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$\overline{d}$-continuity for countable state shifts

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Abstract. For full shifts on finite alphabets, Coelho and Quas [Criteria for $\overline{d}$-continuity. *Trans. Amer. Math. Soc.* **350**(8) (1998), 3257–3268] showed that the map that sends a Hölder continuous potential ϕ to its equilibrium state μ_ϕ is $\overline{d}$-continuous. We extend this result to the setting of full shifts on countable (infinite) alphabets. As part of the proof, we show that the map that sends a strongly positive recurrent potential to its normalization is continuous for potentials on mixing countable state Markov shifts.

Key words: classical ergodic theory, symbolic dynamics
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1. *Introduction*

The concept of $\overline{d}$-distance on the space of invariant measures on a shift space was introduced by Ornstein to study the isomorphism problem for Bernoulli shifts. Ornstein showed that entropy is a complete invariant for the class of Bernoulli shifts: two Bernoulli shifts are isomorphic if and only if they have the same entropy [8, 9]. A key tool in this proof was the $\overline{d}$-metric. Many ergodic properties are well behaved with respect to this metric. The entropy function $\mu \mapsto h_\mu(\sigma)$ is $\overline{d}$-continuous. Moreover, the set of processes that are isomorphic to Bernoulli shifts is $\overline{d}$-closed. The more familiar topology on the space of Borel probability measures is the weak*-topology. The topology coming from the $\overline{d}$-metric refines the weak*-topology.

If S is a countable set, the full shift with alphabet S is the space of all (one-sided or two-sided) sequences with symbols coming from S, together with the left shift map σ. For Markov shifts on finite alphabets, potentials with summable variations have a unique equilibrium state [14]. One can quickly prove that the map that sends a potential to its unique equilibrium state is continuous in the weak*-topology. For full shifts on a finite alphabet, Coelho and Quas proved that the map that sends a potential ϕ to its unique equilibrium state μ_ϕ is continuous with respect to the $\overline{d}$-metric on the space of shift-invariant probability measures and a suitable metric on the space of potentials [4].

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A natural question that arises is whether a similar result holds if we take the alphabet to be countably infinite. In this paper, all shift spaces that we consider are one-sided. The thermodynamic formalism for countable-state Markov shifts (CMSs) was developed by Sarig [2, 12, 13] and by Mauldin and Urbański [7]. In this case, the classical notion of topological pressure must be replaced by the *Gurevich pressure* $P_G(\phi)$, where the partition function is a sum taken only over those periodic orbits that begin in a fixed 1-cylinder.

A useful tool in studying equilibrium states is the Ruelle–Perron–Frobenius (RPF) operator. For every ϕ with summable variations, the Ruelle operator L_ϕ is defined as

$$L_\phi(f)(x) = \sum_{\sigma y = x} e^{\phi(y)} f(y).$$

When $L_\phi(1) = 1$, we call the potential 'normalized'. The generalized RPF theorem proved in [12] relates the modes of recurrence of potentials to the eigendata of the RPF operator corresponding to the eigenvalue $\lambda = e^{P_G(\phi)}$. Some important results relating to the thermodynamic formalism for CMS are summarized in §§2.1 and 2.2. When ϕ is 'positive recurrent', the eigendata of the RPF operator and its dual give rise to an important invariant probability measure corresponding to the potential, called the RPF measure. For getting a result similar to that given by Coelho and Quas for the countable-state case, we need to put an even stronger condition on our potential, namely 'strong positive recurrence'.

THEOREM 1.1. *Let Σ be the full shift on a countably infinite alphabet. Let $0 < \theta < 1$ and let A_θ be the set of θ-weakly Hölder continuous strongly positive recurrent potentials with finite Gurevich pressure. Let d_θ be the metric associated to the Hölder norm on A_θ. Let ϕ be a potential in A_θ. Then, for given $\epsilon > 0$, there exists $\delta > 0$ such that $\overline{d}(m_\phi, m_\tau) < \epsilon$ whenever $d_\theta(\phi, \tau) < \delta$, where $\tau \in A_\theta$, and m_ϕ and m_τ are the corresponding RPF measures.*

The notion of g-functions was introduced to ergodic theory in [6] and is a useful tool in thermodynamic formalism. Conformal measures for g-functions are called g-measures. Corresponding to a strongly positive recurrent potential ϕ, we get a normalized potential, which is the logarithm of a g-function g_ϕ. The RPF measure for ϕ is a g-measure for g_ϕ. Theorem 1.1 follows as a corollary to Theorems 1.2 and 1.4.

The first result gives the continuous dependence of g-measures on their corresponding g-functions.

THEOREM 1.2. *Suppose g is a continuous g-function on a full shift, Σ, on a countably infinite alphabet with the property that*

$$\sum_{n=r}^{\infty} \prod_{i=r}^{n} e^{-V_i(g)} = \infty \tag{1.1}$$

for some $r \geq 1$, where $V_n(g)$ is the nth variation of $\log(g)$. Let ν_g be a g-measure. Then, for every $\epsilon > 0$, there exists $\delta > 0$ such that if h is a g-function on Σ and $d(h, g) := \|\log(g) - \log(h)\|_\infty < \delta$, then for any g-measure ν_h corresponding to h, $\bar{d}(\nu_g, \nu_h) < \epsilon$.

Condition (1.1) is satisfied whenever $\sum_{i=1}^{\infty} V_i(g) < \infty$; in particular, Theorem 1.2 applies whenever $\log(g)$ is weakly Hölder continuous. As a corollary of the proof of

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Theorem 1.2, we get that in this case, the map $g \mapsto \nu_g$ is locally Lipschitz with respect to the metrics d and $\overline{d}$.

THEOREM 1.3. *Suppose g is a continuous g-function on a full shift, Σ, on a countably infinite alphabet such that $\sum_{i=2}^{\infty} V_i(g) < \infty$, where V_n is the nth variation of* $\log(g)$. *Let ν_g be a g-measure. Then, there exists a constant A_g such that if h is any g-function on Σ with $d(h, g) < \ln(2)$ and g-measure ν_h, then*

$$\overline{d}(\nu_g, \nu_h) \leq A_g d(g, h).$$

We prove Theorems 1.2 and 1.3 in §3. The proof of Theorem 1.2 requires the construction of a measure on $\Sigma \times \Sigma$ that is a fixed point for a map using a limiting procedure. In Coelho and Quas' proof, the finiteness of the alphabet plays a critical role. The challenge is that since in our case Σ is not compact, then the set of probability measures on $\Sigma \times \Sigma$ is not weak*-compact, and we cannot use the Schauder–Tychonoff theorem and can no longer use estimates that depend on the number of states. We overcome this obstacle by constructing a sequence of measures that is 'tight' and by defining the metric d on the space of g-functions in terms of $\log g$ instead of g.

The second result gives the continuous dependence of the normalized potentials $\log(g_\phi)$ on the corresponding potentials for countable state Markov shifts. Given a graph G with set of vertices S and a set of directed edges, the topological Markov shift (TMS) associated to G is the collection of infinite admissible paths on this directed graph. If S is a finite set, we call this a subshift of finite type (SFT) and if S is a countable (infinite) set, the corresponding TMS is called a countable-state Markov shift (CMS). Countable-state Markov shifts are of interest as they arise as coding spaces for non-uniformly hyperbolic systems which cannot be coded by an SFT.

THEOREM 1.4. *Let Σ be a mixing CMS. Let $0 < \theta < 1$ and let A_θ be the set of θ-weakly Hölder continuous strongly positive recurrent potentials with finite Gurevich pressure. Let d_θ be the metric associated to the Hölder norm on A_θ and let ϕ be a potential in A_θ. Then, for given $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\|\log(g_\phi) - \log(g_\tau)\|_\infty < \epsilon \quad \textit{whenever } \tau \in A_\theta \textit{ and } d_\theta(\phi, \tau) < \delta.$$

The proof of Theorem 1.4 is based on the spectral gap property (SGP) of the Ruelle operator corresponding to a strongly positive recurrent potential. Section 4 is dedicated to the proof of this result.

Note that Theorem 1.4 holds for CMS, while Theorems 1.1 and 1.2 hold for full shifts only. The extension of Theorems 1.1 and 1.2 to CMS will be explored in a forthcoming paper [**1**].

2. *Background*

2.1. *Setting.* Let S be a countable set (alphabet). The *one-sided full shift* Σ_S with set of states S is the set of sequences

$$\Sigma_S = \{(x_0 x_1 \cdots) \colon x_k \in S \text{ for all } k \in \mathbb{N} \cup \{0\}\}$$

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together with the left shift map $\sigma : \Sigma_S \to \Sigma_S$ defined by $(\sigma(x))_k = x_{k+1}$ for all k. We equip Σ_S with the product topology induced by the discrete topology on S. This topology is metrizable; one metric is given by $d(x, y) = e^{-t(x,y)}$, where $t(x, y) := \inf\{k : x_k \neq y_k\}$. The resulting topology is generated by *cylinder sets*

$$[a_0 \cdots a_{n-1}] := \{x \in X : x_i = a_i, i = 0, \ldots n-1\} \ (n \in \mathbb{N}, a_i \in S).$$

With respect to this topology, Σ_S is a complete separable metric space, but it is not necessarily compact. A *subshift* of Σ_S, also known as a *shift space*, is a closed subset Σ that is shift invariant ($\sigma(\Sigma) = \Sigma$). A *finite word* over the alphabet S is a finite sequence of symbols from S. The *language* of a shift space Σ is the set of finite words that appear in some sequence in Σ. Such words are called *admissible*.

Let $A = (t_{ij})_{S\times S}$ be a matrix of zeroes and ones, with columns and rows that are not all zeroes. The TMS with set of states S and transition matrix A is the shift space

$$\Sigma := \{(x_0, x_1, \ldots) \in \Sigma_S : t_{x_i x_{i+1}} = 1 \text{ for all } i\}.$$

Corresponding to this, we can construct a directed graph with set of vertices S and set of edges $\{a \to b : t_{ab} = 1\}$. We can then think of Σ as the collection of infinite admissible paths on this directed graph. Σ is compact if and only if the S is finite. A *subshift of finite type* is a compact TMS. Non-compact TMSs are also called *countable-state Markov shifts (CMSs)*.

We assume throughout that Σ is *topologically mixing*, that is, for any two states a and b, there is $N(a, b)$ such that for all $n \geq N(a, b)$, there is an admissible word of length $n + 1$ which starts at a and ends at b.

A function $\phi : \Sigma \to \mathbb{R}$ is called a *potential*. The modulus of continuity of ϕ is captured by the decay of variations of ϕ. The *variations* for ϕ are

$$\mathrm{var}_n(\phi) = \sup\{|\phi(x) - \phi(y)| : x, y \in \Sigma, x_i = y_i, 0 \leq i \leq n-1\}.$$

A potential ϕ is uniformly continuous if and only if $\mathrm{var}_n(\phi) \to 0$. Here, ϕ is called the following.

(1) *θ-weakly Hölder continuous* for some $0 < \theta < 1$ if there exists a constant $C_\phi > 0$ such that $\mathrm{var}_n(\phi) \leq C_\phi \theta^n$ for all $n \geq 2$. If, in addition, $\mathrm{var}_1(\phi)$ is finite, ϕ is called *locally Hölder continuous*. Furthermore, if ϕ is bounded, then it is called *θ-Hölder*, where θ is called the *Hölder exponent* of ϕ.

(2) A function with *summable variations* if $\sum_{n\geq 2} \mathrm{var}_n(\phi) < \infty$.

Note that the condition of being weakly Hölder continuous is stronger than summable variations. None of these two regularity conditions imply boundedness, and it is also possible that $\mathrm{var}_1(\phi) = \infty$.

Let $0 < \theta < 1$ and let $\|\cdot\|_\theta$ denote the Hölder norm, which is given by

$$\|\phi\|_\theta := \|\phi\|_\infty + |\phi|_\theta, \tag{2.1}$$

where $\|\cdot\|_\infty$ is the uniform norm and

$$|\phi|_\theta := \sup\{|\phi(x) - \phi(y)|/\theta^{t(x,y)} : x, y \in X, x \neq y\}. \tag{2.2}$$

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Let d_θ be the metric coming from the norm $\|\cdot\|_\theta$. Let A_θ be the set of θ-weakly Hölder continuous strongly positive recurrent potentials with finite Gurevich pressure (see §2.2). Note that it is possible that $d_\theta(\phi, \psi) = \infty$ for potentials ϕ and ψ in A_θ.

2.2. *Thermodynamic formalism for CMS.* We use thermodynamic formalism for CMS developed in [2, 12, 13]. Let ϕ be a potential with summable variations. The *Birkhoff sums* of ϕ are denoted by $S_n(\phi) := \sum_{k=0}^{n-1} \phi \circ \sigma^k$. For any state a, let

$$\mathrm{Per}_n(\Sigma, a) := \{x \in \Sigma : \sigma^n(x) = x, x_0 = a\} \quad \text{and}$$
$$\mathrm{Per}_n^*(\Sigma, a) := \{x \in \mathrm{Per}_n(\Sigma, a) : x_i \neq a \text{ for all } 1 \leq i \leq n-1\}.$$

We also write

$$Z_n(\phi, a) = \sum_{x \in \mathrm{Per}_n(\Sigma, a)} e^{S_n\phi(x)} \quad \text{and}$$
$$Z_n^*(\phi, a) = \sum_{x \in \mathrm{Per}_n^*(\Sigma, a)} e^{S_n\phi(x)}.$$

The limit $P_G(\phi) := \lim_{n\to\infty} 1/n \ln Z_n(\phi, a)$ exists for all states a and is independent of the choice of state a. Here, $P_G(\phi)$ is called the *Gurevich pressure* of ϕ. When $\phi = 0$, we obtain the *Gurevich entropy* $h_G(\Sigma) = P_G(0)$.

It was proved in [13, Theorem 3] that if $\|L_\phi(\mathbb{1})\|_\infty < \infty$, then the Gurevich pressure satisfies a variational principle:

$$P_G(\phi) = \sup \left\{ h_\mu(\sigma) + \int \phi \, d\mu \right\},$$

where the supremum ranges over all invariant probability measures such that the sum is not of the form $\infty - \infty$. A measure μ_ϕ which attains this supremum is called an *equilibrium state* for ϕ.

2.2.1. *Modes of recurrence, RPF operator, and the generalized RPF theorem.* Let $P_G(\phi) < \infty$ and set $\lambda := e^{P_G(\phi)}$. Following Sarig, we say that ϕ is *recurrent* if for some state a, $\sum_{n\geq 1} \lambda^{-n} Z_n(\phi, a)$ diverges, and is *transient* if it converges. Further, if ϕ is recurrent and $\sum_{n\geq 1} n\lambda^{-n} Z_n^*(\phi, a)$ converges, ϕ is called *positive recurrent*. A potential ϕ is said to be *strongly positive recurrent (SPR)* if

$$\limsup_{n\to\infty} \frac{1}{n} \ln Z_n^*(\phi, a) < P_G(\phi). \tag{2.3}$$

(Sarig's original definition of SPR potentials is given later. Climenhaga [3, §8.5] shows that (2.3) is equivalent.) These definitions can be shown to be independent of the choice of the state a.

For every ϕ with summable variations, we associate the RPF operator L_ϕ which is defined as

$$L_\phi(f)(x) = \sum_{\sigma y = x} e^{\phi(y)} f(y).$$

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The generalized RPF theorem proved in [12] relates the modes of recurrence to the eigendata of the RPF operator. We have that ϕ is recurrent if and only if there exists a $\lambda > 0$, a positive continuous function h_ϕ, and a conservative measure ν_ϕ which is finite and positive on cylinders such that

$$L_\phi h_\phi = \lambda h_\phi \quad \text{and} \quad L_\phi^* \nu_\phi = \lambda \nu_\phi.$$

Here, $\lambda = e^{P_G(\phi)}$. Additionally, ϕ is positive recurrent if and only if $\int h_\phi \, d\nu_\phi < \infty$. The measure $dm_\phi = h_\phi \, d\nu_\phi$ is called the RPF measure for ϕ. If ϕ is positive recurrent and its RPF measure has finite entropy, then it is an equilibrium state for ϕ. For subshifts of finite type, this is always the case. Furthermore, if the equilibrium state exists, then it is unique [2].

We now give Sarig's original definition of the strong positive recurrence condition in terms of a *discriminant*. Let Σ be a TMS and $a \in S$. We define an induced system on $[a]$ as $\sigma_a \colon \Sigma_a \to \Sigma_a$, where $\Sigma_a = \{x \in \Sigma \colon x_0 = a, x_i = a \text{ infinitely often}\}$ and $\sigma_a = \sigma^{\tau_a(x)}(x)$, where $\tau_a(x) = \min\{n \geq 1 \colon x_n = a\}$.

This system can be given the structure of a TMS as follows. Let $\overline{S} = \{[a, \xi_1, \ldots, \xi_{n-1}, a] : n \geq 1, \xi_i \neq a\} \setminus \emptyset$. Let $\overline{\Sigma} = \overline{S}^{\mathbb{N}\cup\{0\}}$ and $\overline{\sigma}$ be the left shift map on $\overline{\Sigma}$. Then, $(\overline{\Sigma}, \overline{\sigma})$ is topologically conjugate to (Σ_a, σ_a) and the conjugacy $\pi : \overline{\Sigma} \to \Sigma_a$ is given by

$$\pi([a, \xi_1^0, \ldots \xi_{n_0}^0, a], [a, \xi_1^1, \ldots \xi_{n_1}^1, a], \ldots) := (a, \xi_1^0, \ldots \xi_{n_0}^0, a, \xi_1^1, \ldots \xi_{n_1}^1, a \cdots).$$

Corresponding to every potential function ϕ, we get an induced potential $\overline{\phi} : \overline{\Sigma} \to \mathbb{R}$, which is defined as

$$\overline{\phi} := \left(\sum_{i=0}^{\tau_a - 1} \phi \circ \sigma^i \right) \circ \pi.$$

If ϕ is weakly Hölder continuous, then $\overline{\phi}$ is locally Hölder continuous. The *a-discriminant* of ϕ is defined as

$$\Delta_a[\phi] := \sup\{P_G(\overline{\phi + p}) \colon p \in \mathbb{R} \text{ such that } P_G(\overline{\phi + p}) < \infty\}.$$

A potential ϕ is said to be *strongly positive recurrent* if there exists a state a such that $\Delta_a[\phi] > 0$. If ϕ is strongly positive recurrent, then $P_G(\phi) = 0 \iff P_G(\overline{\phi}) = 0$.

2.3. *$\overline{d}$-metric.* Ornstein introduced the concept of $\overline{d}$-distance on the space of invariant measures on a shift space to study the isomorphism problem for Bernoulli shifts. He also proved that the set of processes which are measure theoretically isomorphic to Bernoulli shifts is $\overline{d}$-closed (see [8, 9]).

Let (Σ, σ) be a CMS. We denote the space of Borel probability measures on Σ by $\mathscr{M}(\Sigma)$ and let $\mathscr{M}^\sigma(\Sigma)$ denote the space of shift invariant Borel probability measures on Σ. For ν_1 and ν_2 in $\mathscr{M}^\sigma(\Sigma)$, a *joining* of (Σ, ν_1) and (Σ, ν_2) is a shift invariant measure on $\Sigma \times \Sigma$ with the property that $\mu(A \times \Sigma) = \nu_1(A)$ and $\mu(\Sigma \times B) = \nu_2(B)$ for any measurable subsets A and B of Σ. Equivalently, a joining of (Σ, ν_1) and (Σ, ν_2) is a shift invariant measure on $\Sigma \times \Sigma$ such that $\pi_1^* \mu = \nu_1$ and $\pi_2^* \mu = \nu_2$, where π_1, π_2 are

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projections onto the two coordinates. We now define the $\bar{d}$-metric in terms of joinings. The following definition is shown to be equivalent to Ornstein's original definition in [11].

Definition 2.1. Let $J(\nu_1, \nu_2)$ be the set of joinings of (Σ, ν_1) and (Σ, ν_2). Let $\delta(x, y)$ be 1 if $x_0 \neq y_0$ and 0 otherwise. Then,

$$\begin{aligned}\bar{d}(\nu_1, \nu_2) &= \inf_{\mu \in J(\nu_1,\nu_2)} \int \delta(x, y)\, d\mu(x, y) \\ &= \inf_{\mu \in J(\nu_1,\nu_2)} \mu(\{(x, y) \in \Sigma \times \Sigma : x_0 \neq y_0\}).\end{aligned}$$

The more familiar topology on the space of Borel probability measures on a metric space Y is the weak* topology. A sequence μ_n in $\mathscr{M}(Y)$ is said to converge weak* to μ if $\int f\, d\mu_n \to \int f\, d\mu$ for all bounded continuous function f on Y. When Y is compact, $\mathscr{M}(Y)$ is compact in the weak* topology. When Y is not compact, it is important to first understand under what conditions does a sequence in $\mathscr{M}(Y)$ have a convergent subsequence.

A subset of $\mathscr{M}(Y)$ is said to be tight if for every $\epsilon > 0$, there exists a compact subset K_ϵ, such that $\mu(K_\epsilon^c) < \epsilon$ for all measures μ in the subset. If Y is a complete separable metric space and μ_n is a tight sequence in $\mathscr{M}(Y)$, then the Helly–Prokhorov theorem gives that it has a subsequence that converges weak* to some μ in $\mathscr{M}(Y)$. Hence, every tight subset of $\mathscr{M}(Y)$ is precompact.

LEMMA 2.1. *For ν_1 and ν_2 in $\mathscr{M}^\sigma(\Sigma)$, $J(\nu_1, \nu_2)$ is weak*-compact and hence the infimum in the above definition of the $\bar{d}$-metric is actually a minimum.*

Proof. Since every finite Borel measure on a complete separable metric space is tight, then for given $\epsilon > 0$, we get compact sets A and B in Σ such that $\nu_1(\Sigma \setminus A) < \epsilon$ and $\nu_2(\Sigma \setminus B) < \epsilon$.

Let $K = A \times B \subset \Sigma \times \Sigma$ and take m in $J(\nu_1, \nu_2)$. Since

$$(\Sigma \times \Sigma) \setminus (A \times B) \subset ((\Sigma \setminus A) \times \Sigma) \cup (\Sigma \times (\Sigma \setminus B)),$$

we conclude that

$$m((\Sigma \times \Sigma) \setminus (A \times B)) \leq \nu_1(\Sigma \setminus A) + \nu_2(\Sigma \setminus B) < 2\epsilon.$$

Therefore, $J(\nu_1, \nu_2)$ is tight; hence, it is pre-compact in the weak*-topology. Also, $J(\nu_1, \nu_2)$ is closed. Indeed, let μ_n be a sequence in $J(\nu_1, \nu_2)$ converging to ν and consider for $f \in C_b(X)$,

$$\int f \circ \pi_1\, d\nu = \lim_{n\to\infty} \int f \circ \pi_1\, d\mu_n = \lim_{n\to\infty} \int f\, d\nu_1 = \int f\, d\nu_1.$$

Hence, $(\pi_1)_*\nu = \nu_1$ and similarly $(\pi_2)_*\nu = \nu_2$. So, ν is in $J(\nu_1, \nu_2)$. Hence, $J(\nu_1, \nu_2)$ is compact. □

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3. *g-function and $\overline{d}$-continuity*

3.1. *Definitions and basic facts.* The notion of g-functions and g-measures was introduced to ergodic theory in [6] and is a useful tool in thermodynamic formalism. In [14], Walters gave a proof of Ruelle's operator theorem using g-measures.

Let Σ be a full shift with a countable set of states S and let σ be the left shift map on Σ. A continuous function $g\colon \Sigma \to (0, 1]$ is called a *g-function* if

$$\sum_{y\in\sigma^{-1}(x)} g(y) = 1 \quad \text{for all } x \in \Sigma.$$

Conformal measures for g-functions are called g-measures, that is, μ is a g-measure if and only if $L_g^*(\mu) = \mu$. For a given g-function, a *g-measure* is a σ-invariant measure μ in the space of Borel probability measures of Σ such that

$$g(x) = \lim_{|A|\to 0, x\in A} \frac{\mu(A)}{\mu(\sigma(A))}, \quad \mu\text{ -a.e}$$

where $|A|$ denotes the diameter of A and a.e. means almost everywhere.

We denote by $[x]^n = \{y : y_k = x_k \text{ for all } k \le n\}$, the cylinder of those points which agree with x for the first $n + 1$ places. The measure μ is a g-measure if

$$\lim_{n\to\infty} \frac{\mu([ix]^{n+1})}{\mu([x]^n)} = g(ix), \quad \mu\text{ -a.e.}$$

We now discuss a probabilistic interpretation of g-functions and g-measures. We can view $g(ix)$ as a Markov transition probability, giving the probability of moving from x to ix and a g-measure is an invariant measure for this Markov process (see [10]). More precisely, let us consider a sequence of random variables $(X_n)_{n\in\mathbb{Z}}$ taking values in the set of states S. Here, X_n are maps from some probability space Ω to S. Suppose the sequence satisfies

$$\mathbb{P}(X_n = i | X_{n-1} = x_1, X_{n-2} = x_2, \ldots) = g(ix).$$

We fix a natural number n and define $\rho_n\colon \Omega \to \Sigma$ by $(\rho_n(\omega))_i = X_{n-i}(\omega)$. The probability distribution on $(X_m)_{m\le n}$ pushes forward to a probability measure on Σ, where $\rho_n^*(\mathbb{P}(A)) = \mathbb{P}(\rho_n^{-1}(A)))$. The g-measures correspond to stationary distributions for the random variables.

For SFTs, continuous functions have at least one g-measure associated to them but uniqueness is not guaranteed. In the cases where g is Hölder continuous or the variations of g are summable, the g-measure is known to be unique (proofs can be found in [6, 14]).

Let ϕ be a recurrent potential and let $\lambda = e^{P_G(\phi)}$. Let h_ϕ be given by the generalized RPF theorem. Then, $g_\phi := e^\phi h_\phi/\lambda h_\phi \circ \sigma$ is a g-function. The RPF measure for ϕ is a g-measure for g_ϕ.

The following lemmas give some results about the RPF operator associated to continuous g-functions.

LEMMA 3.1. *Let g be a continuous g-function on Σ and $L_{\log g} : C(\Sigma) \to C(\Sigma)$ be the RPF operator corresponding to* $\log g$. *Then, $L_{\log g}$ is continuous with respect to the uniform norm on $C_b(\Sigma)$.*

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Proof. Since

$$|f(y)| \le \|f\|_{\sup} \quad \text{for all } y \in \Sigma \quad \text{and}$$
$$\sum_{y \in \sigma^{-1}(x)} g(y) = 1 \quad \text{for all } x \in \Sigma,$$

then

$$\|L_{\log g}(f)\|_{\sup} = \sup_{x \in \Sigma} |L_{\log g}(f)(x)| = \sup_{x \in \Sigma} \left| \sum_{y \in \sigma^{-1}(x)} g(y) f(y) \right| \le \|f\|_{\sup}. \qquad \square$$

The following result about dual operators on Banach spaces shows that $L^*_{\log g}$ is also continuous.

LEMMA 3.2. *Let $T : V \to V$ be a bounded linear operator on a Banach space V. Let $T^* : V^* \to V^*$ be the dual of T, defined by*

$$T^*(f) = f \circ T$$

Then, T^* is continuous with respect to the weak-topology on V^*.*

Proof. Let $(f_\alpha)_\alpha$ be a net in V^* with weak*-limit f, then

$$f_\alpha(Tx) \to f(Tx) \quad \text{for all } x \text{ in } V \implies (f_\alpha \circ T)(x) \to (f \circ T)(x) \quad \text{for all } x \text{ in } V$$
$$\implies f_\alpha \circ T \xrightarrow{\text{weak*}} f \circ T \implies T^*(f_\alpha) \xrightarrow{\text{weak*}} T^*(f). \qquad \square$$

3.2. *Proof of Theorem 1.2.* Let Σ be the full shift with countable alphabet S. We define the following on the space of continuous g-functions:

$$V_n(g) = \sup_{[x]^{n-1} = [y]^{n-1}} |\log(g(x)) - \log(g(y))|,$$
$$d(h, g) = \sup_x |\log(g(x)) - \log(h(x))| = \|\log(g) - \log(h)\|_\infty.$$

Note that when $d(h, g) < \infty$, there exists c in $(0, 1]$ such that

$$g(ix) \ge ch(iy) \quad \text{and} \quad h(iy) \ge cg(ix) \quad \text{for all } i, x, y. \tag{3.1}$$

In the remainder of this section, we give a proof of Theorem 1.2, which extends [4, Theorem 1] to the setting of full shifts on countable alphabets.

Proof of Theorem 1.2. We follow the proof of Coelho and Quas [4]. However, various modifications are needed since the alphabet is infinite and the space is no longer compact. To avoid estimates that depend on the number of states, we use a metric coming from the difference in the logarithms of the g-functions to get multiplicative control. The proof also requires the construction of a joining using a limiting procedure. Since the space of measures we work with is no longer compact, we construct a sequence of measures that is 'tight' to get the joining as a limit point.

We divide the proof into three parts. In §3.2.1, we first define transition probabilities whose marginals are given by the g-functions. In §3.2.2, we use these transition probabilities to get a joining of the corresponding g-measures. Finally, in §3.2.3, we use this joining

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to get an estimate on the $\overline{d}$-distance between the two g-measures to complete the proof. The arguments in §§3.2.1 and 3.2.2 are valid for a general CMS. The arguments in §3.2.3 depend on (3.1), which is only valid for the full shift.

3.2.1. *Transition probabilities.* Let $F : S \times \Sigma \times \Sigma \to (0, 1)$ be defined by

$$F(i, x, y) = \min(g(ix), h(iy)).$$

Let $\Delta : \Sigma \times \Sigma \to [0, 1)$ be given by

$$\Delta(x, y) = 1 - \sum_i F(i, x, y).$$

Let $d(g, h) < \infty$ and set

$$c(g, h) = \inf_{i,x,y} \left(\min \left(\frac{g(ix)}{h(iy)}, \frac{h(iy)}{g(ix)} \right) \right).$$

Then, for all i, x, y,

$$F(i, x, y) \geq c(g, h)g(ix).$$

So, we get that

$$\Delta(x, y) = 1 - \sum_i F(i, x, y) \geq c(g, h) \leq 1 - e^{-(V_1(g)+d(h,g))}.$$

Let $\alpha = 1 - e^{-(V_1(g)+d(h,g))}$. For $[x]^k = [y]^k$,

$$\Delta(x, y) \leq 1 - e^{-(V_{k+1}(g)+d(h,g))}. \tag{3.2}$$

We define $G : \Sigma \times \Sigma \to (0, 1]$ as follows:

$$G(ix, iy) = F(i, x, y) \quad \text{and} \tag{3.3}$$

$$G(ix, jy) = \frac{(g(ix) - F(i, x, y))^+ (h(jy) - F(j, x, y))^+}{\Delta(x, y)} \tag{3.4}$$

for $i \neq j$ and $a^+ = \max(a, 0)$.

LEMMA 3.3. *For G defined by (3.3) and (3.4), we have that*

$$\sum_i G(ix, jy) = h(jy) \quad \textit{for all } j \in S \textit{ and } x, y \in \Sigma \quad \textit{and}$$

$$\sum_j G(ix, jy) = g(iy) \quad \textit{for all } i \in S \textit{ and } x, y \in \Sigma.$$

Proof. We fix x, y and partition the set S as follows: $S = S_g \cup S_h \cup S_0$, where $S_0 = \{i \in S : g(ix) = h(iy)\}$, $S_g = \{i \in S : g(ix) > h(iy)\}$, $S_h = \{i \in S : g(ix) < h(iy)\}$;

$$\begin{aligned} \sum_i F(i, x, y) &= \sum_i \min(g(ix), h(iy)) \\ &= \sum_{i \in S_h \cup S_0} g(ix) + \sum_{i \in S_g} h(ix) \end{aligned}$$

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$$= \sum_{i \in S_h \cup S_0} g(ix) + 1 - \sum_{i \in S_h \cup S_0} h(ix)$$
$$= \sum_{i \in S_h \cup S_0} [g(ix) - h(iy)] + 1.$$

So, we get that

$$\Delta(x, y) = - \sum_{i \in S_h \cup S_0} [g(ix) - h(iy)],$$
$$\Delta(x, y) = \sum_{i \in S_h} [h(iy) - g(ix)]$$
$$= \sum_{i} [h(iy) - F(i, x, y)]^+.$$

Finally, we get that

$$\sum_j G(ix, jy) = \sum_{j \neq i} \frac{(g(ix) - F(i, x, y))^+ (h(jy) - F(j, x, y))^+}{\Delta(x, y)} + F(i, x, y)$$
$$= (g(ix) - F(i, x, y))^+ + F(i, x, y)$$
$$= g(ix) - F(i, x, y) + F(i, x, y)$$
$$= g(ix).$$

Similarly, we have

$$\sum_i G(ix, jy) = h(jy). \qquad \square$$

LEMMA 3.4. *Let $\pi_1 : \Sigma \times \Sigma \to \Sigma$ be the projection onto the first coordinate, that is, $\pi_1(x, y) = x$. Then, the following diagram commutes.*

$$\begin{array}{ccc} \mathscr{M}(\Sigma \times \Sigma) & \xrightarrow{L^*_{\log G}} & \mathscr{M}(\Sigma \times \Sigma) \\ \downarrow{\scriptstyle (\pi_1)_*} & & \downarrow{\scriptstyle (\pi_1)_*} \\ \mathscr{M}(\Sigma) & \xrightarrow{L^*_{\log g}} & \mathscr{M}(\Sigma) \end{array}$$

Similarly, let $\pi_2 : \Sigma \times \Sigma \to \Sigma$ be the projection onto the second coordinate, that is, $\pi_2(x, y) = y$. Then, for all $m \in \mathscr{M}(\Sigma \times \Sigma)$,

$$(\pi_2)_*(L^*_{\log G}(m)) = L^*_{\log g}((\pi_2)_*(m)).$$

In particular, if m in $\mathscr{M}(\Sigma \times \Sigma)$ satisfies $(\pi_1)_*(m) = \nu_g$ and $(\pi_2)_*(m) = \nu_h$, then $(\pi_1)_*(L^*_{\log G} m) = \nu_g$ and $(\pi_2)_*(L^*_{\log G} m) = \nu_h$.

Proof. Consider for $m \in \mathscr{M}(\Sigma \times \Sigma)$ and $f \in C_b(\Sigma)$,

$$\int f \, d((\pi_1)_*(L^*_{\log G}(m))) = \int L_{\log G}(f \circ \pi_1) \, dm.$$

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For $x, y \in \Sigma$,

$$\begin{aligned} L_{\log G}(f \circ \pi_1)(x, y) &= \sum_{i,j} G(ix, jy)((f \circ \pi_1)(ix, jy)) \\ &= \sum_{i,j} G(ix, jy) f(ix) \\ &= \sum_{i} g(ix) f(ix) \quad \text{(by Lemma 3.3)} \\ &= L_{\log g}(f)(x) \\ &= (L_{\log g}(f) \circ \pi_1)(x, y). \end{aligned}$$

Hence, we get that

$$\begin{aligned} \int f \, d((\pi_1)_*(L^*_{\log G}(m))) &= \int L_{\log G}(f \circ \pi_1) \, dm \\ &= \int (L_{\log g}(f) \circ \pi_1) \, dm \\ &= \int f \, d(L^*_{\log g}(\pi_1)_* m). \end{aligned}$$

Hence, $(\pi_1)_*(L^*_{\log G}(m)) = L^*_{\log g}((\pi_1)_*(m))$.

Similarly, $(\pi_2)_*(L^*_{\log G}(m)) = L^*_{\log g}((\pi_2)_*(m))$. Finally, if m in $\mathscr{M}(\Sigma \times \Sigma)$ satisfies $(\pi_1)_*(m) = \nu_g$, then

$$(\pi_1)_*(L^*_{\log G} m) = L^*_{\log g}((\pi_1)_* m) = L^*_{\log g}(\nu_g) = \nu_g.$$

Similarly, we get $(\pi_2)_*(L^*_{\log G} m) = \nu_h$. □

3.2.2. *Joining.* We now produce a joining of ν_g and ν_h. For notational convenience, let $T := L^*_G$. Then, T is an affine map of $\mathscr{M}(\Sigma \times \Sigma)$. Here, T is weak*-continuous (Lemmas 3.2 and 3.1). Our problem now translates to finding a fixed point for T. We do this by defining a tight sequence of measures.

Let

$$P_{\nu_g, \nu_h} = \{m \in \mathscr{M}(\Sigma \times \Sigma) \colon m \text{ has marginals } \nu_g \text{ and } \nu_h\}.$$

Then, take $m_0 = \nu_g \times \nu_h \in P_{\nu_g, \nu_h}$. Define a sequence as

$$m_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k(m_0).$$

Then, by Lemma 3.5, for all n, m_n is in P_{ν_g, ν_h}.

By Lemma 2.1, there is a convergent subsequence, say $(m_{n_j})_j$. Let μ be the weak*-limit of this subsequence. We claim that $T\mu = \mu$. Then, since μ is a fixed point for the dual of the RPF operator, we get that μ is invariant for the map $\sigma \times \sigma$.

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LEMMA 3.5. *Let μ be the weak*-limit of a subsequence of the sequence*

$$m_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k(m_0).$$

Then, μ is a fixed point for T.

Proof. Let $K \subset \Sigma \times \Sigma$ be compact. Then, the weak*-topology on $M(K)$, the space of measures on K, is metrizable. One metric that induces the weak*-topology on $M(K)$ is given by the following. Let $(f_n)_n \subset C(K)$ be a dense subset with $\|f_n\|_\infty = 1$ for all n. We can use this to define a metric d_K by

$$d_K(\mu, \nu) = \sum_{n=1}^{\infty} \frac{|\int_K f_n \, d\mu - \int_K f_n \, d\nu|}{2^n}.$$

This defines a pseudo-metric on $\mathscr{M}(\Sigma \times \Sigma)$ and

$$d_K(T\mu, \mu) \le d_K(T\mu, Tm_{n_j}) + d_K(Tm_{n_j}, m_{n_j}) + d_K(m_{n_j}, \mu).$$

Any continuous function f on K can be extended to a continuous function on $\Sigma \times \Sigma$, as K is compact. Also, μ is the limit of the subsequence (m_{n_j}) in the weak*-topology on $\mathscr{M}(\Sigma \times \Sigma)$, so we get that $d_K(m_{n_j}, \mu) \to 0$. Since T is continuous, $d_K(T\mu, Tm_{n_j}) \to 0$.

For the middle term, observe the following:

$$\begin{aligned}
m_{n_j} &= \frac{1}{n_j}\sum_{k=0}^{n_j-1} T^k(m_0),\\
Tm_{n_j} &= \frac{1}{n_j}\sum_{k=1}^{n_j} T^k(m_0),\\
d_K(Tm_{n_j}, m_{n_j}) &= d_K\left(\frac{1}{n_j}\sum_{k=0}^{n_j-1} T^k(m_0), \frac{1}{n_j}\sum_{i=1}^{n_j} T^i(m_0)\right)\\
&= \sum_{n=1}^{\infty} \frac{|\int_K f_n \, d((1/n_j)\sum_{k=0}^{n_j-1} T^k(m_0)) - \int_K f_n \, d((1/n_j)\sum_{i=1}^{n_j} T^i(m_0))|}{2^n}\\
&= \sum_{n=1}^{\infty} \frac{(1/n_j)|\int_K f_n \, d(m_0) - \int_K f_n \, d(T^{n_j}(m_0))|}{2^n}\\
&\le \sum_{n=1}^{\infty} \frac{1}{2^n n_j}\left(\left|\int_K f_n \, d(m_0)\right| + \left|\int_K f_n \, d(T^{n_j}(m_0))\right|\right)\\
&= \sum_{n=1}^{\infty} \frac{1}{2^n n_j}(m_0(K) + T^{n_j}(m_0)(K))\\
&= \frac{1}{n_j}(m_0(K) + T^{n_j}(m_0)(K)) \le \frac{2}{n_j} \to 0.
\end{aligned}$$

Hence, $d_K(T\mu, \mu) = 0$; therefore, $T\mu|_K = \mu|_K$. Since the compact subset K was arbitrarily chosen, $T\mu = \mu$. □

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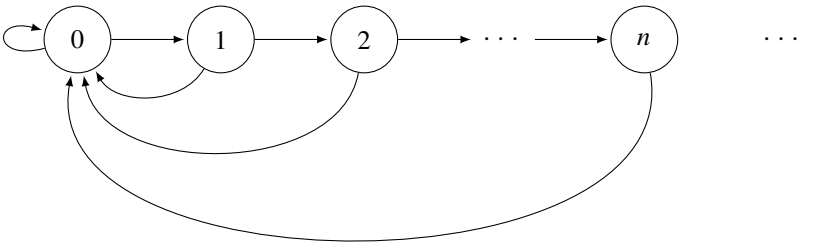


FIGURE 1. Valid transitions for the $\mathbb{Z}$-valued process.

3.2.3. *Estimating* $\overline{d}(\nu_g, \nu_h)$. We want to get an upper bound on $\bar{d}(\nu_g, \nu_h)$. We do this by estimating $\int \delta(x, y)\, d\mu$, where

$$\delta(x, y) = \begin{cases} 1, & x_0 \neq y_0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\pi(x, y) = \inf\{n \geq 0 : x_n \neq y_n\}$, then $\delta(x, y) = 1$ if and only if $\pi(x, y) = 0$.

Let $(X_n)_{n\in\mathbb{Z}}$ and $(Y_n)_{n\in\mathbb{Z}}$ be stochastic processes taking values in the set of states S. Let X_{n-} denote the sequence $(X_{n-1}, X_{n-2} \cdots) \in \Sigma$ and Y_{n-} be similarly defined. Let the transition probabilities for the pair (X_n, Y_n) be given by

$$\mathbb{P}(X_n = i, Y_n = i | X_{n-} = x, Y_{n-} = y) = G(ix, jy),$$

where G is as defined in (3.3) and (3.4).

Define a process $(Z_n)_{n\in\mathbb{Z}}$ taking values in $\mathbb{N} \cup \{0\}$ by

$$Z_n = \pi(X_{n-}, Y_{n-}).$$

To estimate $\int \delta(x, y)\, d\mu$, it is sufficient to estimate $\int \chi_{[0]}\, d\mu'$, where μ' is the induced measure on the $\mathbb{Z}$-valued process. For the $\mathbb{Z}$-valued process, the only valid transitions are from n to 0 or from n to $n + 1$ (see Figure 1).

Let τ be the return time to state 0. Since the proportion of time spent by the above process in the state 0 is the reciprocal of the expected return time to state zero, we have that

$$\int \delta(x, y)\, d\mu = \int \chi_{[0]}\, d\mu' = \frac{1}{\mathbb{E}(\tau)},$$

where $\mathbb{E}(\tau)$ is the expected return time to state 0. Now,

$$\mathbb{E}(\tau) = \sum_{k=1}^{\infty} k\mathbb{P}(\tau = k) = \sum_{k=1}^{\infty} \mathbb{P}(\tau \geq k).$$

Consider

$$\begin{aligned} \mathbb{P}[\tau \geq k] &= \frac{1}{\mu(A_0)} \int_{A_0} \sum_{i_1 \cdots i_k} \mathbb{P}(X_{n+1} \cdots X_{n+k} \\ &= Y_{n+1} \cdots Y_{n+k} = i_1 \cdots i_k | X_{n^-} = x, Y_{n^-} = y)\, d\mu(x, y) \end{aligned}$$

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$$= \frac{1}{\mu(A_0)} \int_{A_0} \sum_{i_1 \cdots i_k} F(i_1, x, y) F(i_2, i_1 x, i_1 y) \cdots$$
$$F(i_k, i_{k-1} \cdots i_2 i_1 x, i_{k-1} \cdots i_2 i_1 y)\, d\mu(x, y).$$

Now, by (3.2), we get that

$$\sum_{i_j} F(i_j, i_{j-1} \cdots i_2 i_1 x, i_{j-1} \cdots i_2 i_1 y) \geq e^{-(V_{j+1}(g)+d(g,h))}.$$

We get that

$$\mathbb{P}[\tau \geq k] \geq \prod_{j=1}^{k} (e^{-(V_{j+1}(g)+d(g,h))}).$$

Hence,

$$\mathbb{E}(\tau) \geq \sum_{n=1}^{\infty} \prod_{i=1}^{n} (e^{-(V_{i+1}(g)+d(h,g))}) = \sum_{n=1}^{\infty} \left(\prod_{i=1}^{n} e^{-V_{i+1}(g)} \right) (e^{-d(h,g)})^n. \tag{3.5}$$

This is a power series in $e^{-d(h,g)}$. As $d(h, g) \to 0$, $e^{-d(h,g)} \to 1$. By Abel's theorem,

$$\sum_{n=1}^{\infty} \prod_{i=1}^{n} (e^{-V_{i+1}(g)})(e^{-d(h,g)})^n \to \sum_{n=1}^{\infty} \prod_{i=1}^{n} e^{-V_{i+1}(g)}.$$

Here, $\sum_{n=1}^{\infty} \prod_{i=1}^{n} e^{-V_{i+1}(g)} = \infty$ is equivalent to the condition (1.1) in the hypothesis, which completes the proof.

We now get Theorem 1.3 as a corollary.

Proof of Theorem 1.3. Let $\sum_{i=1}^{\infty} V_{i+1}(g) \leq L_g < \infty$. Then, by (3.5), we get that

$$\mathbb{E}(\tau) \geq \sum_{n=1}^{\infty} e^{-L_g} (e^{-d(g,h)})^n = e^{-L_g} \frac{e^{-d(g,h)}}{1 - e^{-d(g,h)}} = e^{-L_g} (e^{d(g,h)} - 1)^{-1}.$$

If $d(g, h) \leq \ln(2)$, then $e^{d(g,h)} - 1 \leq 2d(g, h)$, we get

$$\overline{d}(v_g, v_h) \leq \mathbb{E}(\tau)^{-1} \leq e^{L_g} (e^{d(g,h)} - 1) \leq 2e^{L_g} d(g, h). \tag{3.6}$$

□

In particular, if $\log(g)$ is θ-weakly Hölder, then $V_k(g) \leq C_g \theta^k$ for all $k \geq 2$, then

$$\sum_{i=1}^{\infty} V_{i+1}(g) \leq \sum_{i=1}^{\infty} C_g \theta^{i+1} = C_g \frac{\theta^2}{1 - \theta} := L_g.$$

Combining this with (3.6), we get that

$$\overline{d}(v_g, v_h) \leq 2e^{L_g} d(g, h).$$

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4. *Potentials and $\overline{d}$-continuity*

In this section, we will prove Theorem 1.4. Theorem 1.1 then follows by combining this with Theorem 1.2.

We prove Theorem 1.4 by applying the perturbation theory of linear operators to the RPF operator. The proof is based on the spectral gap property (SGP) of the Ruelle operator. For a strongly positive recurrent potential ϕ, there exists a 'rich' Banach space $\mathscr{B}$ of continuous functions on which $L_\phi = \lambda P_\phi + N_\phi$, where $\lambda = P_G(\phi)$ and P_ϕ is a projection whose image is a one-dimensional subspace, see definition (4.1). We first show that the Ruelle operator depends continuously on the potential in the operator norm coming from the Banach space. This is Lemma 4.2. The continuity result follows once we show that the projection varies continuously with the potential. This is Lemma 4.3.

4.1. *Spectral gap property.* We begin by first describing the spectral gap property and the Banach space on which the Ruelle operator associated with an SPR potential acts with a spectral gap. Recall that the Ruelle operator associated to potential ϕ is defined as

$$L_\phi(f)(x) = \sum_{\sigma y = x} e^{\phi(y)} f(y).$$

Let $\mathrm{dom}(L_\phi)$ denote the collection of all functions f such that the sum converges for all x in X. For the following terminology and results, see [5]. For the remainder of the discussion, we will let ϕ be a θ-weakly Hölder continuous potential with finite Gurevich pressure.

Definition 4.1. We say that ϕ has the spectral gap property (SGP) if there is a Banach space $\mathscr{B}$ of continuous functions on X such that:

(1) $\mathscr{B}$ is 'rich'
   (a) $\mathscr{B} \subset \mathrm{dom}(L_\phi)$ and $\{\mathbb{1}_{[a]} : [a] \neq \emptyset\} \subset \mathscr{B}$;
   (b) $f \in \mathscr{B} \implies |f| \in \mathscr{B}$ and $\||f|\|_{\mathscr{B}} \leq \|f\|_{\mathscr{B}}$;
   (c) $\mathscr{B}$-convergence implies convergence on cylinders.

(2) 'Spectral gap': L_ϕ is a bounded operator on $\mathscr{B}$. Furthermore, $L_\phi = \lambda P + N$, where $\lambda = \exp(P_G(\phi))$, $PN = NP$, $P^2 = P$, $\dim(\mathrm{Im}\ P) = 1$ and the spectral radius of N is less than λ.

(3) 'Analytic perturbations': if g is θ-Hölder, then $L_\phi : \mathscr{B} \to \mathscr{B}$ is bounded and $z \to L_{\phi + zg}$ is analytic on some complex neighborhood of zero.

Strong positive recurrence is a necessary and sufficient condition for the spectral gap property.

THEOREM 4.1. [5] *Let X be a topologically mixing countable Markov shift and ϕ be a θ-weakly Hölder continuous potential (for some $0 < \theta < 1$) with finite Gurevich pressure, then ϕ has the spectral gap property if and only if ϕ is strongly positive recurrent.*

Fix $0 < \theta < 1$. Let ϕ be a θ-weakly Hölder continuous potential with finite Gurevich pressure that is strongly positive recurrent. We describe a space $\mathscr{B}$ on which L_ϕ acts with a spectral gap.

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Fix a state a such that $\Delta_a[\phi] > 0$. Without loss of generality, we can assume that $P_G(\phi) = 0$. We induce on the state as described in §2.2.1. Then, $P_G(\overline{\phi}) = 0$. By strong positive recurrence, there exists $\epsilon_a > 0$ such that $0 < P_G(\overline{\phi + 2\epsilon_a}) < \infty$. Also, since $t \to P_G(\overline{\phi + t})$ is continuous, we can choose $\epsilon_a > 0$ small enough such that

$$0 < \theta e^p < 1 \quad \text{where } p := P_G(\overline{\phi + \epsilon_a}). \tag{4.1}$$

Let

$$\psi := \phi + \epsilon_a - p\mathbb{1}_{[a]}. \tag{4.2}$$

Then, ψ is strongly positive recurrent and $P_G(\psi) = 0$. By the generalized RPF theorem, there exists a Borel measure ν_0 which is finite and positive on cylinders, and a positive continuous function $h_0 : X \to \mathbb{R}$, such that

$$L_\psi^* \nu_0 = \nu_0, \; L_\psi h_0 = h_0 \quad \text{and} \quad \int h_0 \, d\nu_0 = 1.$$

Moreover, $\mathrm{var}_1[\log h_0] \le \sum_{k\ge 2} \mathrm{var}_k(\phi)$. Setting $c := \exp(\sum_{k\ge 2} \mathrm{var}_k(\phi))$, we see that for every x,

$$c^{-1} h_0[x_0] \le h_0(x) \le c h_0[x_0], \tag{4.3}$$

where $h_0[x_0] := \sup_{[x_0]} h_0$.

Define for $x, y \in X$,

$$t(x, y) := \min\{n : x_n \ne y_n\} \quad \text{where } \min \emptyset = \infty \quad \text{and}$$
$$s_a(x, y) := \#\{0 \le i \le t(x, y) - 1 : x_i = y_i = a\}.$$

Let $\mathscr{B}$ be the collection of continuous functions $f : X \to \mathbb{C}$ for which

$$\|f\|_{\mathscr{B}} := \sup_{b\in S} \frac{1}{h_0[b]} \left[\sup_{x\in[b]} |f(x)| + \sup_{x\ne y\in[b]} \left\{ \frac{|f(x) - f(y)|}{\theta^{s_a(x,y)}} \right\} \right] < \infty.$$

Here, L_ϕ acts with a spectral gap on $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$.

4.2. *Operator depends continuously on potential.* Let $\|\cdot\|_\theta$ denote the Hölder norm given by $\|\phi\|_\theta := \|\phi\|_\infty + |\phi|_\theta$, where $\|\cdot\|_\infty$ is the uniform norm and $|\phi|_\theta := \sup\{|\phi(x) - \phi(y)|/\theta^{t(x,y)} : x, y \in X, x \ne y\}$.

Let ϕ and τ be locally θ-Hölder-continuous strongly positive recurrent potentials with finite Gurevich pressure. Fix a state a such that $\Delta_a[\phi] > 0$ and $\Delta_a[\tau] > 0$. Without loss of generality, assume that $P_G(\phi) = P_G(\tau) = 0$ (otherwise, we pass to $\phi - P_G(\phi)$ and $\tau - P_G(\tau)$). Consider

$$\|L_\phi - L_\tau\| = \sup_{\|f\|_{\mathscr{B}}=1} \|L_\phi f - L_\tau f\|_{\mathscr{B}}.$$

LEMMA 4.2. *Given ϕ and β as above, for given $\epsilon > 0$, there exists $\delta > 0$ such that $\|L_\phi - L_\tau\| < \epsilon$, whenever $\tau \in A_\theta$ is such that $\|\phi - \tau\|_\theta < \delta$.*

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Proof of Lemma 4.2. Let $f \in \mathscr{B}$ with $\|f\|_{\mathscr{B}} = 1$. Now,

$$\|L_\phi f - L_\tau f\|_{\mathscr{B}} = \sup_{b\in S} \frac{1}{h_0[b]} \sup_{x\in[b]} |L_\phi f(x) - L_\tau f(x)| + \sup_{x,y\in[b],x\neq y} \{|L_\phi f(x) - L_\tau f(x) - L_\phi f(y) + L_\tau f(y)|/\theta^{s_a(x,y)}\}. \tag{4.4}$$

Since h_0 is an eigenfunction for L_ψ, we have that $h_0(x) = \sum_{\sigma(y)=x} e^{\psi(y)} h_0(y)$, where ψ is as given by (4.2). Furthermore, by the definition of the norm $\|\cdot\|_{\mathscr{L}}$ and (4.3),

$$|f(y)| \le \|f\|_{\mathscr{B}} h_0[b] \le c|h_0(y)|. \tag{4.5}$$

By definition, $\psi = \phi + \epsilon_a - p\mathbb{1}_{[a]}$, so

$$\phi \le \psi + p. \tag{4.6}$$

For $x \in [b]$, we have the following calculation:

$$\begin{aligned}\left|\sum_{r\to b} e^{\phi(rx)} f(rx)\right| &\le \sum_{r\to b} e^{\phi(rx)} |f(rx)| \\ &\le \sum_{r\to b} e^{\phi(rx)} c h_0(rx) \quad \text{(using (4.5))} \\ &\le c e^p \sum_{r\to b} e^{\psi(rx)} h_0(rx) \quad \text{(using (4.6))},\end{aligned}$$

and hence,

$$\left|\sum_{r\to b} e^{\phi(rx)} f(rx)\right| \le c e^p h_0(x) \quad \text{(using } L_\psi h_0 = h_0\text{)}. \tag{4.7}$$

We begin by considering the first term in (4.4). Let $g = \tau - \phi$. Fix a state $b \in S$ and consider for $x \in [b]$,

$$\begin{aligned}\left|\frac{L_\phi f(x) - L_\tau f(x)}{h_0(x)}\right| &= \left|\frac{\sum_{\sigma(y)=x} (e^{\phi(y)} - e^{\tau(y)}) f(y)}{h_0(x)}\right| \\ &= \left|\frac{\sum_{\sigma(y)=x} e^{\phi(y)} (1 - e^{g(y)}) f(y)}{h_0(x)}\right| \\ &\le \frac{\sum_{\sigma(y)=x} e^{\phi(y)} |1 - e^{g(y)}| |f(y)|}{h_0(x)} \\ &\le (e^{\|\phi-\tau\|_\infty} - 1) \frac{\sum_{\sigma(y)=x} e^{\phi(y)} |f(y)|}{h_0(x)} \\ &\le c(e^{\|\phi-\tau\|_\infty} - 1) e^p \quad \text{(using (4.7))} \\ &\le c(C\|\phi - \tau\|_\infty) e^p \quad \text{(for } \|\phi - \tau\|_\infty < 1\text{)}.\end{aligned} \tag{4.8}$$

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Since $h_0[b] \geq c^{-1}h_0(x)$ for all $x \in [b]$, we get the following upper bound for the first term in (4.4):

$$\sup_{b\in S} \frac{1}{h_0[b]} \sup_{x\in[b]} |L_\phi f(x) - L_\tau f(x)| \leq C\|\phi - \tau\|_\infty e^p. \tag{4.9}$$

Now, we consider the second term in (4.4). For x, y in $[b]$ with $x \neq y$, consider

$$\begin{aligned} &|L_\phi f(x) - L_\tau f(x) - L_\phi f(y) + L_\tau f(y)|/(h_0[b]\theta^{s_a(x,y)}) \\ &\quad = \left| \frac{\sum_{r\to b} e^{\phi(rx)} f(rx) - e^{\tau(rx)} f(rx) - e^{\phi(ry)} f(ry) + e^{\tau(ry)} f(ry)}{\theta^{s_a(x,y)} h_0[b]} \right|. \end{aligned} \tag{4.10}$$

We add and subtract $e^{\phi(rx)} f(rx) e^{(\tau-\phi)(ry)}$ from the quantity inside the sum in the numerator, and rewrite it as follows:

$$\begin{aligned} &|e^{\phi(rx)} f(rx) - e^{\tau(rx)} f(rx) - e^{\phi(ry)} f(ry) + e^{\tau(ry)} f(ry)| \\ &\quad = |e^{\phi(rx)} f(rx)(e^{(\tau-\phi)(ry)} - e^{(\tau-\phi)(rx)}) + e^{\phi(rx)} f(rx)(1 - e^{(\tau-\phi)(ry)}) \\ &\qquad - e^{\phi(ry)} f(ry)(1 - e^{(\tau-\phi)(ry)})| \\ &\quad = |e^{\phi(rx)} f(rx)(e^{(\tau-\phi)(ry)} - e^{(\tau-\phi)(rx)}) - (1 - e^{(\tau-\phi)(ry)})(e^{\phi(ry)} f(ry) \\ &\qquad - e^{\phi(rx)} f(rx))|. \end{aligned}$$

Using this along with $s_a(x, y) \leq t(x, y)$, we get that

$$|L_\phi f(x) - L_\tau f(x) - L_\phi f(y) + L_\tau f(y)|/(h_0[b]\theta^{s_a(x,y)}) \leq A + B, \tag{4.11}$$

where

$$\begin{aligned} A &= \left| \frac{\sum_{r\to b} e^{\phi(rx)} f(rx)(e^{(\tau-\phi)(ry)} - e^{(\tau-\phi)(rx)})}{\theta^{t(x,y)} h_0[b]} \right|, \\ B &= \left| \frac{\sum_{r\to b} (1 - e^{(\tau-\phi)(ry)})(e^{\phi(ry)} f(ry) - e^{\phi(rx)} f(rx))}{\theta^{t(x,y)} h_0[b]} \right|. \end{aligned}$$

We first estimate the first term in (4.11). Note that for $\|\tau - \phi\|_\infty \leq 1$, there exists C_0 such that

$$\begin{aligned} |e^{(\tau-\phi)(ry)} - e^{(\tau-\phi)(rx)}| &\leq C_0|(\tau - \phi)(ry) - (\tau - \phi)(rx)| \\ &\leq C_0\|\tau - \phi\|_\theta \theta^{t(x,y)}. \end{aligned} \tag{4.12}$$

Consider

$$\begin{aligned} A &\leq C_0\|\tau - \phi\|_\theta \theta^{t(x,y)} \frac{\sum_{r\to b} e^{\phi(rx)}|f(rx)|}{\theta^{t(x,y)} h_0[b]} \quad \text{(using (4.12))} \\ &\leq C_1\|\tau - \phi\|_\theta e^p \quad (h_0[b] = \sup_{[b]} h_0 \text{ using (4.7)}). \end{aligned}$$

Next, let us consider the second term in (4.11). We have that

$$B \leq C_2\|\tau - \phi\|_\infty \frac{\sum_{r\to b} |(e^{\phi(ry)} f(ry) - e^{\phi(rx)} f(rx))|}{\theta^{t(x,y)} h_0[b]}. \tag{4.13}$$

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Now,

$$\begin{aligned}
&|e^{\phi(ry)}f(ry) - e^{\phi(rx)}f(rx)| \\
&\quad = |e^{\phi(ry)}f(ry) - e^{\phi(rx)}f(ry) + e^{\phi(rx)}f(ry) - e^{\phi(rx)}f(rx)| \\
&\quad = |(e^{\phi(ry)} - e^{\phi(rx)})f(ry) + e^{\phi(rx)}(f(ry) - f(rx))|. \qquad (4.14)
\end{aligned}$$

Since

$$|f(rx) - f(ry)| \le \|f\|_{\mathscr{B}} h_0[r]\theta^{t(x,y)} \le h_0[r]\theta^{t(x,y)}, \qquad (4.15)$$

we get

$$\begin{aligned}
\sum_{r\to b} |e^{\phi(rx)}(f(ry) - f(rx))| &\le \sum_{r\to b} |e^{\phi(rx)}h_0[r]\theta^{t(x,y)}| \quad \text{(using (4.15))} \\
&\le c\sum_{r\to b} e^{\phi(rx)}h_0(rx)\theta^{t(x,y)} \quad \text{(using (4.3))} \\
&\le c\theta^{t(x,y)}e^p \sum_{r\to b} e^{\psi(rx)}h_0(rx) \quad \text{(using (4.6))} \\
&\le c\theta^{t(x,y)}e^p h_0(x) \quad (h_0 = L_\psi h_0) \\
&\le c\theta^{t(x,y)}e^p h_0[b] \quad (h_0[b] = \sup_{[b]} h_0). \qquad (4.16)
\end{aligned}$$

Since ϕ is locally Hölder continuous, there exists C_ϕ such that for all x, y and r, we have $|\phi(rx) - \phi(ry)| \le C_\phi\theta^{t(x,y)}$. So,

$$\begin{aligned}
&\left|\sum_{r\to b}(e^{\phi(ry)} - e^{\phi(rx)})f(ry)\right| \\
&\quad \le \sum_{r\to b} e^{\phi(ry)}|(1 - e^{\phi(rx)-\phi(ry)})||f(ry)| \\
&\quad \le ce^p \sum_{r\to b} e^{\psi(ry)}|(1 - e^{\phi(rx)-\phi(ry)})|h_0(ry) \quad \text{(similar to (4.7))} \\
&\quad \le C_3e^p \sum_{r\to b} e^{\psi(ry)}|\phi(rx) - \phi(ry)|h_0(ry) \\
&\quad \le C_3e^pC_\phi\theta^{t(x,y)}h_0(y) \quad (L_\psi h_0 = h_0) \\
&\quad \le C_3e^pC_\phi\theta^{t(x,y)}h_0[b] \quad (h_0[b] = \sup_{[b]} h_0). \qquad (4.17)
\end{aligned}$$

Combining (4.13), (4.14), (4.16), and (4.17), we get the following estimate on the second term in (4.11):

$$\left|\frac{\sum_{r\to b}(1 - e^{(\tau-\phi)(ry)})(e^{\phi(ry)}f(ry) - e^{\phi(rx)}f(rx))}{\theta^{t(x,y)}h_0[b]}\right| \le C_2\|\tau - \phi\|_\infty[ce^p + C_3C_\phi]. \qquad (4.18)$$

Combining (4.4), (4.9), and (4.18), we get

$$\|L_\phi f - L_\tau f\|_{\mathscr{B}} \le C\|\phi - \tau\|_\infty e^p + C_1\|\tau - \phi\|_\theta e^p + C_2\|\tau - \phi\|_\infty[ce^p + C_3C_\phi].$$

This completes the proof. □

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4.3. *Eigenfunctions depend continuously on potential.* Let ϕ and τ be locally θ-Hölder-continuous strongly positive recurrent potentials with finite Gurevich pressure. We extend the result of this section to weakly Hölder potentials at the beginning of §4.4. Let h_ϕ, h_τ be positive continuous functions given by the generalized RPF theorem. Then, $L_\phi h_\phi = h_\phi$ and $L_\tau h_\tau = h_\tau$.

LEMMA 4.3. *Let ϕ be a locally Hölder, SPR potential with finite Gurevich pressure. Given $\alpha > 0$, there exists $\delta > 0$, such that for $\tau \in A_\theta$ such that $d_\theta(\phi, \tau) < \delta$, we get that*

$$\left\| \frac{h_\phi}{h_\phi \circ \sigma} \frac{h_\tau \circ \sigma}{h_\tau} - 1 \right\|_\infty < \alpha. \tag{4.19}$$

Note that (4.19) then holds for all scalar multiples of h_ϕ and h_τ.

Proof of Lemma 4.3. Let $\mathscr{B}$ be the Banach space on which L_ϕ acts with a spectral gap. According to the theory of analytic perturbations of linear operators, there exists an $\epsilon > 0$ such that every $L : \mathscr{B} \to \mathscr{B}$ which satisfies $\|L - L_\phi\| < \epsilon$ has a decomposition of the form $L = \lambda(L)P(L) + N(L)$, where $P(L)$ and $N(L)$ are bounded operators such that $P(L)^2 = P(L)$, $P(L)N(L) = N(L)P(L)$, $\dim(\mathrm{Im}P(L)) = 1$, and the spectral radius of $N(L)$ is less than $\lambda(L)$. Moreover, if $\epsilon > 0$ is small enough, then $L \to P(L), N(L), \lambda(L)$ are analytic on $\{L : \|L - L_\phi\| < \epsilon\}$.

By Lemma 4.2, there exists $\delta > 0$, such that for $\|\phi - \rho\|_\theta < \delta$, we get that $\|L_\phi - L_\rho\| < \epsilon$, so $L_\rho = \lambda_\rho P_\rho + N_\rho$, where $\rho \to P_\rho$ is continuous.

Let $\gamma > 0$ be given, then there exists $\delta > 0$ such that for $\|\rho - \phi\|_\theta < \delta$, $\|P_\rho - P_\phi\| < \gamma/2$. Let τ be such that $\|\tau - \phi\|_\theta < \delta/2$.

By (4.1), $p \to 0$ as $\epsilon_a \to 0$. Further, by (4.2), $\phi - p \le \psi \le \phi + \epsilon_a$, so if we take ϵ_a, $p \in (0, \delta/4)$, we get that $\|\psi - \phi\|_\infty < \delta/2$. Moreover, $\psi - \phi = \epsilon_a - p\mathbb{1}_{[a]}$, so for all $n \ge 1$, $\mathrm{var}_n(\psi - \phi) = 0$. So, $\|\psi - \phi\|_\theta < \delta/2$. We get that $\|P_\psi - P_\phi\| < \gamma/2$. Further, $\|\psi - \tau\|_\theta < \delta$ and $\|P_\psi - P_\tau\| < \gamma$.

Normalize so that the corresponding eigenfunctions h_τ and h_ϕ satisfy $P_\psi h_\phi = P_\psi h_\tau = h_0$. Then,

$$\begin{aligned} \|h_\phi\|_{\mathscr{B}} - \|h_0\|_{\mathscr{B}} \le \|h_\phi - h_0\|_{\mathscr{B}} &= \|P_\phi h_\phi - P_\psi h_\phi\|_{\mathscr{B}} \\ &\le \|P_\phi - P_\psi\| \|h_\phi\|_{\mathscr{B}} < \gamma \|h_\phi\|_{\mathscr{B}} \\ \implies \|h_\phi\|_{\mathscr{B}} &\le \frac{1}{1-\gamma} \|h_0\|_{\mathscr{B}}. \end{aligned}$$

Similarly,

$$\|h_\tau\|_{\mathscr{B}} \le \frac{1}{1-\gamma} \|h_0\|_{\mathscr{B}}.$$

We want to show that $\|(h_\phi/(h_\phi \circ \sigma))((h_\tau \circ \sigma)/h_\tau) - 1\|_\infty$ is small. For $y \in \Sigma$ and for $\gamma \le \frac{1}{2}$, we have that

$$\begin{aligned} \left| \left(\frac{h_\phi}{h_0} - 1 \right)(y) \right| &= \left| \frac{h_\phi(y) - h_0(y)}{h_0(y)} \right| \\ &\le c \|h_\phi - h_0\|_{\mathscr{B}} \end{aligned}$$

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$$\begin{aligned}
&\leq c\gamma \|h_\phi\|_{\mathscr{B}} \\
&\leq c\frac{\gamma}{1-\gamma}\|h_0\|_{\mathscr{B}} \\
&\leq 2\gamma c\|h_0\|_{\mathscr{B}}.
\end{aligned}$$

So,

$$h_0(y)(1-2c\gamma\|h_0\|_{\mathscr{B}}) \leq h_\phi(y) \leq h_0(y)(1+2c\gamma\|h_0\|_{\mathscr{B}}).$$

Similarly,

$$h_0(y)(1-2c\gamma\|h_0\|_{\mathscr{B}}) \leq h_\tau(y) \leq h_0(y)(1+2c\gamma\|h_0\|_{\mathscr{B}}).$$

Given $1 > \alpha > 0$, choose $\beta_1 > 0$ such that $((1+\beta_1)/(1-\beta_1))^2 = 1+\alpha$ and $\beta_2 > 0$ such that $1-\alpha = ((1-\beta_2)/(1+\beta_2))^2$. Now, choose $\gamma \in (0, \frac{1}{2})$ such that $2c\gamma\|h_0\|_{\mathscr{B}} < \min(\beta_1, \beta_2)$.

Consider

$$\begin{aligned}
\frac{h_\phi(y)}{h_\tau(y)} &\leq \frac{(1+2c\gamma\|h_0\|_{\mathscr{B}})}{(1-2c\gamma\|h_0\|_{\mathscr{B}})} \\
&\leq \frac{1+\beta_1}{1-\beta_1}.
\end{aligned}$$

Similarly, $h_\tau(y)/h_\phi(y) \leq (1+\beta_1)/(1-\beta_1)$.

Hence, we get that

$$\frac{h_\phi(y)}{h_\phi \circ \sigma(y)} \frac{h_\tau \circ \sigma(y)}{h_{\tau(y)}} \leq \left(\frac{1+\beta_1}{1-\beta_1}\right)^2 = 1+\alpha.$$

Also,

$$\begin{aligned}
\frac{h_\phi(y)}{h_\tau(y)} &\geq \frac{(1-2c\gamma\|h_0\|_{\mathscr{B}})}{(1+2c\gamma\|h_0\|_{\mathscr{B}})} \\
&\geq \frac{1-\beta_2}{1+\beta_2}.
\end{aligned}$$

Similarly, $h_\tau(y)/h_\phi(y) \geq (1-\beta_2)/(1+\beta_2)$.

So, we get that

$$\frac{h_\phi(y)}{h_\phi \circ \sigma(y)} \frac{h_\tau \circ \sigma(y)}{h_{\tau(y)}} \geq \left(\frac{1-\beta_2}{1+\beta_2}\right)^2 = 1-\alpha.$$

Hence, for all y in Σ,

$$\left| \frac{h_\phi(y)}{h_\phi \circ \sigma(y)} \frac{h_\tau \circ \sigma(y)}{h_{\tau(y)}} - 1 \right| \leq \alpha.$$

Therefore,

$$\left\| \frac{h_\phi}{h_\phi \circ \sigma} \frac{h_\tau \circ \sigma}{h_\tau} - 1 \right\|_\infty < \alpha. \qquad \square$$

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4.4. *Completion of proof.* We begin by extending Lemma 4.3 to weakly Hölder potentials. Let $\tilde{S} = S \times S$ and define $\tilde{\pi} : S^{\mathbb{N}\cup\{0\}} \to \tilde{S}^{\mathbb{N}\cup\{0\}}$ by

$$\tilde{\pi}(x_0x_1x_2\cdots) = (x_0, x_1)(x_1, x_2)\cdots$$

Let $\tilde{\Sigma} = \tilde{\pi}(\Sigma)$. Then, $\tilde{\pi}$ is a topological conjugacy from (Σ, σ) to $(\tilde{\Sigma}, \sigma)$. Given a potential $\phi : \Sigma \to \mathbb{R}$, let $\tilde{\phi} = \phi \circ \tilde{\pi} : \tilde{\Sigma} \to \mathbb{R}$. If $\tilde{x}, \tilde{y} \in \tilde{\Sigma}$ are such that $\tilde{x_1}\cdots\tilde{x_k} = \tilde{y_1}\cdots\tilde{y_k}$, $x = \tilde{\pi}(\tilde{x})$, and $y = \tilde{\pi}(\tilde{y})$, we get that $x_1\cdots x_{k+1} = y_1\cdots y_{k+1}$. So,

$$|\tilde{\phi}(\tilde{x}) - \tilde{\phi}(\tilde{y})| = |\phi(x) - \phi(y)| \le \mathrm{var}_{k+1}(\phi).$$

Therefore, $\mathrm{var}_k(\tilde{\phi}) \le \mathrm{var}_{k+1}(\phi)$. Thus, if ϕ is weakly Hölder continuous, then $\tilde{\phi}$ is locally Hölder continuous. Further, if $L_\phi h_\phi = \lambda h_\phi$, we get that

$$\begin{aligned} L_{\tilde{\phi}}\tilde{h}(\tilde{x}) &= \sum_{r\to x_0} e^{\tilde{\phi}((r,x_0)\tilde{x})}\tilde{h}((r, x_0)\tilde{x}) \\ &= \sum_{r\to x_0} e^{\phi(rx)}h(rx) = L_\phi h(x) \\ &= \lambda h(x) = \lambda\tilde{h}(x). \end{aligned}$$

So, the conclusion of Lemma 4.3 holds for $h_{\tilde{\phi}}$ and $h_{\tilde{\tau}}$ if it holds for h_ϕ and h_τ. Hence, Lemma 4.3 holds for weakly Hölder potentials.

Finally, to complete the proof of Theorem 1.4, consider the g-functions given by $g_\phi = e^\phi h_\phi/(h_\phi \circ \sigma)$ and $g_\tau = e^\tau h_\tau/(h_\tau \circ \sigma)$. We have

$$\|\log(g_\phi) - \log(g_\tau)\|_\infty \le \|\phi - \tau\|_\infty + \left\|\log\left(\frac{h_\phi}{h_\phi\circ\sigma}\frac{h_\tau\circ\sigma}{h_\tau}\right)\right\|_\infty.$$

It follows from Lemma 4.3 that $\|\log(g_\phi) - \log(g_\tau)\|_\infty \to 0$ as $d_\theta(\phi, \tau) \to 0$.

Since $\log(g_\phi)$ and $\log(g_\tau)$ have summable variations, so g_ϕ and g_τ satisfy the condition of Theorem 1.2. The RPF measures m_ϕ and m_τ are g-measures corresponding to g_ϕ and g_τ. This completes the proof of Theorem 1.1.

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