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Beta-expansions of p-adic numbers

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Abstract. In the present article, we introduce beta-expansions in the ring $\mathbb{Z}_p$ of p-adic integers. We characterise the sets of numbers with eventually periodic and finite expansions.

1. *Introduction*

For a real number $\beta > 1$, the *beta-transformation* $T = T_\beta$ is defined for $x \in [0, 1]$ by

$$T(x) = \beta x - \lfloor \beta x \rfloor. \tag{1.1}$$

Denote $T^0(x) := x$ and $T^n(x) := T(T^{n-1}(x))$. By iterating this map, we obtain an expansion

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \cdots,$$

where $x_n = \lfloor \beta T^{n-1}(x) \rfloor$. We will call the sequence

$$d_\beta(x) := \bullet x_1 x_2 \cdots \tag{1.2}$$

the *beta-expansion* of x. This setting was introduced by Rényi and Parry (cf. [**18**, **19**]). For beta-expansions of real numbers, there exist several results for the case when the base β is a Pisot number. Bertrand and Schmidt [**8**, **23**] proved that if β is Pisot, then the set of eventually periodic beta-expansions consists of the non-negative elements of $\mathbb{Q}(\beta)$. Schmidt proved a partial converse of this statement, namely, if the non-negative elements of $\mathbb{Q}$ have eventually periodic expansions, then β is a Pisot or Salem number.

In the case of Pisot numbers, it is proved that the associated shift space is sofic (cf. [**8**, **23**]). Soficness could also be proved for Salem numbers of degree four (cf. [**9**]).

For real beta-expansions there is a long standing conjecture stated by Schmidt [**23**], that the set of eventually periodic beta-expansions corresponds to the rational numbers in

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the interval $[0, 1)$ if and only if β is a Pisot or Salem number. It has been proved in [**9**] that the beta-expansion of 1 is eventually periodic for Salem numbers of degree four. It is conjectured by Boyd [**10**], that this is also true for Salem numbers of degree six, but not for higher degrees.

Following Frougny and Solomyak [**12**], we say that an algebraic number $\beta > 1$ satisfies the finiteness property (F) if for each $x \in \mathbb{Z}[\beta^{-1}] \cap [0, 1)$ there exists a positive integer n such that $T^n(x) = 0$. Property (F) can only hold for Pisot numbers whose associated shift space is of finite type. The quadratic Pisot numbers that satisfy (F) are completely described in [**12**], but from the cubic case on a full characterisation seems to be very hard. Partial results, conditions and algorithms can be found in, among other references, [**1–3**, **6**, **12**, **24**]. Moreover, beta-expansions have been studied in the context of formal Laurent series over finite fields. In this context the conjectures of Schmidt and Boyd have been proved (cf. [**13**, **20**]).

In the present article, we deal with beta-expansions in the ring of the p-adic integers. We characterise the set of numbers with eventually periodic and finite expansions. In particular, in Theorem 4.1, we prove that for β a Pisot–Chabauty number, the set of eventually periodic beta-expansions is $\mathbb{Q}(\beta) \cap \mathbb{Z}_p$. This is the equivalent of the result of Bertrand and Schmidt in the context of p-adic numbers. Furthermore, in Theorem 4.4, we prove an equivalent result to Schmidt's partial converse. In Theorem 5.4, we characterise the set of finite beta-expansions for a family of Pisot–Chabauty numbers. The theory of beta-expansions of p-adic numbers involves techniques from the theory of beta-expansions of real numbers as well as formal Laurent series.

The article is organised as follows: in §2, we introduce some basic definitions and preliminary results. In §3, we define the beta-expansions for p-adic numbers. In §4, we describe the periodic beta-expansions. The main result of this section is Theorem 4.1. In §5, we characterise the set of finite expansions in Theorem 5.4.

2. *Basic definitions and results*

Let p be a prime and

$$\mathbb{A}_p := \{mp^n : m, n \in \mathbb{Z}\} = \mathbb{Z}\left[\frac{1}{p}\right].$$

Then $\mathbb{A}_p \subset \mathbb{Q}$ is a principal ring. The unit group of $\mathbb{A}_p$ is $\{p^k : k \in \mathbb{Z}\}$ and the field of fractions is $\mathbb{Q}$. Define $\nu_p : \mathbb{A}_p \to \mathbb{Z} \cup \{\infty\}$ by

$$\nu_p(x) := \begin{cases} \max\{n \in \mathbb{Z} : p^n | x\} & \text{if } x \neq 0, \\ \infty & \text{if } x = 0. \end{cases}$$

Then ν_p verifies the properties

$$\nu_p(0) = \infty, \quad \nu_p(xy) = \nu_p(x) + \nu_p(y) \tag{2.1}$$

and

$$\begin{aligned} &\nu_p(x + y) \geq \min\{\nu_p(x), \nu_p(y)\} \quad \text{with} \\ &\nu_p(x + y) = \min\{\nu_p(x), \nu_p(y)\} \quad \text{if } \nu_p(x) \neq \nu_p(y). \end{aligned} \tag{2.2}$$

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Therefore, $v_p(\cdot)$ is an exponential valuation on $\mathbb{A}_p$ (cf. [**17**, Ch. II]). The p-adic norm $|\cdot|_p$ on $\mathbb{A}_p$ is defined by

$$|x|_p := \begin{cases} p^{-v_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{2.3}$$

From (2.1) it follows that

$$|xy|_p = |x|_p |y|_p \tag{2.4}$$

and

$$\begin{aligned} |x+y|_p &\leq \max\{|x|_p, |y|_p\} \quad \text{with} \\ |x+y|_p &= \max\{|x|_p, |y|_p\} \quad \text{if } |x|_p \neq |y|_p. \end{aligned} \tag{2.5}$$

Thus, $|\cdot|_p$ is a non-archimedean absolute value on $\mathbb{A}_p$. Let $|\cdot|_\infty$ be the archimedean absolute value. Then $|x|_p$ and $|x|_\infty$ satisfy the product formula

$$\prod_{p \in \mathbb{P} \cup \{\infty\}} |x|_p = 1 \quad \text{for all } x \in \mathbb{Q} \setminus \{0\}$$

where $\mathbb{P}$ denotes the set of primes. The completion of $\mathbb{A}_p$ with respect to $|\cdot|_p$ is the field $\mathbb{Q}_p$ of p-adic numbers. Thus,

$$\mathbb{Z} \subset \mathbb{A}_p \subset \mathbb{Q} \subset \mathbb{Q}_p.$$

Each element $x \in \mathbb{Q}_p$ with $x \neq 0$ admits a unique expansion of the form

$$x = \sum_{n=n_0}^{\infty} x_n p^n \quad \text{such that } n_0 \in \mathbb{Z},\ x_{n_0} \neq 0 \text{ and } x_n \in \{0, \ldots, p-1\}. \tag{2.6}$$

Alternatively, we will use for expansions of the form (2.6) the notation

$$x = \cdots x_2 x_1 x_0 \bullet x_{-1} \cdots x_{n_0}.$$

The x_i are called the p-adic coefficients of x. If we define the extension $v_p : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ by

$$v_p(x) := \begin{cases} n_0 & \text{if } x \neq 0, \\ \infty & \text{if } x = 0, \end{cases}$$

then $|\cdot|_p$ can be defined for $\mathbb{Q}_p$ as in (2.3), and (2.4) as well as (2.5) hold analogously. The ring

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$$

is called the ring of p-adic integers. It easily follows that

$$\mathbb{Z} = \mathbb{A}_p \cap \mathbb{Z}_p = \{x \in \mathbb{A}_p : |x|_p \leq 1\}.$$

Furthermore, $\mathbb{Z}_p$ is compact. The elements of $\mathbb{Z}_p$ can be expressed uniquely in the form (2.6) with $n_0 \geq 0$.

Definition 2.1. Each $x \in \mathbb{Q}_p$ of the form (2.6) has a unique *Artin decomposition*

$$x = \lfloor x \rfloor_p + \{x\}_p$$

with

$$\lfloor x \rfloor_p := \sum_{n \geq 0} x_n p^n \quad \text{and} \quad \{x\}_p := \sum_{n<0} x_n p^n.$$

The number $\lfloor x \rfloor_p \in \mathbb{Z}_p$ is called the *p-adic integer part* and $\{x\}_p \in \mathbb{A}_p \cap [0, 1)$ is called the *p-adic fractional part* of x.

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Remark 2.2. For $x \in \mathbb{R}$, let $\lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \le x\}$ be the real floor function. If $x \in \mathbb{A}_p$, it follows that $\lfloor x \rfloor_p = \lfloor x \rfloor$. However, for $x \in \mathbb{Q} \setminus \mathbb{A}_p$, this identity generally is not true. In any case, for $x \in \mathbb{R} \setminus \mathbb{Q}$ or $x \in \mathbb{Q}_p \setminus \mathbb{Q}$, one of these functions is not defined and therefore, this identity does not hold.

Definition 2.3. An element α is called *algebraic over* $\mathbb{A}_p$, if there is a polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{A}_p[x] \quad \text{with } f(\alpha) = 0. \tag{2.7}$$

If f is irreducible over $\mathbb{A}_p$, then f is called a minimal polynomial of α. If $a_n = p^k$ for some $k \in \mathbb{Z}$ (i.e. if a_n is a unit of $\mathbb{A}_p$), then α is called an *algebraic integer*. Since p^k is a unit of $\mathbb{A}_p$, we can assume without loss of generality, that $a_n = 1$.

It turns out that algebraic elements over $\mathbb{A}_p$ are not necessarily contained in $\mathbb{Q}_p$. In our context, we will only need that $|\cdot|_p$ and $v_p(\cdot)$ can be extended uniquely from $\mathbb{Q}_p$ to all of its algebraic extensions. This follows from the next theorem, which holds generally in non-archimedean fields.

THEOREM 2.4. [**17**, Ch. II, Theorem 4.8] *Let K be a field which is complete with respect to the non-archimedean absolute value $|\cdot|$ and let L/K be an algebraic extension of degree m. Then $|\cdot|$ has a unique extension to L defined by*

$$|\alpha| = \sqrt[m]{|N_{L/K}(\alpha)|},$$

and L is complete with respect to this extension.

We apply Theorem 2.4 to algebraic extensions of $K = \mathbb{Q}_p$. Since $\mathbb{Q}_p$ is complete with respect to $|\cdot|_p$, the absolute value $|\cdot|_p$ can be extended uniquely to each algebraic extension field L of $K = \mathbb{Q}_p$.

Let α be algebraic over $\mathbb{A}_p$ and (2.7) be its minimal polynomial. The *Newton polygon of f* (cf. [**17**, Ch. II]) is defined as the lower convex hull of the set of points

$$\{(0, v_p(a_0)), \ldots, (n, v_p(a_n))\}.$$

The polygon is a sequence of $r \le n$ line segments $E_1, E_2, \ldots, E_r$ with strictly monotonically increasing slopes.

From [**17**, Theorem (4.8), Proposition (6.3) and Proposition (6.4)], the next statement follows.

PROPOSITION 2.5. *Let*

$$f(x) = a_0 + \cdots + a_n x^n, \quad a_0 a_n \neq 0,$$

be a polynomial over the field K, and K be complete with respect to the exponential valuation v. Let w be the unique extension of v to the splitting field L of f.

(i) *If*

$$\overline{(r, v(a_r))(s, v(a_s))}$$

is a line segment of slope $-m$ occurring in the Newton polygon of f, then $f(x)$ has exactly $s - r$ roots $\alpha_1, \ldots, \alpha_{s-r}$ in L with value

$$w(\alpha_1) = \cdots = w(\alpha_{s-r}) = m.$$

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(ii) *Let $E_1, \ldots, E_t$ be the line segments of the Newton polygon. Then, for each E_j, there exists a unique polynomial $g_j(x) \in K[x]$, such that*

$$f(x) = a_n \prod_{j=1}^{t} g_j(x).$$

Thus,

$$g_j(x) = \prod_{w(\alpha_i)=m_j} (x - \alpha_i).$$

Remark 2.6. In this article, we will use the convention that, for algebraic elements α over $\mathbb{A}_p$, we will denote by $\alpha_1, \ldots, \alpha_n$ the non-archimedean conjugates and by $\alpha_{n+1}, \ldots, \alpha_{2n}$ the archimedean conjugates of α.

LEMMA 2.7. *Let $A \subset \mathbb{A}_p$. If A is bounded with respect to $|\cdot|_p$ and $|\cdot|_\infty$, i.e.*

$$\max_{a\in A} |a|_p < \infty \quad \text{and} \quad \max_{a\in A} |a|_\infty < \infty,$$

then A is finite.

Proof. If $|a|_p \le K$ for all $a \in A$, then

$$A \subset \left\{ mp^k : m \in \mathbb{Z},\, k = \left\lfloor -\frac{\log K}{\log p} \right\rfloor \right\}.$$

Therefore, if A is bounded with respect to $|\cdot|_\infty$, it can contain only finitely many points. □

Definition 2.8. [**16**, Ch. II] A Pisot–Chabauty number (for short PC number) is a p-adic number $\alpha \in \mathbb{Q}_p$, such that:
(i) $\alpha_1 := \alpha$ is an algebraic integer over $\mathbb{A}_p$;
(ii) $|\alpha_1|_p > 1$ for one non-archimedean conjugate of α;
(iii) $|\alpha_i|_p \le 1$ for all non-archimedean conjugates α_i, $i \in \{2, \ldots, n\}$ of α;
(iv) $|\alpha_i|_\infty < 1$ for all archimedean conjugates α_i, $i \in \{n+1, \ldots, 2n\}$ of α.
If condition (iv) is replaced by:
(iv)′ $|\alpha_i|_\infty = 1$ for all archimedean conjugates α_i, $i \in \{n+1, \ldots, 2n\}$ of α;
then α is called a Salem–Chabauty number (for short SC number).

Remark 2.9. Note that PC numbers and SC numbers are algebraic integers but not p-adic integers (because their p-adic norm is larger than one and, therefore, they are not contained in $\mathbb{Z}_p$).

Remark 2.10. If an archimedean root is located on the complex unit circle, then its minimal polynomial f is self-reciprocal, i.e. $f(x) = x^n f(1/x)$. If there does not exist any root outside the unit circle, there also cannot exist any root inside the unit circle. Therefore, condition (iv)′ is formulated with equality for all archimedean conjugates of α.

PROPOSITION 2.11. *Let α be a PC number or SC number. Then $\alpha \in \mathbb{Q}_p$.*

Proof. Let α and the minimal polynomial f of α be of the form (2.7). Since there is no non-archimedean conjugate α_j, $j \in \{2, \ldots, n\}$ with $|\alpha|_p = |\alpha_j|_p$, the Newton polygon of f must contain an edge

$$\overline{(i,\, v_p(a_i))(i+1,\, v_p(a_{i+1}))},$$

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with slope $v_p(\alpha) = v_p(a_{i+1}) - v_p(a_i)$ for some $i \in \{0, \dots, n-1\}$. By Proposition 2.5 (i), the minimal polynomial must contain the factor $x - \alpha \in \mathbb{Q}_p[x]$. Thus, $\alpha \in \mathbb{Q}_p$. □

Definition 2.12. Let

$$\mathcal{E}_n := \{(r_1, \dots, r_n) \in \mathbb{R}^n : \\ x^n + r_n x^{n-1} + \cdots + r_1 \text{ has only (real or complex) roots } \alpha \text{ with } |\alpha| < 1\}.$$

Then $\mathcal{E}_n$ is an open set with $(0, \dots, 0) \in \mathcal{E}_n$. By considering the Newton polygon of the minimal polynomial, the following necessary conditions can be derived.

PROPOSITION 2.13. *Let α be an algebraic integer over $\mathbb{A}_p$ and (2.7) be its minimal polynomial with $a_n = 1$. Then α is a PC number if and only if*

$$v_p(a_{n-1}) \le \min_{0 \le j \le n-2} v_p(a_j) \quad \text{and} \quad (a_0, \dots, a_{n-1}) \in \mathcal{E}_n. \tag{2.8}$$

Moreover, α is a SC number if and only if

$$v_p(a_{n-1}) \le \min_{0 \le j \le n-2} v_p(a_j) \quad \text{and} \quad (a_0, \dots, a_{n-1}) \in \partial\mathcal{E}_n. \tag{2.9}$$

In both cases, α is an isolated root with $\alpha \in \mathbb{Q}_p$ and $|\alpha|_p = |a_{n-1}|_p$.

Proof. The proposition follows directly from Proposition 2.5(ii) and Definition 2.8. □

Example 2.14. The PC numbers of degree one admit the form $\{x \in \mathbb{A}_p : |x|_p > 1\}$. In order to obtain PC numbers of arbitrary degree, the following construction may be used. We consider an irreducible polynomial

$$f(x) := p^k x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$$

such that

$$v_p(a_{n-1}) \le \min_{0 \le j \le n-2} v_p(a_j)$$

and k is large enough such that

$$\left(\frac{a_{n-1}}{p^k}, \dots, \frac{a_0}{p^k}\right) \in \mathcal{E}_n.$$

Such a k exists, since $(0, \dots, 0)$ is an inner point of $\mathcal{E}_n$. Then the archimedean roots of f fulfil $|\alpha_i|_\infty < 1$. Since p^k is a unit of $\mathbb{A}_p$, the roots of f are algebraic integers over $\mathbb{A}_p$.

By considering the Newton polygon of f, we confirm that f has one non-archimedean root α_1 with $|\alpha_1|_p > 1$ and $n-1$ non-archimedean roots α_i with $|\alpha_i|_p \le 1$. Therefore, α_1 is a PC number.

Remark 2.15. In the context of the present article, it turns out to be more convenient to write the minimal polynomial of PC numbers or SC numbers in the form

$$x^n - a_1 x^{n-1} - \cdots - a_n, \quad a_i \in \mathbb{A}_p[x]. \tag{2.10}$$

From now on, we will use this notation for the rest of the article.

In Theorem 2.17, a method to compute the p-adic coefficients of PC or SC numbers is given. For the proof, we will need the following auxiliary result.

LEMMA 2.16. *If $z, w \in \mathbb{Q}_p$ with $|z|_p = |w|_p$, then $|z^n - w^n|_p \le |z - w|_p |z|_p^{n-1}$ for all $n \in \mathbb{Z}$.*

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Proof. The statement is trivial for $n = 0$. For $n > 0$, we have

$$\begin{aligned} |z^n - w^n|_p &= |z - w|_p |z^{n-1} + z^{n-2}w + \cdots + w^{n-1}|_p \\ &\le |z - w|_p \max_{0 \le j \le n-1} |z^{n-1-j} w^j|_p \\ &= |z - w|_p |z|_p^{n-1} \end{aligned}$$

and

$$\begin{aligned} |z^{-n} - w^{-n}|_p &= |w^n - z^n|_p |z^{-n} w^{-n}|_p \\ &\le |w - z|_p |w|_p^{n-1} |z^{-n} w^{-n}|_p \\ &= |z - w|_p |z|_p^{-n-1}. \end{aligned}$$

□

THEOREM 2.17. *Let α be a PC number or SC number and (2.10) be its minimal polynomial. Then the recurrence*

$$\begin{aligned} \alpha_0 &:= a_1, \\ \alpha_{k+1} &:= a_1 + \frac{a_2}{\alpha_k} + \cdots + \frac{a_n}{\alpha_k^{n-1}} \quad \text{for } k \ge 1 \end{aligned} \tag{2.11}$$

converges to

$$\lim_{k \to \infty} \alpha_k = \alpha.$$

Proof. First we prove by induction that $|\alpha_k|_p = |a_1|_p$ for all $k \ge 0$. For $k = 0$ this assertion follows from the definition. Let $|\alpha_k|_p = |a_1|_p$. For $j = 2, \ldots, n$, it follows from Proposition 2.13 that $|a_1|_p > 1$ and $|a_1|_p \ge |a_j|_p$. Using (2.5), we easily obtain

$$\left| \frac{a_j}{\alpha_k^{j-1}} \right|_p = \frac{|a_j|_p}{|\alpha_k|_p^{j-1}} \le \frac{|a_1|_p}{|a_1|_p} < |a_1|_p.$$

Thus, $|\alpha_{k+1}|_p = |a_1|_p$. From Lemma 2.16, we get

$$\begin{aligned} |\alpha_{k+1} - \alpha_k|_p &= \left| a_2 \left(\frac{1}{\alpha_k} - \frac{1}{\alpha_{k-1}} \right) + \cdots + a_n \left(\frac{1}{\alpha_k^{n-1}} - \frac{1}{\alpha_{k-1}^{n-1}} \right) \right|_p \\ &\le \max \left(\frac{|a_2|_p}{|a_1|_p^2}, \ldots, \frac{|a_n|_p}{|a_1|_p^n} \right) |\alpha_k - \alpha_{k-1}|_p. \end{aligned}$$

Since $|a_1|_p > 1$ and $|a_1|_p \ge |a_j|_p$, the left factor is constant and less than 1. By Banach's fixed-point theorem, the sequence converges to a limit α with

$$\alpha = a_1 + \frac{a_2}{\alpha} + \cdots + \frac{a_n}{\alpha^{n-1}}.$$

□

Remark 2.18. Proposition 2.11 also follows as a direct consequence of Theorem 2.17. This gives an alternative proof of Proposition 2.11.

Example 2.19. Let $f(x) = x^2 + \frac{1}{2}x + \frac{1}{2}$. Then f has two non-archimedean roots β_1, β_2 with $|\beta_1|_2 = 2$, $|\beta_2|_2 = 1$ and two archimedean roots β_3, β_4 with $|\beta_3|_\infty = |\beta_4|_\infty = (1/\sqrt{2})$. Thus, the dominant non-archimedean root $\beta := \beta_1$ is a PC number. By Theorem 2.17, the recurrence (2.11) converges to

$$\beta = \cdots 11010001001001110001100011011001110011111101001 0 \bullet 1.$$

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3. *Beta-expansions of p-adic numbers*

Let $p \in \mathbb{P}$ and $\beta \in \mathbb{Q}_p$ with $|\beta|_p > 1$ and $\alpha \in \mathbb{Z}_p$. A *representation in base β (or beta-representation)* of α is an infinite sequence $(d_k)_{k\geq 1}$ with $d_k \in \mathbb{A}_p$, such that

$$\alpha = \sum_{k=1}^{\infty} \frac{d_k}{\beta^k}. \tag{3.1}$$

A particular beta-representation, called the *beta-expansion*, can be computed by the following algorithm.

Set $r^{(0)} = \alpha$ and let

$$d_k = \{\beta r^{(k-1)}\}_p, \quad r^{(k)} = \lfloor \beta r^{(k-1)} \rfloor_p$$

for $k \geq 1$. If

$$\mathcal{N} = [0, 1) \cap \{x \in \mathbb{A}_p : |x|_p \leq |\beta|_p\},$$

then $d_k \in \mathcal{N}$ for all $k \geq 1$. The set $\mathcal{N}$ is called the digit set of the beta-expansion. It is finite with $|\beta|_p$ elements. Furthermore, $r^{(k)} \in \mathbb{Z}_p$ for all $k \geq 1$. The above procedure defines a mapping

$$d_\beta : \mathbb{Z}_p \to \mathcal{N}^{\mathbb{N}}$$

from $\mathbb{Z}_p$ to $\mathcal{N}^{\mathbb{N}}$, the set of one-sided infinite sequences over $\mathcal{N}$, by

$$d_\beta(\alpha) := \bullet d_1 d_2 \cdots.$$

It follows that

$$r^{(k)} = \beta^k \left(\alpha - \sum_{i=1}^{k} d_i \beta^{-i} \right).$$

An equivalent definition is obtained by using the beta-transformation $T : \mathbb{Z}_p \to \mathbb{Z}_p$, which is given by the mapping $z \mapsto \lfloor \beta z \rfloor_p$. Note that $T(0) = 0$. For $k \geq 0$, define

$$T^0(x) := x \quad \text{and} \quad T^k(x) := T(T^{k-1}(x)).$$

Then $d_\beta(\alpha) = (d_k)_{k=1}^{\infty}$ if and only if $d_k = \{\beta T^{k-1}(\alpha)\}_p$ for all $k \geq 1$. If s is the one-sided shift defined on $\mathcal{N}^{\mathbb{N}}$, i.e.

$$s(\bullet d_1 d_2 \cdots) := \bullet d_2 d_3 \cdots,$$

we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}_p & \xrightarrow{T} & \mathbb{Z}_p \\ {\scriptstyle d_\beta}\downarrow & & \downarrow{\scriptstyle d_\beta} \\ \mathcal{N}^{\mathbb{N}} & \xrightarrow[s]{} & \mathcal{N}^{\mathbb{N}} \end{array}$$

By iteration, we obtain $d_\beta(T^k(\alpha)) = s^k(d_\beta(\alpha))$ for all $k \geq 0$.

Now let $\alpha \in \mathbb{Q}_p$ with $|\alpha|_p > 1$. Then there is an integer $n > 0$ such that

$$|\beta|_p^{n-1} < |\alpha|_p \leq |\beta|_p^n.$$

If

$$d_\beta(\beta^{-n}\alpha) = \bullet d_{-n} \cdots d_{-1} d_0 d_1 \cdots,$$

we define

$$d_\beta(\alpha) = d_{-n} \cdots d_{-1} d_0 \bullet d_1 d_2 \cdots.$$

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Definition 3.1. The β-expansion $d_\beta(\alpha)$ is called *eventually periodic* if the orbit $\{T^k(\alpha) : k \geq 0\}$ is finite. It is called *finite* if there exists a $k \geq 0$ such that $T^k(\alpha) = 0$ (i.e. if $0 \in \{T^k(\alpha) : k \geq 0\}$). Let

$$\mathbf{Per}(\beta) := \{\alpha \in \mathbb{Z}_p : d_\beta(\alpha) \text{ is eventually periodic}\} \quad \text{and}$$
$$\mathbf{Fin}(\beta) := \{\alpha \in \mathbb{Z}_p : d_\beta(\alpha) \text{ is finite}\}.$$

Then, clearly

$$\mathbf{Fin}(\beta) \subset \mathbf{Per}(\beta).$$

4. *Periodic beta-expansions*

The following theorem provides a p-adic analogue of the famous theorem of Bertrand and Schmidt (cf. [**8**, **23**]).

THEOREM 4.1. *Let β be a PC number. Then* $\mathbf{Per}(\beta) = \mathbb{Q}(\beta) \cap \mathbb{Z}_p$.

Proof. The proof for $\mathbf{Per}(\beta) \subset \mathbb{Q}(\beta) \cap \mathbb{Z}_p$ is trivial. Therefore, we only prove the opposite inclusion $\mathbb{Q}(\beta) \cap \mathbb{Z}_p \subset \mathbf{Per}(\beta)$.

Let $z \in \mathbb{Q}(\beta) \cap \mathbb{Z}_p$. First we prove that the orbit of z under T is bounded with respect to $|\cdot|_p$ and $|\cdot|_\infty$. Let $r_1^{(0)} := z$, $r_1^{(k)} := T^k(z)$ and $r_j^{(k)}$ be the corresponding conjugates (with the numeration defined as in Remark 2.6). If $(d_k)_{k\geq 1}$ is the beta-expansion of $r_1^{(0)}$, then

$$r_j^{(k)} = \beta_j^k \left(r_j^{(0)} - \sum_{i=1}^{k} d_i \beta_j^{-i} \right) \quad \text{for all } j \in \{1, \ldots, 2n\}. \tag{4.1}$$

Since $r_1^{(k)} \in \mathbb{Z}_p$ for all $k \geq 0$, it follows trivially that $|r_1^{(k)}|_p \leq 1$.

We consider the non-archimedean conjugates $r_j^{(k)}$, $j \in \{2, \ldots, n\}$. Since

$$|\beta_j|_p \leq 1 \quad \text{and} \quad |d_\ell|_p \leq |\beta|_p,$$

it follows by (2.2) that

$$\begin{aligned} |r_j^{(k)}|_p &\leq \max\left(|\beta_j^k r_j^{(0)}|_p, \max_{1\leq i\leq k} (|d_i \beta_j^{k-i}|_p) \right) \\ &\leq \max(|r_j^{(0)}|_p, |\beta|_p) < \infty. \end{aligned} \tag{4.2}$$

Therefore, $|r_j^{(k)}|_p$ is bounded for all k and j.

Now we consider the archimedean conjugates $r_j^{(k)}$, $j \in \{n+1, \ldots, 2n\}$. Let

$$\gamma := \max_{n+1\leq j\leq 2n} |\beta_j|_\infty.$$

Since

$$\gamma < 1 \quad \text{and} \quad |d_i|_\infty < 1,$$

it follows from (4.1) that

$$|r_j^{(k)}|_\infty \leq \gamma^k |r_j^{(0)}|_\infty + \sum_{i=1}^{k} \gamma^{k-i} < \infty.$$

We need a technical result.

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LEMMA 4.2. *Define the matrices*

$$B_p := (\beta_j^{-i}) \in \mathbb{Q}_p^{n\times n} \quad \text{and} \quad B_\infty := (\beta_{n+j}^{-i}) \in \mathbb{C}^{n\times n}$$

with $i \in \{0, \dots, n-1\}$ *and* $j \in \{1, \dots, n\}$. *If*

$$R_p^{(k)} := (r_1^{(k)}, \dots, r_n^{(k)}), \quad R_\infty^{(k)} := (r_{n+1}^{(k)}, \dots, r_{2n}^{(k)}),$$

then, for every $k \geq 0$, *there exists a unique n-tuple*

$$W^{(k)} := (w_0^{(k)}, \dots, w_{n-1}^{(k)}) \in \mathbb{A}_p^n$$

such that

$$R_p^{(k)} = q^{-1} W^{(k)} B_p \quad \textit{and} \quad R_\infty^{(k)} = q^{-1} W^{(k)} B_\infty$$

with $q \in \mathbb{N}$.

Proof. The lemma is proved is by induction. For $k = 0$, it follows from

$$\beta_j^n = a_1 \beta_j^{n-1} + \cdots + a_n,$$

that

$$r_j^{(0)} = q^{-1} \sum_{i=0}^{n-1} z_i \beta_j^i = q^{-1} \sum_{i=1}^{n} w_i^{(0)} \beta_j^{-i}. \tag{4.3}$$

If $k > 0$, then

$$\begin{aligned} r_j^{(k)} &= \beta_j r_j^{(k-1)} - d_k \\ &= q^{-1}\left(\beta_j \sum_{i=1}^{n} w_i^{(k-1)} \beta_j^{-i} - q d_k\right) \\ &= q^{-1}\left(w_1^{(k-1)} - q d_k + \sum_{i=1}^{n-1} w_{i+1}^{(k-1)} \beta_j^{-i}\right) \\ &= q^{-1}\left(\sum_{i=1}^{n} w_i^{(k)} \beta_j^{-i}\right). \end{aligned} \tag{4.4}$$

The lemma follows from the definition of the matrices B_p and B_∞. □

We now continue with the proof of Theorem 4.1. On $\mathbb{A}_p^n$, we define the vector norms

$$\|(a_1, \dots, a_n)\|_p := \max_{1\leq i\leq n} |a_i|_p \quad \text{and} \quad \|(a_1, \dots, a_n)\|_\infty := \max_{1\leq i\leq n} |a_i|_\infty.$$

The induced matrix norms on $\mathbb{A}_p^{n\times n}$ are given by

$$\|A\|_p = \max_{1\leq i,j\leq n} |a_{ij}|_p \quad \text{and} \quad \|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|_\infty.$$

Since B_p and B_∞ are invertible, and the $r_j^{(k)}$ are bounded for all $j \in \{1, \dots, 2n\}$, it follows by Lemma 4.2 that

$$\begin{aligned} \max_{0\leq j\leq n-1} |w_j^{(k)}|_p = \|W^{(k)}\|_p &= \|q R_p^{(k)} B_p^{-1}\|_p \\ &\leq \|R_p^{(k)}\|_p \|q B_p^{-1}\|_p < \infty \end{aligned}$$

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and, analogously,

$$\max_{0\le j\le n-1} |w_j|_\infty < \infty.$$

Since $w_j \in \mathbb{A}_p$ for all j, by Lemma 2.7, the w_j must be contained in a finite subset of $\mathbb{A}_p$. Therefore, the $r_1^{(k)}$ must be contained in a finite subset of $\mathbb{Q}(\beta)$ and thus, there exists an m such that $r_1^{(k+m)} = r_1^{(k)}$ for all k large enough. This implies that the beta-expansion of z is eventually periodic. □

Example 4.3. Since $\beta = (1/p)$ is the root of $f(x) = px - 1$, one easily verifies that $(1/p)$ is a PC number. Since $\mathbb{Q}(1/p) = \mathbb{Q}$, Theorem 4.1 implies that the numbers of $\mathbb{Q} \cap \mathbb{Z}_p$ admit eventually periodic p-adic expansions.

THEOREM 4.4. *Let $S \subset \mathbb{N}$ be an infinite set of positive integers, such that $1 \in S$. If $S \subset \mathbf{Per}(\beta)$, then β is a PC number or SC number.*

Proof. Since $1 \in S \subset \mathbf{Per}(\beta)$, it follows that $d_\beta(1) = \bullet d_1 d_2 \cdots$ is eventually periodic. If k is the length of the pre-period and ℓ is the length of the period, then $d_{i+\ell} = d_i$ for all $i \ge k+1$. Therefore,

$$d_\beta(1) = \bullet d_1 \cdots d_k d_{k+1} \cdots d_{k+\ell} d_{k+1} \cdots d_{k+\ell} \cdots$$

and thus,

$$(\beta^\ell - 1)\beta^k \left(1 - \frac{d_1}{\beta} - \cdots - \frac{d_k}{\beta^k}\right) - \beta^\ell \left(\frac{d_{k+1}}{\beta} + \cdots + \frac{d_{k+\ell}}{\beta^\ell}\right) = 0. \tag{4.5}$$

Note that in the case that $d_\beta(1)$ is finite, the second summand of (4.5) is zero. Since $d_i \in \mathbb{A}_p$, it follows that β is an algebraic integer over $\mathbb{A}_p$.

We consider the Newton polygon of (4.5). Since $d_1 = \{\beta \cdot 1\}_p$, it follows that

$$\nu_p(1) = 0, \quad \nu_p(-d_1) = \nu_p(\beta) < 0 \quad \text{and} \quad \nu_p(-d_j) \ge \nu_p(\beta) \quad \text{for } j \ge 2.$$

Therefore, the Newton polygon contains one edge with slope $-\nu_p(\beta) > 0$ and all other edges have slopes ≤ 0. By Proposition 2.5(i), there exists one non-archimedean root α of (4.5) with $\nu_p(\alpha) = \nu_p(\beta) < 0$ and all other non-archimedean roots $\tilde{\alpha}$ have $\nu_p(\tilde{\alpha}) \ge 0$. Since (4.5) is a multiple of the minimal polynomial of β, it follows that $\alpha = \beta$ and $|\tilde{\alpha}|_p \le 1$.

Now we examine the archimedean conjugates of β. Assume that there exists an archimedean conjugate β_j, $j \in \{n+1, \ldots, 2n\}$, such that $|\beta_j|_\infty > 1$. Since S is infinite, there exists an $N \in S$, with

$$N > \frac{1}{|\beta_j|_\infty - 1}. \tag{4.6}$$

It follows that

$$N = \sum_{i=1}^{s} \frac{e_i}{\beta^i} + \frac{r_1^{(s)}}{\beta^s} = \sum_{i=1}^{s} \frac{e_i}{\beta_j^i} + \frac{r_j^{(s)}}{\beta_j^s} \quad \text{for all } s \ge 0.$$

Since $N \in \mathbf{Per}(\beta)$, the sequence $d_\beta(N) = \bullet e_1 e_2 \cdots$ is eventually periodic and thus, the $r_1^{(s)}$ and $r_j^{(s)}$ can take only finitely many values. Therefore, the sequence

$$N = \sum_{i=1}^{\infty} \frac{e_i}{\beta_j^i}$$

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converges. Since $|e_i|_\infty < 1$, it follows that

$$N \leq \frac{1}{|\beta_j|_\infty - 1},$$

which contradicts (4.6). □

5. *Shift radix systems and finite beta-expansions*

Shift radix systems were introduced in [**2**] in order to provide a unified notation for two well-known types of number systems, namely, canonical number systems as well as beta-expansions of real numbers.

Let $n \geq 1$ be an integer and $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$. Let $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the real floor and ceiling functions respectively. To the vector $\mathbf{r}$, we associate the mapping $\tilde{\tau}_\mathbf{r} : \mathbb{Z}^n \to \mathbb{Z}^n$ as follows: if $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{Z}^n$ then let

$$\tilde{\tau}_\mathbf{r}(\mathbf{z}) := (z_2, \ldots, z_n, -\lfloor \mathbf{rz} \rfloor), \tag{5.1}$$

where $\mathbf{rz}$ is the euclidean inner product of $\mathbf{r}$ and $\mathbf{z}$, i.e.

$$\mathbf{rz} = r_1 z_1 + \cdots + r_n z_n.$$

Then $(\mathbb{Z}^n, \tilde{\tau}_\mathbf{r})$ is called *a shift radix system* (for short SRS) on $\mathbb{Z}^n$. Note that $\mathbf{0} = (0, \ldots, 0)$ is a fixed point of $\tilde{\tau}_\mathbf{r}$. We say that the orbit of $\mathbf{z}$ under $\tilde{\tau}_\mathbf{r}$ ends up in $\mathbf{0}$, if there exists a $k \geq 0$ such that $\tilde{\tau}_\mathbf{r}^k(\mathbf{z}) = \mathbf{0}$.

It is an important problem for canonical number systems as well as beta-expansions of real numbers to determine whether each number admits a finite expansion. In SRS language, this translates to the following question.

For which $\mathbf{r} \in \mathbb{Z}^n$ *do all orbits of* $(\mathbb{Z}^n, \tilde{\tau}_\mathbf{r})$ *end up in* $\mathbf{0}$*?*

In the present section, we will prove that beta-expansions of p-adic numbers can be described by a slight variation of this dynamical system, namely by $(\mathbb{Z}^n, \tau_\mathbf{r})$ with

$$\tau_\mathbf{r}(\mathbf{z}) = (z_2, \ldots, z_n, -\lceil \mathbf{rz} \rceil).$$

In Proposition 5.1, we will prove that for a fixed given vector $\mathbf{r} \in \mathbb{R}^n$, both of these systems show exactly the same behaviour. Let

$$\begin{aligned} \mathcal{D}_n &:= \{\mathbf{r} \in \mathbb{R}^d : \forall \mathbf{z} \in \mathbb{Z}^d, \text{ the sequence } (\tilde{\tau}_\mathbf{r}^k(\mathbf{z}))_{k \geq 1} \text{ is eventually periodic}\}, \\ \mathcal{D}_n^0 &:= \{\mathbf{r} \in \mathbb{R}^d : \forall \mathbf{z} \in \mathbb{Z}^d, \exists k > 0 : \tilde{\tau}_\mathbf{r}^k(\mathbf{z}) = \mathbf{0}\}. \end{aligned} \tag{5.2}$$

In [**2**, **3**], it is proved that the inclusions $\mathcal{D}_n^0 \subset \mathcal{D}_n$ and $\mathcal{E}_n \subset \mathcal{D}_n \subset \overline{\mathcal{E}}_n$ hold. It is easy to verify that $\mathcal{D}_1^0 = [0, 1)$ and $\mathcal{D}_1 = [-1, 1]$. However, a full description of $\mathcal{D}_2^0$ and $\mathcal{D}_2$ is as yet unknown. The sets $\mathcal{D}_n^0$ and $\mathcal{D}_n$ have been extensively studied in the last years (cf. [**2–5**, **7**, **11**, **14**, **20**, **25**]). In Figure 1, an approximation of $\mathcal{D}_2^0$ is shown. It is conjectured that $\mathcal{D}_n^0 \subset \mathcal{E}_n$, which is equivalent to $\mathcal{D}_n^0 \cap \partial\mathcal{E}_n = \emptyset$ (cf. [**3**]). Up to now, this conjecture has been proved only for $n \leq 3$ (cf. [**3**, **11**]). For a very recent survey on SRS, we refer to [**15**].

PROPOSITION 5.1. *Let* $\mathbf{r} \in \mathbb{R}^n$. *All orbits of* $(\mathbb{Z}^n, \tau_\mathbf{r})$ *end up in* $\mathbf{0}$ *if and only if* $\mathbf{r} \in \mathcal{D}_n^0$.

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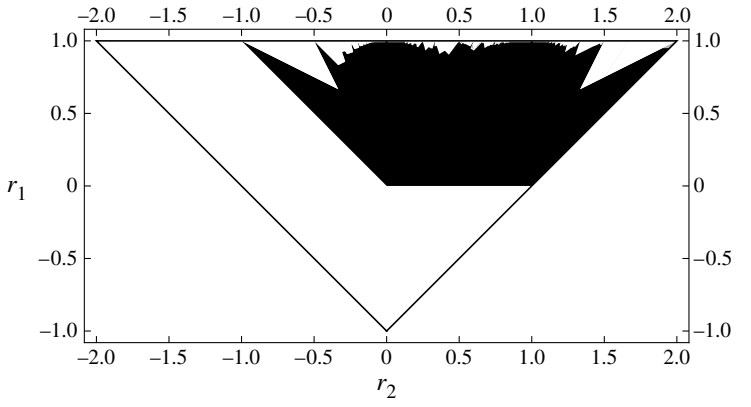


FIGURE 1. The set $\mathcal{D}_2^0$ (in black) inside $\mathcal{D}_2$ (the interior of the triangle).

Proof. At first we show that $\tau_{\mathbf{r}}(\mathbf{z}) = -\tilde{\tau}_{\mathbf{r}}(-\mathbf{z})$ for all $\mathbf{z} \in \mathbb{Z}^n$. Indeed, consider an arbitrary $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{Z}^n$ and note that for any real number x we have $\lceil x \rceil = -\lfloor -x \rfloor$. We obtain

$$\begin{aligned}\tau_{\mathbf{r}}(\mathbf{z}) &= (z_2, \ldots, z_n, -\lceil \mathbf{rz} \rceil) = (z_2, \ldots, z_n, \lfloor -\mathbf{rz} \rfloor) \\ &= -(-z_2, \ldots, -z_n, -\lfloor -\mathbf{rz} \rfloor) = -\tilde{\tau}_{\mathbf{r}}(-\mathbf{z}).\end{aligned}$$

By induction, it follows that

$$\tau_{\mathbf{r}}^k(\mathbf{z}) = -\tilde{\tau}_{\mathbf{r}}^k(-\mathbf{z}). \tag{5.3}$$

Now we easily see that all orbits of $(\mathbb{Z}^n, \tau_{\mathbf{r}})$ end up in $\mathbf{0}$ if and only if all orbits of $(\mathbb{Z}^n, \tilde{\tau}_{\mathbf{r}})$ do so. □

In order to prove Theorem 5.4, which is the main result of this section, we need the following preliminary results.

PROPOSITION 5.2. *If U, $V \in \mathbb{Q}_p$, then*

$$|U - \{U + V\}_p|_p = |\lfloor U + V \rfloor_p - V|_p = \begin{cases} |\lfloor U \rfloor_p|_p & \text{if } |V|_p \le 1, \\ |\{V\}_p|_p & \text{if } |V|_p > 1, \end{cases}$$

or equivalently,

$$v_p(U - \{U + V\}_p) = v_p(\lfloor U + V \rfloor_p - V) = \begin{cases} v_p(\lfloor U \rfloor_p) & \text{if } v_p(V) \ge 0, \\ v_p(\{V\}_p) & \text{if } v_p(V) < 0. \end{cases}$$

Proof. The proof is straightforward. □

PROPOSITION 5.3. *Let β be an arbitrary element of $\mathbb{Q}_p$ with $|\beta|_p > 1$, and let $z \in \mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p$ have purely periodic beta-expansion with period ℓ. Then $z \in \mathbb{A}_p[\beta] \cap \mathbb{Z}_p$.*

Proof. Assume $z \in \mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p$ is purely periodic with period ℓ. Let $d_\beta(z) = \bullet d_1 d_2 \ldots$. Since $z \in \mathbb{A}_p[\beta^{-1}]$, there is an m such that $\beta^{m\ell} z \in \mathbb{A}_p[\beta]$. Therefore,

$$z = \beta^{m\ell} z - d_1 \beta^{m\ell - 1} - \cdots - d_{m\ell} \in \mathbb{A}_p[\beta].$$ □

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Now we are able to state the main result of this section.

THEOREM 5.4. *Let $\beta \in \mathbb{Q}_p$ with $|\beta|_p > 1$. Then*

$$\mathbf{Fin}(\beta) = \mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p, \tag{F}$$

if and only if β is a PC number and its minimal polynomial

$$x^n - a_1 x^{n-1} - \cdots - a_n, \quad a_i \in \mathbb{A}_p[x] \tag{5.4}$$

fulfils

$$\max_{2 \le i \le n} |a_i|_p < |a_1|_p \quad \textit{and} \tag{5.5a}$$

$$-\mathbf{a} \in \mathcal{D}_n^0 \quad \textit{where } \mathbf{a} := (a_n, \ldots, a_1). \tag{5.5b}$$

Proof. We will prove first that each PC number β fulfilling (5.5) has property (F). Since it is trivial that $\mathbf{Fin}(\beta) \subset \mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p$, we will prove only the opposite inclusion. The proof runs in two steps. Condition (5.5*a*) ensures that after some preliminary phase, the dynamical system reduces to a classical SRS. Condition (5.5*b*) ensures that all orbits of this SRS end up in $\mathbf{0}$.

Let $z \in \mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p$. Since β is a PC number, it follows from Theorem 4.1 that

$$\mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p \subset \mathbb{Q}(\beta) \subset \mathbf{Per}(\beta).$$

Thus, z has an eventually periodic expansion. Without loss of generality, we can assume that $z \in \mathbb{A}_p[\beta]$. Otherwise, we replace z by an element from the orbit with purely periodic expansion and apply Proposition 5.3. Let

$$\mathcal{B} = \{1, \beta, \ldots, \beta^{n-1}\} \quad \text{and} \quad \mathcal{V} = \{v_1, \ldots, v_n\}$$

where

$$v_j = \beta^{n-j} - a_1 \beta^{n-j-1} - \cdots - a_{n-j} \tag{5.6a}$$

$$= \frac{a_{n-j+1}}{\beta} + \cdots + \frac{a_n}{\beta^j} \tag{5.6b}$$

for $j \in \{1, \ldots, n\}$. Note $v_n = 1$. Let $\mathbf{v} = (v_1, \ldots, v_n)$. Then, both $\mathcal{B}$ and $\mathcal{V}$ are two different bases of $\mathbb{A}_p[\beta]$ considered as a lattice over $\mathbb{A}_p$. Using (5.6), the coordinates with respect to $\mathcal{V}$ can be computed from the coordinates with respect to $\mathcal{B}$ by a linear system of equations.

In this way, we define a bijection $\varphi : \mathbb{A}_p[\beta] \to \mathbb{A}_p^n$ by

$$\varphi(z) := (z_1, \ldots, z_n).$$

By construction, φ^{-1} is given by

$$\varphi^{-1}((z_1, \ldots, z_n)) = z_1 v_1 + \cdots + z_n v_n.$$

In base $\mathcal{V}$, multiplication by β is represented by the matrix

$$M := \begin{pmatrix} 0 & \cdots & \cdots & 0 & a_n \\ 1 & \ddots & & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & a_2 \\ 0 & \cdots & 0 & 1 & a_1 \end{pmatrix}$$

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such that

$$\varphi(\beta z) = \varphi(z)M.$$

Let $\mathbf{e} := \varphi(1) = (0, \ldots, 0, 1)$. Since

$$T(z) = \beta z - \{\beta z\}_p,$$

the beta-transformation with respect to $\mathcal{V}$ takes the form

$$\sigma : \mathbb{A}_p^n \to \mathbb{A}_p^n \quad \text{with } \mathbf{z} \mapsto \mathbf{z}\, M - \{\mathbf{z}\, M\mathbf{v}^\top\}_p \mathbf{e}. \tag{5.7}$$

Then

$$\varphi(T(z)) = \sigma(\varphi(z)) \quad \text{and} \quad T(\varphi^{-1}(\mathbf{z})) = \varphi^{-1}(\sigma(\mathbf{z})),$$

which is indicated in the following commutative diagram:

$$\begin{array}{ccc} \mathbb{A}_p[\beta] & \xrightarrow{T} & \mathbb{A}_p[\beta] \\ {\scriptstyle\varphi^{-1}}\uparrow\downarrow{\scriptstyle\varphi} & & {\scriptstyle\varphi^{-1}}\uparrow\downarrow{\scriptstyle\varphi} \\ \mathbb{A}_p^n & \xrightarrow[\sigma]{} & \mathbb{A}_p^n \end{array}$$

Substituting M, $\mathbf{v}$ and $\mathbf{e}$ into (5.7), we can express the map σ as

$$\begin{aligned} &\sigma : (z_1, \ldots, z_n) \mapsto (z_2, \ldots, z_{n+1}) \quad \text{with} \\ &z_{n+1} = a_n z_1 + \cdots + a_1 z_n - \{a_n z_1 + \cdots + a_1 z_n + v_1 z_2 + \cdots + v_{n-1} z_n\}_p. \end{aligned}$$

The kth iterate of σ is given by

$$\sigma^k((z_1, \ldots, z_n)) = (z_{k+1}, \ldots, z_{k+n}),$$

where

$$z_{k+n} = U_k - \{U_k + V_k\}_p$$

with

$$\begin{aligned} U_k &:= a_n z_k + \cdots + a_1 z_{k+n-1} \quad \text{and} \\ V_k &:= v_1 z_{k+1} + \cdots + v_{n-1} z_{k+n-1}. \end{aligned} \tag{5.8}$$

From (5.5*a*), (5.6*b*) and $|a_1|_p = |\beta|_p$, it follows that

$$\max_{1 \le i \le n-1} |v_i|_p < 1 \tag{5.9}$$

and therefore,

$$|V_k|_p \le \max_{1 \le i \le n-1} |v_i z_{k+i}|_p < \max_{k+1 \le i \le k+n-1} |z_i|_p.$$

By Proposition 5.2, we get the following implications.

- If $|V_k|_p > 1$, it follows that

$$|z_{k+n}|_p = |V_k|_p < \max_{k+1 \le i \le k+n-1} |z_i|_p.$$

- If $|V_k|_p \le 1$, it follows that

$$|z_{k+n}|_p = |\lfloor U_k \rfloor_p|_p \le 1.$$

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Therefore, there must exist some $k_0 \geq 1$ such that

$$\max_{0 \leq j \leq n-1} |z_{k+j}|_p \leq 1 \tag{5.10}$$

for all $k > k_0$. Since $z_k \in \mathbb{A}_p$, it follows that

$$\sigma^k((z_1, \ldots, z_n)) = (z_{k+1}, \ldots, z_{k+n}) \in \mathbb{Z}^n \tag{5.11}$$

holds for all $k \geq k_0$. From (5.10), it follows by (5.9) that $|V_k|_p < 1$. Therefore,

$$\{U_k + V_k\}_p = \{U_k\}_p$$

and thus,

$$z_{k+n} = U_k - \{U_k + V_k\}_p = \lfloor U_k \rfloor_p.$$

Since $U_k \in \mathbb{A}_p$, it follows that

$$\lfloor U_k \rfloor_p = \lfloor U_k \rfloor = -\lceil -U_k \rceil$$

(cf. Remark 2.2). Thus,

$$\begin{aligned}\sigma((z_k, \ldots, z_{k+n-1})) &= (z_{k+1}, \ldots, z_{k+n-1}, -\lceil -U_k \rceil) \\ &= \tau_{-\mathbf{a}}((z_k, \ldots, z_{k+n-1})).\end{aligned}$$

By (5.3), it follows that

$$\begin{aligned}\sigma^k((z_1, \ldots, z_n)) &= \tau_{-\mathbf{a}}^{k-k_0}(\sigma^{k_0}((z_1, \ldots, z_n))) \\ &= -\tilde{\tau}_{-\mathbf{a}}^{k-k_0}(-\sigma^{k_0}((z_1, \ldots, z_n)))\end{aligned}$$

for all $k \geq k_0$. Since $-\mathbf{a} \in \mathcal{D}_n^0$, there must exist some $k \geq k_0$ with

$$\sigma^k((z_1, \ldots, z_n)) = \mathbf{0}.$$

Thus, each $z \in \mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p$ admits a finite representation with respect to β.

Now we will prove the converse direction of Theorem 5.4. Suppose that

$$\mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p = \mathbf{Fin}(\beta).$$

Since $\mathbb{N} \subset \mathbf{Fin}(\beta) \subset \mathbf{Per}(\beta)$, it follows by Theorem 4.4 that β is a PC number or SC number. Without loss of generality, we can assume that the minimal polynomial has the form (5.4). In order to exclude the case of SC numbers, we distinguish the following cases.

(i) For $n = 1$, there do not exist any SC numbers.

(ii) For $n = 2$, it is proved in [**3**], that $\mathcal{D}_2^0 \cap \partial\mathcal{E}_2 = \emptyset$. If there exists an archimedean root of (5.4) that is located on the complex unit circle, it follows that $-\mathbf{a} \in \partial\mathcal{E}_2$, which contradicts (5.5*b*).

(iii) Let $n > 2$. If (5.4) is the minimal polynomial of an SC number, it must be self-reciprocal (cf. Remark 2.10). Then $a_1 = a_{n-1}$, which contradicts (5.5*a*).

Thus, β is a PC number. We will now prove that each of the converses of (5.5*a*) and (5.5*b*) contradict (F).

In order to prove the necessity of (5.5*a*), we follow the idea of the proofs of [**22**, Lemma 2.4] and [**20**, Theorem 5.4]. Suppose that

$$\max_{2 \leq i \leq n} |a_i|_p \geq |a_1|_p \quad \text{or equivalently} \quad \min_{2 \leq i \leq n} \nu_p(a_i) \leq \nu_p(a_1). \tag{5.12}$$

We will construct an element $z \in \mathbb{A}_p[\beta]$ that does not have a finite representation.

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At first, we will prove that there exists some index $h \in \{1, \ldots, n-1\}$, such that $v_p(v_h) \le 0$. Define

$$h_0 := \max\left\{h \in \{2, \ldots, n\} : v_p(a_h) = \min_{2 \le j \le n} v_p(a_j)\right\}, \tag{5.13}$$

i.e. h_0 is the minimal index $h \in \{2, \ldots, n\}$ such that $v_p(a_h)$ attains its minimal value. Then, by (5.12) and (5.13), it follows that $2 \le h_0 \le n$ and thus,

$$1 \le n - h_0 + 1 \le n - 1. \tag{5.14}$$

By (5.13), it follows that $v_p(a_j) > v_p(a_{h_0})$ for all $j > h_0$ and thus, by (5.6),

$$\begin{aligned} v_p(v_{n-h_0+1}) &= v_p\left(\frac{a_{h_0}}{\beta} + \cdots + \frac{a_n}{\beta^{n-h_0+1}}\right) \\ &= v_p(a_{h_0}) - v_p(\beta) \le v_p(a_{h_0}) - v_p(a_1) \le 0. \end{aligned}$$

If

$$i_0 := \min\left\{i \in \{1, \ldots, n-1\} : v_p(v_i) = \min_{1 \le j \le n-1} v_p(v_j)\right\}, \tag{5.15}$$

then $v_p(v_{i_0}) \le 0$. For $(z_1, \ldots, z_n) \in \mathbb{A}_p^n$, define

$$j_0((z_1, \ldots, z_n)) := \begin{cases} \max\left\{i \in \{2, \ldots, n\} : v_p(z_i) = \min_{i_0+1 \le j \le n} v_p(z_j)\right\} & \text{if } \min_{i_0+1 \le j \le n} v_p(z_j) < 0; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, if $j_0((z_1, \ldots, z_n)) > 0$, it follows that $(z_1, \ldots, z_n) \neq \mathbf{0}$. If

$$\mathbf{z}^{(0)} := (z_1^{(0)}, \ldots, z_n^{(0)}) = (0, \ldots, 0, p^{-1}),$$

then $j_0(\mathbf{z}^{(0)}) = n$ and thus,

$$i_0 + 1 \le j_0(\mathbf{z}^{(0)}) \le n. \tag{5.16}$$

For $k \ge 0$, let

$$\mathbf{z}^{(k)} := (z_1^{(k)}, \ldots, z_n^{(k)}) = (z_{k+1}, \ldots, z_{k+n}) = \sigma^k(\mathbf{z}^{(0)}).$$

We will show that $\mathbf{z}^{(0)}$ has an infinite representation by proving that

$$i_0 + 1 \le j_0(\mathbf{z}^{(k)}) \le n \tag{5.17}$$

for all $k \ge 0$ and thus,

$$\mathbf{z}^{(k)} = \sigma^k(\mathbf{z}^{(0)}) \neq \mathbf{0}.$$

We will prove (5.17) by induction. By (5.16), equation (5.17) holds for $k = 0$. Thus, we can proceed to the induction step. Suppose that (5.17) holds for $k - 1 \ge 0$ and note that

$$\mathbf{z}^{(k)} = (z_1^{(k)}, \ldots, z_n^{(k)}) = (z_2^{(k-1)}, \ldots, z_n^{(k-1)}, z_n^{(k)}). \tag{5.18}$$

Let $j_0 := j_0(\mathbf{z}^{(k-1)})$. We distinguish the following cases.

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(i) Let $j_0 > i_0 + 1$. By (5.18), it follows that

$$\begin{aligned}\min_{i_0+1\le j\le n} \nu_p(z_j^{(k)}) &\le \min\{\nu_p(z_{j_0}^{(k-1)}),\, \nu_p(z_n^{(k)})\}\\ &= \min\{\nu_p(z_{j_0-1}^{(k)}),\, \nu_p(z_n^{(k)})\} \quad \text{(by the definition of } j_0)\\ &< 0.\end{aligned}$$

Thus, $j_0(\mathbf{z}^{(k)}) = j_0 - 1$ or $j_0(\mathbf{z}^{(k)}) = n$. Both of these inequalities imply that

$$i_0 + 1 \le j_0(\mathbf{z}^{(k)}) \le n. \tag{5.19}$$

(ii) Let $j_0 = i_0 + 1$. From (5.8) it follows that

$$\begin{aligned}U_k &:= a_n z_1^{(k-1)} + \cdots + a_1 z_n^{(k-1)} \quad \text{and}\\ V_k &:= v_1 z_2^{(k-1)} + \cdots + v_{n-1} z_n^{(k-1)}.\end{aligned}$$

The definitions of i_0 and j_0 imply that

$$\begin{aligned}\nu_p(v_{i_0}) \le 0, \qquad & \nu_p(v_i) > \nu_p(v_{i_0}) \quad \text{for } i < i_0,\\ & \nu_p(v_i) \ge \nu_p(v_{i_0}) \quad \text{for } i \ge i_0,\end{aligned}$$

and

$$\begin{aligned}\nu_p(z_{j_0}^{(k-1)}) < 0, \qquad & \nu_p(z_j^{(k-1)}) > \nu_p(z_{j_0}^{(k-1)}) \quad \text{for } j > j_0,\\ & \nu_p(z_j^{(k-1)}) \ge \nu_p(z_{j_0}^{(k-1)}) \quad \text{for } j \le j_0.\end{aligned}$$

Hence,

$$\nu_p(v_{i_0} z_{i_0+1}^{(k-1)}) = \nu_p(v_{i_0} z_{j_0}^{(k-1)}) < \nu_p(v_i z_{i+1}^{(k-1)}) \quad \text{for } i \ne i_0.$$

By (2.2), it follows that

$$\nu_p(V_k) = \nu_p(v_1 z_2^{(k-1)} + \cdots + v_{n-1} z_n^{(k-1)}) = \nu_p(v_{i_0} z_{i_0+1}^{(k-1)}) \le \nu_p(z_{j_0}^{(k-1)}),$$

and therefore, by Proposition 5.2,

$$\nu_p(z_n^{(k)}) = \nu_p(U_k - \{U_k + V_k\}_p) = \nu_p(\{V_k\}_p) \le \nu_p(z_{j_0}^{(k-1)}).$$

Thus,

$$i_0 + 1 \le j_0(\mathbf{z}^{(k+1)}) \le n \tag{5.20}$$

and we are also done in this case.

Therefore, the necessity of (5.5*a*) is proved.

Assume now that (F) holds but (5.5*b*) does not hold, i.e. $-\mathbf{a} \notin \mathcal{D}_n^0$. By (5.2), there exists a $\mathbf{z} \in \mathbb{Z}^n$ such that $\tilde{\tau}_{-\mathbf{a}}^k(\mathbf{z})$ does not end up in $\mathbf{0}$. Therefore,

$$\tau_{-\mathbf{a}}^k(-\mathbf{z}) = -\tilde{\tau}_{-\mathbf{a}}^k(\mathbf{z})$$

does not end up in $\mathbf{0}$. Thus, $\varphi^{-1}(-\mathbf{z}) \in \mathbb{A}_p[\beta]$ does not have a finite expansion. This is a contradiction to (F) and thus, (5.5*b*) is a necessary condition. □

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Remark 5.5. Condition (5.5*b*) indicates a relation to canonical number systems. In order to explain this connection in more detail, we recall Example 2.14. If the polynomial

$$\tilde{g}(x) = \tilde{a}_n x^n + \tilde{a}_{n-1} x^{n-1} + \cdots + \tilde{a}_1 x + p^k \in \mathbb{Z}[x]$$

gives raise to a canonical number system, then by [**21**, Theorem 5.3],

$$\left(\frac{\tilde{a}_n}{p^k}, \ldots, \frac{\tilde{a}_1}{p^k} \right) \in \mathcal{D}_n^0.$$

Thus, the archimedean roots of

$$\tilde{f}(x) := x^n \tilde{g}\left(\frac{1}{x}\right) = p^k x^n + \tilde{a}_1 x^{n-1} + \cdots + \tilde{a}_{n-1} x + \tilde{a}_n$$

are strictly inside the complex unit circle. If we define

$$a_j := -\frac{\tilde{a}_j}{p^k}$$

for $j \in \{1, \ldots, n\}$, then the polynomial

$$x^n - a_1 x^{n-1} - \cdots - a_n$$

fulfils (5.5*b*).

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