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Weak limits of powers of Chacon's automorphism

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Abstract. We completely describe the weak closure of the powers of the Koopman operator associated with Chacon's classical automorphism. We show that weak limits of these powers are the ortho-projector to constants and an explicit family of polynomials. As a consequence, we answer negatively the question of α-weak mixing for Chacon's automorphism.

1. *Introduction*

The classical version of Chacon's automorphism, which is the main subject of the present work, was described by Friedman in [**7**]. It is a famous example of a rank-one automorphism, for which we recall briefly the construction by cutting and stacking. We start with a Rokhlin tower of height $h_0 := 1$, called Tower 0. At step n, Tower $n-1$ (of height h_{n-1}) is cut into three subcolumns, and a spacer is inserted above the middle column before stacking all parts to get Tower n, of height $h_n = 3h_{n-1} + 1$. This transformation, denoted hereafter by T and acting on a standard Borel probability space $(X, \mathcal{A}, \mu)$, is known to present an interesting combination of ergodic and spectral properties. It is weakly mixing but not strongly mixing (see [**2**], where the historical version of Chacon's automorphism is constructed with only two subcolumns, but whose arguments also apply in the classical case). Del Junco proved in [**4**] that T has a trivial centralizer, and then improved this result by showing, with Rahe and Swanson, that it has minimal self-joinings [**3**]. The second and fourth authors proved in [**9**] that the convolution powers of its maximal spectral type are pairwise mutually singular. Their method involves the identification, in the weak closure of the powers of the associated Koopman operator $\hat{T}$, of an infinite family of polynomials in $\hat{T}$.

An automorphism S is said to be *α-weakly mixing* (for some $0 \le \alpha \le 1$) if there exists a sequence (k_j) of integers such that $\hat{S}^{k_j}$ converges weakly to $\alpha\Theta + (1-\alpha)\,\mathrm{Id}$,

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where Θ is the ortho-projector to constants. The disjointness of the convolution powers is automatically satisfied in the case of α-weakly mixing transformations with $0 < \alpha < 1$ (see Katok [**8**] and Stepin [**13**]). This property has been applied for numerous counterexamples in ergodic theory [**5**]. The question of α-weak mixing for Chacon's automorphism is a special case of a general problem to tell which operators can be obtained as weak limits of powers of $\hat{T}$. For a recent application of weak limits of powers of the Koopman operator, see [**6**]. Examples of transformations with non-trivial explicit weak closure of powers are given in [**12**].

The purpose of the present paper is to completely describe the weak closure

$$\mathcal{L} := \mathrm{WCl}(\{\hat{T}^{-k}, k \in \mathbb{Z}\}) = \left\{ \lim_{j\to\infty} \hat{T}^{-k_j} \text{ for a sequence of integers } (k_j) \right\}$$

of the powers of $\hat{T}$. Here, and throughout the paper, limits of operators are to be understood in the sense of the weak operator topology on the set of bounded operators on $L^2(X, \mathcal{A}, \mu)$. Our main result, Theorem 5.1, states that $\mathcal{L}$ is reduced to Θ and an explicit family of polynomials in $\hat{T}$. Our result implies in particular that T is not α-weakly mixing for any $0 < \alpha < 1$. Note that partial results in the description of $\mathcal{L}$ have also been given by Ageev [**1**], who gave all polynomials in $\hat{T}$ of degree at most 1 in $\mathcal{L}$.

An essential ingredient in our description of $\mathcal{L}$ is the identification of particular weak limits, along the sequences $(mh_n)_{n\geq 1}$, where h_n is the height of the nth tower in the cutting-and-stacking construction. As observed in [**10**], these weak limits are given by a family of polynomials $(P_m(\hat{T}))$. In §2, we give a definition of these polynomials P_m based on the representation of T as an integral automorphism over the 3-adic odometer (this representation was already used in [**9**]). We provide inductive formulas for these polynomials in §3. These formulas enable us to derive useful results about the asymptotic behavior of their coefficients (§4). Then, by expanding the integers (k_j) along the heights (h_n), we prove that if the weak limit of $\hat{T}^{-k_j}$ is not Θ, then it can be factorized by some polynomial $P_m(\hat{T})$ (Proposition 5.6).

2. *Representation of Chacon's automorphism as integral automorphisms over the 3-adic odometer*

2.1. *Definition of polynomials P_m in the 3-adic group.* Consider the compact group of 3-adic numbers

$$\Gamma := \mathbb{Z}_3 = \{x = (x_0, x_1, x_2, \ldots), x_k \in \{0, 1, 2\}\}.$$

We denote by λ the Haar measure on Γ: under λ, the coordinates $(x_k)_k$ are independent, and uniformly distributed in $\{0, 1, 2\}$.

We introduce two λ-preserving transformations on Γ:

- the shift-map $\sigma : x = (x_0, x_1, \ldots) \in \Gamma \mapsto \sigma x = (x_1, x_2, \ldots) \in \Gamma$; and
- the adding-machine transformation $S : x \in \Gamma \mapsto x + 1 \in \Gamma$, where $1 \in \Gamma$ is identified with the sequence $(1, 0, 0, \ldots)$. (In general, each integer j is identified with an element of Γ, so that $S^j x = x + j$ for all $j \in \mathbb{Z}$ and all $x \in \Gamma$.)

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FIGURE 1. The space X_n.

We define the cocycle $\phi : \Gamma\backslash\{(2, 2, \ldots)\} \to \mathbb{Z}$, where $\phi(x)$ is the first coordinate of x which is different from 2:

$$\phi(x) := \begin{cases} 0 & \text{if } x = 2, \ldots, 20*, \\ 1 & \text{if } x = 2, \ldots, 21*. \end{cases}$$

We set $\phi^{(0)}(x) := 0$, and for $m \geq 1$,

$$\phi^{(m)}(x) := \phi(x) + \phi(Sx) + \cdots + \phi(S^{m-1}x).$$

Let us define π_m as the probability distribution of $\phi^{(m)}$ on $\mathbb{Z}$: $\pi_m(j) := \lambda(\phi^{(m)} = j)$, and the polynomial P_m by

$$P_m(X) := \mathbb{E}_\lambda[X^{\phi^{(m)}}] = \sum_{j=0}^{m} \pi_m(j) X^j.$$

Note that the degree of P_m is strictly less than m as soon as $m > 2$.

2.2. *Integral automorphisms over the 3-adic odometer.* We will make use of the following representations of Chacon's automorphism. For each $n \geq 0$, we define

$$X_n := \{(x, i) : x \in \Gamma, 0 \leq i \leq h_n - 1 + \phi(x)\}$$

(see Figure 1).

We consider the transformation T_n of X_n, defined by

$$T_n(x, i) := \begin{cases} (x, i + 1) & \text{if } i + 1 \leq h_n - 1 + \phi(x), \\ (Sx, 0) & \text{if } i = h_n - 1 + \phi(x). \end{cases}$$

Let us introduce the map $\psi_n : X_n \to X_{n+1}$, defined by

$$\psi_n(x, i) := (\sigma x, x_0 h_n + i + \mathbb{1}_{x_0=2}).$$

Observe that ψ_n is bijective. Moreover, it conjugates the transformations T_n and T_{n+1}. We consider the probability measure μ_n on X_n: for a fixed i and a set $A \subset \{(x, i), x \in \Gamma\}$,

$$\mu_n(A) := \frac{1}{h_n + 1/2} \lambda(\{x \in \Gamma, (x, i) \in A\}).$$

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The transformation T_n preserves μ_n and the map ψ_n sends μ_n to μ_{n+1}. It follows that all the measure-preserving dynamical systems (X_n, T_n, μ_n) are isomorphic.

For $0 \le i \le h_n - 1$, we set $E_{n,i} := \{(x, i) : x \in \Gamma\} \subset X_n$. We have $E_{n,i} = T_n^i E_{n,0}$, hence

$$\{E_{n,0}, \ldots, E_{n,h_n-1}\}$$

is a Rokhlin tower of height h_n for T_n. Moreover, for any $n \ge 0$, and any $0 \le i \le h_n - 1$,

$$\psi_n(E_{n,i}) = E_{n+1,i} \sqcup E_{n+1,h_n+i} \sqcup E_{n+1,2h_n+i+1}.$$

Fix n_0. By composition of the isomorphisms (ψ_n), we can view all these Rokhlin towers inside X_{n_0}. The above formula shows that the towers are embedded in the same way as the towers of Chacon's automorphism. Therefore, $(X_{n_0}, \mu_{n_0}, T_{n_0})$ is isomorphic to (X, μ, T).

2.3. *Weak limits along subsequences* $(mh_n)_{n\ge1}$.

LEMMA 2.1. *Let $m \ge 1$ and $u \ge 0$ be fixed integers. Then,*

$$\sup_{A,B}\left|\mu(T^{mh_n+u}B \cap A) - \sum_{i\in\mathbb{Z}} \pi_m(i+u)\mu(T^{-i}B \cap A)\right| \xrightarrow[n\to\infty]{} 0,$$

where the supremum is taken over any sets A and B which are the union of levels of Tower n.

Proof. We may identify (X, μ, T) with (X_n, μ_n, T_n) and the level j of Tower n with $E_{n,j} = \{(x, j) : x \in \Gamma\} \subset X_n$.

Assume that $A = E_{n,k}$ for $0 \le k \le h_n - 1$ and $B = E_{n,j}$ for $m \le j \le h_n - 1$. We have

$$T^{mh_n}B = T^{mh_n}E_{n,j} = \{(S^m x, j - \phi^{(m)}(x)), x \in \Gamma\}.$$

Then, viewed in X_n,

$$T^{mh_n}B \cap A = S^m(\{x \in \Gamma : \phi^{(m)}(x) = j - k\}) \times \{k\},$$

and

$$\mu(T^{mh_n}B \cap A) = \lambda(\phi^{(m)} = j - k)\mu(E_{n,k}) = \pi_m(j - k)\mu(E_{n,k}).$$

Moreover, $\mu(T^{-i}B \cap A) = \mu(E_{n,k})$ if $k = j - i$, and zero otherwise. We obtain

$$\mu(T^{mh_n}B \cap A) = \sum_{i\in\mathbb{Z}} \pi_m(i)\mu(T^{-i}B \cap A).$$

By additivity of the measure μ, the above equality remains true if A is a union of the levels of Tower n, and if B is a union of levels $j \ge m$ of Tower n. Finally, removing the restriction on the levels in B, we have

$$\left|\mu(T^{mh_n}B \cap A) - \sum_{i\in\mathbb{Z}} \pi_m(i)\mu(T^{-i}B \cap A)\right| \le 2m\mu(E_{n,0}) \xrightarrow[n\to\infty]{} 0.$$

This proves the lemma for $u = 0$. For an arbitrary u, we apply this result to the part of $T^u B$ which remains in the nth tower. We get an extra error term of order $|u|\ \mu(E_{n,0})$. □

As a direct consequence of Lemma 2.1, we recover the result from [**10**].

THEOREM 2.2. *For any $m \ge 1$, we have the weak convergence*

$$\lim_{n\to\infty} \hat{T}^{-mh_n} = P_m(\hat{T}). \tag{1}$$

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$$
\begin{array}{cccccccccccccccc}
0 & 1 & . & 0 & 1 & . & 0 & 1 & . & 0 & 1 & . & 0 & 1 & . & 0 \\
 & & 0 & & & 1 & & & . & & & 0 & & & 1 & \\
 & & & & & & & & 0 & & & & & & &
\end{array}
\begin{array}{l}
\leftarrow \text{Contribution of order 0} \\
\leftarrow \text{Contribution of order 1} \\
\leftarrow \text{Higher orders} \\
\leftarrow \text{The whole sequence}
\end{array}
$$

$$0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0$$

FIGURE 2. Structure of the sequence $(\phi(S^j x))_{j\in\mathbb{Z}}$.

3. *Recurrence formulas for P_m*

3.1. *Description of the sequence $(\phi(S^j x))_{j\in\mathbb{Z}}$.* Let $x \in \Gamma\backslash\{(2, 2, \ldots)\}$. We say that $\text{order}(x) = k \geq 0$ if $x_0 = \cdots = x_{k-1} = 2$ and $x_k \neq 2$.

Since the first digit in the sequence $(\ldots, x-1, x, x+1, \ldots)$ follows a periodic pattern $\cdots 012012012\cdots$, the contribution of points of order zero in the sequence $(\phi(S^j x))_{j\in\mathbb{Z}}$ provides a periodic sequence of blocks 01 separated by one symbol, given by a point of higher order (see Figure 2). To fill in the missing symbols corresponding to positions j such that $\text{order}(x + j) \geq 1$, we observe that if x starts with a 2, then, for all $j \in \mathbb{Z}$,

$$\phi(x + 3j) = \phi(\sigma x + j).$$

Hence, the missing symbols are given by the sequence $(\phi(S^j \sigma x))_{j\in\mathbb{Z}}$.

LEMMA 3.1. *The probability distributions of the random sequences $(\phi(S^j x))_{j\in\mathbb{Z}}$ and $(1 - \phi(S^{-j} x))_{j\in\mathbb{Z}}$ are the same.*

Proof. This is an easy consequence of the above construction of the sequence $(\phi(S^j x))_{j\in\mathbb{Z}}$ and the fact that σ preserves the measure λ. □

LEMMA 3.2. *The coefficients of the polynomial P_m are symmetrical: for all $0 \leq j \leq m$, $\pi_m(j) = \pi_m(m - j)$.*

Proof. The coefficient $\pi_m(m - j)$ is equal to the probability of seeing $(m - j)$ digits equal to 1 when looking at m consecutive terms of the sequence $(\phi(S^j x))_{j\in\mathbb{Z}}$. Thus, $\pi_m(m - j)$ is also equal to the probability of seeing j digits equal to 0 when looking at m consecutive terms. Using Lemma 3.1, we conclude that $\pi_m(m - j) = \pi_m(j)$. □

3.2. *Recurrence formulas for P_m.*

THEOREM 3.3. *For all $m \geq 0$,*

$$\begin{aligned}
P_{3m}(X) &= X^m P_m(X); \\
P_{3m+1}(X) &= \tfrac{1}{3} X^m ((1 + X) P_m(X) + P_{m+1}(X)); \\
P_{3m+2}(X) &= \tfrac{1}{3} X^m (X P_m(X) + (1 + X) P_{m+1}(X)).
\end{aligned}$$

Proof. Let $x \in \Gamma\backslash\{(2, 2, \ldots)\}$. In the computation of $\phi^{(3m)}(x)$, the contribution of the $2m$ points $x + j$ $(0 \leq j \leq 3m - 1)$ of order zero is always m. Because of the structure of the sequence $(\phi(S^j x))_{j\in\mathbb{Z}}$ described in §3.1, the contribution of the other m points is $\phi^{(m)}(\sigma x)$. Hence,

$$\phi^{(3m)}(x) = m + \phi^{(m)}(\sigma x), \tag{2}$$

which proves that $P_{3m} = X^m P_m$.

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Let us compute $\phi^{(3m+1)}(x)$: if $x_0 = 0$, then $\phi^{(3m+1)}(x) = \phi^{(3m)}(x+1)$, which is equal to $m + \phi^{(m)}(\sigma(x+1))$ by (2). Hence, $\phi^{(3m+1)}(x) = m + \phi^{(m)}(\sigma x)$. If $x_0 = 1$, then $\phi^{(3m+1)}(x) = 1 + \phi^{(3m)}(x+1)$. Hence,

$$\phi^{(3m+1)}(x) = 1 + m + \phi^{(m)}(\sigma(x+1)) = 1 + m + \phi^{(m)}(\sigma x).$$

If $x_0 = 2$, then $\phi^{(3m+1)}(x) = \phi(x) + \phi^{(3m)}(x+1)$. Since $\phi(x) = \phi(\sigma x)$, we get

$$\phi^{(3m+1)}(x) = \phi(\sigma x) + m + \phi^{(m)}(\sigma x + 1) = m + \phi^{(m+1)}(\sigma x).$$

$$\phi^{(3m+1)}(x) = \begin{cases} m + \phi^{(m)}(x) & \text{if } x_0 = 0, \\ 1 + m + \phi^{(m)}(\sigma x) & \text{if } x_0 = 1, \\ m + \phi^{(m+1)}(\sigma x) & \text{if } x_0 = 2. \end{cases} \tag{3}$$

Since each digit appears with probability $1/3$ in the first position, and since the distribution of σx conditioned on the first digit is λ, we get

$$P_{3m+1}(X) = \tfrac{1}{3}(\mathbb{E}_\lambda[X^{m+\phi^{(m)}}] + \mathbb{E}_\lambda[X^{m+1+\phi^{(m)}}] + \mathbb{E}_\lambda[X^{m+\phi^{(m+1)}}]).$$

This yields the desired formula for P_{3m+1}.

In the same way, we compute $\phi^{(3m+2)}(x)$:

$$\phi^{(3m+2)}(x) = \begin{cases} 1 + m + \phi^{(m)}(\sigma x) & \text{if } x_0 = 0, \\ 1 + m + \phi^{(m+1)}(\sigma x) & \text{if } x_0 = 1, \\ m + \phi^{(m+1)}(\sigma x) & \text{if } x_0 = 2, \end{cases} \tag{4}$$

and we obtain the recurrence formula for P_{3m+2}. □

3.3. *Recurrence formulas for reduced polynomials.* For $m \geq 0$, let $\ell(m)$ be the highest power of X dividing P_m, so that

$$P_m(X) = X^{\ell(m)} \tilde{P}_m(X),$$

where $\tilde{P}_m$ is the reduced polynomial of order m, $\tilde{P}_m(0) \neq 0$.

Observe that $\ell(m)$ is the minimum value taken by the cocycle $\phi^{(m)}$. Hence, it is easy to see that $m \mapsto \ell(m)$ is non-decreasing, and that

$$s_m := \ell(m+1) - \ell(m) \in \{0, 1\}.$$

Moreover, by Lemma 3.2,

$$\ell(m) + \deg(P_m) = m. \tag{5}$$

Thanks to the recurrence formulas for P_m (see Theorem 3.3), we deduce recurrence formulas for $\ell(m)$:

$$\begin{aligned} \ell(3m) &= m + \ell(m), \\ \ell(3m+1) &= m + \ell(m), \\ \ell(3m+2) &= m + \ell(m+1). \end{aligned} \tag{6}$$

We also get recurrence formulas for the reduced polynomials.

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PROPOSITION 3.4. *Let $m \geq 0$. Then,*

$$\tilde{P}_{3m}(X) = \tilde{P}_m(X);$$
$$3\tilde{P}_{3m+1}(X) = (1+X)\tilde{P}_m(X) + X^{s_m}\tilde{P}_{m+1}(X);$$
$$3\tilde{P}_{3m+2}(X) = X^{1-s_m}\tilde{P}_m(X) + (1+X)\tilde{P}_{m+1}(X).$$

LEMMA 3.5. *Consider the 3-expansion of m: $m = \sum_{j\geq 0} m_j 3^j$, where $m_j \in \{0, 1, 2\}$. Let $i := \inf\{j : m_j \neq 1\}$. Then, $s_m = 1$ if $m_i = 2$. and $s_m = 0$ if $m_i = 0$.*

Proof. By (6), we see that $s_{3m} = 0$ and $s_{3m+2} = 1$. Moreover, $s_{3m+1} = s_m$. □

3.4. *Degree of $\tilde{P}_m$.* Let d_m be the degree of $\tilde{P}_m$: using (5), we get $d_m = \deg(P_m) - \ell(m) = m - 2\ell(m)$. Hence, we easily check that

$$d_{m+1} - d_m = 1 - 2s_m \in \{-1, 1\}, \tag{7}$$

and $s_m = \mathbb{1}_{\{d_m > d_{m+1}\}}$. By Proposition 3.4, we have

$$d_{3m} = d_m, \quad d_{3m+1} = d_m + 1, \quad d_{3m+2} = d_{m+1} + 1 \quad \text{and} \quad d_{3m+3} = d_{m+1}. \tag{8}$$

We thus get an algorithm to compute the degree d_m of $\tilde{P}_m$. Consider the 3-expansion of m: $m = \sum_{j\geq 0} m_j 3^j$. In this expansion, remove all 1s and count the number of blocks of 2s. Then d_m is equal to the number of removed 1s plus twice the number of 2-blocks.

Example: consider m whose expansion in base 3 is 212202. Remove one 1, you get 22202 (two 2-blocks). Hence, $d_m = 1 + 2 \times 2 = 5$.

COROLLARY 3.6. *The first appearance of a reduced polynomial of degree d is observed at*

$$m = 2 + \sum_{j=1}^{d-2} 3^j = \frac{3^{d-1}+1}{2}.$$

4. *Properties of the probability distribution π_m*

4.1. *Unimodality of the distribution π_m.* We set $b_j^{(m)} := \pi_m(j + \ell(m)) = \lambda(\phi^{(m)} = j + \ell(m))$, so that

$$\tilde{P}_m(X) = \sum_{j=0}^{d_m} b_j^{(m)} X^j.$$

Observe that by Lemma 3.2, the coefficients of $\tilde{P}_m$ are symmetric: $b_j^{(m)} = b_{d_m-j}^{(m)}$ for all $0 \leq j \leq d_m$.

LEMMA 4.1. *For any $m \geq 0$, the coefficients $(b_j^{(m)})_{0\leq j\leq [d_m/2]}$ of $\tilde{P}_m$ are increasing.*

Proof. The lemma holds for $\tilde{P}_0 = 1$ and $\tilde{P}_1(X) = (1+X)/2$. Using Proposition 3.4, we prove the lemma by induction on m. Assume the property we want to prove is satisfied for some m and $m+1$, and let us prove it is also true for $3m$, $3m+1$, $3m+2$ and $3m+3$. It obviously holds for $3m$ and $3m+3$, since $\tilde{P}_{3m} = \tilde{P}_m$ and $\tilde{P}_{3m+3} = \tilde{P}_{m+1}$. Observe that the recurrence formulas have the form

$$(\tilde{P}_{3m}, \tilde{P}_{3m+1}, \tilde{P}_{3m+2}, \tilde{P}_{3m+3}) = F(\tilde{P}_m, \tilde{P}_{m+1}),$$

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where F is such that

$$(\tilde{P}_{3m+3},\, \tilde{P}_{3m+2},\, \tilde{P}_{3m+1},\, \tilde{P}_{3m}) = F(\tilde{P}_{m+1},\, \tilde{P}_m).$$

Hence, exchanging if necessary the role of $\tilde{P}_{m+1}$ and $\tilde{P}_m$, we can assume without loss of generality that $d_m < d_{m+1}$ (that is, $s_m = 0$).

By Proposition 3.4, we have $3b_0^{(3m+1)} = b_0^{(m)} + b_0^{(m+1)}$ and for $j \geq 1$

$$3b_j^{(3m+1)} = b_j^{(m)} + b_{j-1}^{(m)} + b_j^{(m+1)}.$$

Recall that $d_{3m+1} = d_m + 1$ (see §3.4) and we assumed that $d_{m+1} = d_m + 1$. Hence, if d_m is even, we have $[d_{3m+1}/2] = [d_m/2] = [d_{m+1}/2]$ and the three terms on the right-hand side (RHS) of the above equation are increasing functions of $j \in \{0, \ldots, [d_{3m+1}/2]\}$. If d_m is odd, only the first term on the RHS may not be increasing for the largest value of j. But in this case, because of the symmetry of the coefficients (see Lemma 3.2), we have $b_{[d_m/2]+1}^{(m)} = b_{[d_m/2]}^{(m)}$. This proves that the property holds for $\tilde{P}_{3m+1}$. A similar argument proves the property for $\tilde{P}_{3m+2}$. □

As a consequence of Lemmas 3.2 and 4.1, we obtain the following proposition.

PROPOSITION 4.2. *For all $m \geq 1$, the probability distribution π_m is symmetric and unimodal.*

4.2. *Asymptotic behavior of π_m when $d_m \to \infty$.* Recall that, for all m, the coefficients of the polynomial P_m are given by the probability distribution π_m on $\mathbb{Z}$ which is symmetric and unimodal (see Proposition 4.2). Recall that d_m is the degree of the reduced polynomial $\tilde{P}_m$, that is, $(d_m + 1)$ is the number of non-zero coefficients of P_m.

For any $m \geq 1$, consider the Fourier transform $\widehat{\pi}_m$, defined by

$$\widehat{\pi}_m(z) := \sum_j \pi_m(j) z^{-j} \quad \text{for all } z \in S^1.$$

Observe that $\widehat{\pi}_m(z) = P_m(1/z) = P_m(z)z^{-m}$. Moreover, we recover $\pi_m(j)$ by the inverse Fourier transform

$$\pi_m(j) = \int_{S^1} z^j \widehat{\pi}_m(z)\, dz.$$

LEMMA 4.3.

$$\sup_{j \in \mathbb{Z}} \pi_m(j) \xrightarrow[d_m \to \infty]{} 0.$$

Proof. Observe that for any $j \in \mathbb{Z}$,

$$\pi_m(j) \leq \int_{S^1} |\widehat{\pi}_m(z)|\, dz = \int_{S^1} |P_m(z)|\, dz.$$

By Theorem 3.3, we have for any $m \geq 3$: if m is a multiple of 3, then $|P_m(z)| = |P_{m/3}(z)|$. Otherwise, there exist $m' < m$ and $m'' < m$ such that

$$|P_m(z)| \leq \frac{1 + |1 + z|}{3} \sup(|P_{m'}(z)|, |P_{m''}(z)|).$$

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Moreover, by (8),

$$d_{m'} \geq d_m - 2 \quad \text{and} \quad d_{m''} \geq d_m - 2.$$

Since $|P_1(z)| \leq 1$ and $|P_2(z)| \leq 1$, we easily prove by induction on m that

$$\text{for all } m \geq 1, \quad |P_m(z)| \leq |\alpha(z)|^{(d_m-2)/2}, \tag{9}$$

where $\alpha(z) := (1 + |1 + z|)/3$. Since $|\alpha(z)| < 1$ for $z \neq 1$, we conclude the proof of the lemma. □

5. *Weak limits of powers of* $\hat{T}$

Recall that $\mathcal{L}$ is the set of all weak limits of powers of $\hat{T}$, and that Θ is the ortho-projector to constants. The purpose of this section is to prove the following theorem.

THEOREM 5.1.

$$\mathcal{L} = \{\Theta\} \cup \{P_{m_1}(\hat{T}) \cdots P_{m_r}(\hat{T})\hat{T}^n, r \geq 0, 1 \leq m_1 \leq \cdots \leq m_r, n \in \mathbb{Z}\}.$$

Moreover, limits of the form $\hat{T}^n$ *for some* $n \in \mathbb{Z}$ *can only be obtained as limits of* $\hat{T}^{-k_j}$ *for* bounded *sequences* (k_j).

5.1. *Correspondence between elements of* $\mathcal{L}$ *and measures on* $\mathbb{Z}$. Any $L \in \mathcal{L}$ is associated with a self-joining ρ by

$$\rho(A \times B) := \langle L\mathbb{1}_A, \mathbb{1}_B \rangle \quad \text{for all } A, B.$$

Since T has minimal self-joinings, ρ is of the form

$$\rho_\nu := \sum_{j\in\mathbb{Z}} \nu(j)\Delta_{-j} + (1 - \nu(\mathbb{Z}))\mu \otimes \mu,$$

where $\Delta_{-j}(A \times B) := \mu(A \cap T^{-j}B)$ and ν is a positive measure on $\mathbb{Z}$ with $\nu(\mathbb{Z}) \leq 1$. Therefore, L has the form

$$L = L_\nu := \sum_{j\in\mathbb{Z}} \nu(j)\hat{T}^j + (1 - \nu(\mathbb{Z}))\Theta.$$

We set, for any positive measure ν on $\mathbb{Z}$ with $\nu(\mathbb{Z}) \leq 1$,

$$\delta(\nu) := \sum_{j\in\mathbb{Z}} |\nu(j + 1) - \nu(j)|.$$

LEMMA 5.2. *For any positive measures* ν *and* ν' *on* $\mathbb{Z}$ *of total mass* ≤ 1, *we have* $\delta(\nu * \nu') \leq \delta(\nu)$.

Proof.

$$\begin{aligned}
\delta(\nu * \nu') &= \sum_{j\in\mathbb{Z}} |\nu * \nu'(j + 1) - \nu * \nu'(j)| \\
&= \sum_{j\in\mathbb{Z}} \left| \sum_k (\nu(j + 1 - k) - \nu(j - k))\nu'(k) \right| \\
&\leq \sum_k \nu'(k) \sum_{j\in\mathbb{Z}} |\nu(j + 1 - k) - \nu(j - k)| \leq \delta(\nu).
\end{aligned}$$

□

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LEMMA 5.3. *Let $(\nu_\ell)_\ell$ be a sequence of positive measures with $\nu_\ell(\mathbb{Z}) \le 1$, such that $\delta(\nu_\ell) \to 0$ as $\ell \to \infty$. Then, $L_{\nu_\ell} \to \Theta$.*

Proof. For any A, B, we have

$$\rho_{\nu_\ell}(A \times B) = \sum_j \nu_\ell(j)\mu(A \cap T^{-j}B) + (1 - \nu_\ell(\mathbb{Z}))\mu(A)\mu(B)$$

and

$$\rho_{\nu_\ell}(A \times TB) = \sum_j \nu_\ell(j+1)\mu(A \cap T^{-j}B) + (1 - \nu_\ell(\mathbb{Z}))\mu(A)\mu(B).$$

Hence,

$$|\rho_{\nu_\ell}(A \times B) - \rho_{\nu_\ell}(A \times TB)| \le \sum_j |\nu_\ell(j+1) - \nu_\ell(j)|\mu(A \cap T^{-j}B) \le \delta(\nu_\ell).$$

It follows that any self-joining ρ which is a limit of a subsequence of (ρ_{ν_ℓ}) satisfies

$$\text{for all } A, B, \quad \rho(A \times B) = \rho(A \times TB).$$

By ergodicity of T, we get that $\rho = \mu \otimes \mu$, and this proves the convergence of L_{ν_ℓ} to Θ. □

LEMMA 5.4.

$$\sup_{1 \le m_1 \le \cdots \le m_r} \delta(\pi_{m_1} * \cdots * \pi_{m_r}) \xrightarrow[r\to\infty]{} 0.$$

Proof. Let us fix $(m_i)_{1 \le i \le r}$ larger than 1. Using the fact that the convolution of symmetric unimodal distributions remains symmetric and unimodal (see [**11**]), we obtain by Proposition 4.2 that $\pi_{m_1} * \cdots * \pi_{m_r}$ is symmetric and unimodal. Thus

$$\delta(\pi_{m_1} * \cdots * \pi_{m_r}) \le 2 \sup_{j \in \mathbb{Z}} \pi_{m_1} * \cdots * \pi_{m_r}(j).$$

Moreover, by (9), we have for all j,

$$|\pi_{m_1} * \cdots * \pi_{m_r}(j)| = \left| \int_{S^1} z^j \prod_{i=1}^{r} \widehat{\pi}_{m_i}(z)\, dz \right| \le \int_{S^1} \beta(z)^r\, dz,$$

where $\beta(z) := \sup(|P_1(z)|, |P_2(z)|, \alpha(z))$. Since $\beta(z) < 1$ if $z \ne 1$, this ends the proof of the lemma. □

5.2. *Factorization in $\mathcal{L}$.*

LEMMA 5.5. *Let (k_j) be a sequence of integers such that $k_j = mh_{n_j} + k'_j$, where $k'_j/h_{n_j} \to 0$ and $\lim_{j\to\infty} \hat{T}^{-k'_j} = L'$. Then,*

$$\lim_{j\to\infty} \hat{T}^{-k_j} = P_m(\hat{T})L'.$$

Proof. Let A and B be unions of levels of a fixed tower. For j large enough, A and B are still unions of levels in Tower n_j, and there exists A_j, the union of levels in Tower n_j, such that

$$\mu(T^{-k'_j}A \triangle A_j) \le |k'_j|/h_{n_j}.$$

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Then we have

$$\begin{aligned}\langle \mathbb{1}_A, \hat{T}^{-k_j}\mathbb{1}_B\rangle &= \mu(T^{-k'_j}A \cap T^{mh_{n_j}}B)\\ &= \mu(A_j \cap T^{mh_{n_j}}B) + O(k'_j/h_{n_j}).\end{aligned}$$

Fix $\varepsilon > 0$. By Lemma 2.1, for j large enough, $\mu(A_j \cap T^{mh_{n_j}}B)$ is within ε of $\sum_{i\in\mathbb{Z}} \pi_m(i)\mu(T^{-i}B \cap A_j)$. The latter expression is equal, up to a correction of order k'_j/h_{n_j}, to

$$\sum_{i\in\mathbb{Z}} \pi_m(i)\mu(T^{-i}B \cap T^{-k'_j}A) = \langle \mathbb{1}_A, \hat{T}^{-k'_j}P_m(\hat{T})\mathbb{1}_B\rangle,$$

which converges to $\langle \mathbb{1}_A, L'P_m(\hat{T})\mathbb{1}_B\rangle$ as $j \to \infty$. □

Let $L \in \mathcal{L}$: there exists a sequence (k_j) of integers such that

$$\lim_{j\to\infty} \hat{T}^{-k_j} = L.$$

If the sequence (k_j) is bounded, then L is of the form $\hat{T}^n$ for some $n \in \mathbb{Z}$. Otherwise, without loss of generality, we can assume that k_j is positive and $k_j \to +\infty$.

Recall that the heights (h_n) of the Rokhlin towers satisfy $h_{n+1} = 3h_n + 1$. We decompose k_j by the greedy algorithm along the integers (h_n):

$$k_j = \alpha_0^j h_{n_j} + \alpha_1^j h_{n_j-1} + \cdots + \alpha_{n_j}^j h_0,$$

where $\alpha_0^j \neq 0$, $0 \le \alpha_\ell^j \le 3$ for all $0 \le \ell \le n_j$. Observe that if $\alpha_\ell^j = 3$, then $\alpha_s^j = 0$ for all $s > \ell$.

Using a diagonal procedure to extract a subsequence if necessary, we can suppose that for all ℓ, $\alpha_\ell^j \to \alpha_\ell$ as j goes to ∞. We have $0 \le \alpha_\ell \le 3$, $\alpha_0 \neq 0$, and if $\alpha_\ell = 3$, then $\alpha_s = 0$ for all $s > \ell$.

PROPOSITION 5.6. *Let $L \in \mathcal{L}$, and let (k_j) and (α_ℓ) be as above. Then one of the following cases holds.*

Case 1. There exists r such that $\alpha_\ell = 2$ for all $\ell > r$, or $\alpha_\ell = 0$ for all $\ell > r$. In this case there exist $m \ge 1$ and a sequence (k'_j) such that

$$\lim_{j\to\infty} \hat{T}^{-k_j} = P_m(\hat{T})L',$$

where $L' = \lim_{j\to\infty} \hat{T}^{-k'_j}$.

Case 2. There exist infinitely many ℓs such that $\alpha_\ell \neq 2$ and infinitely many ℓs such that $\alpha_\ell \neq 0$. In this case $L = \Theta$.

Proof. Since $h_{n+s} = 3^s(h_n + 1/2) - 1/2$, for all $n, s \ge 0$, note that for all $r \ge 0$ and all j large enough,

$$\sum_{\ell=0}^{r} \alpha_\ell h_{(n_j-\ell)} = m_r h_{(n_j-r)} + u_r, \tag{10}$$

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where

$$m_r := \alpha_0 3^r + \alpha_1 3^{r-1} + \cdots + \alpha_r \quad \text{and} \quad u_r := \sum_{\ell=0}^{r-1} \alpha_\ell h_{r-\ell-1}.$$

First assume that $\alpha_\ell = 0$ for all $\ell > r$. Fix some large integer r'. For j large enough, we have $\alpha_\ell^j = \alpha_\ell$, $0 \le \ell \le r$, and $\alpha_\ell^j = 0$ for $r < \ell \le r + r'$. It follows that

$$k_j = \alpha_0 h_{n_j} + \alpha_1 h_{(n_j-1)} + \cdots + \alpha_r h_{(n_j-r)} + k'_j,$$

where $k'_j < h_{(n_j-r-r')} \ll h_{(n_j-r)}$. By (10), we can rewrite k_j as

$$k_j = m_r h_{(n_j-r)} + k'_j + u_r,$$

where $k'_j + u_r \ll h_{(n_j-r)}$. Extracting a subsequence if necessary, we can assume that $\hat{T}^{-(k'_j+u_r)}$ converges to some L'. We conclude using Lemma 5.5.

Assume now that $\alpha_\ell = 2$ for all $\ell > r$. Then, for j large enough,

$$k_j = \alpha_0 h_{n_j} + \alpha_1 h_{(n_j-1)} + \cdots + \alpha_r h_{(n_j-r)} + 2\sum_{\ell=1}^{r'} h_{(n_j-r-\ell)} + k'_j,$$

where $k'_j < h_{(n_j-r-r')}$. Since $2\sum_{\ell=1}^{r'} h_{(n_j-r-\ell)} = h_{(n_j-r)} - h_{(n_j-r-r')} - r'$, we get

$$k_j = \alpha_0 h_{n_j} + \alpha_1 h_{(n_j-1)} + \cdots + (\alpha_r + 1) h_{(n_j-r)} + k''_j,$$

where $|k''_j| = |k'_j - h_{(n_j-r-r')} - r'| \ll h_{(n_j-r)}$. We get the conclusion using the same argument as above. This proves the first part of the proposition.

Assume now that there exist an infinity of r such that $\alpha_r \neq 2$ and an infinity of r such that $\alpha_r \neq 0$. By (10), for all $r \ge 0$, and for all j large enough (depending on r),

$$k_j = m_r h_{(n_j-r)} + u_r + k'_j,$$

where $0 \le k'_j < h_{(n_j-r)}$. We know that d_{m_r} goes to infinity as $r \to \infty$ (see §3.4) by hypotheses on the sequence (α_ℓ). By Lemma 4.3, since π_{m_r} is unimodal, we get $\delta(\pi_{m_r} * \delta_k) \to 0$ as $r \to \infty$ uniformly with respect to $k \in \mathbb{Z}$.

Let us fix A_0 and B_0, some sets which are a union of levels of a fixed tower, say Tower $\overline{n}$. Fix $\varepsilon > 0$. By Lemma 5.3, $P_{m_r}(\hat{T})\,\hat{T}^k \xrightarrow[r\to\infty]{} \Theta$ uniformly with respect to $k \in \mathbb{Z}$. Hence, we can find r large enough so that for all $k \in \mathbb{Z}$,

$$\left| \sum_{i\in\mathbb{Z}} \pi_{m_r}(i+k)\mu(A_0 \cap T^{-i}B_0) - \mu(A_0)\mu(B_0) \right| < \varepsilon. \tag{11}$$

Then, we can choose j large enough to satisfy $\alpha_\ell^j = \alpha_\ell$ for all $\ell \le r$, and

$$\sup_{A,B} \left| \mu(A \cap T^{m_r h_{(n_j-r)}+u_r} B) - \sum_{i\in\mathbb{Z}} \pi_m(i+u_r)\mu(A \cap T^{-i}B) \right| < \varepsilon, \tag{12}$$

where the supremum is taken over any sets A and B which are unions of levels of Tower $(n_j - r)$ (see Lemma 2.1). We also assume that $(n_j - r) \ge \overline{n}$, so that A_0 and B_0 are unions of levels in Tower $(n_j - r)$.

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We want to use (12) in order to estimate

$$\mu(A_0 \cap T^{k_j} B_0) = \mu(A_0 \cap T^{m_r h_{(n_j - r)} + u_r}(T^{k'_j} B_0)).$$

The problem is that, although B_0 is a union of levels in Tower $(n_j - r)$, this is not always the case for $T^{k'_j} B_0$. Therefore, we cut B_0 into four disjoint parts: $B_0 = B_1 \sqcup B_2 \sqcup B_3 \sqcup B_4$, where:

- B_1 is the part of B_0 contained in the first $(h_{(n_j - r)} - k'_j)$ levels of Tower $(n_j - r)$, so that $T^{k'_j} B_1$ is a union of levels in Tower $(n_j - r)$, which is included in $T^{k'_j} B_0$;
- B_2 is the part of B_0 contained in the last $(k'_j - 1)$ levels of Tower $(n_j - r)$, which is not under a spacer, so that $T^{k'_j} B_2$ is included in a union of levels in Tower $(n_j - r)$, which is itself included in $T^{k'_j - h_{(n_j - r)}} B_0$;
- B_3 is the part of B_0 contained in the last $(k'_j - 1)$ levels of Tower $(n_j - r)$, which is under a spacer, so that $T^{k'_j} B_3$ is included in a union of levels in Tower $(n_j - r)$, which is itself included in $T^{k'_j - h_{(n_j - r)} - 1} B_0$; and
- B_4 is the part of B_0 contained in level $h_{(n_j - r)} - k'_j$ of Tower $(n_j - r)$: $\mu(T^{k'_j} B_4) \le 1/h_{(n_j - r)}$.

Using (12) and (11) three times, we get

$$\begin{aligned}
\mu(A_0 \cap T^{k_j} B_0) &\le \sum_{i \in \mathbb{Z}} \pi_m(i + u_r) \mu(A_0 \cap T^{-i} T^{k'_j} B_0) \\
&\quad + \sum_{i \in \mathbb{Z}} \pi_m(i + u_r) \mu(A_0 \cap T^{-i} T^{k'_j - h_{(n_j - r)}} B_0) \\
&\quad + \sum_{i \in \mathbb{Z}} \pi_m(i + u_r) \mu(A_0 \cap T^{-i} T^{k'_j - h_{(n_j - r)} - 1} B_0) \\
&\quad + 3\varepsilon + 1/h_{(n_j - r)} \\
&\le 3\mu(A_0)\, \mu(B_0) + 6\varepsilon + 1/h_{(n_j - r)}.
\end{aligned}$$

Hence, the self-joining ρ defined by

$$\rho(A \times B) := \lim_{j \to \infty} \mu(A \cap T^{k_j} B) = \langle \mathbb{1}_A, L \mathbb{1}_B \rangle \quad \text{for all } A, B$$

satisfies $\rho \le 3\mu \otimes \mu$. By ergodicity of $\mu \otimes \mu$, we conclude that $\rho = \mu \otimes \mu$ and $L = \Theta$. □

Proof of Theorem 5.1. First, fix integers $n \in \mathbb{Z}$, $r \ge 1$ and $1 \le m_1 \le \cdots \le m_r$. Using Lemma 5.5 r times, we easily construct a sequence (k_j) such that

$$\lim_{j \to \infty} \hat{T}^{-k_j} = \prod_{i=1}^{r} P_{m_i}(\hat{T}) \hat{T}^n.$$

Conversely, we now want to prove that any $L \in \mathcal{L}$ has this form. For $L \in \mathcal{L}$, set

$$r_L := \sup\{0\} \cup \left\{ r \ge 1 : \exists (m_i)_{i \le r}, \exists L' \in \mathcal{L} \text{ s.t. } L = \prod_{i=1}^{r} P_{m_i}(\hat{T}) L' \right\}.$$

Observe that if $L = \Theta$, then $r_L = \infty$ (because for all m, $P_m(\hat{T})\Theta = \Theta$). Conversely, we prove that if $r_L = \infty$, then $L = \Theta$. Suppose that $r_L = \infty$. Then, for any r, we can

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write $L = \prod_{i=1}^{r} P_{m_i}(\hat{T})L'$ with $L' \in \mathcal{L}$. We know that L and L' are of the form L_ν and $L_{\nu'}$ respectively. Moreover, we have

$$\nu = \pi_{m_1} * \cdots * \pi_{m_r} * \nu'.$$

By Lemma 5.2, $\delta(\nu) \leq \delta(\pi_{m_1} * \cdots * \pi_{m_r})$, but the RHS goes to 0 as $r \to \infty$, by Lemma 5.4, so that $\delta(\nu) = 0$. We conclude that $\nu(\mathbb{Z}) = 0$ and $L = \Theta$.

Assume now that $r_L < \infty$. Let (k_j) be such that $L = \lim_j \hat{T}^{-k_j}$. If the sequence (k_j) is bounded, then $L = \hat{T}^n$ for some $n \in \mathbb{Z}$. Otherwise, since $L \neq \Theta$ (because $r_L < \infty$), we are in case 1 of Proposition 5.6. Hence, there exist $L' \in \mathcal{L}$ and $m \geq 1$ such that $L = P_m(\hat{T})L'$. We then have $r_L \geq r_{L'} + 1$, and we can prove by induction on r_L that L is of the form

$$L = \prod_{i=1}^{r} P_{m_i}(\hat{T})\hat{T}^n. \qquad \square$$

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