Ergod. Th. & Dynam. Sys. (2014), **34**, 1057–1076 © *Cambridge University Press, 2013*
doi:10.1017/etds.2012.186

IP-rigidity and eigenvalue groups

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(*Received* 18 *June* 2012 *and accepted in revised form* 13 *November* 2012)

Abstract. We examine the class of increasing sequences of natural numbers which are IP-rigidity sequences for some weakly mixing probability-preserving transformation. This property is closely related to the uncountability of the eigenvalue group of a corresponding non-singular transformation. We give examples, including a super-lacunary sequence which is not IP-rigid.

1. *Introduction*

Let $(X, \mathcal{B}, m)$ be a standard, continuous, probability space, that is, $(X, \mathcal{B})$ is a Polish space equipped with its Borel sets and a non-atomic $m \in \mathcal{P}(X)$ (the collection of probability measures on $(X, \mathcal{B})$). We will denote by $\mathtt{PPT} = \mathtt{PPT}(X, \mathcal{B}, m)$ the collection of invertible, probability-preserving transformations of $(X, \mathcal{B}, m)$. This is a Polish space when equipped with the *coarse topology* with basic neighborhoods of the form

$$U(T_0, f_1, \ldots, f_N; \epsilon) := \{T \in \mathtt{PPT} : \|f_j \circ T^s - f_j \circ T_0^s\|_{L^2(m)} < \epsilon \ \forall 1 \le j \le N, s = \pm 1\},$$

where $T_0 \in \mathtt{PPT}(X, \mathcal{B}, m)$ and $f_1, \ldots, f_N \in L^2(m)$.

Equipped with the coarse topology, $\mathtt{PPT}(X, \mathcal{B}, m)$ is a topological group under composition. It is embedded into the Polish, topological group $\mathcal{U}(L^2(m))$ of unitary operators on $L^2(m)$ equipped with the strong operator topology by the *Koopman representation* $U_T f := f \circ T (T \in \mathtt{PPT}(X, \mathcal{B}, m),\ f \in L^2(m))$. Accounts of the spectral theory of unitary operators can be found in [**KT**] and [**N2**].

1.1. *Recurrence and rigidity.*

A sequence $q \in \mathbb{N}^{\mathbb{N}}(\uparrow) := \{q \in \mathbb{N}^{\mathbb{N}} : \ q_n < q_{n+1}$ for all $n \ge 1\}$ is called a sequence of *recurrence* for $T \in \mathtt{PPT}(X, \mathcal{B}, m)$ if $\limsup_{n\to\infty} \mu(A \cap T^{-q_n} A) > 0$ for each $A \in \mathcal{B}$ of positive measure.

Rigidity is a stronger version of recurrence.

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A sequence $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ is called a *rigidity sequence* for $T \in \mathtt{PPT}(X, \mathcal{B}, m)$ if $\mu(T^{q_n} A \triangle A) \to 0$ for each $A \in \mathcal{B}$; equivalently,

$$T^{q_n} \xrightarrow[n\to\infty]{\mathtt{PPT}} \mathtt{Id}. \tag{⚔}$$

We denote by σ_T the *restricted spectral type* of T (i.e., $U_T|_{L^2(m)_0}$). It is the equivalence class of all maximal spectral measures. Recall that the *spectral measure* of $f \in L^2(m)_0$ is the unique finite positive Borel measure μ on the additive circle $\mathbb{T}$ satisfying $\widehat{\mu}(k) := \int \chi_k(t)\, dt = \int_X f \circ T^k \cdot \overline{f}\, dm$, $k \in \mathbb{Z}$, where $\chi_k(t) := e^{2\pi i k t}$. It follows that (⚔) is equivalent to σ_T having the *Dirichlet property* along q, that is,

$$\chi_{q_n} \xrightarrow[n\to\infty]{L^2(\mathbb{T},\sigma_T)} 1. \tag{‡}$$

1.2. *Gaussian automorphism.* Let $\mu_0 \in \mathcal{P}(\mathbb{T})$ be a continuous, Borel, probability measure, and let μ be the symmetrization of μ_0 defined by $\mu(A) := \frac{1}{2}(\mu_0(A) + \mu_0(-A))$; then $\widehat{\mu}(n) \in \mathbb{R}$, and there is a real-valued Gaussian process with spectral measure μ.

The *Gaussian automorphism* with spectral measure μ is the shift $(X, \mathcal{B}, m, T)$ of this process. It is (see, e.g., [**CFS**]) weakly mixing and indeed its spectral type is $\sigma_T = \sum_{n\geq 0} \mu^{n*}$, where μ^{n*} denotes the n-fold convolution of μ with itself.

It follows that (⚔) holds for the Gaussian automorphism with spectral measure μ if and only if (‡) is satisfied with σ_T replaced by μ.

Rigidity sequences for non-trivial transformations must be sparse. In particular, unless $T \in \mathtt{PPT}$ is purely periodic, any rigidity sequence for T has at most finite intersection with each of its translates, and therefore has zero Banach density.

Additional properties of rigidity sequences are studied in [**BJLR**] and [**EG**] including the rigidity properties of **lacunary** and **super-lacunary** sequences, a sequence $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ being called *lacunary* if $(q_{n+1}/q_n) \geq \lambda > 1$ for all $n \geq 1$, and *super-lacunary* if $(q_{n+1}/q_n) \xrightarrow[n\to\infty]{} \infty$.

1.3. *Rigid factors, mild mixing and IP-sets.* Let $T \in \mathtt{PPT}(X, \mathcal{B}, m)$ and let $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$. It is well known that the collection of sets

$$\mathcal{R}(q) := \left\{ A \in \mathcal{B} : m(A \Delta T^{q_n} A) \xrightarrow[n\to\infty]{} 0 \right\} \tag{🔔}$$

is a T-invariant σ-algebra. It corresponds to the maximal factor of T which is rigid along q. The transformation $T \in \mathtt{PPT}(X, \mathcal{B}, m)$ is called *mildly mixing* if it has no non-trivial, rigid factor along any $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ (as in [**FW**]).

Since the spectral type σ_S of a factor S of T is absolutely continuous with respect to σ_T, it is evident that T has some non-trivial rigid factor if and only if there exists a *Dirichlet measure* $\mu \ll \sigma_T$ (that is, one satisfying (‡) along some $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$).

An *IP-set* is a collection of 'finite-sum sets' of the form

$$\mathtt{FS}\,(q) := \{q(F) : F \in \mathcal{F}\},$$

where $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$, and for

$$F \in \mathcal{F} := \{\text{finite, non-empty subsets of } \mathbb{N}\},\ q(F) := \sum_{j\in F} q_j.$$

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This notion appears in combinatorics, ultrafilter theory, topological dynamics and ergodic theory (see [**Fu, HS**]).

As shown in [**Fu**], $T \in \mathtt{PPT}(X, \mathcal{B}, m)$ is mildly mixing if and only if there exists $K \subset \mathbb{N}$ which intersects with every finite-sum set, such that

$$m(A \cap T^{-n}B) \xrightarrow[n\to\infty, n\in K]{} m(A)m(B) \quad \text{for all } A, B \in \mathcal{B};$$

equivalently (see [**HMP1**]), T is not mildly mixing if and only if there exists $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$, such that

$$\inf_{n\in \mathtt{FS}(q)} |\widehat{\mu}(n)| > 0.$$

The considerations involved give rise to the notion of IP-convergence.

1.4. *IP-convergence.* Let $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$. We will say that a sequence $a : \mathbb{N} \to Z$ (a metric space) *converges IP to* $L \in Z$ *along* $\mathtt{FS}(q)$,

$$\text{written } a(n) \xrightarrow[n\to\infty]{\mathtt{FS}(q)} L \quad \text{in } Z, \text{ if}$$

$$a(q(F)) \xrightarrow[F\in\mathcal{F}, \min F\to\infty]{Z} L.$$

This paper is about IP-rigidity.

1.5. *IP-rigidity.* We will say that $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ is an *IP-rigidity sequence for* T and that T is *IP-rigid along* q if

$$T^n \xrightarrow[n\to\infty]{\mathtt{FS}(q)} \mathtt{Id} \text{ in } \mathtt{PPT}.$$

Let

$$\mathtt{IPRWM} := \{q \in \mathbb{N}^{\mathbb{N}}(\uparrow) : \exists T \in \mathtt{PPT}(X, \mathcal{B}, m), \text{ weakly mixing and IP-rigid along } q\}.$$

Any rigid transformation is IP-rigid on some subsequence (see [**Fu**]). On the other hand, if a transformation is IP-rigid on $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$, then it is rigid along much thicker subsequences (see §6).

Similarly to (**ℵ**), for $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$, the collection

$$\mathcal{R}_{\mathtt{IP}}(q) := \left\{A \in \mathcal{B} : m(A \Delta T^n A) \xrightarrow[n\to\infty]{\mathtt{FS}(q)} 0\right\}$$

is a T-invariant σ-algebra. It corresponds to the maximal factor of T which is IP-rigid along q. As above, T has a non-trivial factor, IP-rigid along q, if and only if $\lim_{N\to\infty} \inf_{F\in\mathcal{F}, \min F\geq N} |\widehat{\sigma_T}(q(F))| > 0$.

The existence of IP-Dirichlet measures along $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ is related to the groups

$$G_p(q) := \left\{t \in \mathbb{T} : \sum_{n=1}^{\infty} \|q_n t\|^p < \infty\right\} \quad (0 < p < \infty) \quad \text{and}$$

$$G_\infty(q) := \left\{t \in \mathbb{T} : \|q_n t\| \xrightarrow[n\to\infty]{} 0\right\},$$

where, for $x \in \mathbb{R}$, $\|x\| := \min_{n\in\mathbb{Z}} |x - n|$.

These groups are discussed in connection with diophantine approximations in [**E, ET**], and their ergodic significance is discussed in [**AN, HMP2**].

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1.6. *Results.*

PROPOSITION 1. *Suppose that* $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$*; then*

$$|G_1(q)| > \aleph_{\mathbf{0}} \Longrightarrow q \in \texttt{IPRWM}.$$

Proposition 1 (which is folklore; see also §3.1.1 in [**BJLR**], the proof of Proposition 3.2 therein) can be proved using Propositions 2.1 and 2.2 (below).

THEOREM 2. *If* $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$*, then*

$$q \in \texttt{IPRWM} \Longrightarrow |G_2(q)| > \aleph_{\mathbf{0}}.$$

Theorem 2 also provides an answer to a question in [**BJLR**]:
☺ *if* $q \in \texttt{IPRWM}$*, then some irrational rotation is rigid along* q, because if $|G_2(q)| > \aleph_0$, then there exists $a \in G_2(q)\backslash\mathbb{Q} \subset G_\infty(q)\backslash\mathbb{Q}$. It follows from [**BJLR**] that rotation of $\mathbb{T}$ by α is rigid along q. □

The converse of Theorem 2 holds for **arithmetic** sequences and **non-arithmetic** sequences. Unfortunately, there are also many sequences which are neither arithmetic nor non-arithmetic.

A sequence $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ is called *arithmetic* if it is either:

- *multiplicative* in the sense that $q_n | q_{n+1}$ for all $n \geq 1$; or it is the
- *principal denominator sequence* of some $\alpha \in \mathbb{T}\backslash\mathbb{Q}$, being defined by $q_0 = 1$, $q_1 = a_1$, $q_{n+1} := a_{n+1}q_n + q_{n-1}$, where $\alpha = [0; a_1, a_2, \ldots]$ is the continued fraction expansion of α.

The following proposition characterizing arithmetic members of IPRWM is an extension of a result from [**BJLR**, Section 3.4.5], where only implication (c)⇒(a) has been proved.

PROPOSITION 3. *Let* $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ *be arithmetic. The following statements are equivalent.*

$$\text{(a) } \overline{\lim_{n\to\infty}} \frac{q_{n+1}}{q_n} = \infty, \quad \text{(b) } |G_1(q)| > \aleph_{\mathbf{0}},$$
$$\text{(c) } q \in \texttt{IPRWM}, \quad \text{(d) } |G_2(q)| > \aleph_{\mathbf{0}}.$$

A sequence $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ is called *non-arithmetic* if, for n large, we have

- $q_{n+1} = a_n q_n + \Delta_n$, where $a_n \in \mathbb{N}$, $\Delta_n \in \mathbb{Z}\backslash\{0\}$ with $\sup_n |\Delta_n| < \infty$.

Non-arithmetic sequences with $\Delta_n \equiv 1$ were introduced in [**ET**], and are called here *Erdos–Taylor* sequences.

PROPOSITION 4. *If* $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$*, is non-arithmetic, then*

$$\text{(i) } \sum_{n\geq 1}\left(\frac{q_n}{q_{n+1}}\right)^2 < \infty \iff \text{(ii) } q \in \texttt{IPRWM} \iff \text{(iii) } |G_2(q)| > \aleph_{\mathbf{0}}.$$

We will see that there are super-lacunary Erdos–Taylor sequences $q^{(0)}$ and $q^{(1)} \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ satisfying

$$q^{(0)} \in \texttt{IPRWM} \quad \text{and} \quad G_1(q^{(0)}) = \{0\} \quad \text{and} \quad G_2(q^{(1)}) = \{0\}.$$

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1.7. *Eigenvalue groups and Theorem 2.* Groups of the form G_2 appear as eigenvalue groups (see [**AN**]). Eigenvalue groups and rigidity are related as follows.

An ergodic probability-preserving transformation S is not mildly mixing (i.e., has a rigid factor) if and only if there is a conservative, ergodic, non-singular transformation T such that $S \times T$ is not ergodic (see [**FW**]).

By Keane's ergodic multiplier theorem (see, e.g., §2.7 of [**A1**]), this situation is characterized by $\sigma_S(e(T)) > 0$, where σ_S is the restricted spectral type of S and $e(T)$ is the group of eigenvalues of T.

We prove Theorem 2 in §5 by considering a dyadic cocycle (see below) associated with $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ over the dyadic adding machine. The eigenvalue group of the Mackey range (as in [**Z**, pp. 76–77]) of this cocycle is $G_2(q)$.

In case q is a *growth sequence*, as in [**A2**], that is, $q_n > \sum_{1 \le k < n} q_k$, then the Mackey range preserves an equivalent, σ-finite measure and is isomorphic to the appropriate dyadic tower over the dyadic adding machine (defined in [**A2**]).

1.8. *Organization of the paper.* In §2, we establish the basic results on Dirichlet sets and measures and begin to consider membership of IPRWM.

In §3, we consider the class of arithmetic sequences, and prove Proposition 3.

In §4, we prove Proposition 4 for non-arithmetic sequences and give our main examples.

The proofs of Propositions 3 and 4 both use Theorem 2, which is established in §5. In §6, we make some quantitative remarks on the growth of rigid sequences for transformations IP-rigid along some (particular) $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$.

2. *Dirichlet sets and measures*

2.1. *Dirichlet sets.* A *Dirichlet set* is a subset $\Gamma \subset \mathbb{T}$ of the form

$$\Gamma(q) = \left\{ t \in \mathbb{T} : \chi_{q_n}(t) \xrightarrow[n\to\infty]{} 1 \right\},$$

where $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ and $\chi_n(t) := e^{2\pi i n t}$.

An *IP-Dirichlet set* is a subset $\Gamma \subset \mathbb{T}$ of the form

$$\Gamma(\mathrm{FS}(q)) = \left\{ t \in \mathbb{T} : \chi_n(t) \xrightarrow[n\to\infty]{\mathrm{FS}(q)} 1 \right\},$$

where $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$.

PROPOSITION 2.1. *For $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$,*

$$\Gamma(\mathrm{FS}(q)) = G_1(q).$$

Proof sketch of $\subseteq$. It suffices to show that, for $t \in \mathbb{R}$,

$$\|nt\| \xrightarrow[n\to\infty]{\mathrm{FS}(q)} 0 \Rightarrow \sum_{n\ge 1} \|q_n t\| < \infty.$$

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For $x \in \mathbb{R}$, let $\lfloor x \rceil$ be the nearest integer to x (if there are two, take the lesser one), and let

$$\langle x \rangle := x - \lfloor x \rceil, \quad \text{then} \quad |\langle x \rangle| = \|x\| \leq \tfrac{1}{2}.$$

Fix $t \in \mathbb{R}$ such that $\|nt\| \xrightarrow[n\to\infty]{\text{FS}(q)} 0$ and let $K > 0$ be such that

$$\|q(F)t\| < \tfrac{1}{16} \quad \text{for all } F \in \mathcal{F}, \min F \geq K;$$

then $\langle q(F)t \rangle = \sum_{n \in F} \langle q_n t \rangle$ for all $F \in \mathcal{F}$, $\min F \geq K$, and it follows that

$$\sum_{n \geq K, \langle q_n t \rangle \geq 0} \|q_n t\| \leq \frac{1}{16}, \qquad \sum_{n \geq K, \langle q_n t \rangle < 0} \|q_n t\| \leq \frac{1}{16} \quad \text{and} \quad \sum_{n \geq K} \|q_n t\| < \frac{1}{8}. \qquad \square$$

2.2. *Dirichlet measures.* A probability measure $\mu \in \mathcal{P}(\mathbb{T})$ is called

- a *Dirichlet* measure if
$$\|\chi_{q_n} - 1\|_{L^2(\mu)} \xrightarrow[n\to\infty]{} 0$$
for some $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$, in which case μ is called *Dirichlet along* q, and
- an *IP-Dirichlet* measure if
$$\|\chi_n - 1\|_{L^2(\mu)} \xrightarrow[n\to\infty]{\text{FS}(q)} 0$$
for some $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$, in which case μ is called *IP-Dirichlet along* q.

Evidently, for $\mu \in \mathcal{P}(\mathbb{T})$:

- $\chi_{n_k} \xrightarrow[k\to\infty]{L^2(\mu)} 1$ if and only if $\widehat{\mu}(n_k) \xrightarrow[k\to\infty]{} \mu(\mathbb{T})$,
- if μ is Dirichlet (IP-Dirichlet) along q, then so is any $\nu \ll \mu$,
- if $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ and $\mu(G_1(q)) = 1$, then by Proposition 2.1, μ is IP-Dirichlet along q,
- if μ is Dirichlet, then there exists $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ such that $\sum_{n \geq 1} \|\chi_{q_n} - 1\|_{L^2(\mu)} < \infty$, whence $\mu(G_1(q)) = 1$, and μ is IP-Dirichlet along q.

By Proposition 2.1, a totally atomic measure $\mu \in \mathcal{P}(\mathbb{T})$ is IP-Dirichlet along $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ only if $\mu(G_1(q)) = 1$. The examples in §4 (below) show that this is false for continuous measures $\mu \in \mathcal{P}(\mathbb{T})$.

PROPOSITION 2.2.

$$\texttt{IPRWM} = \{q \in \mathbb{N}^{\mathbb{N}}(\uparrow) : \exists \mu \in \mathcal{P}(\mathbb{T}) \text{ continuous and IP-Dirichlet along } q\}.$$

Proof of $\subseteq$. Suppose that $q \in \texttt{IPRWM}$ and that $(X, \mathcal{B}, m, T)$ is a weakly mixing, probability-preserving transformation such that $T^n \xrightarrow[n\to\infty]{\text{FS}(q)} \texttt{Id}$. Fix $f \in L^2(m)$, $\int_X f dm = 0$, $\int_X |f|^2 dm = 1$. The spectral measure of f, $\mu \in \mathcal{P}(\mathbb{T})$, is continuous and IP-Dirichlet along q.

Proof of $\supseteq$. Suppose that $\mu_0 \in \mathcal{P}(\mathbb{T})$ is continuous and IP-Dirichlet along q. Let μ be the symmetrization of μ_0 (also continuous and IP-Dirichlet along q), and let $(X, \mathcal{B}, m, T)$ be the shift of the Gaussian process with spectral measure μ. Each μ^{K*} is continuous and IP-Dirichlet along q (since $\widehat{\mu^{K*}}(n) = \widehat{\mu}(n)^K \xrightarrow[n\to\infty]{\text{FS}(q)} 1$). Every $\nu \ll \sigma_T$ is also IP-Dirichlet along q and T is IP-rigid along q. Thus, $q \in \texttt{IPRWM}$. $\square$

It follows from Proposition 2.2 that if $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ is an IP-rigidity sequence for some $T \in \texttt{PPT}$ not of discrete spectrum (i.e., σ_T is not totally atomic), then $q \in \texttt{IPRWM}$.

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Proof of Proposition 1. Suppose that $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ and that $|G_1(q)| > \aleph_0$, then $G_1(q)$ is an uncountable Borel subset of $\mathbb{T}$ and supports a continuous probability measure $\mu \in \mathcal{P}(\mathbb{T})$.

By Proposition 2.1, $\mu(\Gamma(\mathtt{FS}(q))) = 1$ and μ is IP-Dirichlet along q.

By Proposition 2.2, $q \in \mathtt{IPRWM}$. □

We complete this section with a 'mixed' multiplicative finite-sum condition for membership in IPRWM, which is an extension of a similar finite-sum result from [**BJLR**].

PROPOSITION 2.3. *Suppose that $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ and that there exists $S \subset \mathbb{N}$ infinite, such that $\sum_{n\in S}(q_n/q_{n+1}) < \infty$, and $q_n | q_{n+1}$ for $n \notin S$; then $q \in \mathtt{IPRWM}$.*

If $\mathbb{N}\backslash S$ is finite, then $|G_1(q)| > \aleph_0$ by [**ET**, Theorem 5], whence $q \in \mathtt{IPRWM}$ by Proposition 1.

Proof. Assume (without loss of generality) that $q_1 = 1$. We construct a weakly mixing $T \in \mathtt{PPT}(X, \mathcal{B}, m)$ which is IP-rigid along q by cutting and stacking as in [**N1**, Ch. 7].

To this end, we construct a nested sequence of Rokhlin towers $(\tau_n)_{n\geq 1}$ of intervals, where τ_n has height q_n.

Let τ_1 be [0, 1]. To construct τ_{n+1} from τ_n:

- if $n \notin S$ and $q_{n+1} = a_n q_n$, $a_n \in \mathbb{N}$, $a_n \geq 2$, then we cut τ_n into a_n columns and stack; and
- if $n \in S$ and $q_{n+1} = a_n q_n + r_n$, $a_n, r_n \in \mathbb{N}$, $1 \leq r_n < q_n$, we cut τ_n into a_n columns and put one spacer interval above the $\lfloor a_n/2 \rfloor$th column from the left and $r_n - 1$ spacer intervals above the last column in the right, and then we stack.

The tower τ_n is a stack of q_n intervals of length $\prod_{j=1}^{n-1} 1/a_j$, called *levels of* τ_n.

It follows from [**N1**, §7.31] that the constructed transformation T preserves a finite measure m. A standard argument, as in [**BJLR**, proof of Proposition 3.10], shows that T is weakly mixing.

Next, we show that if A is a union of levels in some τ_K, then

$$\sum_{n=1}^{\infty} m(A \Delta T^{q_n} A) < \infty.$$

To see this, we note first that A is also a union of levels in every τ_n, $n \geq K$. Fix $n \geq K$ and write

$$S \cap [K, \infty) = \{s_1 < s_2 < s_3 < \cdots\}.$$

Since $1/(a_{s_\ell} + 1) \leq q_{s_\ell}/q_{s_\ell+1} \leq 1/a_{s_\ell}$, our assumptions imply $\sum_{\ell=1}^{\infty} 1/a_{s_\ell} < \infty$.

To estimate $m(A \Delta T^{-q_n} A)$ for $n \in (s_{\ell-1}, s_\ell]$, we consider the appearance of the tower τ_n as 'τ_n-stalks' inside $\tau_{s_\ell+1}$.

A *τ_n-stalk* in $\tau_{s_\ell+1}$ is a union $\mathfrak{s} = \bigcup_{k=0}^{q_n-1} T^k B$ of levels of $\tau_{s_\ell+1}$, where B is contained in the base of τ_n.

Let $\mathfrak{s} \in \mathcal{B}$ be a τ_n-stalk in $\tau_{s_\ell+1}$; then

$$m(\mathfrak{s} \cap A) = \prod_{j=n}^{s_\ell} \frac{1}{a_j} \cdot m(A).$$

By construction, $\tau_{s_\ell+1}$ consists entirely of τ_n-stalks and spacer stalks added in $\tau_{s_\ell+1}$. Thus, all points of A, except those contained in the two τ_n-stalks preceding the spacer

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stalks added in $\tau_{s_\ell+1}$, return to A at time q_n, so

$$m(A\Delta T^{-q_n}A) = 2m(A\setminus T^{-q_n}A) = 4\prod_{j=n}^{s_\ell}\frac{1}{a_j}\cdot m(A) \le \frac{4}{2^{s_\ell-n}}\cdot\frac{1}{a_{s_\ell}}\cdot m(A).$$

Thus, writing $s_0 := K-1$, we have

$$\begin{aligned}\sum_{n=K}^{\infty} m(A\Delta T^{-q_n}A) &= \sum_{\ell=1}^{\infty}\sum_{n\in(s_{\ell-1},s_\ell]} m(T^{q_n}(A)\Delta A)\\ &\le \sum_{\ell=1}^{\infty}\sum_{n\in(s_{\ell-1},s_\ell]}\frac{4}{2^{s_\ell-n}}\cdot\frac{1}{a_{s_\ell}}\cdot m(A)\\ &\le 4m(A)\sum_{\ell=1}^{\infty}\frac{1}{a_{s_\ell}}\\ &< \infty.\end{aligned}$$

It follows from this that for A a union of levels in some tower τ_n and $F=\{n_1<n_2<\cdots<n_k\}\in\mathcal{F}$,

$$\begin{aligned}m(A\Delta T^{q(F)}A) &= m\left(A\Delta T^{\sum_{j=1}^{k} q_{n_j}}A\right)\\ &\le m(A\Delta T^{q_1}A) + m\left(T^{q_1}A\Delta T^{\sum_{j=1}^{k} q_{n_j}}A\right)\\ &= m(A\Delta T^{q_1}A) + m\left(A\Delta T^{\sum_{j=2}^{k} q_{n_j}}A\right)\\ &\le\\ &\vdots\\ &\le \sum_{j=1}^{k} m(A\Delta T^{q_j}A)\\ &\le \sum_{n=\min F}^{\infty} m(T^{q_n}(A)\Delta A)\\ &\xrightarrow[F\in\mathcal{F},\ \min F\to\infty]{} 0.\end{aligned}$$

The collection of measurable sets $\mathcal{C}$ satisfying this last convergence is a σ-algebra and

$$\mathcal{C}\supset\sigma\left(\bigcup_{n\ge1}\{\text{unions of levels in }\tau_n\}\right)=\mathcal{B}.$$

Thus T is IP-rigid along q. □

Remark 2.4. The proof of Proposition 2.3 establishes the following proposition.

Suppose that $q\in\mathbb{N}^{\mathbb{N}}(\uparrow)$, $p>0$ and that there exists $S\subset\mathbb{N}$ infinite, such that $\sum_{n\in S}(q_n/q_{n+1})^p<\infty$, and $q_n|q_{n+1}$ for $n\notin S$, then there exists $T\in\mathtt{PPT}(X,\mathcal{B},m)$ weakly mixing and a dense collection $\mathcal{A}\subset\mathcal{B}$ such that

$$\sum_{n\ge1} m(A\Delta T^{q_n}A)^p<\infty \quad \textit{for all } A\in\mathcal{A}. \tag{$\Upsilon(p)$}$$

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As above ($\Uparrow(1)$) $\Longrightarrow$ IP-rigidity along q, whence $\sigma_T(G_2(q)^c) = 0$ by Theorem 2. Using the spectral theorem, one sees that ($\Uparrow(\frac{1}{2})$) $\Longrightarrow \sigma_T(G_1(q)^c) = 0$.

3. *Arithmetic sequences*

In this section, we prove Proposition 3. The implications (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (a) follow from Proposition 1, [**E**, Theorems 2 and 16] (respectively). None of these uses arithmeticity.

We turn to the remaining implication (a) $\Longrightarrow$ (b).

LEMMA 3.1. *If $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ is multiplicative and $\sup_{n\geq 1} q_{n+1}/q_n = \infty$, then $|G_1(q)| > \aleph_0$.*

Cf. [**ET**, Theorem 3].

Proof. Suppose that $q_{n+1} = a_{n+1}q_n$, where $a_n \geq 2$ for all $n \geq 1$.

Since $\sup_{n\geq 1} a_n = \infty$, there exists a subsequence $(n_k)_{k\geq 1}$ such that $a_{n_{k+1}}/a_{n_k} \geq 3$ for all $k \geq 1$. Define $t : \Omega := \{0, 1\}^{\mathbb{N}} \to [0, 1]$ by

$$t(\omega) := \sum_{k=1}^{\infty} \frac{\omega_k}{q_{n_k}}.$$

Since

$$\sum_{k=L+1}^{\infty} \frac{\omega_k}{q_{n_k}} \leq \frac{1}{q_{n_L}} \sum_{j=1}^{\infty} \frac{1}{a_{n_L+1}a_{n_L+2}\cdots a_{n_j}} \leq \frac{1}{q_{n_L}} \sum_{j=1}^{\infty} \frac{1}{3^j} < \frac{1}{q_{n_L}},$$

we have $t : \Omega := \{0, 1\}^{\mathbb{N}} \to [0, 1]$ is strictly increasing (with respect to lexicographic order on Ω), whence it is injective and $|t(\Omega)| > \aleph_0$.

It suffices to show that $t(\Omega) \subset G_1(q)$.

To see this, fix $\omega \in \Omega$. For $N \geq 1$, we have

$$q_N t(\omega) = \sum_{k=1}^{\infty} \frac{q_N \omega_k}{q_{n_k}} = \sum_{k\geq 1, n_k \geq N} \frac{\omega_k}{a_{N+1}a_{N+2}\cdots a_{n_k}} \mod 1,$$

whence

$$\|q_N t(\omega)\| = \sum_{k\geq 1, n_k \geq N} \frac{\omega_k}{a_{N+1}a_{N+2}\cdots a_{n_k}} \leq \sum_{k\geq 1, n_k \geq N} \frac{1}{a_{N+1}a_{N+2}\cdots a_{n_k}} =: \Delta_N.$$

For $n_{K-1} < N \leq n_K$,

$$\Delta_N = \sum_{k\geq K} \frac{1}{a_{N+1}a_{N+2}\cdots a_{n_k}} \leq \sum_{k\geq K} \frac{1}{a_{N+1}a_{N+2}\cdots a_{n_K} \cdot a_{n_k}} \leq \frac{1}{2^{n_K - N} a_{n_K}}.$$

Thus,

$$\begin{aligned}\sum_{N=n_1}^{\infty} \|q_N t(\omega)\| &\leq \sum_{N=n_1}^{\infty} \Delta_N = \sum_{k=2}^{\infty} \sum_{N=n_{k-1}+1}^{n_k} \sum_{\nu=k}^{\infty} \frac{1}{a_{N+1}a_{N+2}\cdots a_{n_\nu}} \\ &\leq \sum_{k=2}^{\infty} \sum_{N=n_{k-1}+1}^{n_k} \frac{1}{2^{n_k - N} a_{n_k}} \leq \sum_{k=2}^{\infty} \frac{2}{a_{n_k}} \leq 4,\end{aligned}$$

and $t(\omega) \in G_1(q)$. □

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LEMMA 3.2. *Let* $q = q(\alpha) \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ *be the principal denominator sequence of* $\alpha \in (0, 1)\backslash\mathbb{Q}$.

If $\sup_{n\geq 1} q_{n+1}/q_n = \infty$, *then* $|G_1(q)| > \aleph_0$.

Proof. As shown in [**IN**], for any $t \in [0, 1]$, there is a unique sequence $(\omega_n)_{n\geq 1} \in \prod_{n\geq 1}\{0, 1, \ldots, a_n\}$ such that

- $\omega_k \leq a_k$, $\omega_k = a_k \Rightarrow \omega_{k+1} = 0$, and
- $t = \sum_{n=1}^{\infty} \omega_n \langle q_n\alpha\rangle \bmod 1$.

Since $\sup_{n\geq 1} a_n = \sup_{n\geq 1} q_{n+1}/q_n = \infty$, we can choose a subsequence a_{n_k} such that $\sum_{k\geq 1} 1/a_{n_k} < \infty$, and define

$$t : \Omega = \{0, 1\}^{\mathbb{N}} \to \mathbb{T} \quad \text{by } t(\omega) := \sum_{k=1}^{\infty} \omega_k \langle q_{n_k-1}\alpha\rangle \mod 1.$$

By the above, $t : \Omega \to \mathbb{T}$ is injective. It follows that $|t(\Omega)| > \aleph_0$ and it suffices to show that $t(\Omega) \subset G_1(q)$.

We claim that $\sup_{\omega\in\Omega} \sum_{n=1}^{\infty} \|q_n t(\omega)\| < \infty$.

Fix $K \geq 1$ and consider $n_K - 1 \leq N \leq n_{K+1} - 1$. Then

$$\begin{aligned}\|q_N t(\omega)\| &\leq \sum_{k=1}^{\infty} \|\omega_{n_k-1} q_N q_{n_k-1}\alpha\| \\ &\leq \sum_{k=1}^{\infty} \|q_N q_{n_k-1}\alpha\| \\ &\leq \sum_{k=1}^{K} \frac{q_{n_k-1}}{q_{N+1}} + \sum_{k=K+1}^{\infty} \frac{q_N}{q_{n_k}}.\end{aligned}$$

Using the fact that $(q_{j+n}/q_j) \geq \sqrt{2}^{n-2}$ for all $j, n \geq 1$, we have, for some absolute constant C,

$$\frac{q_{n_k-1}}{q_{N+1}} = \frac{q_{n_k-1}}{q_{n_K-1}} \cdot \frac{q_{n_K-1}}{q_{n_K}} \cdot \frac{q_{n_K}}{q_{N+1}} \leq \frac{C}{\sqrt{2}^{n_K-n_k}} \cdot \frac{1}{a_{n_K}} \cdot \frac{C}{\sqrt{2}^{N-n_K}} \quad \text{for } k \leq K,$$

and

$$\frac{q_N}{q_{n_k}} = \frac{q_N}{q_{n_{K+1}-1}} \cdot \frac{q_{n_{K+1}-1}}{q_{n_{K+1}}} \cdot \frac{q_{n_{K+1}}}{q_{n_k}} \leq \frac{C}{\sqrt{2}^{n_{K+1}-N}} \cdot \frac{1}{a_{n_{K+1}}} \cdot \frac{C}{\sqrt{2}^{n_k-n_{K+1}}} \quad \text{for } k > K.$$

Therefore,

$$\begin{aligned}\sum_{N=n_K-1}^{n_{K+1}-2} \|q_N t(\omega)\| &\leq \sum_{N=n_K-1}^{n_{K+1}-2} \left(\sum_{k=1}^{K} \frac{q_{n_k-1}}{q_{N+1}} + \sum_{k=K+1}^{\infty} \frac{q_N}{q_{n_k}}\right) \\ &\leq C^2 \sum_{N=n_K-1}^{n_{K+1}-2} \left(\sum_{k=1}^{K} \frac{1}{a_{n_K}\sqrt{2}^{N-n_k}} + \sum_{k=K+1}^{\infty} \frac{1}{a_{n_{K+1}}\sqrt{2}^{n_k-N}}\right) \\ &\leq C^3 \sum_{N=n_K-1}^{n_{K+1}-2} \left(\frac{1}{a_{n_K}\sqrt{2}^{N-n_K}} + \frac{1}{a_{n_{K+1}}\sqrt{2}^{n_{K+1}-N}}\right) \\ &\leq C^4 \left(\frac{1}{a_{n_K}} + \frac{1}{a_{n_{K+1}}}\right),\end{aligned}$$

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and

$$\sum_{N=1}^{\infty} \|q_N t(\omega)\| \le C^4 \sum_{K=1}^{\infty} \left(\frac{1}{a_{n_K}} + \frac{1}{a_{n_{K+1}}} \right) < \infty. \qquad \square$$

Hence, $q \in \texttt{IPRWM}$.

The proof of Proposition 3 is now complete. $\square$

4. *Super-lacunary sequences*

Suppose that $b = (q_1, q_2, \dots) \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ is super-lacunary, i.e., $(q_{n+1}/q_n) \xrightarrow[n\to\infty]{} \infty$.

As in [**E**, Theorem 17], we fix $N \ge 1$ with $(q_{n+1}/q_n) > 10$ for all $n \ge N$ and let

$$E := \bigcap_{n \ge N} E_n,$$

where

$$E_n := \left\{ t \in [0, 1] : \|q_n t\| \le \frac{4q_n}{q_{n+1}} \right\}.$$

Now,

$$E_n \supseteq \bigcup_{k=1}^{q_n - 1} I_{k,n},$$

where

$$I_{k,n} := \left[\frac{k}{q_n} - \frac{4}{q_{n+1}}, \frac{k}{q_n} + \frac{4}{q_{n+1}} \right].$$

For each $n \ge N$, the intervals $\{I_{k,n} : 1 \le k < q_n\}$ are disjoint, and each interval $I_{k,n}$ contains at least five disjoint intervals of the form $I_{k',n+1}$. It follows that E contains a Cantor set and $|E| > \aleph_0$. Thus, the proof is as follows (cf. [**ET**, Theorem 5]).

PROPOSITION 4.1. *Suppose that $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ and $p > 0$; then*

$$\sum_{n \ge 1} \left(\frac{q_n}{q_{n+1}} \right)^p < \infty \Longrightarrow |G_p(q)| > \aleph_0.$$

Proof. Let $N \ge 1$ and E be as above; then, for $t \in E$,

$$\sum_{n \ge 1} \|q_n t\|^p \le N + 4^p \sum_{n \ge N} \left(\frac{q_n}{q_{n+1}} \right)^p < \infty. \qquad \square$$

PROPOSITION 4.2. *Suppose that $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ and that*

$$\sum_{n \ge 1} \left(\frac{q_n}{q_{n+1}} \right)^2 < \infty, \quad \textit{then } q \in \texttt{IPRWM}.$$

Proof. By Proposition 2.2, it suffices to construct a continuous probability in $\mathbb{T}$ which is IP-Dirichlet along q.

To this end, let $N \ge 1$, E and $\{I_{k,n} : 1 \le k < q_n\}$ be as above. As above, for each $n \ge N$, the intervals $\{I_{k,n} : 1 \le k < q_n\}$ are disjoint, and each interval $I_{k,n}$ contains at least five disjoint intervals of the form $I_{k',n+1}$.

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Thus, we may choose

$$I_n(\omega) = I_{k_n(\omega),n} \quad (n \geq 1,\ \omega \in \{0,1\}^n),$$

such that

$$I_{n+1}(\omega,\epsilon) \subset I_n(\omega) \quad \text{for all } n \geq 1,\ \omega \in \{0,1\}^n \text{ and } \epsilon = 0,1,$$

and

$$X_{n+1}(\omega,0) < X_n(\omega) < X_{n+1}(\omega,1) \quad \text{where } X_n(\omega) := \frac{k_n(\omega)}{q_n}.$$

Next, for $\omega \in \Omega := \{0,1\}^{\mathbb{N}}$,

$$X_n(\omega_1,\ldots,\omega_n) \xrightarrow[n\to\infty]{} X(\omega), \quad \text{where } \bigcap_{n\geq 1} I_n(\omega_1,\ldots,\omega_n) = \{X(\omega)\},$$

and

$$\begin{aligned} q_n X(\omega) &= q_n X_n(\omega) + q_n(X_{n+1}(\omega) - X_n(\omega)) + q_n(X(\omega) - X_{n+1}(\omega)) \\ &= k_n(\omega) + \xi_n(\omega) + \theta_n(\omega), \end{aligned}$$

where

$$\begin{aligned} \xi_n(\omega) &:= q_n(X_{n+1}(\omega) - X_n(\omega)) \quad \text{and} \\ \theta_n(\omega) &:= q_n(X(\omega) - X_{n+1}(\omega)). \end{aligned}$$

Note that $|\theta_n(\omega)| \leq \mathcal{E}_n := (4q_n/q_{n+2})$ and that, by assumption, $\sum_{n\geq 1} \mathcal{E}_n < \infty$.

For $n \geq 1$, $\omega \in \{0,1\}^n$, there exists a unique $p_{n,\omega} : \{0,1\} \to (0,1)$, such that

$$p_{n,\omega}(0) + p_{n,\omega}(1) = 1 \quad \text{and} \quad X_{n+1}(\omega,0)\,p_{n,\omega}(0) + X_{n+1}(\omega,1)\,p_{n,\omega}(1) = X_n(\omega).$$

Define $P : \{\texttt{cylinders}\} \to (0,1)$ by

$$P([a_1,a_2,\ldots,a_n]) := \frac{1}{2}\prod_{k=1}^{n-1} p_{k,(a_1,a_2,\ldots,a_k)}(a_k).$$

It follows that P is additive and, by standard extension theory, there exists $\mathbb{P} \in \mathcal{P}(\Omega)$ extending P.

Denoting expectation with respect to $\mathbb{P}$ by $\mathbb{E}$, and writing $\omega = (\omega_1,\omega_2,\ldots,\omega_n) \in \{0,1\}^n$, we have

$$\mathbb{E}(X_{n+1}\|\omega_1,\omega_2,\ldots,\omega_n) = X_{n+1}(\omega,0)\,p_{n,\omega}(0) + X_{n+1}(\omega,1)\,p_{n,\omega}(1) = X_n(\omega),$$

whence

$$\mathbb{E}(\xi_n\|\omega_1,\omega_2,\ldots,\omega_n) = q_n(\mathbb{E}(X_{n+1}\|\omega_1,\omega_2,\ldots,\omega_n) - X_n(\omega)) = 0,$$

and $\mathbb{E}(\xi_n) = 0$.

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For $n, k \geq 1$,

$$\begin{aligned}\mathbb{E}(\xi_n \xi_{n+k}) &= q_n q_{n+k} \mathbb{E}(\mathbb{E}(\xi_n \xi_{n+k} \| \omega_1, \omega_2, \ldots, \omega_{n+k})) \\ &= q_n q_{n+k} \mathbb{E}(\xi_n(\omega) \mathbb{E}(\xi_{n+k} \| \omega_1, \omega_2, \ldots, \omega_{n+k})) = 0 \quad \text{and}\end{aligned}$$

$$\mathbb{E}(\xi_n^2) =: \Delta_n \leq \frac{16 q_n^2}{q_{n+1}^2}.$$

By assumption, $\sum_{n \geq 1} \Delta_n < \infty$, so $\sum_{n \in J} \xi_n$ converges in $L^2(\mathbb{P})$ for every $J \subset \mathbb{N}$ and

$$\mathbb{E}\left(\left(\sum_{n \in J} \xi_n\right)^2\right) = \sum_{n \in J} E(\xi_n^2) = \sum_{n \in J} \Delta_n.$$

The measure $\mu := \mathbb{P} \circ X^{-1} \in \mathcal{P}(\mathbb{T})$ is continuous. We claim that it is IP-Dirichlet along q.

To check this, let $K \geq 1$, $F \subset \mathbb{N} \cap [K, \infty)$ be finite and write $\Xi_F := \sum_{n \in F} \xi_n$; then

$$\begin{aligned}\|\chi_{q(F)} - 1\|_{L^2(\mu)} &\leq \left\| \left\langle \sum_{N \in F} q_N t \right\rangle \right\|_{L^2(\mu)} \\ &= \left\| \left\langle \sum_{N \in F} (\xi_N + \theta_N) \right\rangle \right\|_{L^2(\mathbb{P})} \\ &\leq \|\langle \Xi_F \rangle\|_{L^2(\mathbb{P})} + \sum_{N \in F} \mathcal{E}_N.\end{aligned}$$

Next,

$$\begin{aligned}\|\langle \Xi_F \rangle\|_{L^2(\mathbb{P})}^2 &= \mathbb{E}(\langle \Xi_F \rangle^2) \\ &= \mathbb{E}(1_{[|\Xi_F| \leq \frac{1}{2}]} \langle \Xi_F \rangle^2) + \mathbb{E}(1_{[|\Xi_F| > \frac{1}{2}]} \langle \Xi_F \rangle^2) \\ &\leq \mathbb{E}(\Xi_F^2) + \tfrac{1}{4} \mathbb{P}[|\Xi_F| > \tfrac{1}{2}] \quad \because \langle \Xi_F \rangle^2 \leq \tfrac{1}{4}, \\ &\leq 2\mathbb{E}(\Xi_F^2) \quad \text{by Tchebychev's inequality} \\ &= 2 \sum_{N \in F} \mathbb{E}(\xi_N^2) \leq 2 \sum_{N=K}^{\infty} \Delta_N.\end{aligned}$$

Thus,

$$\begin{aligned}\|\chi_{q(F)} - 1\|_{L^2(\mu)} &\leq \|\langle \Xi_F \rangle\|_{L^2(\mathbb{P})} + \sum_{N=K}^{\infty} \mathcal{E}_N \\ &\leq \sqrt{2 \sum_{N=K}^{\infty} \Delta_N} + \sum_{N=K}^{\infty} \mathcal{E}_N \\ &\xrightarrow[K \to \infty]{} 0,\end{aligned}$$

proving that μ is IP-Dirichlet along q. □

Remark. The converses to Propositions 4.1 and 4.2 are false. It is easy to construct $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ multiplicative, super-lacunary so that $\sum_{n \geq 1} (q_n / q_{n+1})^p = \infty$ for all $p > 0$. By Proposition 3, $|G_1(q)| > \aleph_0$ and $T \in \texttt{IPRWM}$.

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4.1. *Non-arithmetic sequences.* We begin with a strong converse to Proposition 4.1 for non-arithmetic sequences. For $d \in \mathbb{N}$, let

$$\mathbb{Z}_d := \{\tfrac{k}{d} : 0 \le k \le d-1\} \le \mathbb{T}.$$

PROPOSITION 4.3. *Suppose that $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ is a non-arithmetic sequence with $q_{n+1} = a_n q_n + \Delta_n$, where $1 \le |\Delta_n| \le d$ for n large, and let $p > 0$; then*

$$\sum_{n\ge 1}\left(\frac{q_n}{q_{n+1}}\right)^p = \infty \Longrightarrow G_p(q) \subset \mathbb{Z}_{d!}.$$

This was stated in [**ET**] for $p = 1$ and q, the Erdos–Taylor sequence associated with $(2, 3, \dots)$. See also [**P**, Théorème 2].

Proof. Let $t \in \mathbb{T}\backslash\mathbb{Z}_{d!}$; then $E := \min\{\|kt\| : 1 \le |k| \le d\} > 0$, and we claim that

$$\|q_n t\| < \frac{E}{2a_n} \Longrightarrow \|q_{n+1}t\| > \frac{E}{2}$$

since, in this case, $q_n t = K + (\theta/a_n)$, where $K \in \mathbb{Z}$ and $|\theta| < E/2$, whence $q_{n+1}t = \Delta_n t + a_n K + \theta$ and

$$\|q_{n+1}\| = \|\Delta_n t + \theta\| \ge \|\Delta_n t\| - \|\theta\| > \frac{E}{2}.$$

Consequently,

- either $\|q_n t\| \ge (E/2a_n)$ eventually and $\sum_{n\ge1} \|q_n t\|^p = \infty$; or
- $\|q_{n+1}t\| > (E/2)$ infinitely often and $\sum_{n\ge1} \|q_n t\|^p = \infty$.

Either way, $t \notin G_p(q)$. □

Proof of Proposition 4. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) follow from Proposition 4.2, Theorem 2 and Proposition 4.3, respectively. □

4.2. *Examples.* It follows from Proposition 4 that

- if $q^{(0)} \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ is the Erdos–Taylor sequence associated with $(2, 3, \dots)$, then

$$\text{(a) } q^{(0)} \in \texttt{IPRWM} \quad \text{and} \quad \text{(b) } G_1(q^{(0)}) = \{0\},$$

and that

- if $q^{(1)} \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ is the Erdos–Taylor sequence associated with $(a_2, a_3, \dots)$, where $a_n \to \infty$ and $\sum_{n\ge1}(1/a_n^2) = \infty$ (e.g., $a_n := \lfloor\sqrt{n}\rfloor$), then $q^{(1)}$ is super-lacunary, $G_2(q^{(1)}) = \{0\}$ and $q^{(1)}$ is not a sequence of IP-rigidity for any probability-preserving transformation other than the identity.

5. *Proof of Theorem 2*

We prove Theorem 2 using dyadic cocycles over the dyadic odometer.

Let $\Omega := \{0, 1\}^{\mathbb{N}}$, let $P \in \mathcal{P}(\Omega)$ be the symmetric product measure $P = \prod(\frac{1}{2}, \frac{1}{2})$, and let $\tau : \Omega \to \Omega$ be the dyadic odometer defined by

$$\tau(1, \dots, 1, 0, \omega_{\ell+1}, \dots) = (0, \dots, 0, 1, \omega_{\ell+1}, \dots),$$

where $\ell = \ell(\omega) := \min\{n \ge 1 : \omega_n = 0\}$.

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The *dyadic cocycle* $\varphi : \Omega \to \mathbb{Z}$ *associated with* $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ is defined by

$$\varphi(\omega) := q_{\ell(\omega)} - \sum_{k=1}^{\ell(\omega)-1} q_k,$$

and its *skew product* $\tau_\varphi : \Omega \times \mathbb{Z} \to \Omega \times \mathbb{Z}$ is defined by $\tau_\varphi(x, n) = (\tau(x), n + \varphi(x))$.

Define $Q : \Omega \times \mathbb{Z} \to \Omega \times \mathbb{Z}$ by $Q(x, n) := (x, n + 1)$ and fix $p \in \mathcal{P}(\Omega \times \mathbb{Z})$, $p \sim P \times \#$, where # is the counting measure on $\mathbb{Z}$.

There is (see [**Z**, pp. 76–77]) an ergodic, non-singular transformation $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}), \mathfrak{p}, \mathfrak{T})$ of a standard probability space and a map $\pi : \Omega \times \mathbb{Z} \to \mathfrak{X}$ such that

$$p \circ \pi^{-1} = \mathfrak{p}, \quad \pi^{-1}\mathcal{B}(\mathfrak{X}) = \{A \in \mathcal{B}(\Omega \times \mathbb{Z}) : \tau_\varphi A = A\} \quad \text{and} \quad \pi \circ Q = \mathfrak{T} \circ \pi.$$

The ergodic, non-singular transformation $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}), \mathfrak{p}, \mathfrak{T})$ is called the *Mackey range of* (τ, φ).

In case q is a growth sequence, equivalently $\varphi : \Omega \to \mathbb{N}$, there is a σ-finite, T-invariant measure $\mathfrak{m} \sim \mathfrak{p}$ with respect to which the Mackey range $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}), \mathfrak{m}, \mathfrak{T})$ is isomorphic to the tower over $(\Omega, \mathcal{B}(\Omega), P, \tau)$ with height function φ (also known as the *dyadic tower with growth sequence* q in [**A2**]).

The collection of eigenvalues of the Mackey range is

$$e(\mathfrak{T}) := \{t \in \mathbb{T} : \exists F \in L^\infty(\mathfrak{p}),\ F \not\equiv 0,\ F \circ \mathfrak{T} = e^{2\pi i t} F\},$$

and it follows from the definitions that

$$e(\mathfrak{T}) = \mathcal{T}(\tau, \varphi) := \{s \in \mathbb{T} : \exists f \in L^\infty(P),\ f \not\equiv 0,\ f \circ \tau = e^{2\pi i s\varphi} f\}.$$

It it is shown in [**AN**, §2] (see also [**A1**, Theorem 2.6.3]) that

$$\mathcal{T}(\tau, \varphi) = G_2(q). \tag{☞}$$

Although, formally, (☞) was only stated for growth sequences in [**AN, A1**], the proofs do not use this condition and apply to arbitrary $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$.

Consider the Polish group $\mathfrak{B}(\mu) := \{f \in L^2(\mu) : |f| \equiv 1\}$ equipped with $L^2(\mu)$-distance.

LEMMA 5.1. *If the probability* $\mu \in \mathcal{P}(\mathbb{T})$ *is IP-Dirichlet along* q*, then there exists* $\mathcal{X} : \Omega \to \mathfrak{B}(\mu)$ *continuous such that*

$$\sup_{\omega \in \Omega} \|\chi_{q(K(\omega)\cap[1,n])} - \mathcal{X}(\omega)\|_{L^2(\mu)} \xrightarrow[n\to\infty]{} 0, \tag{☠}$$

where $K(\omega) := \{n \geq 1 : \omega_n = 1\}$.

Proof. Suppose that $\mu \in \mathcal{P}(\mathbb{T})$ is IP-Dirichlet along q. Fix $\omega \in \Omega$. We claim that the sequence $n \mapsto \chi_{q(K(\omega)\cap[1,n])}$ is Cauchy in $L^2(\mu)$. To see this, let

$$\mathcal{E}_n := \sup_{F \in \mathcal{F}, \min F \geq n} \|\chi_{q(F)} - 1\|_{L^2(\mu)};$$

then, by assumption, $\mathcal{E}_n \xrightarrow[n\to\infty]{} 0$. Evidently,

$$\|\chi_{q(K(\omega)\cap[1,n])} - \chi_{q(K(\omega)\cap[1,n+k])}\|_{L^2(\mu)} = \|\chi_{q(K(\omega)\cap[n+1,n+k])} - 1\|_{L^2(\mu)} \leq \mathcal{E}_n,$$

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whence there exists $\mathcal{X} : \Omega \to \mathfrak{B}(\mu)$ such that

$$\chi_{q(K(\omega)\cap[1,n])} \xrightarrow[n\to\infty]{L^2(\mu)} \mathcal{X}(\omega) \quad \text{uniformly in } \omega \in \Omega.$$

For $\omega \in \Omega$,

$$\|\chi_{q(K(\omega)\cap[1,n])} - \mathcal{X}(\omega)\|_{L^2(\mu)} \xleftarrow[k\to\infty]{} \|\chi_{q(K(\omega)\cap[1,n])} - \chi_{q(K(\omega)\cap[1,n+k])}\|_{L^2(\mu)} \le 2\mathcal{E}_n,$$

proving (☄). Clearly, for each $n \ge 1$, $\omega \mapsto \chi_{q(K(\omega)\cap[1,n])}$ is continuous ($\Omega \to \mathfrak{B}(\mu)$) and so continuity of $\mathcal{X} : \Omega \to \mathfrak{B}(\mu)$ follows from the uniformity of the convergence. □

Note that the converse of Lemma 5.1 is also true.

Completion of the proof. Now suppose that $\mu \in \mathcal{P}(\mathbb{T})$ is IP-Dirichlet along q. By Lemma 5.1, there exists $\mathcal{X} : \Omega \to \mathfrak{B}(\mu)$ satisfying (☄).

We claim that

$$\mathcal{X}(\tau\omega) = \chi_{\varphi(\omega)}\mathcal{X}(\omega). \tag{☢☢}$$

To see this, note that $\chi_{q(K(\omega)\cap[1,n])}(t) = \prod_{k=1}^{n} \chi_{\omega_k q_k}(t) =: \mathcal{X}_n(\omega, t)$. For $n > \ell(\omega)$,

$$\frac{\mathcal{X}_n(\tau\omega)}{\mathcal{X}_n(\omega)} = \frac{\chi_{q(K(\tau\omega)\cap[1,n])}}{\chi_{q(K(\omega)\cap[1,n])}} = \prod_{k=1}^{n} \frac{\chi_{(\tau\omega)_k q_k}}{\chi_{\omega_k q_k}} = \chi_{\varphi(\omega)}.$$

Since $(\mathcal{X}_n(\tau\omega)/\mathcal{X}_n(\omega)) \xrightarrow[n\to\infty]{\mathfrak{B}(\mu)} (\mathcal{X}(\tau\omega)/\mathcal{X}(\omega))$, this proves (☢☢).

By (☄), there exists $n_J \to \infty$ such that

$$\sum_{J\ge1} \|\mathcal{X}_{n_J} - \mathcal{X}\|_{L^2(P\times\mu)} \le \sum_{J\ge1} \sup_{\omega\in\Omega} \|\chi_{q(K(\omega)\cap[1,n_J])} - \mathcal{X}(\omega)\|_{L^2(\mu)} < \infty$$

(where $\mathcal{X}(\omega, t) := \mathcal{X}(\omega)(t)$). Hence, $\mathcal{X}_{n_J} \to \mathcal{X}$ $P \times \mu$-almost everywhere, and by Fubini's theorem, there exists $\Lambda \in \mathcal{B}(\mathbb{T})$, $\mu(\Lambda) = 1$, such that

$$\prod_{k=1}^{n_J} \chi_{\omega_k q_k}(t) \xrightarrow[J\to\infty]{} \mathfrak{X}_t(\omega) := \mathcal{X}(\omega)(t) \text{ for all } t \in \Lambda \quad \text{and} \quad P\text{-almost every } \omega \in \Omega.$$

By (☢☢), for $t \in \Lambda$,

$$\mathfrak{X}_t \circ \tau = e^{2\pi i t\varphi}\mathfrak{X}_t \quad P\text{-almost everywhere}$$

and $t \in \mathcal{T}(\tau, \varphi)$. By (☞), $t \in G_2(q)$. □

As shown in [**A3**] (see also Remark 6.1 below), measures on $G_2(q)(q \in \mathbb{N}^{\mathbb{N}}(\uparrow))$ have rigidity sequences which are 'large' with respect to q. However, note the following example.

Example 5.2. There exist $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ and $\mu \in \mathcal{P}(G_2(q))$ non-atomic, not IP-Dirichlet along q.

Construction: Define $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ by $q_n := \prod_{k=1}^{n} a_k$ with $a_k := k + 1$.

Consider the mapping $t : \Omega := \prod_{k\ge1}\{0, 1\} \to [0, 1]$ defined by

$$t(\omega) = t(\omega_1, \omega_2, \dots) := \sum_{n\ge1} \frac{\omega_n}{q_n}.$$

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This is injective and Borel measurable, so $t(\Omega)$ is an uncountable Borel set in $[0, 1]$. We claim that

$$t(\Omega) \subset G_2(q). \qquad (☣)$$

Proof of (☣). For $\omega \in \Omega$ and $N \geq 1$, we have

$$q_N t(\omega) = \frac{\omega_{N+1}}{a_{N+1}} + \frac{1}{a_{N+1}} \sum_{k\geq 2} \frac{\omega_{N+k}}{a_{N+2} \dots a_{N+k}} \mod 1.$$

Now,

$$\left| \sum_{k\geq 2} \frac{\omega_{N+k}}{a_{N+2} \cdots a_{N+k}} \right| \leq \sum_{k\geq 2} \frac{1}{a_{N+2} \dots a_{N+k}} < \frac{1}{a_{N+1}}.$$

Thus, we have

$$\langle q_N t(\omega) \rangle = \frac{\omega_{N+1}}{a_{N+1}} + \theta_N(\omega) \quad \text{where } \theta_N \leq \frac{1}{a_{N+1}^2}, \qquad (✷)$$

whence $|\langle q_N t(\omega)\rangle|^2 \leq 2/N^2$ and

$$\sum_{k=1}^{\infty} |\langle q_k t(\omega)\rangle|^2 \leq \frac{\pi}{3}. \ (☣). \qquad \square$$

Now define $P \in \mathcal{P}(\Omega)$ by $P := \prod_{k\geq 1}(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)$ and set $\mu := P \circ t^{-1}$; then $\mu \in \mathcal{P}(G_2(q))$. We now show that μ is not IP-Dirichlet along q.

Fix $1 < \lambda < e^{\frac{1}{3}}$; then, by (✷),

$$\left| \sum_{\lambda^N < k < \lambda^{N+1}} \langle q_N t(\omega)\rangle - \sum_{\lambda^N < k < \lambda^{N+1}} \frac{\omega_{N+1}}{a_{N+1}} \right| \leq \sum_{\lambda^N < k < \lambda^{N+1}} \frac{2}{k^2} \xrightarrow[N\to\infty]{} 0,$$

and setting $s_n := \sum_{j=1}^{n} \omega_j$, $\kappa_n := \lceil \lambda^n \rceil$, $\ell_n := \lfloor \lambda^{n+1} \rfloor$, we have

$$\begin{aligned} \sum_{\lambda^N < k < \lambda^{N+1}} \frac{\omega_k}{a_k} &= \sum_{\kappa_N \leq k \leq \ell_N} \frac{s_k - s_{k-1}}{k+1} \\ &= \sum_{\kappa_N \leq k \leq \ell_N} \frac{s_k}{k+1} - \sum_{\kappa_N - 1 \leq k \leq \ell_N - 1} \frac{s_k}{k+2} \\ &= \frac{s_{\ell_N}}{\ell_N + 1} - \frac{s_{\kappa_N}}{\kappa_N + 2} + \sum_{\kappa_N \leq k \leq \ell_N - 1} \frac{s_k}{(k+1)(k+2)}. \end{aligned}$$

By the strong law of large numbers, we have

$$\frac{s_N}{N} \xrightarrow[N\to\infty]{} \frac{1}{2} \quad \text{almost surely.}$$

Writing $\Gamma_N \approx \Delta_N$ as $N \to \infty$ to mean $\Gamma_N \approx \Delta_N \xrightarrow[N\to\infty]{} 0$, we have for P-almost every $\omega \in \Omega$, as $N \to \infty$:

$$\begin{aligned} \sum_{\lambda^N < k < \lambda^{N+1}} \langle q_N t(\omega)\rangle &\approx \sum_{\lambda^N < k < \lambda^{N+1}} \frac{\omega_k}{a_k} \\ &= \frac{s_{\ell_N}}{\ell_N + 1} - \frac{s_{\kappa_N}}{\kappa_N + 2} + \sum_{\kappa_N \leq k \leq \ell_N - 1} \frac{s_k}{(k+1)(k+2)} \\ &\approx \sum_{\kappa_N \leq k \leq \ell_N - 1} \frac{1}{2k} \\ &\longrightarrow \log \lambda. \end{aligned}$$

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Thus,

$$\begin{aligned}\sup_{F\in\mathcal{F},\min F>\lambda^N}\|\chi_{q(F)}-1\|^2_{L^2(\mu)} &\geq \mathbb{E}\left(\left|\exp[2\pi i\sum_{\lambda^N<k<\lambda^{N+1}}q_N t(\omega)]-1\right|^2\right)\\ &\geq 4\mathbb{E}\left(\left|\sum_{\lambda^N<k<\lambda^{N+1}}\langle q_N t(\omega)\rangle\right|^2\right)\\ &\longrightarrow 2\log\lambda. \qquad \square\end{aligned}$$

6. *Remarks on the thickness of rigidity sequences*

Remark 6.1. Rigidity sequences for weakly mixing transformations can be arbitrarily 'large' within the limitation of density zero.

It follows from the definitions that for $q\in\mathbb{N}^{\mathbb{N}}(\uparrow)$, a growth sequence,

$$|\mathtt{FS}(q)\cap[1,n]|\asymp 2^{c(n)},$$

where $c(n)=\min\{k\geq 1: q_k\geq n\}$, and if $T\in\mathtt{PPT}(X,\mathcal{B},m)$ is IP-rigid along q, one might expect a rigid sequence of at least this thickness.

Indeed, in this case, by Theorem 2, $\sigma_T(e(\mathfrak{T})^c)=0$, where $\mathfrak{T}$ is the dyadic tower with growth sequence q (since, by [**AN**], $e(\mathfrak{T})=G_2(q)$). By [**A2**, Theorem 4], there exists $L\subset\mathbb{N}$ with

$$T^n\xrightarrow[n\to\infty,n\in L]{\mathtt{PPT}}\mathtt{Id}\quad\text{and}\quad\frac{|L\cap[1,n]|}{2^{c(n)}}\xrightarrow[n\to\infty]{}\infty.$$

By the corollary in [**A2**], for all $a(n)>0$, $a(n)/n\xrightarrow[n\to\infty]{}0$, there is a weakly mixing $T\in\mathtt{PPT}(X,\mathcal{B},m)$ and $L\subset\mathbb{N}$ such that

$$T^n\xrightarrow[n\to\infty,n\in L]{\mathtt{PPT}}\mathtt{Id}\quad\text{and}\quad\frac{|L\cap[1,n]|}{a(n)}\xrightarrow[n\to\infty]{}\infty.$$

For more on this phenomenon, see [**BJLR**, §3].

Remark 6.2. Let $q\in\mathbb{N}^{\mathbb{N}}(\uparrow)$ be a growth sequence and suppose that $\mathtt{H\text{-}dim}(G_1(q))>\alpha$ (where $\alpha\in(0,1)$ and $\mathtt{H\text{-}dim}$ denotes Hausdorff dimension).

We claim that there exists $T\in\mathtt{PPT}$ weakly mixing and IP-rigid along q with the property that for any sequence $L\subset\mathbb{N}$ along which T is rigid:

$$\sum_{n=1}^{\infty}\frac{|L\cap[1,n]|}{n^{2-\alpha}}<\infty. \qquad (\text{⚽})$$

Note that it follows from this that $\sum_{n=1}^{\infty}2^{c(n)}/n^{2-\alpha}<\infty$.

Proof of (⚽). As in the proof of [**A3**, Theorem 1], it follows from Frostman's theorem ([**Fr**], see also [**KS**]) that there exists $\mu\in\mathcal{P}(G_1(q))$ such that

$$\sum_{n=1}^{\infty}\frac{|\widehat{\mu}(n)|}{n^{1-\alpha}}<\infty.$$

Let $T\in\mathtt{PPT}(X,\mathcal{B},m)$ be the associated Gaussian automorphism. By Proposition 1, T is IP-rigid along q.

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Suppose that T is rigid along $L \subset \mathbb{N}$. In particular, there exists $N \geq 1$ such that $|\widehat{\mu}(n)| \geq \frac{1}{2}$ for all $n \in L$, $n > N$.

It follows that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{|L \cap [1,n]|}{n^{2-\alpha}} &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} 1_L(k) \frac{1}{n^{2-\alpha}} \\ &= \sum_{k=1}^{\infty} 1_L(k) \sum_{n=k}^{\infty} \frac{1}{n^{2-\alpha}} \\ &\leq \frac{1}{1-\alpha} \sum_{k=1}^{\infty} 1_L(k) \frac{1}{k^{1-\alpha}} \\ &\leq \frac{N}{1-\alpha} + \frac{2}{1-\alpha} \sum_{k=N+1}^{\infty} \frac{|\widehat{\mu}(k)|}{k^{1-\alpha}} \\ &< \infty. \ (\text{⚽}).\end{aligned}$$

□

Remark 6.3. The condition (⚽) is sharp. In §5 of [**A3**], for all $\alpha \in (0, 1)$, a growth sequence $q^{(\alpha)} \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ is exhibited with

$$\texttt{H-dim}\,(G_1(q^{(\alpha)})) = \alpha \text{ and } 2^{c^{(\alpha)}(n)} \gg n^{1-\alpha},$$

whence, if $T \in \texttt{PPT}(X, \mathcal{B}, m)$ is such that $\sigma_T(G_2(q^{(\alpha)})^c) = 0$, then (again by Theorem 4 in [**A2**]) there exists $L \subset \mathbb{N}$ rigid for T with $|L \cap [1, n]|/n^{1-\alpha} \to \infty$, and therefore

$$\sum_{n=1}^{\infty} \frac{|L \cap [1,n]|}{n^{2-\alpha}} = \infty.$$

6.1. *Update.* Since submission, [**G**] appeared, which, *inter alia*, shows that the converse to Theorem 2 is false. In particular ([**G**, Theorem 1.5]), if $q \in \mathbb{N}^{\mathbb{N}}(\uparrow)$ is given by $\{q_n : n \geq 1\} = \bigcup_{\ell \geq 2} \{p_\ell, 2p_\ell, \ldots, \ell^2 p_\ell\}$, where $p_1 = 1$ and $p_{\ell+1} = ((\ell^2(\ell^2+1))/2)p_\ell$ for all $l \geq 1$, then $|G_2(q)| > \aleph_0$, but $\texttt{FS}(q)$ has positive density, whence $q \notin \texttt{IPRWM}$.

Acknowledgements. Aaronson and Hosseini would like to thank the Nicolaus Copernicus University, and Aaronson the Leverhulme Foundation and University of Surrey, for hospitality provided when this paper was being written. Aaronson's research was partially supported by Israel Science Foundation grant No. 1114/08. Lemańczyk's research was supported by Polish National Science Center grant DEC-2011/03/B/ST1/00407.

References

[A1] J. Aaronson. *An Introduction to Infinite Ergodic Theory (Mathematical Surveys and Monographs, 50)*. American Mathematical Society, Providence, RI, 1997.

[A2] J. Aaronson. Rational ergodicity, bounded rational ergodicity and some continuous measures on the circle, a collection of invited papers on ergodic theory. *Israel J. Math.* **33**(3–4) (1979), 181–197.

[A3] J. Aaronson. The eigenvalues of non-singular transformations. *Israel J. Math.* **45** (1983), 297–312.

https://doi.org/10.1017/etds.2012.186 Published online by Cambridge University Press

[AN] J. Aaronson and M. G. Nadkarni. L_∞ eigenvalues and L_2 spectra of non-singular transformations. *Proc. Lond. Math. Soc.* (3) **55** (1987), 538–570.

[BJLR] V. Bergelson, A. del Junco, M. Lemańczyk and J. Rosenblatt. Rigidity and non-recurrence along sequences. *Preprint*, submitted, arxiv.org/pdf/1103.0905.

[CFS] I. P. Cornfeld, S. V. Fomin and Y. G. Sinai. *Ergodic Theory*. Springer, New York, 1982.

[E] H. G. Eggleston. Sets of fractional dimensions which occur in some problems of number theory. *Proc. Lond. Math. Soc.* (2) **54** (1952), 42–93.

[EG] T. Eisner and S. Grivaux. Hilbertian Jamison sequences and rigid dynamical systems. *J. Funct. Anal.* **261** (2011), 2013–2052.

[ET] P. Erdős and S. J. Taylor. On the set of points of convergence of a lacunary trigonometric series and the equidistribution properties of related sequences. *Proc. Lond. Math. Soc.* (3) **7** (1957), 598–615.

[Fr] O. Frostman. Potentiel d'equilibre et capacité des ensembles avec quelques applications la théorie des fonctions. *Meddelanden Mat. Sem. Univ. Lund.* 3 **115s** (1935).

[Fu] H. Furstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory*. Princeton University Press, Princeton, NJ, 1981.

[FW] H. Furstenberg and B. Weiss. The finite multipliers of infinite ergodic transformations. *The Structure of Attractors in Dynamical Systems (Proc. Conf., North Dakota State University, Fargo, ND, 1977) (Lecture Notes in Mathematics, 668)*. Springer, Berlin, 1978, pp. 127–132.

[G] S. Grivaux. IP-Dirichlet measures and IP-rigid dynamical systems: an approach via generalized Riesz products. *Preprint*, arXiv:1209.2884.

[HS] N. Hindman and D. Strauss. *Algebra in the Stone–Čech compactification*. De Gruyter, Berlin, 1998.

[HMP1] B. Host, J.-F. Mela and F. Parreau. *Analyse harmonique des mesures (Asterisque, 135–136)*. Société Mathématique de France, Paris, 1986, 261 pp.

[HMP2] B. Host, J.-F. Mela and F. Parreau. Nonsingular transformations and spectral analysis of measures. *Bull. Soc. Math. France* **119**(1) (1991), 33–90.

[IN] Sh. Ito and H. Nakada. Approximations of real numbers by the sequence $\{n\alpha\}$ and their metrical theory. *Acta Math. Hungar.* **52** (1988), 91–100.

[KS] J. P. Kahane and R. Salem. *Ensembles parfaites et series trigonometriques*. Hermann, Paris, 1963.

[KT] A. Katok and J.-P. Thouvenot. Spectral properties and combinatorial constructions in ergodic theory. *Handbook of Dynamical Systems.* Vol. 1B. Elsevier, Amsterdam, 2006, pp. 649–743.

[N1] M. G. Nadkarni. *Basic Ergodic Theory*. Hindustan Book Agency, New Delhi, 1998.

[N2] M. G. Nadkarni. *Spectral Theory of Dynamical Systems*. Hindustan Book Agency, New Delhi, 1998.

[P] F. Parreau. Ergodicité et pureté des produits de Riesz. *Ann. Inst. Fourier* **40** (1990), 391–405.

[Z] R. J. Zimmer. *Ergodic Theory and Semisimple Groups (Monographs in Mathematics, 81)*. Birkhäuser, Basel, 1984.

https://doi.org/10.1017/etds.2012.186 Published online by Cambridge University Press