Central force fields and Kepler's laws

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In this survey we give very simple proofs of Kepler's Laws and other facts about central force fields using only Newton's second law, Newton's law of universal gravitation, basic notions of vector calculus, and an elementary double integral.

Hopefully, this article will help undergraduate students of mathematics and engineering who wish to understand these fundamental scientific discoveries.

1. *Introduction*

In many textbooks (see, for instance, [1, 2, 3, 4, 5]), Kepler's Laws are obtained using conservation of energy and angular momentum, differential equations, mobile reference systems, or notions not so well-defined such as differentials or 'infinitesimal elements'. Some of the arguments appear to be rather involved if one is not accustomed to them, whereas the proof of Kepler's Laws may actually be obtained from quite simple facts.

We use here elementary notions with which every first year mathematics or engineering student is familiar.

A *central force field* in three dimensions is a vector field of the form

$$\mathbf{F}(\mathbf{r}) = f(r)\,\mathbf{r}, \tag{1}$$

where $\mathbf{r} = \langle x, y, z\rangle$ is the position vector of an arbitrary point $(x, y, z) \in \mathbb{R}^3$, $r = |\mathbf{r}|$, and $f \in C(0, \infty)$. In a central force field the forces are radial and their magnitude depends only on the distance to the origin of coordinates.

Consider a point particle with mass m moving with a non-radial speed so that the case of one-dimensional motion is excluded. Then the position of this particle is given at time t by the vector $\mathbf{r}(t)$, and its velocity and acceleration are defined as

$$\mathbf{v}(t) = \mathbf{r}'(t), \qquad \mathbf{a}(t) = \mathbf{v}'(t).$$

Denote by

$$\hat{\mathbf{r}}(t) = \frac{\mathbf{r}(t)}{r(t)} \tag{2}$$

the unit position vector.

In what follows, we omit the variable t if no confusion is possible. We have

$$\mathbf{v} = \mathbf{r}' = r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}'. \tag{3}$$

Define $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ (which, multiplied by the mass m, is known in

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mechanics as the *angular momentum*). Then,

$$\mathbf{h} = r\hat{\mathbf{r}} \times \left(r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}'\right) = r^2\hat{\mathbf{r}} \times \hat{\mathbf{r}}'. \tag{4}$$

Differentiating the identity $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$ with respect to time we get $2\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = 0$, which implies that $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ are orthogonal.

2. *Properties of central force fields*

In this section we give a few basic properties of central force fields.

The proof of the following result uses only the vector $\mathbf{h}$ and Newton's second law, and is taken from [6, 7].

Proposition 1: The motion of a point mass particle under a central force field is planar.

Proof: By differentiation,

$$\mathbf{h}' = (\mathbf{r} \times \mathbf{v})' = \underbrace{\mathbf{v} \times \mathbf{v}}_{=\mathbf{0}} + \mathbf{r} \times \mathbf{a}. \tag{5}$$

Newton's second law states that

$$\mathbf{F} = m\mathbf{a}. \tag{6}$$

Using (1),

$$f(r)\mathbf{r} = m\mathbf{a} \qquad \Rightarrow \qquad \mathbf{a} = \frac{f(r)}{m}\mathbf{r},$$

so the acceleration and the position vector have the same direction. From (5),

$$\mathbf{h}' = \mathbf{r} \times \frac{f(r)}{m}\mathbf{r} = \frac{f(r)}{m}\mathbf{r} \times \mathbf{r} = \mathbf{0},$$

which implies that $\mathbf{h}$ is a constant vector. Since $\mathbf{h} = \mathbf{r} \times \mathbf{v}$, the vectors $\mathbf{r}(t)$ and $\mathbf{v}(t)$ must always be contained in the plane normal to $\mathbf{h}$, so the motion is planar.

Proposition 2: The trajectory of a point mass particle under a central force field admits an equation in polar coordinates in the form $r = r(\theta)$.

Proof: Since $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ are orthogonal, we have $|\hat{\mathbf{r}} \times \hat{\mathbf{r}}'| = |\hat{\mathbf{r}}||\hat{\mathbf{r}}'|$. From (4) we can write the modulus of $\mathbf{h}$ as

$$h = |\mathbf{h}| = r^2|\hat{\mathbf{r}}||\hat{\mathbf{r}}'| = r^2|\hat{\mathbf{r}}'|. \tag{7}$$

This modulus can only be zero in three cases: if the speed is always zero, if the position is always the zero vector (which also requires the speed to be zero), or if the position and the velocity have the same direction. As we suppose that the particle is moving (the case of a stationary particle is trivial) and radial motion has been discarded, we can assert that $h \neq 0$, which implies that both r and $|\hat{\mathbf{r}}'|$ cannot be zero for any value of t.

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Letting $v = |\mathbf{v}|$ and using again the orthogonality of $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$, we have

$$v^2 = \mathbf{v}\cdot\mathbf{v} = \left(r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}'\right)\cdot\left(r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}'\right) = r'^2 + r^2\left|\hat{\mathbf{r}}'\right|^2.$$

By Proposition 1, the motion is planar and we can select a polar axis which will be the origin of the polar angle θ. On the plane containing the trajectory of the particle, we can write $\hat{\mathbf{r}} = \langle\cos\theta, \sin\theta\rangle$. By differentiating with respect to θ, we obtain

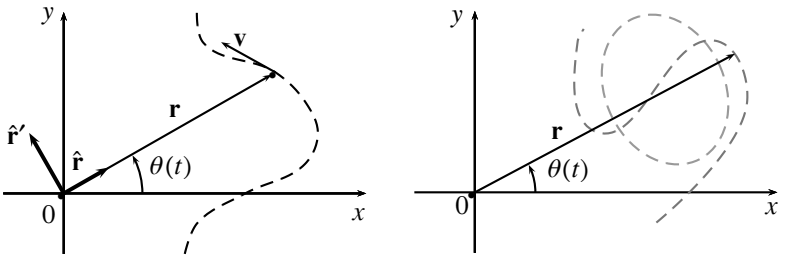
$$\frac{d\hat{\mathbf{r}}}{d\theta} = \langle -\sin\theta, \cos\theta\rangle \Rightarrow \left|\frac{d\hat{\mathbf{r}}}{d\theta}\right| = 1. \tag{8}$$

By the chain rule,

$$\hat{\mathbf{r}}'(t) = \frac{d\hat{\mathbf{r}}}{d\theta}\,\theta'(t). \tag{9}$$

Since $\hat{\mathbf{r}}'(t) \neq \mathbf{0}$ for every t, we conclude that $\theta'(t) \neq 0$ for every t. By the physical nature of the problem, we can assume that the angular velocity $\theta'(t)$ is continuous, so $\theta'(t)$ is either positive for all t or negative for all t. Therefore, $\theta(t)$ must be monotonic and so it has an inverse $t(\theta)$. This allows us to write $\mathbf{r}(t) = \mathbf{r}\left(t(\theta)\right) = \mathbf{r}(\theta)$, hence $r = r(\theta)$, that is, the trajectory admits an equation in polar coordinates.

This gives some information about the shape of the trajectory. Certain curves must be discarded as being impossible (see Figure 1).



(a) Particle moving in a central field (b) Two impossible trajectories

FIGURE 1: Trajectories. In (b), for certain values of θ there are several possible values of r, which implies that $\theta(t)$ is non-monotonic

We cannot fail to mention Bertrand's theorem, a fundamental result on central force fields:

Theorem [8]: Let $\mathbf{F}(\mathbf{r})$ be a central force field. All bounded orbits of $\mathbf{F}(\mathbf{r})$ are closed if, and only if, the field is either

$$\mathbf{F}(\mathbf{r}) = -k_1\mathbf{r} \qquad \text{or} \qquad \mathbf{F}(\mathbf{r}) = -\frac{k_2}{r^3}\mathbf{r} \qquad \text{with } k_1, k_2 > 0.$$

In other words, $\mathbf{F}(\mathbf{r})$ is either the radial harmonic oscillator or a gravitational field.

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Proof (of the 'if' part): In the case of the harmonic oscillator $\mathbf{F}(\mathbf{r}) = -k\mathbf{r}$, applying Newton's second law to a particle with mass m we obtain

$$m\mathbf{r}'' = -k\mathbf{r}.$$

Introduce the constant $\omega^2 = \dfrac{k}{m}$ and decompose the differential equation into its three components:

$$x'' + \omega^2 x = 0,$$

$$y'' + \omega^2 y = 0,$$

$$z'' + \omega^2 z = 0.$$

From a first course in differential equations, or simply by substitution, we know that their general solution is

$$x(t) = A_1 \cos(\omega t + \phi_1),$$

$$y(t) = A_2 \cos(\omega t + \phi_2),$$

$$z(t) = A_3 \cos(\omega t + \phi_3),$$

where A_i and ϕ_i $(i = 1, 2, 3)$ are constants. Hence, the particle describes a closed trajectory since x, y, z are periodic functions of period $\frac{2\pi}{\omega}$.

In Law 1 we shall show that all bounded trajectories of $\mathbf{F}(\mathbf{r}) = -\frac{k}{r^3}\mathbf{r}$ are closed. These two results prove the 'if' part of Bertrand's theorem.

The proof of the 'only if' part is much harder and beyond the scope of this paper.

3. *Gravitational field and Kepler's Laws*

Johannes Kepler*, key figure in the scientific revolution, was an astronomer who stated the laws about the planets motion which are named after him. He joined Tycho Brahe†, also an astronomer, who had built the best astronomic observation centre of his time. After Brahe's death, Kepler could work with the results of all the observations about the planets motion made by Brahe during his life, a huge amount of data which Kepler studied meticulously. Thanks to this he inferred his laws, publishing the first and the second ones in *Astronomia nova* (1609) and the third one in *Harmonices mundi* (1619). However, Kepler never proved these results analytically, and it was Newton who did that using his discoveries about gravitation and mechanics in *Philosophiæ Naturalis Principia Mathematica* (1687).‡

* Weil der Stadt, Germany, December 27, 1571 - Regensburg, Germany, November 15, 1630.

† Knudstrup castle, Scania, December 14, 1546 - Prague, October 24, 1601.

‡ See Wikipedia and [9].

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Law 1: Suppose a particle of mass m moves in a gravitational field. Then the particle describes a conic with one focus at the centre of the field.

Proof: The proof is taken from [6, 7]. A gravitational field centred at the origin has the form

$$\mathbf{F}(\mathbf{r}) = -\frac{k}{r^3}\mathbf{r}.$$

The force acting on a particle of mass m is

$$-\frac{km}{r^3}\mathbf{r}.$$

Therefore, by Newton's second law (6), the acceleration of the particle is

$$\mathbf{a} = -\frac{k}{r^3}\mathbf{r}.$$

From the definition of $\mathbf{h}$, we get

$$h^2 = \mathbf{h}\cdot(\mathbf{r}\times\mathbf{v}).$$

Using the triple product properties we can rewrite this as

$$h^2 = \mathbf{r}\cdot(\mathbf{v}\times\mathbf{h}). \tag{10}$$

On the other hand, by differentiating $\mathbf{v}\times\mathbf{h}$, we obtain

$$(\mathbf{v}\times\mathbf{h})' = \mathbf{a}\times\mathbf{h} = -\frac{k}{r^3}\mathbf{r}\times\mathbf{h} = -\frac{k}{r^3}\mathbf{r}\times(\mathbf{r}\times\mathbf{v}).$$

Recall that, given three vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$, we have

$$\mathbf{x}_1\times(\mathbf{x}_2\times\mathbf{x}_3) = (\mathbf{x}_1\cdot\mathbf{x}_3)\mathbf{x}_2 - (\mathbf{x}_1\cdot\mathbf{x}_2)\mathbf{x}_3.$$

Therefore

$$\begin{aligned}
(\mathbf{v}\times\mathbf{h})' &= -\frac{k}{r^3}[(\mathbf{r}\cdot\mathbf{v})\mathbf{r} - (\mathbf{r}\cdot\mathbf{r})\mathbf{v}] \\
&= -\frac{k}{r}\left[(\hat{\mathbf{r}}\cdot\mathbf{v})\hat{\mathbf{r}} - (\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})\mathbf{v}\right] \text{ (by (2))} \\
&= -\frac{k}{r}\left(r'\hat{\mathbf{r}} - \mathbf{v}\right) \text{ (by (3) and the orthogonality of } \hat{\mathbf{r}} \text{ and } \hat{\mathbf{r}}') \\
&= -\frac{k}{r}\left[r'\hat{\mathbf{r}} - (r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}')\right] \text{ (again by (3))} \\
&= -\frac{k}{r}(-r\hat{\mathbf{r}}') = k\hat{\mathbf{r}}'.
\end{aligned}$$

Integrating,

$$\mathbf{v}\times\mathbf{h} = k\hat{\mathbf{r}} + \mathbf{c}, \tag{11}$$

where $\mathbf{c}$ is an integration constant vector. Combining (10) and (11) yields:

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$$h^2 = \mathbf{r} \cdot (k\hat{\mathbf{r}} + \mathbf{c}) = kr + \mathbf{r} \cdot \mathbf{c} = kr + rc \cos\gamma,$$

where $c = |\mathbf{c}|$ and γ is the angle between $\mathbf{c}$ and $\mathbf{r}$. Solving for r,

$$r = \frac{h^2}{k + c\cos\gamma} = \frac{\varepsilon d}{1 + \varepsilon\cos\gamma} \tag{12}$$

where we have introduced the parameters

$$\varepsilon = \frac{c}{k} \qquad \text{and} \qquad d = \frac{h^2}{c}. \tag{13}$$

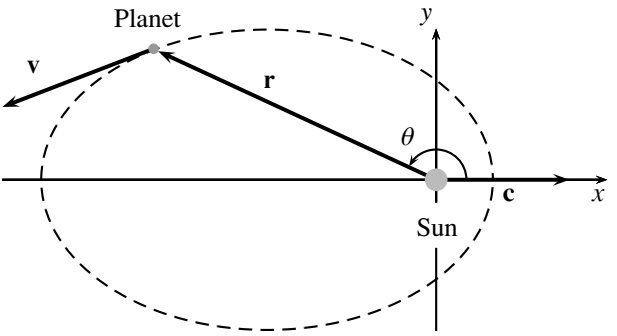


FIGURE 2: Trajectory of a planet around the Sun

Choose a coordinate system on the plane of motion so that $\mathbf{c}$ has the direction of the positive abscissa axis (see Figure 2). Then γ is the polar angle θ, and it is easy to see that (12) represents a conic in polar coordinates. Indeed, since

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad x = r\cos\theta,$$

we obtain from (12):

$$\sqrt{x^2 + y^2} = \frac{\varepsilon d}{1 + \varepsilon\dfrac{x}{\sqrt{x^2 + y^2}}}$$

$$\sqrt{x^2 + y^2} + \varepsilon x = \varepsilon d$$

$$x^2 + y^2 = \varepsilon^2 (d - x)^2$$

$$\frac{\sqrt{x^2 + y^2}}{|d - x|} = \varepsilon, \tag{14}$$

which is the equation of a conic with focus at the origin and directrix $x = d$.

The parameter $\varepsilon > 0$ is called *eccentricity*. Obviously, if $0 < \varepsilon < 1$, the conic is an ellipse, if $\varepsilon = 1$, it is a parabola, and if $\varepsilon > 1$, it is a

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hyperbola. From (14), we obtain:

$$x^2 + y^2 = \varepsilon^2 d^2 - 2d\varepsilon^2 x + \varepsilon^2 x^2$$

$$\Rightarrow y^2 + (1 - \varepsilon^2)x^2 + 2d\varepsilon^2 x = \varepsilon^2 d^2.$$

In the case $0 < \varepsilon < 1$, this equation may be written in the form

$$\frac{y^2}{(\varepsilon d)^2} + \frac{x^2}{\left(\frac{\varepsilon d}{\sqrt{1 - \varepsilon^2}}\right)^2} + \frac{2x}{d} = 1. \tag{15}$$

Let $a = \dfrac{\varepsilon d}{\sqrt{1 - \varepsilon^2}}$ be the semi-major axis of the ellipse (15), whose semi-minor axis is εd. Then

$$\text{Area}\,(\text{Ellipse}) = \pi a \varepsilon d = \pi a^2 \sqrt{1 - \varepsilon^2}. \tag{16}$$

This formula will be needed in Law 3.

Note that Law 1 is satisfied by the Solar System where, leaving out interaction between planets and relativistic phenomena, we can assume that the Sun remains stationary and the planets orbit around it. Therefore, we can conclude that the trajectory of the planets are ellipses, since experimentally it is clear that these trajectories are bounded. However, other bodies whose trajectories are parabolas or hyperbolas could be found (comets, for example).

Law 2: In a central force field, the position vector of a mass particle sweeps out equal areas in equal times.

Proof: Consider the domain D swept out by the position vector between two arbitrary points A and B of the trajectory (see Figure 3).

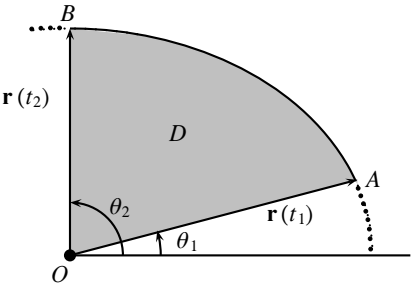


FIGURE 3: Area swept by the position vector

The area of D is

$$\text{Area}\,(D) = \int\int_D dx\,dy = \int\int_{D'} r\,dr\,d\theta,$$

where D' is the domain D in polar coordinates. By Fubini's theorem and

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Proposition 2, we have

$$\text{Area}(D) = \int_{\theta_1}^{\theta_2} d\theta \int_0^{r(\theta)} r\, dr = \frac{1}{2}\int_{\theta_1}^{\theta_2} r(\theta)^2\, d\theta = \frac{1}{2}\int_{t_1}^{t_2} r(\theta(t))^2 \theta'(t)\, dt, \quad (17)$$

where we have made the change of variable $\theta \to t$ with $\theta = \theta(t)$. We have assumed $\theta'(t) > 0$ for all t. From (7), (8) and (9), we have

$$\frac{h}{r^2} = |\hat{\mathbf{r}}'| = \theta'(t).$$

Hence,

$$\text{Area}(D) = \frac{1}{2}\int_{t_1}^{t_2} h\, dt = \frac{h}{2}(t_2 - t_1). \quad (18)$$

If $\theta'(t) < 0$ for all t, then $\theta(t_1) = \theta_2$ and $\theta(t_2) = \theta_1$, so the last integral in (17) should be replaced by

$$\frac{1}{2}\int_{t_2}^{t_1} r(\theta(t))^2 \theta'(t) dt = \frac{1}{2}\int_{t_1}^{t_2} r(\theta(t))^2 |\theta'(t)|\, dt = \frac{1}{2}\int_{t_1}^{t_2} h\, dt = \frac{h}{2}(t_2 - t_1).$$

Therefore the area swept out is proportional to the time spent in the motion.

In [10, p. 334] the Green-Riemann formula is used, instead of the double integral, to compute the area of D.

Although this result may seem not to have a clear physical interpretation, it implies that a planet moves faster when it is closer to the Sun and slower when it is further from the Sun (see Figure 4).

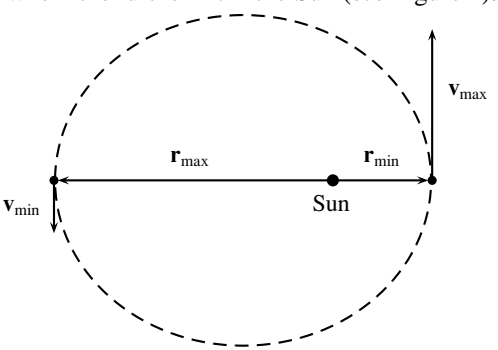


FIGURE 4: Planet speed

Law 3: Suppose that a point mass particle travels in a gravitational field describing bounded orbits (ellipses as seen in Law 1). Then the ratio between the square of the orbital period and the cube of the semi-major axis of the ellipse is constant.

Proof: We use the results of Law 1. From (12), the minimum and maximum distances of the particle to the centre of the field are, respectively,

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$$r_{\min} = r\Big|_{\gamma=0} = \frac{\varepsilon d}{1+\varepsilon}, \qquad r_{\max} = r\Big|_{\gamma=\pi} = \frac{\varepsilon d}{1-\varepsilon}.$$

Since the trajectory is an ellipse with semi-major axis a, we have

$$2a = r_{\min} + r_{\max} = \varepsilon d\left(\frac{1}{1+\varepsilon} + \frac{1}{1-\varepsilon}\right) = \frac{2\varepsilon d}{1-\varepsilon^2}$$

$$\Rightarrow \qquad \sqrt{1-\varepsilon^2} = \sqrt{\frac{\varepsilon d}{a}} = \frac{h}{\sqrt{ka}} \qquad \text{by (13).}$$

Let T be the time spent by the particle to describe the ellipse. By (16) and (18):

$$\frac{1}{2}hT = \text{Area}\,(\text{Ellipse}) = \pi a^2\sqrt{1-\varepsilon^2} = \pi a^2\frac{h}{\sqrt{ka}}$$

$$\Rightarrow \qquad T = \frac{2\pi a^2}{\sqrt{ka}} \qquad \Rightarrow \qquad T^2 = \frac{4\pi^2 a^3}{k},$$

so the square of the orbital period is proportional to the cube of the semi-major axis.

In the particular case of the Solar System, as the gravitational field is

$$\mathbf{F}(\mathbf{r}) = -\frac{GM}{r^3}\mathbf{r},$$

where G is the universal gravitation constant and M is the mass of the Sun, the proportionality constant in Law 3 is $\dfrac{4\pi^2}{k} = \dfrac{4\pi^2}{GM}$.

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References

1. A. P. French, *Newtonian mechanics (M.I.T. Introductory Physics Series)*, W. W. Norton & Co., New York (1971).
2. D. Kleppner and R. Kolenkow, *An introduction to mechanics* (2nd edn.), Cambridge University Press (2014).
3. D. J. McGill and W. W. King, *Engineering mechanics – dynamics*, PWS-KENT Publ. Co. (1989).
4. D. Morin, *Introduction to classical mechanics with problems and solutions*, Cambridge University Press (2008).
5. W. F. Riley and L. D. Sturges, *Engineering mechanics, dynamics*, John Wiley & Sons, Inc., New York (1993).
6. D. Metzler, Derivation of Kepler's first law, video accessed 2018 at https://www.youtube.com/watch?v=SvOc3LUJcwA

https://doi.org/10.1017/mag.2018.58 Published online by Cambridge University Press

7. J. Stewart, *Calculus* (2nd edn.), Brooks/Cole (1991).
8. J. Bertrand, Théorème relatif au mouvement d'un point attiré vers un centre fixe, *C. R. Acad. Sci.* **LXXVII** (1873) pp. 849-853.
9. L. Lederman and D. Teresi, *The god particle: if the universe is the answer, what is the question?*, Houghton Mifflin Harcourt, New York (1993).
10. B. Braden, The surveyor's area formula, *The College Math. J.* **17** (1986) pp. 326-337.

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