THE SHOT EFFECT WITH SPACE CHARGE

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1. Introduction

The technical importance of constructing valve amplifiers disturbed by as little background noise as possible has led to a very searching investigation of the causes of this defect. Some causes—bad contacts, changes in the surface of the filament, secondary ions from the grid, and so forth—though technically speaking not trivial, are of minor physical interest; but it has long been recognized that there are two which cannot be dismissed so easily.

One of these is the *Johnson effect*. It was observed by Johnson (3) that there is a fluctuation of potential difference between the terminals of a conductor arising from the thermal motion of the charges in the conductor. The theory, due to Schottky (12, 14), Nyquist (8) and Moullin and Ellis (6), leads to the equation

$$V^2 = \frac{kT}{C}, \tag{1}$$

where V^2 is the mean square of the potential difference, T is the temperature, C is the capacity of the conductor, and k is Boltzmann's constant; and this is amply confirmed by experiment. These fluctuations are of course very small, but they share in the amplification and produce an audible effect.

The other significant cause of disturbance is the *shot effect*. Electrons are shot off from the cathode at a rate which is on the average constant; actually, however, it fluctuates and the anode receives electrons at a rate which is sometimes above the average and sometimes below. Its potential varies accordingly and this gives rise to disturbance in the same way as the Johnson effect. The mean square of the fluctuation voltage was calculated by Schottky (12) on the assumption that the electrons are received at random. If the amplifier has a uniform response over a frequency range df, the result is found to be

$$V^2 = 2I\epsilon R^2 df, \tag{2}$$

where I is the anode current, R is the resistance through which it flows, and ϵ is the electronic charge. If the valve is operated without space charge and at frequencies which are not too low, this result is confirmed by experiment*, but an anomalous shot effect is found if either of these conditions is not satisfied.

At low frequencies—less than about 1 kilocycle per sec.—the voltage may be

* Cf. e.g. Williams, N. H. and Vincent (19).

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enormously greater than (2). This is called the *flicker effect**. It is believed to be due to changes in the emitting surface, but it is little understood and is not considered here.

The effect of space charge, on the other hand, is to diminish the shot effect. There can be no doubt that in some way the presence of the space charge smooths out† the fluctuations in the anode current, but the mechanism of this process is by no means clear and has been much debated. One factor which must certainly be taken into account is the fact that the "slope resistance" ρ of the valve is now finite‡. Moullin and Ellis (6) argue that the effect of this is that (2) is replaced by

$$V^2 = 2Ie\left(\frac{R\rho}{R+\rho}\right)^2 df, \tag{3}$$

but they are dissatisfied (7) with their derivation of this formula, pointing out that their argument ought to apply equally well to the case of a current flowing along a wire, whereas experience shows that in this case the shot effect is zero.

A different method of introducing ρ has been put forward recently by Rowland (11). Moullin and Ellis had pointed out, as indeed had others, that there tends to be periodicity in the arrival of large groups of electrons, since the arrival of one large group lowers the anode potential and so diminishes the probability of further arrivals until the effect of the large group has died away. They did not, however, allow for this process in their calculations. Rowland attempts to do so and concludes that in (2) R^2 should be replaced by $R^2(\rho - R)/(\rho + R)$, which agrees with the Moullin-Ellis formula when R/ρ is small. He obtains this result by a modification of the method by which he had previously discussed Campbell's theorem (10). As has been indicated elsewhere (16), I believe that this method is erroneous, but Rowland's general contention that the introduction of ρ is to be attributed to the group periodicity in the arrivals at the anode is supported by the theory which is put forward below.

Considerations of a different kind have been advanced by Llewellyn (4, 5), and adopted in a modified form by Schottky (13, 14). The dispatch of the electrons from the cathode is now more to the fore. If the space charge approaches saturation most of these electrons are returned to the cathode, so that fluctuations in the number emitted by the cathode produce much smaller fluctuations in the number received by the anode. Llewellyn holds that this diminishes the shot effect, and he arrives at the formula

$$V^2 = 2I_c\left(\frac{\partial I}{\partial I_c}\right)^2 e\left(\frac{R\rho}{R+\rho}\right)^2 df, \tag{4}$$

* Johnson (2).

† The phrase "smooths out", which will be frequently used, is intended to imply that the fluctuations are diminished, not necessarily that they are removed altogether.

‡ ρ is defined as the ratio of a small increase in anode potential to the resulting increase in anode current, the other factors being kept constant.

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where I_c is the current emitted by the cathode. Since $\partial I/\partial I_c$ tends to zero as saturation is approached, (4) implies that with perfect saturation the shot effect is zero. Llewellyn maintains that this is the case, and that the observed fluctuations of potential are really due to a Johnson effect, the electron stream being treated as equivalent to a conductor of resistance ρ. The question then arises, what is the effective temperature of this resistance? Llewellyn himself (4) gave a thermodynamic argument leading to the conclusion that the cathode temperature should be taken, but it was found by Pearson (9) that half the cathode temperature gave good agreement with experiment, and Schottky's theory (14) supports this. On the other hand Moullin and Ellis (7) maintain that, on their view of the Johnson effect, an electron stream cannot be treated as a conductor in this way; and the recent experimental work of F. C. Williams (17, 18) is not consistent with Llewellyn's theory, whatever the effective temperature of the stream is assumed to be.

Williams found also that the Moullin-Ellis formula (3) gives values which are sometimes much too large. He believes, however, that the factor $(R\rho)^2(R+\rho)^{-2}$ really represents the dependence of the shot voltage on R and ρ and accordingly replaces (3) by the equation

$$V^2 = 2Ie\Lambda\left(\frac{R\rho}{R+\rho}\right)^2 df, \tag{5}$$

where Λ is a factor which is unity in the extreme cases when almost all or almost none of the electrons released from the cathode reach the anode. For intermediate cases Λ is less than unity (except in cases when the flicker effect is present) and may be as small as $\frac{1}{20}$. Williams gives curves showing the dependence of Λ on various factors but does not give an expression for it nor a theoretical reason for its existence.

It will be noticed that Llewellyn's theory is concerned with conditions at the cathode and in the space charge, while Moullin and Ellis and Rowland concentrate rather on the conditions at the anode*. The theory put forward below endeavours to take both points of view into account. It is based on the following principles:

(i) The fluctuations in the stream of electrons are smoothed on their way to the anode by the passage through the space charge.

(ii) The anode has a power of regulating the supply of electrons it accepts. This process applies a further smoothing to the fluctuations.

The theory leads to Williams's formula (5) with a factor Λ which behaves in the right way, but its precise form has not been determined.

* This introduction must not be regarded as a complete account of the subject, but only of those aspects which have most bearing on the rest of the paper. In particular two other theories, that of Thatcher and N. H. Williams (15) and that of Schottky (13, 14), have not been discussed. To do so would lengthen the paper unduly. These theories involve different ideas and lead to formulae which bear no resemblance to (5).

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2. The smoothing factor

We start by considering the theory of the simple shot effect. Let electrons be shot off at random at an average rate λ per sec. and let $W_n(t)$ denote the probability that n have been shot off in the interval $(0, t)$. It is not difficult to show that*

$$W_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

The average number of events in time t is accordingly

$$\bar{n} = \sum_{n=0}^{\infty} n W_n(t) = \lambda t,$$

and the mean square of this number is

$$\overline{n^2} = \sum_{n=0}^{\infty} n^2 W_n(t) = \lambda^2 t^2 + \lambda t,$$

so that the mean square fluctuation (the shot effect) is

$$\overline{(n-\bar{n})^2} = \overline{n^2} - (\bar{n})^2 = \lambda t.$$

Suppose now that we cut off half the stream of electrons, say by interposing a screen. It appears at first sight as if $W_n(t)$ should now be associated with $\frac{1}{2}n$ instead of n, so that the mean square fluctuation should be one-quarter of its former value. Obviously there is something wrong here, since it is clear physically that halving the electron stream must halve the shot effect, and a little examination shows where the error lies. We are forgetting that a number of electrons must be an integer, so that, if n is odd, it cannot be correct to associate $\frac{1}{2}n$ with $W_n(t)$. To put the matter in another way, when we halve the electron stream we are automatically assuming that the electrons are shot off in pairs, and, when this fact is allowed for, the extra factor 2 needed to change $\frac{1}{4}$ into $\frac{1}{2}$ appears†.

This is a most important point, neglect of which has led to error in the past, as it is very easy to overlook it in the course of a calculation. The proper procedure in the case of a fluctuating stream which for some reason is reduced in size is to consider separately the factor which reduces the average stream and the factor which reduces the fluctuations. If the average stream is reduced from I to αI and the fluctuations of size x become fluctuations of size αx, then there has been no smoothing of the fluctuations and the shot effect is also reduced by the factor

* Bateman (**1**); cf. Whittaker (**16**).

† Cf. Whittaker (**16**), § 1.

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$\alpha^{-1}\alpha^2 = \alpha$. If, however, the fluctuations become of size $\alpha\beta x$, where $\beta < 1$, the factor β represents a true smoothing and the shot effect is reduced by the factor $\alpha^{-1}(\alpha\beta)^2 = \alpha\beta^2$. These facts can be summed up in the following statement:

A reduction in the size of the current does not alter the ratio of the shot effect to the current unless the fluctuations are smoothed. If the fluctuations are smoothed by a factor β, the ratio is diminished by the factor β^2.

3. THE AUTOMATIC CONTROL OF THE ANODE

Let the number of electrons submitted to the anode in time dt be $\phi(t)\,dt$. The number accepted by the anode is then

$$\chi(t) = \phi(t)\,U(t), \tag{6}$$

where $U(t)$, the "acceptance factor", is unity if no electrons have been accepted up to date and in general is less than unity.

An electron which arrived at time τ reduces the anode potential at time t by an amount*

$$-Be^{-k(t-\tau)} \qquad \left(k = \frac{1}{CR}\right),$$

where C is the capacity of the anode circuit and B is a constant.

The reduction of the anode potential due to all the previous electrons is therefore

$$-B\int_{-\infty}^{t} e^{-k(t-\tau)}\chi(\tau)\,d\tau.$$

The consequent reduction of the anode current is proportional to this, so we may assume that†

$$U(t) = 1 - K\int_{-\infty}^{t} e^{-k(t-\tau)}\chi(\tau)\,d\tau,$$

where K is a constant whose value is considered later.

On substituting for $\chi(t)$ from (6) we obtain the integral equation

$$U(t) = 1 - K\int_{-\infty}^{t} e^{-k(t-\tau)}\phi(\tau)\,U(\tau)\,d\tau. \tag{7}$$

If $\phi(t)$ has the constant value ϕ_0, this has the solution $U(t) = U_0$, where

$$U_0 = 1 - K\int_{-\infty}^{t} e^{-k(t-\tau)}\phi_0\,U_0\,d\tau = 1 - \frac{K\phi_0 U_0}{k}, \tag{8}$$

so that

$$U_0 = \frac{k}{k + K\phi_0}. \tag{9}$$

* This is the usual assumption and is true if the electron behaves like a large charge. On this point see Rowland (**10**), § 7.

† A rather similar assumption is made by Rowland (**11**).

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The rate of acceptance corresponding to a rate of incidence ϕ_0 is therefore

$$\chi_0 = \frac{k\phi_0}{k+K\phi_0}. \tag{10}$$

Suppose now that the rate of incidence changes from ϕ_0 to a value ϕ_1, which persists during the interval $(0, L)$. To find the appropriate solution of (7) put

$$u(t) = e^{kt}\,U(t),$$

so that the equation becomes

$$u(t) = e^{kt} - K\int_{-\infty}^{t} \phi(\tau)\,u(\tau)\,d\tau,$$

and differentiate this with respect to t. It becomes

$$u'(t) = ke^{kt} - K\phi(t)\,u(t),$$

and the solution of this gives

$$U(t) = \frac{k}{k+K\phi_1} + \frac{kK(\phi_1-\phi_0)}{(k+K\phi_0)\,(k+K\phi_1)}\,e^{-(k+K\phi_1)t} \qquad (0 \leqslant t \leqslant L),$$

so that
$$\int_0^L \chi(t)\,dt = \frac{k\phi_1 L}{k+K\phi_1} + \frac{kK(\phi_1-\phi_0)\phi_1}{(k+K\phi_0)\,(k+K\phi_1)^2}\{1-e^{-(k+K\phi_1)L}\}. \tag{11}$$

Now k is of the order 10^7. If therefore L is, say, 10^{-6}, the exponential is negligible and the modulus of the ratio of the first term to the second is

$$L(k+K\phi_1)\frac{k+K\phi_0}{K\,|\,\phi_1-\phi_0\,|} \geqslant 10\,\frac{\phi_0}{|\,\phi_1-\phi_0\,|}.$$

If we consider only fluctuations $|\,\phi_1-\phi_0\,|$ which are not larger than the average value ϕ_0, it is therefore sufficient to retain the first term in (11), so that

$$\int_0^L \chi(t)\,dt \sim \frac{k\phi_1 L}{k+K\phi_1}. \tag{12}$$

Physically this means that the rate of acceptance adjusts itself to the new rate of incidence in about 10^{-7} sec.

The difference between the number received in $(0, L)$ and the number which would have been received if the rate ϕ_0 had been continued is thus

$$\frac{k\phi_1 L}{k+K\phi_1} - \frac{k\phi_0 L}{k+K\phi_0} = \frac{k^2(\phi_1-\phi_0)\,L}{(k+K\phi_1)\,(k+K\phi_0)}.$$

The fluctuation $|\,\phi_1-\phi_0\,|\,L$ in the incident stream has therefore been reduced by the factor

$$\frac{k^2}{(k+K\phi_1)\,(k+K\phi_0)} = \frac{kU_0}{k+K\phi_1}.$$

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On the other hand it follows from (10) that the average incident current itself has been reduced by the factor U_0. Hence there is a smoothing factor

$$\frac{k}{k+K\phi_1} \sim \frac{k}{k+K\phi_0}$$

on the average.

The constant K can be determined from the consideration that, when the incident current has its average value ϕ_0, the effect of a single electron at time τ is to diminish the anode current by an amount

$$-\frac{1}{C\rho} e^{-k(t-\tau)}.$$

Now, by (8), the diminution in the anode current due to the $\chi_0 d\tau$ electrons accepted in time $d\tau$ is

$$-\phi_0 K e^{-k(t-\tau)} \chi_0 d\tau,$$

so that

$$\phi_0 K = \frac{1}{C\rho}.$$

The smoothing factor due to what may be called the automatic control of the anode is therefore

$$\frac{k}{k+K\phi_0} = \frac{\rho}{R+\rho}. \tag{13}$$

4. THE SHOT EFFECT

The shot effect is obtained by inserting the smoothing factors in (2). If the smoothing factor due to the passage through the space charge is denoted by $\sqrt{A}$ the result is

$$V^2 = 2I\epsilon R^2 \left(\frac{\rho}{R+\rho}\right)^2 (\sqrt{A})^2 df,$$

which is Williams's formula (5). The precise form of A seems difficult to determine, but it can be seen that it is equal to unity in the extreme cases. This is evidently so if there is no space charge. If, on the other hand, there is saturated space charge, an increase in I_c does not cause any increase in I, so that the fluctuations which originate at the cathode have no influence on those in the anode current. The fluctuations submitted to the anode are now unsmoothed fluctuations in the adjacent parts of the space charge, so that $A = 1$. In intermediate cases part of the stream from the cathode reaches the anode and the fluctuations in it are smoothed on the way, so that $A < 1$. The process is presumably somewhat analogous to that investigated in § 3, but I have not been able to construct a satisfactory theory.

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5. The shot effect for showers

Some points arising from the previous paper (16) may be dealt with briefly.

Dr C. G. Darwin has pointed out to me that the problem of showers can be treated by the Darwin-Fowler method of statistical mechanics. He had used this method in 1912 to discuss fluctuations in α-ray counts (the problem of random single events), but did not publish his results.

Dr A. C. Aitken has given me the following rule for constructing the polynomials $w_n(t)$.

Write down the determinants

$$\lambda_1, \quad \begin{vmatrix} \lambda_1 & \lambda_2 \\ -1 & \lambda_1 \end{vmatrix}, \quad \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ -1 & \lambda_1 & \lambda_2 \\ 0 & -1 & \lambda_1 \end{vmatrix}, \quad \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ -1 & \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & -1 & \lambda_1 & \lambda_2 \\ 0 & 0 & -1 & \lambda_1 \end{vmatrix}, \quad \ldots,$$

evaluate them, replace λ_i by $\lambda_i D^{-1}$, and operate on 1; e.g. the third determinant gives

$$(\lambda_1^3 D^{-3} + 2\lambda_1\lambda_2 D^{-2} + \lambda_3 D^{-1})\, 1 = \frac{\lambda_1^3}{3!} t^3 + \lambda_1\lambda_2 t^2 + \lambda_3 t = w_3(t).$$

Section 3 of the previous paper could have been shortened if it had been noticed that

$$\sum_0^\infty x^n w_n(t) = e^{t\lambda(x)} \qquad (|x| < 1), \tag{14}$$

where

$$\lambda(x) = \sum_1^\infty \lambda_n x^n. \tag{15}$$

This can be deduced from equations (10) of (16) without difficulty. Successive differentiation of (14) with respect to x gives (20) and (21) of (16) and the analogous equations

$$\sum_0^\infty (n - \mu T)^3 W_n(T) = T \sum_1^\infty n^3 \lambda_n, \tag{16}$$

$$\sum_0^\infty (n - \mu T)^4 W_n(T) = T \sum_1^\infty n^4 \lambda_n + 3\nu^2 T^2, \tag{17}$$

provided that the series on the right converge. Thus in the case of the fourth-order fluctuation the showers "interfere" with each other—the mean fluctuation is not the sum of the mean fluctuations of the showers of different sizes taken separately.

Some properties of the polynomials are worth noting.

(a) If the λ's have general complex values, the polynomials are the most general, such that

$$w_n(t) \text{ is of degree } n, \tag{18}$$

and

$$w_n(t_1 + t_2) = w_n(t_1)\, w_0(t_2) + w_{n-1}(t_1)\, w_1(t_2) + \ldots + w_0(t_1)\, w_n(t_2). \tag{19}$$

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(*b*) If $\{w_n(t,\lambda)\}$ denotes the set of polynomials formed with constants $\lambda_1, \lambda_2, \ldots$ and $\lambda_1', \lambda_2', \ldots$ is any other set of constants, then

$$w_n(t,\lambda+\lambda') = w_n(t,\lambda)\,w_0(t,\lambda') + w_{n-1}(t,\lambda)\,w_1(t,\lambda') + \ldots + w_0(t,\lambda)\,w_n(t,\lambda'). \quad (20)$$

(*c*) If $\lambda_1 = \lambda_2 = \ldots = 1$, then (14) is

$$\sum_0^\infty x^n w_n(t) = e^{xt/(1-x)},$$

so that

$$\sum_1^\infty x^{n-1} w_n'(t) = (1-x)^{-1} e^{xt/(1-x)} = \sum_0^\infty \frac{x^n}{n!} L_n(-t),$$

where $L_n(t)$ is the Laguerre polynomial. Thus

$$L_n(t) = n!\, w_{n+1}'(-t), \quad (21)$$

and

$$w_n(t) = \frac{1}{n!} L_n(-t) - \frac{1}{(n-1)!} L_{n-1}(-t). \quad (22)$$

(*d*) If $\lambda_n = 1/n$, (14) is

$$\sum_0^\infty x^n w_n(t) = e^{-t\log(1-x)} = (1-x)^{-t},$$

so that

$$w_n(t) = \frac{1}{n!} t(t+1)\ldots(t+n-1). \quad (23)$$

(*a*) and (*b*) are easily deduced from (14) and have obvious interpretations in terms of probability.

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