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On Toeplitz sections in sequence spaces

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1. *Introduction.* The concept of sectional convergence (AK) in FK-spaces was investigated by Zeller in (20). In (5) and (6), Garling investigated convergent and bounded sections in more general topological sequence spaces. Many of the results hold for Toeplitz sections in sequence spaces. A topological sequence space has the property of Toeplitz sectional convergence (TK) if and only if the unit sequences $e^k = (\delta_{jk})_{j=1}^{\infty}$ form a Toeplitz basis. In section 3, we present characterizations of Toeplitz sectional boundedness (TB) and functional Toeplitz sectional convergence (FTK) in terms of β_T- and γ_T-duality. In section 4, we apply our results to summability fields. These results are related to the Hardy–Bohr property of multipliers for Cesàro summable sequences of positive order. In section 5, we characterize the properties TK and TB in FK-spaces by factorization statements.

Let T_α be the matrix of the series-sequence Cesàro method of order α. For $\alpha = 0$, the properties $T_\alpha B$, $FT_\alpha K$, and $T_\alpha K$ correspond to the properties of sectional boundedness (AB), functional sectional convergence (FAK) and sectional convergence (AK) ((5), (9), (20)). For $\alpha = 1$, the Cesàro sections are useful in the study of Fourier series ((3), (7), (8), (9)).

Some of the results of section 3 for the property TB were presented to the American Mathematical Society at the 1971 meeting in Atlantic City. Analogous results for the property FTK were later obtained through close work with Professor Goes (see (9)). Meyers (13) has also investigated Toeplitz sections in FK-spaces. His work includes results related to Theorems 6 and 7. Parts of section 4 for the case when T^{-1} has finite columns and $A = T$ contains results of Ruckle (15). I would like to thank Professors Goes and Zeller for helpful suggestions. This research was in part supported by the Alexander von Humboldt Foundation.

2. *Definitions.* Let $T = (t_{nk})$ be a row-finite infinite matrix of real (or complex) numbers and let ω be the space of all real (or complex) valued sequences $x = (x_k)_{k=1}^{\infty}$. The coordinatewise product $(x_k y_k)$ of two sequences x and y is denoted by xy. The sequence consisting of the n-th row $(t_{n1}, t_{n2}, t_{n3}, \ldots)$ of T is denoted by t^n. For each x in ω, $t^n x$ is called the n-th T-section of x. Let E be a Hausdorff locally convex sequence space (l.c.s.s.) and let E' be the space of continuous linear functionals on E. For simplicity, each sequence space E considered here will be assumed to contain all unit sequences $e^k = (\delta_{kj})_{j=1}^{\infty}$, $k = 1, 2, 3, \ldots$, but our results can be extended to (gap) spaces E which have the property $t^n x \in E$ for all $x \in E$, $n \in Z^+$. A K-space is a l.c.s.s. on which the linear

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functionals $x \to x_k$, $k \in Z^+$, are continuous. We define $E_{TB} = \{x \in \omega : \{t^n x\}_{n=1}^{\infty}$ is a bounded subset of $E\}$, $E_{FTK} = \{x \in \omega$: for every $f \in E'$, $\lim_n f(t^n x)$ exists$\}$,

$$E_{TK} = \{x \in E : \lim_n t^n x = x\},$$

and E_{AD} is the closure of the span of the unit sequences e^k, $k \in Z^+$. A sequence x in E has the property TB (T-sectional boundedness) if $x \in E_{TB}$, the property FTK (functional T-sectional convergence) if $x \in E_{FTK}$, and the property TK (T-sectional convergence) if $x \in E_{TK}$. If $E \subset E_{TB}$ ($E \subset E_{FTK}$, $E = E_{TK}$, $E = E_{AD}$) then E is called a TB-space (FTK-space, TK-space, AD-space).

The matrix T is called an Sp_a-matrix (a real or complex) if $\lim_n t_{nk} = a$, $k \in Z^+$. For the finite sequences of a K-space to have the property TK it is necessary that T be an Sp_1-matrix. The summability field of T, $c_T = \{x \in \omega : T - \Sigma x_k = \lim_n \Sigma t_{nk} x_k$ exists$\}$, and the boundedness domain of T, $m_T = \{x \in \omega : \sup_n |\Sigma t_{nk} x_k| < \infty\}$, are FK-spaces (19). If E is a l.c.s.s., then the β_T-dual of E is $E^{\beta_T} = \{x \in \omega$: for every $y \in E$, $xy \in c_T\}$, the γ_T-dual of E is $E^{\gamma_T} = \{x \in \omega$: for every $y \in E$, $xy \in m_T\}$, and $E^{\phi} = \{x \in \omega$: for some $f \in E'$, $x_k = f(e^k)$, $k \in Z^+\}$. E is called a β_T-space (γ_T-space) if $E = E^{\beta_T \beta_T}$ ($E = E^{\gamma_T \gamma_T}$). Clearly, for every sequence space E and $X = \beta_T$ or $X = \gamma_T$, we have $E \subset E^{XX}$ and $E^X = E^{XXX}$.

3. *β_T- and γ_T-duality.* Here we characterize the properties TB and FTK in terms of β_T- and γ_T-duality and we investigate the spaces E_{TB} and E_{FTK}.

THEOREM 1. *For each l.c.s.s. E, $E_{TB} = (E^{\phi})^{\gamma_T}$.*

Proof. By the Banach–Mackey Theorem a subset of a l.c.s.s. is bounded if and only if it is weakly bounded. Hence, $x \in E_{TB} \Leftrightarrow$ for every $f \in E'$, $\sup_n |f(t^n x)| < \infty \Leftrightarrow$ for every $y \in E^{\phi}$, $\sup_n |\Sigma t_{nk} x_k y_k| < \infty$.⊣

COROLLARY 1. *For every l.c.s.s. E, E_{TB} is a γ_T-space.*

COROLLARY 2. *Let E be a l.c.s.s. The following statements are equivalent:*

(*a*) *E is a TB-space;*

(*b*) *$E^{\phi} \subset E^{\gamma_T}$;*

(*c*) *$E^{\gamma_T \gamma_T} \subset E_{TB}$.*

Proof. If E is a TB-space, then, for every $x \in E$ and $y \in E^{\phi}$, $xy \in m_T$. Thus (*a*) $\Rightarrow$ (*b*). If $E^{\phi} \subset E^{\gamma_T}$, then $E^{\gamma_T \gamma_T} \subset (E^{\phi})^{\gamma_T} = E_{TB}$. Thus (*b*) $\Rightarrow$ (*c*). Finally, (*c*) $\Rightarrow$ (*a*) since $E \subset E^{\gamma_T \gamma_T}$.⊣

COROLLARY 3. *Let S be another row-finite matrix and suppose $m_T = m_S$. Then a l.c.s.s. E is a TB-space if and only if it is an SB-space.*

THEOREM 2. *For each l.c.s.s. E, $E_{FTK} = (E^{\phi})^{\beta_T}$.*

Proof. We have $x \in E_{FTK} \Leftrightarrow$ for every $y \in E^{\phi}$, $\lim_n \Sigma t_{nk} x_k y_k$ exists $\Leftrightarrow$ for every $y \in E^{\phi}$, $xy \in c_T$.⊣

COROLLARY 1. *For each l.c.s.s. E, E_{FTK} is a β_T-space.*

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COROLLARY 2. *Let E be a l.c.s.s. The following statements are equivalent:*

(*a*) *E is an FTK-space;*

(*b*) $E^{\phi} \subset E^{\beta_T}$;

(*c*) $E^{\beta_T \beta_T} \subset E_{FTK}$.

Proof. Similar to Theorem 1, Corollary 2.⊣

COROLLARY 3. *Let S be another row-finite matrix and suppose $c_T = c_S$. Then a l.c.s.s E is an FTK-space if and only if it is an FSK-space.*

The proof of the following Lemma uses only elementary topological techniques and is omitted.

LEMMA. *Let $F = \{f_1, f_2, f_3, \ldots\}$ be a countable subset of a topological space X and let g be a continuous function from X into a Hausdorff space Y. If f is a limit point of F and $\lim_n g(f_n) = y$, then $g(f) = y$.*

THEOREM 3. *Suppose E is a barrelled K-space and T is an Sp_a-matrix with $a \neq 0$. Then $E^{\beta_T} \subset E^{\gamma_T} \subset E^{\phi}$.*

Proof. That $E^{\beta_T} \subset E^{\gamma_T}$ is clear. Let $y \in E^{\gamma_T}$. Define $f_n(x) = \Sigma t_{nk} x_k y_k$ for each $x \in E$. Since t^n is a finite sequence and E is a K-space, f_n is a continuous linear functional on E for each n. Since, for each $x \in E$, $xy \in m_T$, the set $F = \{f_1, f_2, \ldots\}$ is a bounded subset of E' for the weak topology $\sigma(E', E)$. Since E is barrelled, F is $\sigma(E', E)$ relatively compact ((16), p. 141). Every infinite relatively compact set has a limit point. Let f be a limit point of F. Clearly, $\lim_n f_n(e^k) = a y_k$ for each k. By the Lemma, $f(e^k) = a y_k$ for each k. Hence, $y \in E^{\phi}$.⊣

COROLLARY. *Suppose E is a barrelled K-space and T is an Sp_a-matrix with $a \neq 0$. Then $E_{TB} \subset E^{\gamma_T \gamma_T}$ and $E_{FTK} \subset E^{\beta_T \beta_T}$.*

THEOREM 4. *Suppose E is a barrelled K-space and T is an Sp_a-matrix with $a \neq 0$. The following statements are equivalent:*

(*a*) *E is a TB-space;*

(*b*) $E^{\phi} = E^{\gamma_T}$;

(*c*) $E^{\gamma_T \gamma_T} = E_{TB}$.

The same holds if TB and γ_T are replaced by FTK and β_T, respectively.

Proof. The equivalence of the statements follows from Corollary 2 of Theorem 1, Theorem 3, and the Corollary of Theorem 3. The equivalence of the statements with FTK and β_T follows from Corollary 2 of Theorem 2, Theorem 3, and the Corollary of Theorem 3.⊣

THEOREM 5. *Suppose E is a barrelled K-space and T is an Sp_a-matrix with $a \neq 0$. The following statements are equivalent:*

(*a*) *E is an FTK-space;*

(*b*) $E^{\phi} = E^{\beta_T} = E^{\gamma_T}$;

(*c*) *E is a TB-space and $E^{\beta_T} = E^{\gamma_T}$.*

Proof. If E is a TB-space, $E^{\phi} = E^{\gamma_T}$. Hence, (*c*) ⇒ (*b*). By Theorem 4, $E^{\phi} = E^{\beta_T}$

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implies E is an FTK-space. Hence $(b) \Rightarrow (a)$. Finally, (a) implies E is a TB-space which implies, by Theorem 4, $E^{\phi} = E^{\gamma_T}$. Also, by Theorem 4, (a) implies $E^{\phi} = E^{\gamma_T}$. Hence $(a) \Rightarrow (c)$.⊣

DEFINITION. *Let p be a continuous seminorm on a l.c.s.s. E. Then p_T is the seminorm on E_{TB} defined by $p_T(x) = \sup_n p(t^n x)$. If E is a l.c.s.s. defined by the collection $\mathscr{P}$ of seminorms p, we define the topology on E_{TB} by the collection $\mathscr{P}_T$ of seminorms p_T.*

THEOREM 6. *If E is a K-space and T has no column of zeros, then E_{TB} is a sequentially complete K-space.*

Proof. E_{TB} is a K-space since, for the seminorms $p(x) = |x_k|$, we have

$$p_T(x) = \sup_n |t_{nk}|\, p(x)$$

with $\sup_n |t_{nk}| \neq 0$. Let x^n be a Cauchy sequence in E_{TB} with coordinatewise limit x. For each continuous seminorm p on E and each n, there exists m such that

$$p(t^n x - t^n x^m) \leqslant 1.$$

Then $p(t^n x) \leqslant p(t^n x - t^n x^m) + p(t^n x^m) \leqslant 1 + p_T(x^m)$. Since $p_T(x) \leqslant 1 + \sup_m p_T(x^m) < \infty$, it follows that $x \in E_{TB}$. For each $\epsilon > 0$, there exists N such that $p_T(x^n - x^m) < \epsilon/2$ whenever $n, m > N$. Also for each k, there exists $m > N$ such that $p(t^k x - t^k x^m) < \epsilon/2$ since x^m converges coordinatewise to x. Hence, for all $n > N$,

$$p(t^k x - t^k x^n) \leqslant p(t^k x - t^k x^m) + p(t^k x^m - t^k x^n) < \epsilon/2 + p_T(x^m - x^n) < \epsilon.$$

Thus, for all $n > N$, $p_T(x - x^n) < \epsilon$. Hence, $\lim_n x^n = x$ in E_{TB}.⊣

COROLLARY 1. *If E is a metrizable K-space and T has no column of zeros, then E_{TB} is an FK-space.*

Proof. A K-space is metrizable if and only if its topology is generated by a countable number of seminorms. An FK-space is a sequentially complete metrizable K-space.⊣

The β_T- and γ_T-duals of an FK-space need not be FK-spaces. For example, let $E = \bigcap_{r>1} \ell^r$. However, by Theorem 4 and the above corollary, we have the following:

COROLLARY 2. *Suppose T is an Sp_a-matrix with $a \neq 0$. If E is an FK-space and a TB-space, then $E^{\gamma_T \gamma_T}$ is an FK-space.*

This statement is not true for all metrizable K-spaces which are TB-spaces as can be seen by the space of finite sequences under the topology generated by the seminorms $p_k(x) = |x_k|$ and any Sp_1-matrix T.

THEOREM 7. *If E is a K-space and T has no column of zeros, then E_{FTK} is a sequentially complete K-space under the topology induced by E_{TB}.*

Proof. Let x^n be a Cauchy sequence in E_{FTK} with limit x in E_{TB}. We show that

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$x \in E_{FTK}$. Let $f \in E'$. Given $\epsilon > 0$, choose n such that $\sup_k |f(t^k x - t^k x^n)| < \epsilon/3$ and then choose N such that $|f(t^k x^n - t^j x^n)| < \epsilon/3$ whenever $j, k > N$. Then, for $j, k > N$,

$$|f(t^k x - t^j x)| \leqslant |f(t^k x - t^k x^n)| + |f(t^k x^n - t^j x^n)| + |f(t^j x^n - t^j x)| < \epsilon.\dashv$$

COROLLARY 1. *If E is a metrizable K-space and T has no column of zeros, then E_{FTK} is an FK-space.*

COROLLARY 2. *Suppose T is an Sp_a-matrix with $a \neq 0$ and E is an FK-space. If E is a TB-space, then $(E^{\gamma_T})^{\beta_T}$ is an FK-space. If E is an FTK-space, then $E^{\beta_T \beta_T}$ is an FK-space.*

Proof. If E is a TB-space, then $E_{FTK} = (E^{\phi})^{\beta_T} = (E^{\gamma_T})^{\beta_T}$. If E is an FTK-space, then $E^{\beta_T \beta_T} = E_{FTK}$.$\dashv$

PROPOSITION 1. *Suppose T is an Sp_a-matrix with $a \neq 0$ and E is a BK-space. Then E^{ϕ} is a BK-space and $(E^{\phi})_{TB} = (E_{AD})^{\gamma_T} \subset E^{\phi}$.*

Proof. If E is a BK-space, then E_{AD} is a BK-space with the norm of E. $(E_{AD})'$ can be identified with E^{ϕ} and hence E^{ϕ} is a BK-space. Let $f_{t^n\lambda} \in E'$ be defined by

$$f_{t^n\lambda}(x) = \Sigma t_{nk} \lambda_k x_k.$$

Then $\lambda \in (E_{AD})^{\gamma_T}$ if and only if $\sup_n |f_{t^n\lambda}(x)| = \sup_n |\Sigma t_{nk} \lambda_k x_k| < \infty$ for all x in E_{AD}. By uniform boundedness, this is equivalent to the condition that $\{t^n\lambda\}$ is bounded in E^{ϕ}. Thus $(E^{\phi})_{TB} = (E_{AD})^{\gamma_T} \subset (E_{AD})^{\phi} = E^{\phi}$.$\dashv$

Under the conditions of Proposition 1, if E is an AD-space, then E^{γ_T} is a TB-space. However, E^{β_T} need not be an FTK-space. For example, let $cs = \{x \in \omega : \Sigma x_k \text{ exists}\}$, $bv = \{x \in \omega : \Sigma |x_k - x_{k+1}| < \infty\}$, and $t^n = \sum_1^n e^k$. Then $(cs)^{\beta_T} = bv$ but $(bv)_{FTK} = bv \cap c_0$.

PROPOSITION 2. *Let T be an Sp_1-matrix. Suppose E is a BK-space and an AD-space. Then E is a TK-space if and only if E^{ϕ} is a TB-space.*

Proof. An FK-space is a TK-space if and only if it is a TB-space and an AD-space ((13), (21)). By Theorem 4 and Proposition 1, an AD-space E is a TB-space if and only if $(E^{\phi})_{TB} = E^{\gamma_T} = E^{\phi}$.$\dashv$

PROPOSITION 3. *If there exists a BK-space which is also a TB-space, then $\ell \subset c_T$.*

Proof. The proof is similar to that of (3·8) of (4).$\dashv$

This statement is not true for FK-spaces. For example, ω is an FK-space with the property TK for every Sp_1-matrix T. The condition $\ell \subset c_T$ is equivalent to the condition $\sup_{n,k} |t_{nk}| < \infty$.

4. *Applications to summability fields.* In this section, we assume A and T are row-finite reversible Sp_1-matrices and we examine the properties TB, FTK, and TK for the summability field c_A.

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DEFINITION. *If E and F are sequence spaces, the multiplier space $(E \to F)$ is the space of all sequences x such that, for all y in E, $xy \in F$. The multiplier spaces $(E \to c_T)$ and $(E \to m_T)$ are the β_T- and γ_T-duals of E, respectively.*

The Hardy–Bohr Theorem states that for T_α, the matrix of the series-sequence Cesàro method of order $\alpha \geqslant 0$, we have $c_{T_\alpha}{}^\phi = (c_{T_\alpha} \to c_{T_\alpha})$ ((1), (2)). Using this result, Zeller (21) proved that c_{T_α} is a $T_\alpha K$-space. We prove, in part, that if either $A = T$, or $\ell \subset c_A$, or c_A is an AD-space, then the condition c_A is a TK-space is equivalent to the condition $c_A{}^\phi = (c_A \to c_T)$.

PROPOSITION 4. *Let A and T be reversible row-finite Sp_1-matrices. Suppose c_A is a TB-space. Then $c_A \subset c_T$.*

Proof. If c_A is an AD-space, then $c_A \subset c_T$ ((21), Satz 4, Corollary or (4), Corollary (3·7)). Suppose c_A is not an AD-space. Then there exists a non-zero $f \in c_A'$ of the form $f(x) = \alpha(A - \Sigma x_k) + \Sigma\lambda_n(Ax)_n$, where $(Ax)_n = \Sigma a_{nk}x_k$ and $\lambda \in \ell$, such that $f(e^k) = 0$ for all k. If $\alpha = 0$, then $f(e^k) = 0$ implies $\lambda_k = 0$ for all k ((4), Theorem (3·2)). This contradicts $f \neq 0$. Thus $\alpha \neq 0$. Since $f(t^m x) = \alpha(\Sigma t_{mk}x_k) + \Sigma\lambda_n(At^m x)_n = 0$ for all m and $\lim_m \Sigma\lambda_n(At^m x)_n = \Sigma\lambda_n(Ax)_n$ for all $x \in c_A$ ((4), Lemma (3·1)), we have

$$T - \Sigma x_k = \lim_m \Sigma t_{mk}x_k = -\alpha^{-1}\Sigma\lambda_n(Ax)_n$$

for all $x \in c_A$. ⊣

THEOREM 8. *Suppose A and T are reversible row-finite Sp_1-matrices. The following statements are equivalent:*

(a) *c_A is a TB-space;*

(b) *c_A is an FTK-space;*

(c) $\sup_{m,n} |\Sigma t_{nk}a_{mk}x_k| < \infty$ *for all $x \in c_A$;*

(d) $c_A{}^\phi = (c_A \to c_T)$;

(e) $c_A{}^\phi = (c_A \to m_T)$;

(f) $\sup_{m,n} \sum_j |\sum_k t_{mk}a_{nk}a_{kj}^{-1}| < \infty$, *where (a_{kj}^{-1}) denotes the entries of A^{-1};*

(g) *m_A is a TB-space.*

Proof. (a) ⇔ (c) is clear by the definition of the norm on c_A. (a) ⇔ (e), and (b) ⇔ (d) follow from Theorem 4. (b) ⇒ (a) is clear. (a) ⇒ (b): Let $f \in c_A'$. Then

$$f(x) = \alpha(A - \Sigma x_k) + \Sigma\lambda_m(Ax)_m,$$

where $\lambda \in \ell$, for $x \in c_A$. Then $f(t^n x)$ converges as $n \to \infty$ since $A - \Sigma(t^n x)_k = \sum_k t_{nk}x_k$ tends to $T - \Sigma x_k$ (Proposition 4) and $\Sigma\lambda_m(At^n x)_m$ tends to $\Sigma\lambda_m(Ax)_m$ ((4), Lemma (3·1)). (a) ⇔ (f): For $\lambda_i = a_{mi}t_{ni}$ we have $\lambda x = (\lambda A^{-1})(Ax)$. Thus,

$$\sum_i a_{mi}t_{ni}x_i = \sum_j (\sum_k a_{mk}t_{nk}a_{kj}^{-1})(\sum_i a_{ji}x_i). \qquad (*)$$

Since A is reversible, every element of c, the space of convergent sequences, can be written in the form $(\Sigma a_{ji}x_i)_{j=1}^\infty$ for some $x \in c_A$. Thus, the mapping

$$f_n\colon (\sum_i a_{ji}x_i)_{j=1}^\infty \to (\sum_i a_{mi}t_{ni}x_i)_{m=1}^\infty$$

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takes c into c and $\|f_n\| = \sup_m \sum_j |\sum_k a_{mk}t_{nk}a_{kj}^{-1}|$. The set of mappings $\{f_n\}$ is bounded if and only if c_A is a TB-space. $(g) \Rightarrow (a)$ is clear since c_A is a closed subspace of m_A. Conversely, from (*), $\|t^n x\|_{m_A} \leqslant \sup_m \sum_j |\sum_k a_{mk}t_{nk}a_{kj}^{-1}| \, \|x\|_{m_A}$. Thus, by (f), $m_A \subset (m_A)_{TB}$.⊣

If A is a reversible row-finite Sp_1-matrix and $\ell \subset c_A$, then $c_A{}^\phi \subset c$ and

$$c_A{}^\phi = \{\lambda \in \omega : \sum_j |\sum_k \lambda_k a_{kj}^{-1}| + \sup_k |\lambda_k| < \infty\}. \tag{**}$$

The dual norm on $c_A{}^\phi$ is given by

$$\|\lambda\|_\phi = \sum_j |\sum_k \lambda_k a_{kj}^{-1}| + \lim_k |\lambda_k| \tag{***}$$

((18), 6·4, p. 92). The characterization (**) and Theorem 8 (d), (e) give Hardy–Bohr criteria for multipliers in $(c_A \to c_T)$ and $(c_A \to m_T)$ when c_A has the property TB.

THEOREM 9. *Let A and T be reversible row-finite Sp_1-matrices. If $\ell \subset c_A$ or c_A is an AD-space, then the following statements are equivalent:*

(a) c_A is a TB-space;

(b) c_A is a TK-space;

(c) $c_A{}^\phi$ is a TB-space.

Proof. Suppose c_A is an AD-space. Then $(a) \Leftrightarrow (b)$ by (21), and $(b) \Leftrightarrow (c)$ by Proposition 2. Suppose $\ell \subset c_A$. By (3·4) of (4), we have $(a) \Leftrightarrow (b)$. $(b) \Rightarrow (c)$ follows from Proposition 2. $(c) \Rightarrow (b)$: By (*) and (***), we have

$$\|t^n x\|_{m_A} \leqslant \sup_m \sum_j |\sum_k a_{mk}t_{nk}a_{kj}^{-1}| \, . \, \|x\|_{m_A} = \|(t_{nk}a_{mk})_{k=1}^\infty\|_\phi \, . \, \|x\|_{m_A}.$$

Since $\|(a_{mk})_{k=1}^\infty\|_\phi = 1$ for all m and since $c_A{}^\phi$ is a TB-space, we have by uniform boundedness, $\sup_n \|(t_{nk}a_{mk})_{k=1}^\infty\|_\phi < \infty$. Thus $\sup_n \|t^n x\|_{m_A} < \infty$.⊣

The above theorem shows that for reversible row-finite Sp_1-matrices, the conditions $\ell \subset c_A$ and c_A is a TB-space imply that c_A is a TK-space. The converse is not true. For example, the space $C_{-1} = cs \cap \{x \in \omega : x_k = o(1/k)\}$ of Cesàro summable sequences of order -1 is an AK-space which does not contain ℓ ((4), section 4). Thus, for c_A with the property TB, the condition TK (or the condition AD) is strictly weaker than the condition $\ell \subset c_A$. However, the condition TB is not equivalent to TK. For example, the summability field of the triangular Sp_1-matrix with entries $2^n - 1$ on the diagonal and entries 1 below, is an AB-space but not an AK-space.

THEOREM 10. *Suppose T is a reversible (row-finite) Sp_1-matrix. The following statements are equivalent:*

(a) c_T is a TB-space;

(b) c_T is a TK-space;

(c) $c_T{}^\phi$ is a Banach algebra under coordinatewise multiplication.

Proof. By Proposition 3 and Theorem 9, $(a) \Leftrightarrow (b)$. If c_T is a TB-space, then by Theorem 8, $c_T{}^\phi = (c_T \to c_T)$. The dual norm makes $c_T{}^\phi$ a Banach algebra. Thus $(a) \Rightarrow (c)$.

If $c_T{}^\phi$ is a Banach algebra, then $c_T{}^\phi \subset m = \{x \in \omega : \sup_k |x_k| < \infty\}$ ((12), Lemma 2, p. 100). Thus $\ell \subset c_T$. For the dual norm (***) on $c_T{}^\phi$, we have $\|t^n\|_\phi = 1$ for all n. Hence,

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for the norm that makes $c_T{}^\phi$ a Banach algebra, we have $\|t^n x\| \leqslant \|t^n\| . \|x\| \leqslant K\|x\|$. Consequently, $c_T{}^\phi$ is a TB-space. By Theorem 9, (a) follows.⊣

Summability fields satisfying the conditions of the above theorem are called *sum spaces* by Ruckle(15). By Proposition 4 and (13), Theorem (5·2), the strongest form of sectional convergence that the summability field c_T can have is TK. The summability fields of the series-sequence Cesàro methods of order $\alpha \geqslant 0$, are sum spaces(21). More generally, if B is a matrix satisfying the mean value criterion and P is a weighted arithmetic mean, then for any $k = 0, 1, 2, \ldots$, the series-sequence summability field of BP^k is a sum space (10). Other results of this type appear in (11), (13), and (14).

5. *Factorization in FK-spaces.* In this section we characterize the properties TK and TB for an FK-space E in terms of the factorization

$$E = c_T{}^\phi \cdot E = \{\lambda x \in \omega : \lambda \in c_T{}^\phi, x \in E\}$$

when c_T is a sum space. In order to simplify notation, we will write

$$q = c_T{}^\phi, \quad q_0 = q \cap c_0,$$

where c_0 is the space of null sequences. For the matrix T_α $(\alpha \geqslant 0)$, the space

$$q = \{\lambda \in \omega : \|x\|_q = \Sigma k^\alpha |\Delta^{\alpha+1}\lambda_k| + \sup_k |\lambda_k| < \infty\},$$

where

$$\Delta^{\alpha+1}\lambda_k = \sum_{j=k}^{\infty} \binom{j-k-\alpha-2}{j-k} \lambda_j$$

is the space of bounded sequences which have bounded variation of order α.

PROPOSITION 5. *Let T be a reversible row-finite Sp_1-matrix and suppose c_T is a TK-space. Then $q = q_{TB}$, $q_0 = q_{TK}$, and $q = q_0 + span\{e\}$, where $e = (1, 1, 1, \ldots)$.*

Proof. $q = q_{TB}$ follows from Propositions 1 and 2. q_{AD} is a BK-space with the properties TB and AD. Hence, $q_{AD} = q_{TK}$ ((13), (21)). The dual norm on $c_T{}^\phi$ is given by (***) and $c_T{}^\phi$ is isomorphic to $\ell \oplus F$ ($F = \mathbb{R}$ or $\mathbb{C}$) where $t^n \in c_T{}^\phi$ corresponds to

$$e^n + 0 \in \ell \oplus F.$$

The closure of the set $\{t^n\}_{n=1}^\infty$ in q is q_0 and since $\{t^n\}_{n=1}^\infty$ consists of finite sequences, $q_0 \subset q_{AD}$. Conversely, $q_{TK} \subset c_0$ since, by (***), $q \subset c$. Finally, $q = q_0 + \text{span}\{e\}$ is obvious from $q \subset c$.⊣

THEOREM 11. *Suppose T is a reversible row-finite Sp_1-matrix, c_T is a TK-space, and E is an FK-space. Then the following statements are equivalent:*

(a) E is a TB-space;
(b) $q \cdot E \subset E_{TB}$;
(c) $q_0 \cdot E \subset E_{TK}$;
(d) $q \cdot E = E$;
(e) $q_0 \cdot E = E_{TK}$.

Proof. $(a) \Rightarrow (b)$: Let E be a TB-space, $x \in E$, $\lambda \in q$. For each continuous seminorm p on E and for $t^n \lambda x = (t^n \lambda T^{-1})(Tx)$, we have

$$p(t^n \lambda x) = p(\sum_m (\sum_k t_{nk} \lambda_k t_{km}^{-1}) t^m x) \leqslant \sum_m |\sum_k t_{nk} \lambda_k t_{km}^{-1}| \, p_T(x) \leqslant \sup_n \|t^n \lambda\|_\phi \, p_T(x).$$

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Thus $\lambda x \in E_{TB}$. $(b) \Rightarrow (a)$ is immediate since $e \in q$. $(b) \Rightarrow (c)$: For each $x \in E$, $\lambda \to \lambda x$ defines a continuous matrix map (19) from q_0 into E_{TB}. Since q_0 is a TK-space, λx has the property TK for each $\lambda \in q_0$, $x \in E$. That is, $q_0 \cdot E \subset (E_{TB})_{TK}$. $(E_{TB})_{TK} \subset E_{TK}$ since, for each continuous seminorm p, $p(t^n x - t^m x) \leqslant p_T(t^n x - t^m x)$. Conversely, from $E \subset E_{TB}$ it follows that $E_{TK} \subset (E_{TB})_{TK}$. Thus, $q_0 \cdot E \subset (E_{TB})_{TK} = E_{TK}$. $(c) \Rightarrow (d)$: $q_0 \cdot E \subset E_{TK}$ implies $q \cdot E = (q_0 + \operatorname{span}\{e\}) \cdot E = E$. $(d) \Rightarrow (a)$: If $E = q \cdot E$, then for each $x \in E$, $\lambda \to \lambda x$ defines a continuous matrix map from q into E. Since the image of a bounded set is bounded and q is a TB-space, λx has the property TB for each $\lambda \in q$, $x \in E$. Thus $E = q \cdot E$ is a TB-space. $(a) \Rightarrow (e)$: If E is a TB-space, then $E_{TK} = E_{AD}$. Hence E_{TK} is an FK-space generated by seminorms of the form p_T. By (c) above, $q_0 \cdot E_{TK} \subset E_{TK}$. As in the proof of $(a) \Rightarrow (b)$ above, $p_T(\lambda x) \leqslant K \|\lambda\|_\phi p_T(x)$ for all $\lambda \in q_0$, $x \in E_{TK}$. Thus, by Theorem (2·1) of (17), $E_{TK} = q_0 \cdot E_{TK}$. By (c), $E_{TK} = q_0 \cdot E_{TK} \subset q_0 \cdot E \subset E_{TK}$. Finally, $(d) \Rightarrow (c)$ is immediate.⊣

COROLLARY 1. *Suppose T is a reversible row-finite Sp_1-matrix and c_T is a TK-space. Every FK-space which is a β_T-space or a γ_T-space is a TB-space.*

Proof. Since, for every E, $q \cdot E^{\beta_T} = E^{\beta_T}$ and $q \cdot E^{\gamma_T} = E^{\gamma_T}$, the statement follows from Theorem 11, $(a) \Leftrightarrow (d)$.⊣

COROLLARY 2. *Suppose T is a reversible row-finite Sp_1-matrix, c_T is a TK-space, and E is an FK-space. Then E is a TK-space if and only if $E = q_0 \cdot E$.*

Proof. By Theorem 11, E is a TK-space $\Leftrightarrow q_0 \cdot E = E_{TK} = E \Leftrightarrow E = q_0 \cdot E$.⊣

PROPOSITION 6. *Suppose T is a reversible row-finite Sp_1-matrix and c_T is a TK-space. Then, for any sequence space E, we have $E^{\beta_T \beta_T} \subset (E^{\beta_T})^{\gamma_T} = E^{\gamma_T \gamma_T}$.*

Proof. The inclusions $E^{\beta_T \beta_T} \subset (E^{\beta_T})^{\gamma_T}$ and $E^{\gamma_T \gamma_T} \subset (E^{\beta_T})^{\gamma_T}$ are clear. It remains to prove $(E^{\beta_T})^{\gamma_T} \subset E^{\gamma_T \gamma_T}$. Since $q_0 \cdot m_T = c_T$, we have $q_0 \cdot E^{\gamma_T} \subset E^{\beta_T}$. Let $x \in (E^{\beta_T})^{\gamma_T}$ and $y \in E^{\gamma_T}$. Then the matrix map f defined by $f(\lambda) = \lambda xy$ takes q_0 into m_T. Since $\{t^n\}_{n=1}^\infty$ is a bounded subset of q_0, $\{t^n xy\}_{n=1}^\infty$ is a bounded subset of m_T. Thus

$$xy \in (m_T)_{TB} = m_T.$$

Hence $x \in (E^{\gamma_T} \to m_T) = E^{\gamma_T \gamma_T}$.⊣

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