n).$$

To save the notation, we define $\Xi_1(B_0, 1) = \{B_0\}$. Then the first sublevel of the first level $\mathbb{G}_1$ is just

$$\mathbb{G}_1(B_0, 1) = \bigcup_{B(x,r)\in\Xi_1(B_0,1)} \bigcup_{B(x_n,r_n)\in K_{G,B(x,r)}} B^{\mathbf{a}}(x_n, r_n).$$

To remind the reader, we re-state some notation: $B(x, r^a)$ denotes a ball with radius r^a for a real positive number a, while $B^{\mathbf{a}}(x, r)$ denotes a rectangle with side-length $(r^{a_1}, \ldots, r^{a_d})$ for a d-dimensional vector $\mathbf{a} = (a_1, \ldots, a_d)$ and $\mathbb{G}_1(B_0, 1)$ is also used to denote the collection of rectangles contained in it.

It should be emphasized that since $a_d > 1$, when applying the $K_{G,B}$-lemma to B_0, we require G is so large that $\mathcal{L}(B^{\mathbf{a}}(x_n, r_n)) \ll r_n^d/\eta$ for any $n \geqslant G$. In this way, the rectangle $B^{\mathbf{a}}(x_n, r_n)$ takes a small part in the ball $B(x_n, r_n)$. Thus there is enough space left in

$$\frac{1}{2}B_0 \setminus \bigcup_{B^{\mathbf{a}}(x_n,r_n)\in\mathbb{G}_1(B_0,1)} 4B^{\mathbf{a}}(x_n, r_n).$$

This enables us to construct the next sub-level of the first level $\mathbb{G}_1$. This is the main idea for constructing the sublevels of $\mathbb{G}_1$. Now we give the details.

Estimate the measure of the set

$$R_1(B_0, 1) := \frac{1}{2}B_0 \setminus \bigcup_{B^{\mathbf{a}}(x_n,r_n)\in\mathbb{G}_1(B_0,1)} 4B^{\mathbf{a}}(x_n, r_n).$$

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Since $\mathcal{L}(B^{\mathbf{a}}(x_n, r_n)) \ll r_n^d/\eta$ for any $n \geqslant G$, by the packing properties of the $K_{G,B}$-Lemma, we get

$$\sum_{B^{\mathbf{a}}(x_n,r_n)\in\mathbb{G}_1(B_0,1)} \mathcal{L}\left(B^{\mathbf{a}}(x_n, r_n)\right) \ll \frac{1}{\eta} \sum_{B(x_n,r_n)\in K_{G,B_0}} r_n^d \ll \frac{1}{\eta}\mathcal{L}\left(\frac{1}{2}B_0\right).$$

Moreover, by the fact that $\lim_{n\to\infty} \mathcal{L}(B^{\mathbf{a}}(x_n, r_n))/r_n^d = 0$, we can choose G sufficiently large such that

$$\sum_{B^{\mathbf{a}}(x_n,r_n)\in\mathbb{G}_1(B_0,1)} \mathcal{L}(4B^{\mathbf{a}}(x_n, r_n)) \leqslant \frac{1}{2\eta}\mathcal{L}\left(\frac{1}{2}B_0\right).$$

Now let $r > 0$ be sufficiently small compared with the sidelength of the rectangles in the previous sublevel. At first, for each $x \in R_1(B_0, 1)$, take a ball $B(x, 3r)$ centered at x with radius $3r$. Secondly, using the $5r$ covering lemma to get a collection $\Xi_1(B_0, 2)$ of well separated balls $B(x, r)$ in the sense that for any two distinct centers x and x',

$$B(x, 3r) \bigcap B(x', 3r) = \varnothing,\ B(x, 3r) \subset B_0,$$

and moreover

$$\sum_{B(x,r)\in\Xi_1(B_0,2)} \mathcal{L}(B(x, r)) \approx \mathcal{L}\left(\frac{1}{2}B_0\right). \tag{4·1}$$

At the same time, since r is sufficiently small, we also have that for any $B(x, r) \in \Xi_1(B_0, 2)$,

$$B(x, 3r) \bigcap \bigcup_{B^{\mathbf{a}}(x_n,r_n)\in\mathbb{G}_1(B_0,1)} 3B^{\mathbf{a}}(x_n, r_n) = \varnothing.$$

Thirdly, applying $K_{G,B}$-lemma to each $B(x, r) \in \Xi_1(B_0, 2)$ to get a collection of balls $B(x_n, r_n)$. Then by the packing property of the $K_{G,B}$-lemma, we have

$$\sum_{B(x_n,r_n)\in K_{G,B(x,r)}} r_n^d \approx r^d. \tag{4·2}$$

Thus combining (4·1) and (4·2), we have

$$\sum_{B(x,r)\in\Xi_1(B_0,2)} \sum_{B(x_n,r_n)\in K_{G,B(x,r)}} r_n^d \approx \mathcal{L}\left(\frac{1}{2}B_0\right). \tag{4·3}$$

At last, shrinking each ball $B(x_n, r_n)$ obtained above to a rectangle $B^{\mathbf{a}}(x_n, r_n)$. Then set the second sublevel of $\mathbb{G}_1$ as

$$\mathbb{G}_1(B_0, 2) := \bigcup_{B(x,r)\in\Xi_1(B_0,2)} \bigcup_{B(x_n,r_n)\in K_{G,B(x,r)}} B^{\mathbf{a}}(x_n, r_n).$$

It should also be noticed that

$$\begin{aligned}\sum_{B^{\mathbf{a}}(x_n,r_n)\in\mathbb{G}_1(B_0,2)} \mathcal{L}\left(B^{\mathbf{a}}(x_n, r_n)\right) &= \sum_{B(x,r)\in\Xi_1(B_0,2)} \sum_{B(x_n,r_n)\in K_{G,B(x,r)}} \mathcal{L}\left(B^{\mathbf{a}}(x_n, r_n)\right) \\ &\ll \frac{1}{\eta} \sum_{B(x,r)\in\Xi(B_0,2)} \sum_{B(x_n,r_n)\in K_{G,B(x,r)}} r_n^d \approx \frac{1}{\eta}\mathcal{L}\left(\frac{1}{2}B_0\right). \end{aligned} \tag{4·4}$$

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Hence, when applying the $K_{G,B}$-lemma to each $B(x,r)\in\Xi_1(B_0,2)$, we can require that G is large enough that

$$\sum_{B^{\mathbf{a}}(x_n,r_n)\in\mathbb{G}_1(B_0,2)}\mathcal{L}\left(4B^{\mathbf{a}}(x_n,r_n)\right)\leqslant\frac{1}{2\eta}\mathcal{L}\left(\frac{1}{2}B_0\right).$$

Now we construct other sublevels by induction. Assume that for each $1\leqslant\ell<\ell_1$, the sublevels $\mathbb{G}_1(B_0,\ell)$ have been constructed of the form

$$\mathbb{G}_1(B_0,\ell)=\bigcup_{B(x,r)\in\Xi_1(B_0,\ell)}\ \bigcup_{B(x_n,r_n)\in K_{G,B(x,r)}}B^{\mathbf{a}}(x_n,r_n),$$

where $\Xi_1(B_0,\ell)$ is a collection of balls and

$$\sum_{B(x,r)\in\Xi_1(B_0,\ell)}\ \sum_{B(x_n,r_n)\in K_{G,B(x,r)}}r_n^d\approx\mathcal{L}\left(\frac{1}{2}B_0\right),\tag{4·5}$$

$$\sum_{B^{\mathbf{a}}(x_n,r_n)\in\mathbb{G}_1(B_0,\ell)}\mathcal{L}(4B^{\mathbf{a}}(x_n,r_n))\leqslant\frac{1}{2\eta}\mathcal{L}\left(\frac{1}{2}B_0\right).\tag{4·6}$$

Now we construct the $(\ell+1)$th sublevel of the first level. Estimate the measure of the set

$$R_1(B_0,\ell):=\frac{1}{2}B_0\setminus\bigcup_{j=1}^{\ell}\ \bigcup_{B^{\mathbf{a}}(x_n,r_n)\in\mathbb{G}_1(B_0,j)}4B^{\mathbf{a}}(x_n,r_n)\ .$$

By (4·6), we have

$$\sum_{j=1}^{\ell}\ \sum_{B^{\mathbf{a}}(x_n,r_n)\in\mathbb{G}_1(B_0,j)}\mathcal{L}(4B^{\mathbf{a}}(x_n,r_n)))\leqslant\frac{\ell}{2\eta}\mathcal{L}\left(\frac{1}{2}B_0\right)\leqslant\frac{1}{2}\mathcal{L}\left(\frac{1}{2}B_0\right).$$

This means that there is still large space left in B_0.

Now construct $\mathbb{G}_1(B_0,\ell+1)$ with the same way as for $\mathbb{G}_1(B_0,2)$. Let $r>0$ be sufficiently small compared with the side-length of the rectangles in all the previous sublevels. For each $x\in R_1(B_0,\ell)$, take a ball $B(x,3r)$ centered at x with radius $3r$. Secondly, using the $5r$ covering lemma to get a collection $\Xi_1(B_0,\ell+1)$ of well separated balls $B(x,r)$ in the sense that for any two distinct centers x and x',

$$B(x,3r)\cap B(x',3r)=\varnothing,\ B(x,3r)\subset B_0,$$

$$B(x,3r)\cap\bigcup_{j=1}^{\ell}\ \bigcup_{B^{\mathbf{a}}(x_n,r_n)\in\mathbb{G}_1(B_0,j)}3B^{\mathbf{a}}(x_n,r_n)=\varnothing,$$

and moreover

$$\sum_{B(x,r)\in\Xi_1(B_0,\ell+1)}\mathcal{L}(B(x,r))\approx\mathcal{L}\left(\frac{1}{2}B_0\right).$$

Thirdly, apply $K_{G,B}$-lemma to each $B(x,r)\in\Xi_1(B_0,\ell+1)$ to get a collection of balls $B(x_n,r_n)$. Finally, shrink each ball $B(x_n,r_n)$ obtained above to a rectangle $B^{\mathbf{a}}(x_n,r_n)$. Then set the $(\ell+1)$th sublevel of the first level as

$$\mathbb{G}_1(B_0,\ell+1):=\bigcup_{B(x,r)\in\Xi_1(B_0,\ell+1)}\ \bigcup_{B(x_n,r_n)\in K_{G,B(x,r)}}B^{\mathbf{a}}(x_n,r_n).$$

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We can also obtain the estimate that

$$\sum_{B(x,r)\in\Xi_1(B_0,\ell+1)}\ \sum_{B(x_n,r_n)\in K_{G,B(x,r)}}\mathcal{L}\left(4B^{\mathbf{a}}(x_n,r_n)\right)\leqslant\frac{1}{2\eta}\mathcal{L}\left(\frac{1}{2}B_0\right).$$

Finally, define the first level $\mathbb{G}_1$ as

$$\begin{aligned}\mathbb{G}_1=\mathbb{G}_1(B_0)=\bigcup_{\ell=1}^{\ell_1}\mathbb{G}_1(B_0,\ell)&=\bigcup_{\ell=1}^{\ell_1}\ \bigcup_{B^{\mathbf{a}}(x_n,r_n)\in\mathbb{G}_1(B_0,\ell)}B^{\mathbf{a}}(x_n,r_n)\\&=\bigcup_{\ell=1}^{\ell_1}\ \bigcup_{B(x,r)\in\Xi_1(B_0,\ell)}\ \bigcup_{B(x_n,r_n)\in K_{G,B(x,r)}}B^{\mathbf{a}}(x_n,r_n).\end{aligned}$$

Finally, we give a remark on the separation properties of the rectangles in $\mathbb{G}_1$: for any two different rectangles $B^{\mathbf{a}}(x_n,r_n)$ and $B^{\mathbf{a}}(x_m,r_m)$ in $\mathbb{G}_1=\bigcup_{1\leqslant j\leqslant\ell_1}\mathbb{G}_1(B_0,j)$, if they are in the same sublevel $\mathbb{G}_1(B_0,j)$ for some j, we have

$$B(x_n,3r_n)\cap B(x_m,3r_m)=\varnothing;$$

if they are in different sublevels, say $B^{\mathbf{a}}(x_n,r_n)\in\mathbb{G}_1(B_0,j)$ and $B^{\mathbf{a}}(x_m,r_m)\in\mathbb{G}_1(B_0,j')$ with $j<j'$, then

$$3B^{\mathbf{a}}(x_n,r_n)\cap B(x_m,r_m)=\varnothing.$$

General levels

Let $C_k=B^{\mathbf{a}}(x_{n_k},r_{n_k})\in\mathbb{G}_k$. Divide C_k into cubes $\{B_k^{(i)}\}_{1\leqslant i\leqslant t_k(C_k)}$ as we did in the proof of Theorem 1·2. Let $\ell_{k+1}=\lfloor\eta/\mathcal{L}(C_k)\rfloor$. Then replace the role of B_0 by $B_k^{(i)}$ for each $1\leqslant i\leqslant t_k(C_k)$ and do the same construction to $B_k^{(i)}$ as we did for B_0, we get a collection of rectangles:

$$\mathbb{G}_{k+1}(B_k^{(i)})=\bigcup_{j=1}^{\ell_{k+1}}\ \bigcup_{B(x,r)\in\Xi_{k+1}(B_k^{(i)},j)}\ \bigcup_{B(x_n,r_n)\in K_{G,B(x,r)}}B^{\mathbf{a}}(x_n,r_n),$$

where the definition of $\Xi_{k+1}(B_k^{(i)},j)$ is similar to $\Xi_1(B_0,j)$.

We define the desired Cantor set as

$$\mathbb{G}_\infty=\bigcap_{k\geqslant1}\ \bigcup_{B^{\mathbf{a}}(x_n,r_n)\in\mathbb{G}_k}B^{\mathbf{a}}(x_n,r_n).$$

In general, we have:

LEMMA 4·1.

(i) Volume condition

$$\begin{aligned}\sum_{B(x,r)\in\Xi_{k+1}(B_k^{(i)},j)}\ \sum_{B(x_n,r_n)\in K_{G,B(x,r)}}r_n^d&\approx\sum_{B(x,r)\in\Xi_{k+1}(B_k^{(i)},j)}\mathcal{L}(B(x,r))\\&\approx\mathcal{L}(B_k^{(i)}),\ \textit{for all}\ 1\leqslant j\leqslant\ell_{k+1};\end{aligned}$$

(ii) Separation condition: *for any two different* $B^{\mathbf{a}}(x_n,r_n)$ *and* $B^{\mathbf{a}}(x_m,r_m)$ *in* $\bigcup_{1\leqslant j\leqslant\ell_{k+1}}\mathbb{G}_{k+1}(B_k^{(i)},j)$, *if they are in the same sublevel* $\mathbb{G}_{k+1}(B_k^{(i)},j)$ *for some* j, *we have*

$$B(x_n,3r_n)\cap B(x_m,3r_m)=\varnothing;$$

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if they are in different sublevels, say $B^{\mathbf{a}}(x_n, r_n) \in \mathbb{G}_{k+1}(B_k^{(i)}, j)$ *and* $B^{\mathbf{a}}(x_m, r_m) \in \mathbb{G}_{k+1}(B_k^{(i)}, j')$ *with* $j < j'$, *then*

$$3B^{\mathbf{a}}(x_n, r_n) \cap B(x_m, r_m) = \varnothing.$$

Remark 3. It should be mentioned that by the separation condition in Lemma 4·1, when a ball $B(x, r)$ can intersect two elements in $\mathbb{G}_{k+1}(B_k^{(i)})$, say $B^{\mathbf{a}}(x_n, r_n)$ and $B^{\mathbf{a}}(x_m, r_m)$, we have:

(i) if they are in the same sublevel $\mathbb{G}_{k+1}(B_k^{(i)}, j)$ for some j, we have

$$r \geqslant \max\{r_n, r_m\};$$

(ii) if they are in different sublevels, say $B^{\mathbf{a}}(x_n, r_n) \in \mathbb{G}_{k+1}(B_k^{(i)}, j)$ and $B^{\mathbf{a}}(x_m, r_m) \in \mathbb{G}_{k+1}(B_k^{(i)}, j')$ with $j < j'$, then

$$r \geqslant r_n^{a_d}.$$

4·1. *Mass distribution on* $\mathbb{G}_\infty$

We define a probability measure supported on $\mathbb{G}_\infty$ as follows. At first let $\mu(B_0) = 1$. For each rectangle $B^{\mathbf{a}}(x_{n_1}, r_{n_1}) \in \mathbb{G}_1$, define

$$\begin{aligned}\mu\left(B^{\mathbf{a}}(x_{n_1}, r_{n_1})\right) &= \frac{r_{n_1}^d}{\sum_{B^{\mathbf{a}}(x_m, r_m) \in \mathbb{G}_1} r_m^d} \\ &= \frac{r_{n_1}^d}{\sum_{\ell=1}^{\ell_1} \sum_{B(x,r) \in \Xi_1(B_0, \ell)} \sum_{B(x_m, r_m) \in K_{G, B(x,r)}} r_m^d}.\end{aligned}$$

Assume that the measure of the rectangles in the $(k-1)$-th level $\mathbb{G}_{k-1}$ of the Cantor set has been defined. Now we define the measure on the rectangles in the kth level $\mathbb{G}_k$. For each $C_k = B^{\mathbf{a}}(x_{n_k}, r_{n_k}) \in \mathbb{G}_k$, let $C_{k-1} = B^{\mathbf{a}}(x_{n_{k-1}}, r_{n_{k-1}}) \in \mathbb{G}_{k-1}$ be the mother rectangle in $\mathbb{G}_{k-1}$ for which $C_k \subset C_{k-1}$. Then define

$$\begin{aligned}\mu\left(B^{\mathbf{a}}(x_{n_k}, r_{n_k})\right) &= \frac{r_{n_k}^d}{\sum_{i=1}^{t_{k-1}(C_{k-1})} \sum_{B^{\mathbf{a}}(x_n, r_n) \in \mathbb{G}_k(B_{k-1}^{(i)})} r_n^d} \times \mu(C_{k-1}) \\ &= \frac{r_{n_k}^d}{\sum_{i=1}^{t_{k-1}(C_{k-1})} \sum_{\ell=1}^{\ell_k} \sum_{B(x,r) \in \Xi_k(B_{k-1}^{(i)}, \ell)} \sum_{B(x_n, r_n) \in K_{G, B(x,r)}} r_n^d} \times \mu(C_{k-1}). \quad (4{\cdot}7)\end{aligned}$$

Then μ can be expanded uniquely into a probability measure supported on $\mathbb{G}_\infty$.

Finally, by Lemma 4·1, we state an estimate on the denominator of the formula in (4·7):

$$\sum_{i=1}^{t_{k-1}(C_{k-1})} \sum_{\ell=1}^{\ell_k} \sum_{B(x,r) \in \Xi_k(B_{k-1}^{(i)}, \ell)} \sum_{B(x_n, r_n) \in K_{G, B(x,r)}} r_n^d \approx \ell_k \mathcal{L}(C_{k-1}). \quad (4{\cdot}8)$$

4·2. *Mass distribution principle*

We estimate the μ-measure of an arbitrary ball $B(x, r)$ and then apply the mass distribution principle to arrive at the desired result.

At first, we estimate the measure of a rectangle $C_k = B^{\mathbf{a}}(x_{n_k}, r_{n_k}) \in \mathbb{G}_k$. Let $C_{k-1} \in \mathbb{G}_{k-1}$ be the mother rectangle in $\mathbb{G}_{k-1}$ containing C_k. By (4·8) and then the choice of $\ell_k = \lfloor \eta / \mathcal{L}(C_{k-1}) \rfloor$, we have that

$$\mu(B^{\mathbf{a}}(x_{n_k}, r_{n_k})) \approx \frac{r_{n_k}^d}{\ell_k \mathcal{L}(C_{k-1})} \times \mu(C_{k-1}) \ll \frac{r_{n_k}^d}{\eta}. \quad (4{\cdot}9)$$

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Recall that

$$s = \min\left\{\frac{d + ja_j - \sum_{1\leqslant i\leqslant j} a_i}{a_j} : 1 \leqslant j \leqslant d\right\} \leqslant \frac{d}{a_1}$$

and that the diameter of C_k is equivalent to $r_{n_k}^{a_1}$. We have

$$\mu(C_k) \ll \frac{r_{n_k}^d}{\eta} \ll \frac{\left(r_{n_k}^{a_1}\right)^s}{\eta}. \tag{4·10}$$

Now we turn to the μ-measure of an arbitrary ball $B(x, r)$. Assume that r is small enough. If $B(x, r)$ can intersect at most one rectangle $B_k \in \mathbb{G}_k$ for each $k \geqslant 1$, then

$$\mu(B(x, r)) \leqslant \mu(B_k) \longrightarrow 0, \text{ as } k \longrightarrow \infty.$$

So, in this case, there is nothing to prove.

Now let k be the integer such that $B(x, r)$ can intersect only one rectangle in $\mathbb{G}_k$, while $B(x, r)$ intersects at least two rectangles in $\mathbb{G}_{k+1}$. Let $C_k = B^{\mathbf{a}}(x_{n_k}, r_{n_k})$ be the unique rectangle in $\mathbb{G}_k$ for which the ball $B(x, r)$ intersects. Then

$$\mu\left(B(x, r)\right) = \mu\left(B(x, r) \bigcap C_k\right).$$

We divide three cases to estimate the measure of the ball $B(x, r)$ according to the range of r compared with the side-length of C_k.

Case (a) When $r > r_{n_k}^{a_1}$, i.e. r is larger than the diameter of C_k. In this case, by (4·10), it follows that

$$\mu\left(B(x, r)\right) \leqslant \mu(C_k) \ll \left(r_{n_k}^{a_1}\right)^s \frac{1}{\eta} \leqslant \frac{r^s}{\eta}.$$

Case (b) $r_{n_k}^{a_d} < r \leqslant r_{n_k}^{a_1}$, i.e. r is larger than the sidelengh of the cubes $\{B_k^{(i)}\}$ partitioning C_k. In this case, $B(x, r) \bigcap C_k$ may contain many $B_k^{(i)}$. With the same argument as in Section 3, the number of $B_k^{(i)}$ for which $B(x, r) \bigcap C_k \neq \varnothing$ is

$$\ll t := \left(\prod_{i:r_{n_k}^{a_i}\geqslant r} r \cdot \prod_{i:r_{n_k}^{a_i}<r} r_{n_k}^{a_i}\right) / (r_{n_k}^{a_d})^d.$$

As a result, together with (4·9), we have

$$\begin{aligned}\mu(B(x, r)) &\leqslant \sum_{B_k^{(i)}:B_k^{(i)}\bigcap B(x,r)\neq\varnothing} \mu(B_k^{(i)}) \ll \frac{t}{t_k(C_k)}\mu(C_k)\\ &\ll \left(\prod_{i:r_{n_k}^{a_i}\geqslant r} \frac{r}{r_{n_k}^{a_i}}\right) \cdot \frac{r_{n_k}^d}{\eta}.\end{aligned}$$

Then, by Lemma 2·2, it follows that

$$\mu(B(x, r)) \ll \frac{r^s}{\eta}.$$

Case (c) $r \leqslant r_{n_k}^{a_d}$. In this case, the ball $B(x, r)$ can intersect only one $B_k^{(i)}$ for some $1 \leqslant i \leqslant t_k(C_k)$, since the cubes $B_k^{(i)}$ are well separated. So all the rectangles in $\mathbb{G}_{k+1}$ having nonempty intersections with $B(x, r)$ are all contained in $\mathbb{G}_{k+1}(B_k^{(i)})$.

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Now we consider the rectangles which can have nonempty intersection with $B(x,r)$. Let ℓ_0 be the smallest integer such that there exists at least one rectangle $B^{\mathbf{a}}(x_{n_{k+1}}, r_{n_{k+1}}) \in \mathbb{G}_{k+1}(B_k^{(i)}, \ell_0)$ intersecting $B(x,r)$.

At first, we consider the case when the ball $B(x,r)$ can intersect elements in other sublevels. Assume that $B^{\mathbf{a}}(x'_{n_{k+1}}, r'_{n_{k+1}}) \in \mathbb{G}_{k+1}(B_k^{(i)}, \ell)$ for some $\ell > \ell_0$ is an element having nonempty intersection with $B(x,r)$. Since

$$B(x'_{n_{k+1}}, 3r'_{n_{k+1}}) \bigcap \mathbb{G}_{k+1}(B_k^{(i)}, \ell_0) = \varnothing$$

and

$$B(x,r) \bigcap B(x'_{n_{k+1}}, r'_{n_{k+1}}) \neq \varnothing,\ B(x,r) \bigcap \mathbb{G}_{k+1}(B_k^{(i)}, \ell_0) \neq \varnothing,$$

we have that

$$r \geqslant r'_{n_{k+1}}.$$

This means that for any $\ell > \ell_0$ and for any rectangle $B^{\mathbf{a}}(x'_{n_{k+1}}, r'_{n_{k+1}}) \in \mathbb{G}_{k+1}(B_k^{(i)}, \ell)$ intersecting $B(x,r)$, we have

$$B(x'_{n_{k+1}}, r'_{n_{k+1}}) \subset B(x, 3r).$$

Therefore,

$$\sum_{B^{\mathbf{a}}(x'_{n_{k+1}}, r'_{n_{k+1}}) \in \mathbb{G}_{k+1}(B_k^{(i)}, \ell),\, B^{\mathbf{a}}(x'_{n_{k+1}}, r'_{n_{k+1}}) \cap B(x,r) \neq \varnothing} \left(r'_{n_{k+1}}\right)^d \ll r^d.$$

As a result, by (4·8) and then (4·10),

$$\begin{aligned}
&\sum_{\ell > \ell_0} \sum_{B^{\mathbf{a}}(x'_{n_{k+1}}, r'_{n_{k+1}}) \in \mathbb{G}_{k+1}(B_k^{(i)}, \ell),\, B^{\mathbf{a}}(x'_{n_{k+1}}, r'_{n_{k+1}}) \cap B(x,r) \neq \varnothing} \mu\left(B^{\mathbf{a}}(x'_{n_{k+1}}, r'_{n_{k+1}})\right) \\
&\qquad \ll \sum_{\ell > \ell_0} \frac{r^d}{\ell_{k+1} \mathcal{L}(C_k)} \times \mu(C_k) \leqslant \frac{r^d}{\mathcal{L}(C_k)} \cdot \mu(C_k) \\
&\qquad \ll \frac{r^d}{\mathcal{L}(C_k)} \cdot \frac{r_{n_k}^d}{\eta},
\end{aligned}$$

which is less than r^s/η by Lemma 2·3.

This means the contributions of the measure of the rectangles in other sublevels with nonempty intersection with $B(x,r)$ to the measure of $B(x,r)$ is less than r^s/η. Thus in the following we need only focus on the contributions of the measure of the rectangles in the sublevel $\mathbb{G}_{k+1}(B_k^{(i)}, \ell_0)$ to the measure of $B(x,r)$.

Now we distinguish two cases to estimate the measure of $B(x,r)$.

Case (i) $B(x,r)$ can intersect only one element $B^{\mathbf{a}}(x_{n_{k+1}}, r_{n_{k+1}})$ in $\mathbb{G}_{k+1}(B_k^{(i)}, \ell_0)$.

In this case, since $B(x,r)$ should intersect at least one rectangle in $\mathbb{G}_{k+1}(B_k^{(i)}, \ell)$ for some $\ell > \ell_0$, then by Remark 3, we have

$$r > r_{n_{k+1}}^{a_d}.$$

Using the same arguments as in *Case* a and *Case* b in this subsection, we obtain that

$$\mu\left(B(x,r) \bigcap B^{\mathbf{a}}(x_{n_{k+1}}, r_{n_{k+1}})\right) \ll \frac{r^s}{\eta}.$$

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Case (ii) $B(x,r)$ can intersect more than one rectangle in $\mathbb{G}_{k+1}(B_k^{(i)},\ell_0)$. Denote by $\mathfrak{T}$ the collection of those rectangles $B^{\mathbf{a}}(x_{n_{k+1}},r_{n_{k+1}})\in\mathbb{G}_{k+1}(B_k^{(i)},\ell_0)$ intersecting $B(x,r)$. Since $\sharp\mathfrak{T}\geqslant 2$ and these rectangles are well separated, we must have

$$B(x_{n_{k+1}},r_{n_{k+1}})\subset B(x,3r)$$

for each $B^{\mathbf{a}}(x_{n_{k+1}},r_{n_{k+1}})\in\mathfrak{T}$. Moreover, the balls

$$\{B(x_{n_{k+1}},r_{n_{k+1}})\colon B^{\mathbf{a}}(x_{n_{k+1}},r_{n_{k+1}})\in\mathfrak{T}\}$$

are disjoint. As a result,

$$\sum_{B^{\mathbf{a}}(x_{n_{k+1}},r_{n_{k+1}})\in\mathfrak{T}} r_{n_{k+1}}^d\approx\sum_{B^{\mathbf{a}}(x_{n_{k+1}},r_{n_{k+1}})\in\mathfrak{T}}\mathcal{L}\left(B(x_{n_{k+1}},r_{n_{k+1}})\right)\ll r^d.$$

Therefore, by the choice of $\ell_{k+1}=\lfloor\eta/\mathcal{L}(C_k)\rfloor$ and Lemma 2·3, we have that

$$\begin{aligned}\mu\left(B(x,r)\right)&\ll\sum_{B^{\mathbf{a}}(x_{n_{k+1}},r_{n_{k+1}})\in\mathfrak{T}}\frac{r_{n_{k+1}}^d}{\ell_{k+1}\mathcal{L}(C_k)}\cdot\mu(C_k)+\frac{r^s}{\eta}\\&\ll\frac{r^d}{\ell_{k+1}\mathcal{L}(C_k)}\cdot\frac{r_{n_k}^d}{\eta}+\frac{r^s}{\eta}\ll\frac{r^s}{\eta}.\end{aligned}$$

This completes the proof.

5. *Applications to Diophantine approximation*

5·1. *Simultaneous Diophantine approximation*

Let $\tau=(\tau_1,\dots,\tau_d)$ be a vector of strictly positive numbers and W_τ be the set of points $x=(y_1,\dots,y_d)\in[0,1]^d$ for which the inequalities

$$\|qy_i\|<q^{-\tau_i},\quad 1\leqslant i\leqslant d,$$

hold for infinitely many integers $q\in\mathbb{N}$.

COROLLARY 5·1. *Suppose that* $1/d\leqslant\tau_1\leqslant\cdots\leqslant\tau_d$, *then*

$$\dim_{\mathrm{H}}W_\tau=\min\left\{\frac{d+1+j\tau_j-\sum_{i=1}^j\tau_i}{1+\tau_j}\colon 1\leqslant j\leqslant d\right\}.$$

Proof. It is clear that

$$W_\tau=\bigcap_{q_0=1}^{\infty}\bigcup_{q\geqslant q_0}\bigcup_{0\leqslant p_1,\cdots,p_d\leqslant q}E(p_1,\dots,p_d;q),$$

where

$$E(p_1,\dots,p_d;q):=\left\{(y_1,\dots,y_d)\in[0,1]^d\colon|y_i-p_i/q|<q^{-1-\tau_i},\,1\leqslant i\leqslant d\right\}.$$

Fix an integer j with $1\leqslant j\leqslant d$. We use a collection of cubes to cover the rectangle $E(p_1,\dots,p_d;q)$ instead of using itself. Then, to cover this rectangle, it needs

$$4^j\prod_{1\leqslant i\leqslant j}\frac{q^{-1-\tau_i}}{q^{-1-\tau_j}}$$

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cubes of side-length $q^{-(1+\tau_j)}$. As a result,

$$\mathcal{H}^s(W_\tau) \leqslant \liminf_{q_0\to\infty} \sum_{q\geqslant q_0} 4^j (q+1)^d \cdot \prod_{1\leqslant i\leqslant j} \frac{q^{-1-\tau_i}}{q^{-1-\tau_j}} \cdot \left(q^{-1-\tau_j}\right)^s.$$

Thus for any $s > s_j$, we have $\mathcal{H}^s(W_\tau) < \infty$. This gives that $\dim_{\mathrm{H}} W_\tau \leqslant s_j$.

The Khintchine–Groshev theorem [**7**, **10**] (or Dirichlet theorem [**17**, **18**]) states that the Lebesgue measure of the set of points $x = (y_1, \ldots, y_d) \in [0,1]^d$ for which the inequalities

$$\|qy_i\| < q^{-\frac{1}{d}}, \quad 1 \leqslant i \leqslant d,$$

hold for infinitely many integers $q \in \mathbb{N}$ is one. Let q be a positive integer and $(p_1, \ldots, p_d)$ integer vectors in $[0, q]^d$. If we treat the rational vectors $(p_1/q, \ldots, p_d/q)$ and positive numbers $q^{-(1+\frac{1}{d})}$ as the centers and radii of the balls $B(x_n, r_n)$ in Theorem 1·2, respectively, then the desired lower bound of $\dim_{\mathrm{H}} W_\tau$ is easily deduced on combining the Khintchine–Groshev theorem and Theorem 1·2.

Remark 4. Let Q be an infinite subset of $\mathbb{N}$ and $\tau_1 + \cdots + \tau_d \geqslant 1$. In [**16**], Rynne showed a general result, that is the Hausdorff dimension of the set

$$\left\{x = (y_1, \cdots, y_d) \in [0,1]^d \colon \|qy_i\| < q^{-\tau_i},\ 1 \leqslant i \leqslant d,\ \text{i.o. } q \in Q\right\}$$

is $\min_{1\leqslant j\leqslant d}\{\frac{d+\nu(Q)+j\tau_j-\sum_{i=1}^j \tau_i}{1+\tau_j}\}$, where $\nu(Q) = \inf\{\nu \in [0,1] \colon \sum_{q\in Q} q^{-\nu} < \infty\}$.

5·2. *High dimensional Duffin–Schaeffer conjecture*

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a positive function (without the monotonicity assumption). We use W_ψ to denote the collection of points $y = (y_1, \ldots, y_d) \in [0,1]^d$ such that

$$|y_i - p_i/q| < \psi(q)/q, 1 \leqslant i \leqslant d$$

for infinitely many $q \in \mathbb{N}$ and $(p_1, \ldots, p_d) \in \mathbb{Z}^d$ with $(p_i, q) = 1,\ 1 \leqslant i \leqslant d$. It is conjectured that when

$$\sum_{q\in\mathbb{N}} \left(\frac{\psi(q)\phi(q)}{q}\right)^d = \infty, \tag{5·1}$$

where ϕ is the Euler's totient function, one would have $\mathcal{L}(W_\psi) = 1$. When $d = 1$, this is the famous Duffin–Schaeffer conjecture, and it is still a major open problem in metric number theory. For $d \geqslant 2$, the conjecture was formally stated by Sprindžuk [**18**] and solved by Pollington and Vaughan [**15**].

Combining our main result and that in [**15**], we have the following Hausdorff measure version of the Duffin–Schaeffer conjecture.

COROLLARY 5·2. *Let* $d \geqslant 2$. *Assume that* (5·1) *holds. For any real vector* $\mathbf{a} = (a_1, \ldots, a_d)$ *with* $1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_d$, *define*

$$W_\psi^{\mathbf{a}} = \left\{y \in [0,1]^d \colon |y_i - p_i/q| < (\psi(q)/q)^{a_i},\ 1 \leqslant i \leqslant d,\ \textit{for infinitely many } q \in \mathbb{N} \textit{ and } (p_1, \ldots, p_d) \in \mathbb{Z}^d \textit{ with } (p_i, q) = 1,\ 1 \leqslant i \leqslant d\right\}.$$

Then

$$\mathcal{H}^s(W_\psi^{\mathbf{a}}) = \mathcal{H}^s([0,1]^d), \quad s = \min\left\{\frac{d + ja_j - \sum_{i=1}^j a_i}{a_j} \colon 1 \leqslant j \leqslant d\right\}.$$

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Proof. When $a_1 = \cdots = a_d = 1$, we have $s = d$. Then it follows from the result of Pollington & Vaughan. When $a_d > 1$, we have $s < d$. So, $\mathcal{H}^s([0,1]^d) = \infty$. Then by Theorem 1·3, the desired result follows.

6. *Final remark*

The setting of Beresnevich and Velani [**1**] is that under a full measure theoretic statement of limsup sets generated by balls, one derives the size of limsup sets generated by shrunk balls. Informally speaking, it is from balls to balls. While our setting is from balls to rectangles.

So, there is still another natural setting, namely from rectangles to rectangles. Assume that the set

$$W^{\mathbf{1}} := \left\{ x \in [0,1]^d : x \in \prod_{i=1}^{d} B(x_{i,n}, r_{i,n}),\ \text{i.o. } n \in \mathbb{N} \right\}$$

is of full Lebesgue measure, where $B(x_{i,n}, r_{i,n})$ denotes the 1-dimensional interval $(x_{i,n} - r_{i,n}, x_{i,n} + r_{i,n})$. What can we say about the Hausdorff measure or dimension of the set

$$W^{\mathbf{a}} := \left\{ x \in [0,1]^d : x \in \prod_{i=1}^{d} B(x_{i,n}, r_{i,n}^{a_i}),\ \text{i.o. } n \in \mathbb{N} \right\}$$

for $\mathbf{a} = (a_1, \ldots, a_d)$ with $1 \leqslant a_1 \leqslant \cdots \leqslant a_d$? It is also hoped that one could derive some mass transference principles. But in this setting, it seems that the situation is a little subtle. The dimension of $W^{\mathbf{a}}$ also depends on the relative positions of the rectangles $\prod_{i=1}^{d} B(x_{i,n}, r_{i,n})$.

We cite a special case of a result due to Hill and Velani [**13**]. Let $2 \leqslant b_d \leqslant \cdots \leqslant b_1$ be integers. Define a sequence of rectangles:

$$\prod_{i=1}^{d} B\left(\sum_{\ell=1}^{n} \frac{\epsilon_{i,\ell}}{b_i^{\ell}}, \frac{1}{b_i^{n}} \right), \quad 0 \leqslant \epsilon_{i,\ell} < b_i, 1 \leqslant i \leqslant d, n \in \mathbb{N}. \tag{6·1}$$

Fix $\eta \geqslant 0$. Let $\mathbf{a} = (a_1, \ldots, a_d)$ with $a_i = 1 + \eta/\log b_i$. Consider the shrunk rectangles:

$$\prod_{i=1}^{d} B\left(\sum_{\ell=1}^{n} \frac{\epsilon_{i,\ell}}{b_i^{\ell}}, \left(\frac{1}{b_i^{n}}\right)^{a_i} \right), \quad 0 \leqslant \epsilon_{i,\ell} < b_i, 1 \leqslant i \leqslant d, n \in \mathbb{N}. \tag{6·2}$$

Denote by $E^{\mathbf{1}}$ and $E^{\mathbf{a}}$ respectively the sets of points which fall into the rectangles in (6·1) and the rectangles in (6·2) infinitely many times. It is clear that $\mathcal{L}(E^{\mathbf{1}}) = 1$. However the dimension of $E^{\mathbf{a}}$ is given by (see [**13**])

$$\dim_{\mathrm{H}} E^{\mathbf{a}} = \min_{1 \leqslant i \leqslant d} \left\{ \frac{i \log b_i - \sum_{j : b_j > b_i e^{\eta}} (\log b_j - \log b_i - \eta) + \sum_{j<i} \log b_j}{\eta + \log b_i} \right\}.$$

The term

$$\min_{1 \leqslant i \leqslant d} \left\{ \frac{i \log b_i + \sum_{j<i} \log b_i}{\eta + \log b_i} \right\}$$

is natural, since this can be obtained by using the same argument as in Corollary 5·1 for the upper bound of W_τ. But this term is not the exact dimensional number of $E^{\mathbf{a}}$ when η is small. The extra negative term comes from the reason that when we cover one rectangle in (6·2) by balls, these balls may also cover parts of other rectangles in (6·2).

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This means that the relative position of the rectangles in (6·2) also functions when determining the dimension of $E^{\mathbf{a}}$. So, it seems that without additional technical assumptions, it is hard to get a transference principle in the general setting from rectangles to rectangles.

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