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# A structure theorem for foliations on non-compact 2-manifolds

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*Abstract*. In this work, singular orientable foliations which admit non-trivial recurrent leaves on 2-manifolds of finite or infinite genus are considered. A structure theorem for such foliations is given. The theorem is similar to Gutierrez's structure theorem (C. Gutierrez. Smoothing continuous flows on 2-manifolds and recurrences. *Ergod. Th. & Dynam. Sys.* **6** (1986), 17–44) for flows on compact surfaces.

## 1. *Introduction*

In this paper we study the structure of non-trivial recurrent leaves. The results obtained here have been very useful in the smoothing of the continuous singular foliations on arbitrary genus 2-manifolds [**L1**]. From the famous example of Denjoy [**D**], it is clear that it is very important to understand the dynamical structure of non-trivial recurrence. For compact 2-manifolds, the study of non-trivial recurrence, together with the smoothing problems for orientable singular foliations, was carried out by Gutierrez [**G1**]. However, recent work shows that infinite-genus surfaces can support orientable foliations whose recurrent leaves have dynamics which cannot appear on compact surfaces (see [**GHL**]). Besides, the existence of minimal foliations on those surfaces has already been proven by Beniere (see [**B**]). Our main theorem generalizes Gutierrez's structure theorem [**G1**] for orientable singular foliations on 2-manifolds of infinite genus. First, we need some definitions to state the theorem.

Let $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a map whose domain of definition ($\mathrm{Dom}(T)$) and image ($\mathrm{Im}(T)$) are open and dense subsets of $\mathbb{R}/\mathbb{Z}$. We say that $T$ is a *generalized interval exchange transformation* (or, for short, a GIET) if $T$ homeomorphically takes each connected component of its domain of definition onto a connected component of its image. A GIET is said to be *affine* (respectively *isometric*) if it is affine (respectively isometric) when restricted to every connected component of its domain of definition. If $T$ is an isometric GIET such that $\mathbb{R}/\mathbb{Z} \setminus \mathrm{Dom}(T)$ is at most finite, then we shall say that $T$ is an *interval exchange transformation* (IET).

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Let $F, G : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be two transformations whose domains of definition are open subsets of $\mathbb{R}/\mathbb{Z}$. We shall say that $F$ is *semi-conjugate* to $G$ if there is a monotone continuous map $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ of degree one such that $(h \circ F)(x) = (G \circ h)(x)$ for all points $x \in \mathrm{Dom}(F)$ satisfying the condition $h(x) \in \mathrm{Dom}(G)$. If $h$ is a homeomorphism, then we shall say that the transformation $F$ is *conjugate* to $G$. Notice that our definitions of semi-conjugation and conjugation are not exactly the same as the usual ones because the maps involved are not defined everywhere. Let $M$ be a foliated manifold. A leaf $L$ will be called a *non-trivial recurrent leaf* if $L \subset \overline{\overline{L} \setminus L}$, and it will be called an *exceptional leaf* if $\overline{L}$ has the local structure of a Cantor set. Here, $\overline{L}$ denotes the topological closure of $L$ in $M$. A leaf that is homeomorphic to a circle is called a *compact leaf*.

A non-compact 2-manifold $M$ is said to be of *finite genus* if there exists a compact manifold with boundary $N \subset M$ such that $M \setminus N$ is homeomorphic to a subset of the plane $\mathbb{R}^2$. If not, $M$ is said to be of *infinite genus*. The *genus* of a compact 2-manifold with boundary $N$ is defined to be the genus of the compact manifold obtained by attaching a disc to each boundary component of $N$. An invariant set $\Gamma$ to foliation of $M$ is said to be of *finite type* if there exists a finite-genus sub-manifold of $M$ containing $\Gamma$.

Our main result is the following.

THEOREM. *Let $\mathcal{F}$ be an orientable singular foliation on a smooth Hausdorff 2-manifold $M$, with a countable basis, of finite or infinite genus. Then there is a set, at most countable, of circles $\{C_i\}$ transversal to $\mathcal{F}$ such that if $\Lambda_i$ denotes the union of the leaves passing through $C_i$ and $T_i : C_i \to C_i$ denotes the forward Poincaré map induced by $\mathcal{F}$, then the following conditions are satisfied.*

(1) *$i \neq j$ implies $\Lambda_i \cap \Lambda_j = \emptyset$; moreover, the non-trivial recurrent leaves of $\mathcal{F}$ are contained in $\overline{\bigcup_i \Lambda_i}$.*

(2) *If $\Lambda_i$ is of finite type, then there exists a non-trivial recurrent leaf $L$ such that either:*
   (2.1) *$\overline{L \cap C_i} = C_i$ and $T_i$ is conjugate to an IET, or*
   (2.2) *$\overline{L \cap C_i}$ is a Cantor set and $T_i$ is semi-conjugate to an IET.*

(3) *If $\Lambda_i$ is not of finite type, then:*
   (3.1) *there is a recurrent leaf densely intersecting $C_i$ and $T_i$ is conjugate to an affine GIET; or*
   (3.2) *the points in $C_i$ belonging to recurrent leaves form a residual dense subset of $C_i$ and every recurrent leaf intersecting $C_i$ is exceptional; moreover, the map $T_i$ is conjugate to an affine GIET; or else*
   (3.3) *the circle $C_i$ meets an exceptional leaf $L$ and the union of the wandering points with the compact leaves contains an open and dense subset of $C_i$; also, $T_i$ is semi-conjugate to an affine GIET; in the particular case that $\overline{L}$ is a minimal compact set, the set of wandering points is open and dense in $C_i$ and $T_i$ is semi-conjugate to an IET.*

In [**L3**] is proved that an affine GIET may not be conjugated to an isometric GIET; therefore, our result cannot be improved in the sense of Gutierrez's structure theorem.

Now we wish to mention other results related to the structure of non-trivial recurrence. Gardiner [**Ga**] showed that flows on a compact orientable 2-manifolds admit transversal circles which permit a decomposition of the manifolds into components in such a way

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that the closure of each non-trivial recurrent trajectory is contained in some of these components. Similar results for singular foliations with saddle-type singularities were proven by G. Levitt [**Le**]. The structure theorem of Aranson and Zhuzhoma [**AZ**] gives an estimate of the number of sets determined by the closure of non-trivial recurrent half-leaves in the case of orientable (and a 'class' of non-orientable) foliations on compact surfaces with finite singularities.

This paper is organized as follows. In §3, the manifold is turned compact by its ends. The structure of non-trivial recurrent leaves in the region, invariant by the foliation, of finite or infinite genus is studied separately by introducing the notion of point of finite type. In §4, we generalize the equivalence relation introduced by Gutierrez in [**G1**] and, in the last part of the section, we give a precise description of the local structure of the non-trivial recurrent leaves which intersect no circle $C_i$ of the theorem. In §5, we prove a fundamental proposition: every continuous injective map with a dense set of non-trivial recurrent points is conjugate to an affine GIET.

## 2. *Preliminaries*

Let $M$ be a 2-manifold and $\mathcal{F}$ an orientable foliation of $M$ with a closed set of singularities $\text{sing}(\mathcal{F}) \subset M$. A leaf of $\mathcal{F}$ that is homeomorphic to a circle is called a compact leaf. The leaf of $\mathcal{F}$ passing through the point $p \in M \setminus \text{sing}(\mathcal{F})$ will be denoted by $L(p)$. Let $L$ be a non-compact leaf; a point $p \in L$ divides $L$ into two half-leaves. Thus fixed, an orientation of $\mathcal{F}$, $L^+(p)$ (respectively $L^-(p)$) denotes the component of $L \setminus \{p\}$ which keeps the positive (respectively negative) orientation of $\mathcal{F}$ beginning in $p$. If $q \in L^+(p)$, $L(p,q)$ denotes the segment of the half-leaf in $L^+(p)$ with endpoints $p, q$. Let $L^+(p)$ (respectively $L^-(p)$) be a half-leaf and let $\gamma : L^+(p) \to [0, +\infty)$ (respectively $\gamma : L^-(p) \to (-\infty, 0]$) be a parametrization of $L^+(p)$ (respectively $L^-(p)$) with $\gamma(0) = p$, which is a homeomorphism in the intrinsic topology of $L^+(p)$ (respectively $L^-(p)$). The *$\omega$-limit set* of $L^+(p)$ (respectively *$\alpha$-limit set* of $L^-(p)$) is the set of points $q \in M$ such that each neighborhood of $q$ contains points of $L^+(p)$ (respectively $L^-(p)$) corresponding to arbitrarily large values of the parameter. The set of *$\omega$-limit points* (respectively *$\alpha$-limit points*) of $L^+(p)$ (respectively $L^-(p)$) will be denoted by $\omega(L^+(p))$ (respectively $\alpha(L^-(p))$). Such a set is closed, invariant and non-empty when $M$ is a compact 2-manifold. We say that a point $p$ or a leaf $L(p)$ is *$\omega$-recurrent* (respectively *$\alpha$-recurrent*) if $L(p) \subset \omega(L^+(p))$ (respectively $L(p) \subset \alpha(L^-(p))$) and we say that $L(p)$ is recurrent if it is $\omega$- and $\alpha$-recurrent. Thus, $L(p)$ is a non-trivial recurrent leaf if $L(p)$ is not a compact leaf. We say that $L(p)$ is a *proper* leaf when $\overline{L(p)} \setminus L(p)$ is closed in $M$, *locally dense* if $\overline{L(p)}$ has non-empty interior, and *exceptional* if $L(p)$ is non-proper and nowhere dense. Let $B$ be a subset of $M$. The *saturated set* of $B$ by the foliations $\mathcal{F}$ (or, in short, $\text{Sat}(B)$) is the set formed by the union of leaves $L(p)$, with $p \in B \setminus \text{sin}(\mathcal{F})$, and the set $\text{sing}(\mathcal{F}) \cap B$. A set $B \subset M$ which does not contain singularities of $\mathcal{F}$ is a *minimal set* of $\mathcal{F}$ if it is a *closed*, non-empty and invariant set (i.e. $\text{Sat}(B) = B$), which is minimal (in the sense of inclusion) for these properties.

A segment $\Sigma \subset M \setminus \text{sing}(\mathcal{F})$ is called a segment transversal to the foliation $\mathcal{F}$ if every point $p \in \Sigma$ has a neighborhood $U$ with local coordinates $(x; y) : U \to \mathbb{R}^2$ such that the leaves in $U$ are determined by the equation $y =$ constant and the intersection $U \cap \Sigma$ is

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given by the equation $x =$ constant. A closed curve $C \subset M \setminus \text{sing}(\mathcal{F})$ is called a transversal circle if each segment in $C$ is transversal to $\mathcal{F}$. Let $C_1$ (respectively $C_2$) be a segment or a circle transversal to $\mathcal{F}$. The *forward Poincaré map* (respectively *backward Poincaré map*) induced by $\mathcal{F}$ is the mapping which associates to each point $p \in C_1$ (respectively $q \in C_2$), the first point where $L^+(p)$ (respectively $L^-(q)$) intersects $C_2$ (respectively $C_1$).

Open, closed and half-open subintervals of $C$, with endpoints $a$ and $b$, will be denoted as usual: $(a, b)$, $[a, b]$, $[a, b)$.

A segment $\Sigma$ which is an open interval transversal to the foliation $\mathcal{F}$ will be said to be *wandering* if every leaf of $\mathcal{F}$ intersects $\Sigma$ at most once. A point $p \in M \setminus \text{sing}(\mathcal{F})$ is said to be wandering if there is a wandering transversal open interval containing $p$. For convention, every singularity of $\mathcal{F}$ will be considered as a non-wandering point. Let $C$ be a topological circle and let $T : I \to I$ be a map defined in $I \in \{\mathbb{R}/\mathbb{Z}, C\}$. The positive (respectively negative) orbit $O^+(p)$ (respectively $O^-(p)$) of $T$ through $p$ will be the set $\{T^n(p) : n \in \mathbb{Z}^+$(respectively $\mathbb{Z}^-$) and $p \in \text{Dom}(T^n)\}$. The orbit $O(p)$ of $T$ through $p$ will be the set $O^-(p) \cup \{p\} \cup O^+(p)$. The point $p \in I$ will be said to be *recurrent* ($T$-*recurrent*) if it is an accumulation point of its orbit $O(p)$. A point $p$ is *non-trivial recurrent* if it is a recurrent point such that $O(p) \subset \overline{O(p)} \setminus O(p)$.

## 3. *The compactification*

We begin this section by stating a fundamental theorem, proved by Richard [**R**], which allows a concrete representation of a non-compact 2-manifold.

THEOREM 3.1. [**R**] *Every 2-manifold $M$ is homeomorphic to a surface formed from a sphere $S^2$ by first removing a closed totally disconnected set $\beta(M)$ from $S^2$, then removing the interior of a finite or infinite sequence $D_1, D_2, \ldots$ of non-overlapping closed discs in $S^2 \setminus \beta(M)$, and finally suitably identifying the boundaries of these discs in pairs (it may be necessary to identify the boundary of one disc with itself to produce an odd 'cross cap'). The sequence $D_1, D_2, \ldots$ 'approaches $\beta(M)$' in the sense that, for any open set $U$ in $S^2$ containing $\beta(M)$, all but a finite number of the $D_i$ are contained in $U$.*

By means of the homeomorphism given by Richard's theorem above, let $\overline{M} = M \cup \beta(M)$ be the space which is the usual compactification of $M$ by its ends. Using the fact that $M$ and $\beta(M)$ have countable bases, it can be shown that $\overline{M}$ is a compact connected Hausdorff space with a countable basis. Notice that if $M$ has infinite genus, then $\overline{M}$ is not a 2-manifold. The set $\beta(M)$ is called the *space of ends* of $M$. The elements of $\beta(M)$ belonging to the accumulation set of the sequence $D_i$, $i = 1, 2, \ldots$. will be called *non-flat ends*. If $\mathcal{F}$ is a foliation of $M$, then the compactification of $M$ by its ends induces a foliation on $\overline{M} \setminus \beta(M)$ which will be denoted by $\tilde{\mathcal{F}}$. The union of the closed totally disconnected set $\beta(M)$ with the singularities of $\mathcal{F}$ will be called singularities of $\tilde{\mathcal{F}}$ and will be denoted by $\text{sing}(\tilde{\mathcal{F}})$. We say that $p \in \overline{M}$ is a *flat (respectively not-flat) singularity* of $\tilde{\mathcal{F}}$ if $p$ is a flat (respectively not-flat) end of $M$.

*Definition.* Let $N$ be a 2-manifold and $\mathcal{G}$ be a foliation of $N$. A set $F \in N \setminus \text{sing}(\mathcal{G})$ is said to be of finite type if and only if there exists a neighborhood of $F$ whose saturated by

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$\mathcal{G}$ is contained in a sub-manifold of $N$ (possibly with boundary) of finite genus. A point $p \in N \setminus \mathrm{sing}(\mathcal{G})$ is of finite type if $\{p\}$ is.

Here and subsequently, the following notations will be used: $\Omega(\tilde{\mathcal{F}})$ will denote the non-wandering point set of $\overline{M} \setminus \mathrm{sing}(\tilde{\mathcal{F}})$, $\Omega_1(\tilde{\mathcal{F}})$ are the elements of $\Omega(\tilde{\mathcal{F}})$ of finite type, and $\Omega_2(\tilde{\mathcal{F}})$ is the set $\Omega(\tilde{\mathcal{F}}) \setminus \Omega_1(\tilde{\mathcal{F}})$. Notice that it follows from the definition that $\Omega_2(\tilde{\mathcal{F}})$ is a closed subset of $\overline{M}$, clearly invariant by $\tilde{\mathcal{F}}$.

Let $\Delta \subset M$ be a minimal subset of $\mathcal{F}$. If $\tilde{\Delta} \subset \overline{M}$ denotes the set induced by $\Delta$ and the usual compactification of $M$ by its ends, then it is clear that $\Delta$ is compact (respectively not-compact) in $M$ if, and only if, $\tilde{\Delta}$ is disjoint of (respectively intersects) the set $\mathrm{sing}(\tilde{\mathcal{F}})$.

The following lemma is a consequence of Peixoto [**P**] and the foliation theory (see [**CN**]).

LEMMA 3.2. *For every point $p$ belonging to a non-trivial recurrent leaf of $\tilde{\mathcal{F}}$ and for every open neighborhood $V$ of $p$, there exists a circle $C$ transversal to $\tilde{\mathcal{F}}$ passing through $p$ such that:*

(1) $\mathrm{Sat}(V) \supset \mathrm{Sat}(C)$*;*

(2) *if $L(p)$ is the leaf passing through $p$, then either $\overline{L(p) \cap C} = C$ or $\overline{L(p) \cap C}$ is a Cantor set.*

## 4. *An equivalence relation*

Let $C$ be a circle transversal to $\tilde{\mathcal{F}}$ and $T : C \to C$ be the forward Poincaré map induced by the oriented foliation $\tilde{\mathcal{F}}$. Given $x, y \in C$, we say that $x \sim y$ if there exist closed subintervals $A, B$ of $C$ (with $A$ containing $\{x, y\}$) and an open disc $D$ punctured by a set, at most countable, of open discs such that the following hold.

- If $a, b$ are the endpoints of $A$, then $a, b \in \mathrm{Dom}(T)$ and $T(a), T(b)$ are endpoints of $B$.
- Let $p \in A$. If $p \in \mathrm{Dom}(T)$, then $T(p) \in B$, $L(p, T(p)) \subset D$ and $T_{|\mathrm{Dom}(T) \cap A} : \mathrm{Dom}(T) \cap A \to B$ is strictly monotone. If $p \in A \setminus \mathrm{Dom}(T)$, the positive leaf starting at $p$ is contained in $D$.
- Let $q \in B$. If $q \in \mathrm{Dom}(T^{-1})$, then $T^{-1}(q) \in A$, $L(T^{-1}(q), q) \subset D$ and $T^{-1}_{|\mathrm{Dom}(T^{-1}) \cap B} : \mathrm{Dom}(T^{-1}) \cap B \to A$ is strictly monotone. If $q \in B \setminus \mathrm{Dom}(T^{-1})$, the negative leaf starting at $q$ is contained in $D$.

It is not difficult to see that $\sim$ determines an equivalence relation on $C$. The equivalence relation $\sim$ will be called the relation associated to $T$.

We remark that the relation defined above is 'larger' than the one defined by Gutierrez in [**G1**]; i.e. if $x, y \in C$ are equivalent according to the definition of Gutierrez, then they are equivalent according to the relation $\sim$ defined above. Clearly, the converse is false.

4.1. *The map $T_C$.* Let $C$ be a circle transversal to $\tilde{\mathcal{F}}$ and let $\mathrm{Rec}(C)$ be defined as the set of elements of $C$ that belong to non-trivial recurrent leaves of $\tilde{\mathcal{F}}$. If $\mathrm{Rec}(C) \neq \emptyset$, consider the set $\mathcal{A}(C)$ made up of all the closed sub-intervals $[a, b]$ of $C$ that satisfy one of the following two conditions:

- $[a, b]$ is the closure of a connected component of $C \setminus \overline{\mathrm{Rec}(C)}$;
- $a = b$ and $a$ does not belong to the closure of any connected component of $C \setminus \overline{\mathrm{Rec}(C)}$.

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Let $T : C \to C$ be the forward Poincaré map on $C$ induced by the oriented foliation $\tilde{\mathcal{F}}$. The mapping $T_C : \mathcal{A}(C) \to \mathcal{A}(C)$ is defined as follows: if $[a, b], [c, d] \in \mathcal{A}(C)$ we will write $T_C([a, b]) = [c, d]$ if, and only if, we have one of the following conditions:

- $a = b \in \mathrm{Rec}(C)$ and $[c, d] = \{T(a)\}$;
- if either $a = b \notin \mathrm{Rec}(C)$ or $a \neq b$, and there are sequences $\{p_n\}, \{q_n\}$ in $\mathrm{Dom}(T) \cap \mathrm{Rec}(C)$ with $a \sim b \sim p_n \sim q_n$ such that $a = \lim p_n$, $b = \lim q_n$ and $\{c, d\} = \{\lim T(p_n), \lim T(q_n)\}$.

Here, $\sim$ is the relation associated to $T$. On the other hand, as no point in $\mathrm{Rec}(C)$ is isolated, the elements of $\mathcal{A}(C)$ are disjoint, closed subsets of $C$ and, therefore, $\mathcal{A}(C)$ is a partition of $C$. In what follows, $\mathcal{A}(C)$ will be considered with the quotient topology.

As an immediate consequence of the introduced definition, one has the following lemma.

LEMMA 4.1.

(1) *The equivalence class of the relation $\sim$ associated to $T$ are open and connected subsets of $C$.*
(2) *$\mathcal{A}(C)$ is homeomorphic to $C$ and $T_C$ is a continuous and injective map.*

LEMMA 4.2. *If $C$ is a circle transversal to $\tilde{\mathcal{F}}$ so that every point in $C$ is of finite type, then the relation $\sim$ associated to $T$ (respectively the set $\mathcal{A}(C) \setminus \mathrm{Dom}(T_C)$) has finitely many equivalence classes (respectively finitely many elements).*

*Proof.* If every point in $C$ is of finite type, then from the compactness of $C$ the closure of the saturate of $C$ by $\tilde{\mathcal{F}}$ is contained in a topological sub-manifold (possibly with boundary) of finite genus. Under these conditions, the proof of the assertion follows in the same manner as in [**G1**, Lemma 3.4]. □

If $\overline{M}$ and $\tilde{\mathcal{F}}$ are as above, then the following theorem is a consequence of Lemma 3.2 and Gutierrez's structure theorems [**G1**].

THEOREM 4.3. *There exists in $\overline{M} \setminus \mathrm{sing}(\tilde{\mathcal{F}})$ a set, at most countable, of circles $\{C_i\}_{i \in \Delta_1}$ transversal to $\tilde{\mathcal{F}}$ such that:*

(1) *every $C_i$ is of finite type and $\mathrm{Sat}(C_i) \cap \mathrm{Sat}(C_j) = \emptyset$, for all $i \neq j$;*
(2) *each locally dense leaf passing by a non-trivial recurrent point $p$ of finite type intersects some circle $C_i$, $i \in \Delta_1$. Moreover, we have that for each recurrent point $q \in C_i$, $\overline{L(p) \cap C_i} = \overline{L(q) \cap C_i} = C_i$ and the forward Poincaré map $T_i : C_i \to C_i$ induced by $\tilde{\mathcal{F}}$ is conjugate to an IET;*
(3) *each exceptional leaf passing by a non-trivial recurrent point $p$ of finite type intersects some circle $C_i$, $i \in \Delta_1$. Moreover, we have that for each recurrent point $q \in C_i$, $\overline{L(p) \cap C_i} = \overline{L(q) \cap C_i}$ is a Cantor set and the forward Poincaré map $T_i : C_i \to C_i$ induced by $\tilde{\mathcal{F}}$ is semi-conjugate to an IET.*

*Proof.* Let $L(p_1)$ be a non-trivial recurrent leaf passing through a point $p_1$ of finite type. It follows by the condition on $p_1$ and Lemma 3.2 that there exists in $\overline{M} \setminus \mathrm{sing}(\tilde{\mathcal{F}})$ an open neighborhood $U(p_1)$ of $p_1$ such that $\overline{\mathrm{Sat}(U(p_1))}$ is contained in a topological 2-manifold (possibly with boundary) of finite genus. Thus, from Gutierrez's structure theorem, there is a transversal circle $C_1 \subset U(p_1)$ satisfying the following property.

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(G): For all $p$, $q$ recurrent points in $C$, either $\overline{L(p)\cap C} = \overline{L(q)\cap C} = C$ and $T : C \to C$ (the forward Poincaré map induced by $\tilde{\mathcal{F}}$) is conjugate to a standard IET, or $\overline{L(p)\cap C} = \overline{L(q)\cap C}$ is a Cantor set and $T : C \to C$ is semi-conjugate to a standard IET.

Now, suppose that there exists a non-trivial recurrent leaf $L(p_2)$ of $\tilde{\mathcal{F}}$ with $p_2$ of finite type and $L(p_2)\cap C_1 = \emptyset$ (in the contrary case the theorem follows with $\Delta_1 = \{1\}$). Since $\mathrm{Sat}(C_1)$ is contained in a topological 2-manifold (possibly with boundary) of finite genus, is follows that $L(p_2)\cap\overline{\mathrm{Sat}(C_1)} = \emptyset$. Then, from Lemma 3.2, there is a transversal circle $C_2$ which contains $p_2$, so that $\mathrm{Sat}(C_1)\cap\mathrm{Sat}(C_2) = \emptyset$ and $\overline{\mathrm{Sat}(C_2)}$ is contained in a topological 2-manifold, possibly with boundary, of finite genus. It follows from Gutierrez's structure theorem that one can consider $C_2$ so that it satisfies the property (G). Repeating the previous argument, one can obtain a family of circles with those properties. Thus, let $D$ be the set whose elements have the form $\bigcup_{i\in I_n} C_i$; $I_n \subset \mathbb{N}$ (i.e. countable union) of transversal circles $\{C_i\}$ so that $\mathrm{Sat}(C_i)\cap\mathrm{Sat}(C_j) = \emptyset$ for all $i \neq j$, and every $C_i$ is of finite type and satisfies the property (G). Since the set $\{\mathrm{Sat}(C_i)\}_i$ forms a family of open sets pairwise disjoint on $\overline{M}$, which has a countable basis, then this family of circles is at most countable. Therefore, $D$ with the partial order relation '$\subseteq$' determined by the usual inclusion of sets will be inductively ordered. Thus, from Zorn's lemma we have that there are $\Delta_1 \subseteq \mathbb{N}$ and circles $C_i$, $i \in \Delta_1$, so that $\bigcup_{i\in\Delta_1} C_i$ is a maximal element of $D$.

Now, we claim that each non-trivial recurrent leaf $L(p)$, with $p$ of finite type, meets some circle of the countable family $\{C_i\}_{i\in\Delta_1}$. Suppose that the assertion is false. It is easy to check that for every circle $C_i$, $i \in \Delta_1$, $L(p)\cap\overline{\mathrm{Sat}(C_i)} = \emptyset$. On the other hand, if $L(p)\cap\overline{\bigcup_{i\in\Delta_1}\mathrm{Sat}(C_i)} \neq \emptyset$, then the latter equality implies that $p$ is not of finite type, and therefore $L(p)\cap\overline{\bigcup_{i\in\Delta_1}\mathrm{Sat}(C_i)} = \emptyset$. The latter allows us to consider, in $\overline{M}\setminus\mathrm{sing}(\tilde{\mathcal{F}})$, an open neighborhood of $p$ disjoint of $\bigcup_{i\in\Delta_1}\mathrm{Sat}(C_i)$ and a circle $\tilde{C}$ transversal to $\tilde{\mathcal{F}}$ such that $\mathrm{Sat}(\tilde{C})\cap\mathrm{Sat}(C_i) \neq \emptyset$ for all $i \in \Delta_1$, and $\tilde{C}\cup\bigcup_{i\in\Delta_1} C_i$ belongs to $D$. But this contradicts the maximality of $\bigcup_{i\in\Delta_1} C_i$ in $D$, and therefore the proof is complete. □

The following result of Cherry [**C**] is one of the first theorems concerning the structure of orientable foliations on surfaces that admit recurrent leaves. Aranson and Zhuzhoma have obtained a similar theorem concerning non-orientable foliations [**AZ**].

THEOREM 4.4. [**C**] *Let $C$ be a circle transversal to $\tilde{\mathcal{F}}$. If $L^+(p)$ (respectively $L^-(p)$) is $\omega$-(respectively $\alpha$-) recurrent half-leaf of $\tilde{\mathcal{F}}$, then $\overline{L^+(p)}$ (respectively $\overline{L^-(p)}$) contains a continuum of non-compact recurrent leaves each one of which is everywhere dense in $\overline{L^+(p)}$ (respectively $\overline{L^-(p)}$).*

Now, let us consider the recurrent leaves which lie on sub-manifolds of $\overline{M}\setminus\mathrm{sing}(\tilde{\mathcal{F}})$ which are invariant under $\mathcal{F}$ and have infinite genus. We begin supposing that $\Omega_2(\tilde{\mathcal{F}})$ is a set with non-empty interior ($\mathrm{int}(\Omega_2(\tilde{\mathcal{F}})) \neq \emptyset$).

LEMMA 4.5. *If $\Omega_2(\tilde{\mathcal{F}})$ is a set with non-empty interior then:*

(1) *there are no compact leaves contained in* $\overline{\mathrm{int}(\Omega_2(\tilde{\mathcal{F}}))}$*;*
(2) *if $C$ is a circle transversal to $\tilde{\mathcal{F}}$ such that $C \subset \mathrm{int}(\Omega_2(\tilde{\mathcal{F}}))$, then the set $\{p \in C : \omega(L^+(p))$ or $\alpha(L^-(p))$ is a singularity of $\tilde{\mathcal{F}}\}$ is a dense subset of $C$.*

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*Proof.* (1) Let $L$ be a compact leaf (i.e. homeomorphic to a circle) of $\tilde{\mathcal{F}}$. Take $V \subset \overline{M} \setminus \mathrm{sing}(\tilde{\mathcal{F}})$ a neighborhood of $L$ homeomorphic to a cylinder or to a Möbius band (notice that $M$ can be non-orientable). Let $W \subset V$ be a sufficiently small neighborhood of $L$. From the structure of the foliation restricted to $W$, it is easy to see that if $p \in W$ is not a wandering point, then $p$ belongs to a compact leaf. But, by definition of $\Omega_2(\tilde{\mathcal{F}})$, this is impossible in the interior of $\Omega_2(\tilde{\mathcal{F}})$. Therefore, the first assertion in the lemma follows.

(2) Suppose the contrary; i.e. suppose that there is an interval $I \subset C$ such that no point in $I$ has a singularity of $\tilde{\mathcal{F}}$ as $\omega$- or $\alpha$-limit set. From Lemma 3.2, there exists a circle $\tilde{C}$ transversal to $\tilde{\mathcal{F}}$ such that $\mathrm{Sat}(\tilde{C}) \subset \mathrm{Sat}(I)$ and no point in $\tilde{C}$ has as $\omega$- or $\alpha$-limit set a singularity. If we denote by $T$ the forward Poincaré map in $\tilde{C}$ induced by $\tilde{\mathcal{F}}$, then $T$ will be defined in all $\tilde{C}$. Thus, $\mathrm{Sat}(\tilde{C}) = \bigcup_{p\in\tilde{C}} L(p, T(p))$ is a compact connected sub-manifold without boundary of $M$. But this is clearly impossible since $\mathrm{int}(\Omega_2(\tilde{\mathcal{F}})) \neq \emptyset$ implies that $M$ is a manifold of infinite genus. Therefore, (2) follows and the proof is complete. □

LEMMA 4.6. *If $C$ is a circle transversal to $\tilde{\mathcal{F}}$ contained in the interior of $\Omega_2(\tilde{\mathcal{F}})$, then the points of $C$ belonging to $\omega$- or $\alpha$-recurrent leaves form a residual dense subset of $C$.*

*Proof.* Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable base of open intervals in $C$. For each $n \in \mathbb{N}$ define

$$A_n = \{p \in \overline{U}_n : \#\{L(p) \cap U_n\} < n\},$$

where, as before, $L(p)$ denotes the leaf passing through $p$ and $\#\{L(p) \cap U_n\}$ denotes the number of intersections of $L(p)$ with the interval $U_n$. We claim that for each $n$, $A_n$ is a closed set with empty interior. In fact, the first assertion follows immediately from the theorem on continuous dependence of leaves on initial conditions. Now, we will assume that the second assertion is false. Let $I$ be an open interval contained in $C$ such that $I \subset \overline{U_n}$ and $\#\{L(x) \cap U_n\} < n$ for all $x \in I$. On the other hand, as $C \subset \mathrm{int}(\Omega_2(\tilde{\mathcal{F}}))$ we have that there exists a leaf (non-compact by Lemma 4.5) intersecting $I$ in at least two points. Let $L$ be such a leaf. Let $p \in I \cap L$ and $q \in C \cap L^+(p)$ be such that $\#\{L(p, q) \cap U_n\} = \#\{L(p) \cap U_n\} < n$ (the case $q \in L^-(p)$ will be similar). Again, according to the theorem on the continuous dependence of leaves on initial conditions, there exists a neighborhood $V(L(p, q)) \subset \overline{M} \setminus \mathrm{sing}(\tilde{\mathcal{F}})$ of the segment of leaf $L(p, q)$ with the following property: any leaf $L_2$ passing through a point sufficiently close to $p$ contains points $p_2, q_2 \in C$ such that $L_2(p_2, q_2) \subset V(L(p, q))$ and

$$\#\{L_2(p_2, q_2) \cap U_n\} = \#\{L(p, q) \cap U_n\} < n.$$

As $I \subset \mathrm{int}(\Omega_2(\tilde{\mathcal{F}}))$, for every open interval $J \subset V(L(p, q)) \cap I$ containing $p_2$, we have that there is a leaf $L$ (non-compact) intersecting $J$ in at least two points. But a piece of the compact segment of $L$ is contained in $V(L(p, q))$; therefore $J$ can be taken so that

$$\#\{L_2(p_2, q_2) \cap U_n\} < \#\{L \cap U_n\}.$$

Repeating the previous arguments with point $p$, now in $L \cap J$, and leaf $L(p)$ being $L$, it is possible to find a leaf $L_k(p_k)$ passing through $p_k \in I$ so that

$$n < \#\{L_k(p_k) \cap U_n\},$$

which contradicts the fact that $I \subset A_n$. Therefore $\mathrm{int}(A_n) = \emptyset$.

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As $C$ is a Baire space, then $R(C) = \bigcap_{n=1}^{\infty}(C \setminus A_n)$ is a residual set dense in $C$. We now claim that every element of $R(C)$ belongs to an $\omega$- or $\alpha$-recurrent leaf of $\tilde{\mathcal{F}}$. In fact, let $x \in R(C)$ and let $U \subset C$ be an arbitrary open interval of $x$. Of course, there is an increasing sequence of integers $\{n_k\}_{k=1}$ such that $U \supsetneq U_{n_k} \supsetneq U_{n_{k+1}}$ and $\{x\} = \bigcap_k U_{n_k}$, where $\{U_{n_k}\}_{k\in\mathbb{N}}$ is a sub-sequence of elements of the base $\{U_n\}_{n\in\mathbb{N}}$. Let $L(x)$ be the leaf (non-compact by Lemma 4.5) passing through $x$. From the definition of $R(C)$, we have that for all $k$

$$\#\{L(x) \cap U_{n_k}\} \geq n_k.$$

It is easy to check that this last inequality and the theorem on the continuous dependence of leaves on initial conditions imply that either $L(x) \subset \omega(L^+(x))$ or $L(x) \subset \alpha(L^-(x))$, which completes the proof. □

The previous Lemma and Theorem 4.4 immediately imply the following corollary.

COROLLARY 4.7. *If $C$ is as in Lemma 4.6, then the points of $C$ belonging to non-trivial recurrent leaves form a residual dense subset of $C$.*

THEOREM 4.8. *If $\Omega_2(\tilde{\mathcal{F}})$ is a set with non-empty interior* ($\mathrm{int}(\Omega_2(\tilde{\mathcal{F}})) \neq \emptyset$) *then there exists a set, at most countable, of circles $\{C_i\}_{i\in\Delta_2}$ transversal to $\tilde{\mathcal{F}}$ such that:*

(1) $C_i \subset \mathrm{int}(\Omega_2(\tilde{\mathcal{F}}))$ *for all* $i \in \Delta_2$;
(2) $\mathrm{Sat}(C_i) \cap \mathrm{Sat}(C_j) = \emptyset$ *for all* $i \neq j$;
(3) *for each* $i \in \Delta_2$, *the set of elements* $p \in C_i$ *such that* $\omega(L^+(p))$ *or* $\alpha(L^-(p))$ *is a singularity of* $\tilde{\mathcal{F}}$, *is dense in* $C_i$;
(4) *the points in* $C_i$ *belonging to recurrent leaves form a residual dense subset of* $C_i$. *Besides, for each* $i \in \Delta_2$, *we have one of the following alternatives. Either:*
  (4.1) *there is a recurrent leaf intersecting* $C_i$ *densely , or*
  (4.2) *any recurrent leaf intersecting* $C_i$ *is exceptional;*
(5) $\mathrm{int}(\Omega_2(\tilde{\mathcal{F}})) \subset \overline{\bigcup_{i\in\Delta_2} \mathrm{Sat}(C_i)}$.

*Proof.* We begin assuming that there exists a locally dense leaf passing through a point $p \in \mathrm{int}(\Omega_2(\tilde{\mathcal{F}}))$. Notice that by Corollary 4.7, $\mathrm{int}(\Omega_2(\tilde{\mathcal{F}}))$ contains a non-trivial recurrent leaf. From Lemma 3.2, it follows that there exists a circle $C_1$ transversal to $\tilde{\mathcal{F}}$ such that:

$\overline{L(p) \cap C_1} = C_1$; and
$C_1 \subset \mathrm{int}(\Omega_2(\tilde{\mathcal{F}}))$.

Now, if $\mathrm{int}(\Omega_2(\tilde{\mathcal{F}})) \setminus \overline{\mathrm{Sat}(C_1)}$ is an empty set, then we have from Lemma 4.5 and Corollary 4.7 that the proof is complete. On the other hand, if this difference is not null, then we proceed as follows.

(i) Consider a point $p_2 \in \mathrm{int}(\Omega_2(\tilde{\mathcal{F}})) \setminus \overline{\mathrm{Sat}(C_1)}$ such that the leaf passing through $p_2$ is locally dense (otherwise, go to step (iv) considering $\Delta_d = \{1\}$). Proceeding as in the beginning, we claim that there is a circle $C_2$ transversal to $\tilde{\mathcal{F}}$ such that:

$\overline{L(p_2) \cap C_2} = C_2$; and
$C_2 \subset \mathrm{int}(\Omega_2(\tilde{\mathcal{F}})) \setminus \overline{\mathrm{Sat}(C_1)}$.

In fact, notice that this last sentence implies that $\mathrm{Sat}(C_1) \cap \mathrm{Sat}(C_2) = \emptyset$. On the other hand, if $\mathrm{int}(\Omega_2(\tilde{\mathcal{F}})) \setminus \overline{\mathrm{Sat}(C_1) \cup \mathrm{Sat}(C_2)} = \emptyset$, then the proof is complete. In the contrary case, the previous argument can be inductively repeated to obtain a family $\{C_i\}$ of circles transversal

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to $\tilde{\mathcal{F}}$ satisfying items (1)–(3) and (4.1) of the theorem. As $\overline{M}$ has a countable basis and $\{\mathrm{Sat}(C_i)\}$ is a family of pairwise disjoint open set of $\overline{M}$, then we have the following.

(ii) Any family $C_i, i = 1, 2, \ldots$ of circles transversal to $\tilde{\mathcal{F}}$ verifying items (1)–(3) and (4.1) of the theorem, is at most countable.

Let $D$ denote the set formed by all possible countable unions of transversal circles to $\tilde{\mathcal{F}}$ which satisfy items (1)–(3) and (4.1) of the theorem.

If '$\subseteq$' denotes the usual inclusion of a set, then a trivial verification shows that $D$ is a partially ordered set with partial order $\subseteq$. Therefore, one can proceed as in the proof of Theorem 4.3 and claim the following.

(iii) There is a countable set $\Delta_d$ and circles $C_i$, $i \in \Delta_d$, satisfy items (1)–(3) and (4.1) of the theorem so that $\bigcup_{i\in\Delta_d} C_i$ is a maximal element of $D$. Notice that if $\mathrm{int}(\Omega_2(\tilde{\mathcal{F}})) \setminus \overline{\bigcup_{i\in\Delta_d} \mathrm{Sat}(C_i)} = \emptyset$, then the proof is complete. Now we will suppose the contrary.

(iv) Let $q_1 \in \mathrm{int}(\Omega_2(\tilde{\mathcal{F}})) \setminus \overline{\bigcup_{i\in\Delta_d} \mathrm{Sat}(C_i)}$. From Lemma 3.2, there exists a circle $C_i'$ transversal to $\tilde{\mathcal{F}}$ containing $q_1$ such that $C_i' \subset \mathrm{int}(\Omega_2(\tilde{\mathcal{F}})) \setminus \overline{\bigcup_{i\in\Delta_d} \mathrm{Sat}(C_i)}$. But, from Corollary 4.7, there is a non-trivial recurrent leaf intersecting $C_i'$ which has to be an exceptional leaf. Thus, without loss of generality, we can assume that $L(q_1)$ is an exceptional recurrent leaf.

If $E$ denotes the set formed by all possible countable unions of transversal circles to $\tilde{\mathcal{F}}$ that satisfy items (1)–(3) and (4.2) of the theorem, then, proceeding as in the case of set $D$, we have the following.

(v) There exists a countable set $\Delta_e$ and circles $C_i'$, $i \in \Delta_e$, that satisfy items (1)–(3) and (4.2) of the theorem so that $\bigcup_{i\in\Delta_e} C_i'$ is a maximal element of $E$ with respect to the partial order defined by the usual inclusion of set '$\subseteq$'.

Finally, considering $\Delta_2 = \Delta_d \cup \Delta_e$, $\bigcup_{i\in\Delta_d} C_i$ and $\bigcup_{i\in\Delta_e} C_i'$ obtained in (iii) and (v) respectively, we have that, under a suitable reordering, $\mathrm{int}(\Omega_2(\tilde{\mathcal{F}})) \subset \overline{\bigcup_{i\in\Delta_2} \mathrm{Sat}(C_i) \cup \mathrm{Sat}(C_i')}$, which completes the proof. □

Examples of 2-manifolds supporting a foliation $\mathcal{F}$ such that $\mathrm{int}(\Omega(\mathcal{F})) \neq \emptyset$, where each element of $\Omega(\mathcal{F})$ is not of finite type, can be found in [**B**] and [**GHL**]. In both cases, there is a global circle transversal to the foliations. In [**B**], the circle satisfies item (4.1) (of the last theorem) and in [**GHL**], item (4.2).

Let $C$ be a circle transversal to $\tilde{\mathcal{F}}$ and $T : C \to C$ be the forward Poincaré map induced by $\tilde{\mathcal{F}}$. Let $\mathcal{A}(C)$ and $T_C$ be as defined in §4. If $C$ is the circle obtained in the previous theorem, then it is immediate that each element of $\mathcal{A}(C)$ is a one-point set. Moreover, $\mathcal{A}(C) = C$ and $T_C(x) = T(x)$ for all $x \in \mathrm{Dom}(T)$. Therefore, the next corollary follows immediately from (4.1) or (4.2) of the previous theorem.

COROLLARY 4.9. *If $C$ is a circle obtained in Theorem 4.8, then we have that:*

(1) *the forward Poincaré map in $C$ is conjugate to $T_C$;*
(2) *$\mathcal{A}(C) \setminus \mathrm{Dom}(T_C)$ is a compact totally disconnected set with an infinite number of elements. Besides, $T_C$ admits a dense subset of non-trivial recurrent points.*

Let $\Sigma$ be a transversal section in $\overline{M} \setminus \mathrm{sing}(\tilde{\mathcal{F}})$. In what follows, $\mathrm{Per}(\Sigma)$ will denote the points in $\Sigma$ belonging to compact leaves (i.e. homeomorphic to circles).

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THEOREM 4.10. *Let $\{C_i\}_{i\in\Delta_3}$ be the family of circles transversal to $\tilde{\mathcal{F}}$ obtained in Theorems 4.3 and 4.8. If there exists a non-trivial recurrent leaf disjoint of $\overline{\bigcup_{i\in\Delta_3}\mathrm{Sat}(C_i)}$, then there is a family, at most countable, $\{\tilde{C}_j\}_{j\in\Delta_4}$ of circles transversal to $\tilde{\mathcal{F}}$ so we have the following.*

(1) *Each circle $\tilde{C}_j$ contains a point $p$ belonging to a recurrent leaf $L(p)$ such that $p$ is not of finite type and $\overline{L(p)\cap\tilde{C}_j}$ is a Cantor set. Besides, the wandering points, together with the set $\mathrm{Per}(\tilde{C}_j)$, contain an open and dense subset of $\tilde{C}_j$.*

(2) *If $L$ is a non-trivial recurrent leaf of $\tilde{\mathcal{F}}$ disjoint from $\overline{\bigcup_{i\in\Delta_3}\mathrm{Sat}(C_i)}$ such that $\overline{L}\cap\mathrm{sing}(\tilde{\mathcal{F}})=\emptyset$, then $L$ intersects a circle $\tilde{C}_j$ (for some $j\in\Delta_4$) which admits an open and dense subset of wandering points. Moreover, for this circle, $\mathcal{A}(\tilde{C}_j)\setminus\mathrm{Dom}(T_{\tilde{C}_j})$ is a finite set and $T_{\tilde{C}_j}$ has a dense orbit.*

(3) *$\tilde{C}_j\subset\overline{M}\setminus\overline{\bigcup_{i\in\Delta_3}\mathrm{Sat}(C_i)}$, for all $j\in\Delta_4$, $\mathrm{Sat}(\tilde{C}_j)\cap\mathrm{Sat}(\tilde{C}_k)=\emptyset$, for all $j\neq k$ and any non-trivial recurrent leaf disjoint of $\overline{\bigcup_{i\in\Delta_3}\mathrm{Sat}(C_i)}$ is contained in $\overline{\bigcup_{j\in\Delta_4}\mathrm{Sat}(\tilde{C}_j)}$.*

(4) *For each $\tilde{C}_j$, $j\in\Delta_4$, the forward Poincaré map on $\tilde{C}_j$ induced by $\tilde{\mathcal{F}}$ is semi-conjugate to $T_{\tilde{C}_j}$, $\mathcal{A}(\tilde{C}_j)\setminus\mathrm{Dom}(T_{\tilde{C}_j})$ is a compact and totally disconnected set and $T_{\tilde{C}_j}$ admits a dense subset of non-trivial recurrent points.*

*Proof.* We begin assuming that there exists a point $p\in\Omega(\tilde{\mathcal{F}})\setminus\overline{\bigcup_{i\in\Delta_3}\mathrm{Sat}(C_i)}$ such that $L(p)$ is a non-trivial recurrent leaf of $\tilde{\mathcal{F}}$ (otherwise we have that $\Delta_4=\emptyset$). It follows from Theorems 4.3 and 4.8 that $L(p)$ is not locally dense. Therefore, from Lemma 3.2, we have that through $p$ there exists a circle $\tilde{C}\subset\overline{M}\setminus\overline{\bigcup_{i\in\Delta_3}\mathrm{Sat}(C_i)}$ transversal to $\tilde{\mathcal{F}}$, such that $\overline{L(p)\cap\tilde{C}}$ is a Cantor set. Moreover, the wandering points in $\tilde{C}$, together with $\mathrm{int}(\mathrm{Per}(\tilde{C}))$, contain an open and dense subset of $\tilde{C}$. It is easy to check that $p$ cannot be of finite type. On the other hand, if $\overline{L(p)}$ is a set disjoint of $\mathrm{sing}(\tilde{\mathcal{F}})$ (i.e. $\overline{L(p)}$ is a compact minimal set), then, although the saturate of a neighborhood of $p$ cannot be contained in a sub-manifold of $\overline{M}\setminus\mathrm{sing}(\tilde{\mathcal{F}})$ of finite genus, it follows from $\overline{L(p)}\cap\mathrm{sing}(\tilde{\mathcal{F}})=\emptyset$ and from a similar argument used in compact manifolds that we can obtain the circle $\tilde{C}$ so that the wandering points set is open and dense in $\tilde{C}$. Besides that, $\mathcal{A}(\tilde{C}_j)\setminus\mathrm{Dom}(T_{\tilde{C}_j})$ is formed by a finite number of elements.

On the other hand, following similar arguments to those used in Theorems 4.3 and 4.8, it can be proved that there exists at most a countable set of circles $\{\tilde{C}_i\}_{i\in\Delta_4}$ as defined above, so that $\tilde{C}_j\subset\overline{M}\setminus\overline{\bigcup_{i\in\Delta_4}\mathrm{Sat}(C_i)}$ for all $j\in\Delta_4$, $\mathrm{Sat}(\tilde{C}_j)\cap\mathrm{Sat}(\tilde{C}_k)=\emptyset$ for all $j\neq k$ and any non-trivial recurrent leaf disjoint of $\overline{\bigcup_{i\in\Delta_3}\mathrm{Sat}(C_i)}$ is contained in $\overline{\bigcup_{j\in\Delta_4}\mathrm{Sat}(\tilde{C}_j)}$.

On the other hand, it is clear that if $\mathrm{Per}(\tilde{C}_j)$ is a set with non-empty interior ($\mathrm{int}(\mathrm{Per}(\tilde{C}_j))\neq\emptyset$), then $\mathrm{int}(\mathrm{Per}(\tilde{C}_j))$ is formed by a union of open sub-intervals of $\tilde{C}_j$, pairwise disjoint, so that in each sub-interval its points have equal $T$-period. If $I$ is one of these sub-intervals and $n_I>0$ is the smallest integer such that $T^{n_I}(I)=I$, then the boundary of $\{L(q);\,q\in I\}$ is a subset of the set formed by compact leaves, singularities and separatrices joining the singularities. Now, let $h:\tilde{C}_j\to\mathcal{A}(\tilde{C}_j)$ be the natural quotient map.

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The definitions for $h$ and $T_{\tilde{C}_j}$ imply that the following diagram:

$$\begin{array}{ccc} \tilde{C}_j & \xrightarrow{T} & \tilde{C}_j \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{A}(\tilde{C}_j) & \xrightarrow[T_{\tilde{C}_j}]{} & \mathcal{A}(\tilde{C}_j) \end{array}$$

is commutative. Thus, for each $j \in \Delta_4$, the forward Poincaré map on $\tilde{C}_j$ is semi-conjugate to $T_{\tilde{C}_j}$. We remark that if $I$ is a connected component of $\text{int}(\text{Per}(\tilde{C}_j))$ and $n_i$ is its $T$-period, then $T_{\tilde{C}_j}$ is not defined in $h(T^i(I)) = [x_i] \in \mathcal{A}(\tilde{C}_j)$ for some $0 < i \leqslant n_I$. Therefore, from the diagram above follows the assertion that $\mathcal{A}(\tilde{C}_j) \setminus \text{Dom}(T_{\tilde{C}_j})$ is a compact and totally disconnected set. Since $T_{\tilde{C}_j}$ admits a dense subset of non-trivial recurrent points, then the closure of any non-trivial recurrent leaf $\overline{L(p)}$ contains a continuous set of non-trivial recurrent leaves which are dense in $\overline{L(p)}$ (see Theorem 4.4), thus the proof is complete. □

As an immediate consequence of Theorems 4.3, 4.8 and 4.10, we have the following corollary.

COROLLARY 4.11. *If $L(p)$ is a non-trivial recurrent leaf of $\tilde{\mathcal{F}}$ passing through $p$ and does not intersect any of the circles $C_i$ obtained in Theorems 4.3, 4.8 and 4.10, then $L(p)$ necessarily is an exceptional leaf and $p$ is not of finite type. Moreover, through $p$ there exists a section $\Sigma(p)$ transversal to $\tilde{\mathcal{F}}$ so that either the set $\Upsilon(\Sigma(p))$ formed by the recurrent points in $\Sigma(p)$ belonging to leaves intersecting some of these circles $C_i$ is dense in $\Sigma(p)$ or the wandering points together with the set* $\text{int}(P(\Sigma(p)))$ *is open and dense in* $\Sigma(p) \setminus \Upsilon(\Sigma(p))$.

## 5. *Proof of the main theorem*

From Theorems 4.3, 4.8, 4.10 and Corollary 4.9 we only need to prove that when the circle $C$ is given by Theorem 4.8 or satisfies (4) of Theorem 4.10 (respectively (2) of Theorem 4.10), the mapping $T_C$ is conjugate to an affine GIET (respectively to an IET). Notice that when $C$ satisfies item (2) of Theorem 4.10, the assertion follows from of [**G2**, Lemma 5]. Therefore, only the two first cases are left to be proved. This will be a consequence of the following fundamental proposition.

PROPOSITION 5.1. *Let $C \in \{S^1, \mathbb{R}/\mathbb{Z}\}$ and let $T : C \to C$ be a continuous injective map defined everywhere except in a compact, totally disconnected set of points. If $T$ admits a dense subset of non-trivial recurrent points, then $T$ is conjugate to an affine GIET.*

Let us divide the proof into a sequence of lemmas and claims.

CLAIM 5.2. *If $T : C \to C$ is as in the previous proposition, then $C$ admits a dense subset of recurrent points such that each one contains in its $\omega$- and $\alpha$-limit set, points where $T$ is not defined.*

In fact, it is clear that we only need to prove that in $C$ there exists a dense subset of recurrent points such that each one contains in its $\omega$- or $\alpha$-limit set points where $T$ is not defined.

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Similarly to the proof in item (2) of Theorem 4.10, if we suppose that this last assertion is false, then we can prove that in $C$ there exists a compact minimal set and wandering points, which is impossible under the conditions in $T$. Thus the claim follows.

Note that, given a compact totally disconnected subset $K$ of $C$, an easy computation shows that for each value $\varepsilon > 0$, $K$ admits a finite covering by open intervals, pairwise disjoint, such that each interval has a length shorter than $\varepsilon$.

Let $\{\varepsilon_n : n = 1, 2, \dots\}$ be a decreasing sequence in the interval $(0, 1)$ converging to 0. Let $K$ be the set of points where the map $T$ is not defined. For each $n$, let $t_n$ be a positive integer such that $\{\mathcal{U}_i(\varepsilon_n)\}_{i=1}^{t_n}$ denotes a finite covering of $K$ formed by open intervals in $C$, pairwise disjoint, of length shorter than $\varepsilon_n$.

Here and subsequently each iterate finite $T^i(p); i = m_p, m_p + 1, \dots, n_p - 1, n_p$ of any point $p \in C$ will be called a *truncate orbit* of $T$.

Considering the previous notations, for each $n \in \mathbb{Z}^+ \setminus \{0\}$ we will define a set $\Theta(n)$ with the following properties:

(i) $\Theta(n)$ is made of a finite set of truncate orbits of $T$;
(ii) if $q \in \Theta(n)$ and $T(q) \notin \Theta(n)$, then $q \in \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n)$;
(iii) if $q \in \Theta(n)$ and $T^{-1}(q) \notin \Theta(n)$, then $T^{-1}(q) \in \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n)$;
(iv) $\Theta(n) \subsetneq \Theta(n+1)$ for all $n$ and $\bigcup_{n=1}^{\infty} \Theta(n)$ is dense in $C$.

There are only two cases to consider: either there is a dense orbit in $C$ or not. Consider the first case. Let $p$ be a point in $C \setminus \bigcup_{i=1}^{t_1} \mathcal{U}_i(\varepsilon_1)$ such that the orbit of $T$ through $p$ is dense in $C$. For each $n \in \mathbb{Z}^+ \setminus \{0\}$, fix the smallest positive integers $m_n, n_n$ such that $m_n > m_{n-1}$, $n_n > n_{n-1}$ (where $m_0 = n_0 = 0$), and the set $\Theta(n) = \{T^i(p) : i = -m_n, -m_n + 1, \dots, n_n - 1, n_n\}$ satisfies the properties (i)–(iv) and besides intersects the interior of each connected component of $C \setminus \bigcup_{i=1}^{t_1} \mathcal{U}_i(\varepsilon_1)$. When there is not a dense orbit of $T$, $\Theta(n)$ will be defined proceeding inductively on $n$ as follow.

Let $n = 1$. Of course, $C \setminus \bigcup_{i=1}^{t_1} \mathcal{U}_i(\varepsilon_1)$ is a finite union of closed intervals, pairwise disjoint, such that each component is contained in the domain of continuity of $T$; i.e. there are finitely many intervals $J_k^{(1)} \subset \mathrm{Dom}(T)$ so that $\bigcup_{k=1}^{k_1} J_k^{(1)} = C \setminus \bigcup_{i=1}^{t_1} \mathcal{U}_i(\varepsilon_1)$ for some integer $k_1 > 0$. For each $k = 1, 2, \dots, k_1$, split $J_k^{(1)}$ into a union of $s_1 = 2$ intervals of equal lengths. Let us denote these intervals by $J_{k,s}^{(1)}$, $s = 1, \dots, s_1$. It is clear from the conditions on $T$ that, for each $k \in \{1, \dots, k_1\}$ and each $s \in \{1, \dots, s_1\}$, we can take a $T$-recurrent point $p_{k,s}^{(1)}$ in the interior of $J_{k,s}^{(1)}$ so that for each $k \in \{1, 2, \dots, k_1\}$, $O(p_{k,r}^{(1)}) \cap O(p_{k,t}^{(1)}) = \emptyset$ for all $r \neq t$ with $r, t \in \{1, 2, \dots s_1\}$, and $O^+(p_{k,s}^{(1)}) \cap \bigcup_{i=1}^{t_1} \mathcal{U}_i(\varepsilon_1) \neq \emptyset$, and $O^-(p_{k,s}^{(1)}) \cap \bigcup_{i=1}^{t_1} \mathcal{U}_i(\varepsilon_1) \neq \emptyset$, for all $s \in \{1, 2, \dots, s_1\}$ (see Claim 5.2).

Let $n_1^+ := n_1^+(p_{k,s}^{(1)})$, $n_1^- := n_1^-(p_{k,s}^{(1)})$ be positive integers such that

$$n_1^+ = \min_{r \geq 1} \left\{ \{T^i(p_{k,s}^{(1)})\}_{i=1}^r \cap J_{k,s}^{(1)} \neq \emptyset \text{ and } T^r(p_{k,s}^{(1)}) \in \bigcup_{i=1}^{t_1} \mathcal{U}_i(\varepsilon_1) \right\},$$

$$n_1^- = \min_{r \geq 1} \left\{ \{T^{-i}(p_{k,s}^{(1)})\}_{i=1}^r \cap J_{k,s}^{(1)} \neq \emptyset \text{ and } T^{-r}(p_{k,s}^{(1)}) \in \bigcup_{i=1}^{t_1} \mathcal{U}_i(\varepsilon_1) \right\}.$$

With these notations, we define the set $\Theta(1)$ as follows:

$$\Theta(1) = \{T^i(p_{k,s}^{(1)}) : k = 1, \dots, k_1, s = 1, \dots, s_1, i = -n_1^- + 1, \dots, n_1^+\}.$$

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Let us supposedly define the set $\Theta(n-1)$ for $n \geq 2$. To define $\Theta(n)$, the following two steps will be taken.

*First step.* For $\varepsilon_n$ fixed above, let

$$\widetilde{\varepsilon}_n = \min_{x \neq y} \left\{ d(x, y) : x \in \Theta(n-1) \text{ and } y \in \Theta(n-1) \cup \bigcup_{i=1}^{t_n} \partial\{\mathcal{U}_i(\varepsilon_n)\} \right\},$$

where $d(\cdot, \cdot)$ is the usual distance in $C$ and $\partial\{\cdot\}$ denotes the boundary (endpoints) of the set $\{\cdot\}$.

Now, for each $n$, let us set the following notations.

- $\widetilde{J}^n(q)$ will denote the intervals of radius shorter than $\widetilde{\varepsilon}_n/2$, centered in the point $q \in \Theta(n-1)$. Notice that these intervals are pairwise disjoint.
- $J_k^n$; $k = 1, 2, \ldots, k_n$ will be the connected components of $C \setminus \big[ \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n) \cup \{\widetilde{J}^n(q) : q \in \Theta(n-1)\}\big]$.

Similar to the case $n = 1$, for all $n \geq 2$, let us split up each interval $J_k^n$ into a finite union of $s_n = 2^n$ intervals of equal lengths. Let us denote these intervals by $J_{k,s}^{(n)}$, $s = 1, 2, \ldots, s_n$.

*Second step.* From conditions on $T$, for each $k \in \{1, \ldots, k_n\}$ and $s \in \{1, \ldots, s_n\}$, let us take points $p_{k,s}^{(n)} \in \operatorname{int}(J_{k,s}^{(n)})$ so that the $T$-orbit of $p_{k,s}^{(n)}$ is recurrent with $O(p_{k,r}^{(n)}) \cap O(p_{k,t}^{(n)}) = \emptyset$ for all $t \neq r$, $O^+(p_{k,s}^{(n)}) \cap \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n) \neq \emptyset$ and $O^-(p_{k,s}^{(n)}) \cap \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n) \neq \emptyset$ (see Claim 5.2). On the other hand, for each $k \in \{1, \ldots, k_n\}$ and $s \in \{1, \ldots, s_n\}$, $n_n^+ := n_n^+(p_{k,s}^{(n)})$ and $n_n^- := n_n^-(p_{k,s}^{(n)})$ are defined as follows:

$$n_n^+ = \min_{r \geq 1} \left\{ \{T^i(p_{k,s}^{(n)})\}_{i=1}^r \cap J_{k,s}^{(n)} \neq \emptyset \text{ and } T^r(p_{k,s}^{(n)}) \in \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n) \right\},$$

$$n_n^- = \min_{r \geq 1} \left\{ \{T^{-i}(p_{k,s}^{(n)})\}_{i=1}^r \cap J_{k,s}^{(n)} \neq \emptyset \text{ and } T^{-r}(p_{k,s}^{(n)}) \in \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n) \right\}.$$

For construction, $\Theta(n-1)$ is made up of a finite number of truncate orbits of $T$ which are generated by $p_{k,s}^{(n-1)}$ and $p_{k,s}^{(n-2)}$, $k = 1, 2, \ldots, k_{n-1}$ and $s = 1, 2, \ldots, s_{n-1}$ (here the truncate orbit of point $p_{k,s}^{(0)}$ is the empty set). Thus, for each $q \in \{p_{k,s}^{(n-1)}, p_{k,s}^{(n-2)}\}$, let $j_n^+(q)$ (respectively $j_n^-(q)$) be the least positive index $m$ so that it satisfies the following properties:

- $\{T^i(q)\}_{i=1}^m \cap \tilde{J}^n(q) \neq \emptyset$ (respectively $\{T^{-i}(q)\}_{i=1}^m \cap \tilde{J}^n(q) \neq \emptyset$);
- $T^m(q) \in \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n)$ (respectively $T^{-m}(q) \in \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n)$ );
- $\{T^i(q)\}_{i=0}^m$ (respectively $\{T^{-i}(q)\}_{i=0}^m$), intersects $\tilde{J}^n(x) \setminus \{x\}$ for each $x \in O(q) \cap \Theta(n-1)$.

Finally, let $\Sigma(n)$ be the set formed by the union of truncate orbit $T^j(q)$; $j = -j_n^-(q), \ldots, j_n^+(q)$ such that $q \in \{p_{k,s}^{(n-1)}, p_{k,s}^{(n-2)}\}$, where $k \in \{1, 2, \ldots, k_{n-1}\}$ and $s \in \{1, 2, \ldots, s_{n-1}\}$. The set $\Theta(n)$ is defined as:

$$\Theta(n) = \Sigma(n) \cup \{T^j(p_{k,s}^{(n)}) : j = -n_n^- + 1, \ldots, n_n^+; k = 1, \ldots, k_n; s = 1, \ldots, s_n\}.$$

For construction, $\Theta(n) \subset \Theta(n+1)$ for all $n \in \mathbb{Z}^+ \setminus \{0\}$. On the other hand, since $s_n \to +\infty$ and $\varepsilon_n \to 0$ (therefore $\tilde{\varepsilon}_n \to 0$) as $n \to +\infty$, then one can conclude that $\bigcup_{n=1} \Theta(n)$ is a dense subset of $C$. Therefore, the definition of $\Theta(n)$ is complete.

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*An infinite partition $\mathcal{P}$ on $C$.* We will begin defining, by recurrence on $n$, a sequence $\{\mathcal{P}(n) : n = 1, 2, \dots\}$ of finite partitions of $C$. Given $n \in \mathbb{Z}^+ \setminus \{0\}$, for each $x \in \bigcup_{i=1}^{t_{n+1}} \partial\{\mathcal{U}_i(\varepsilon_{n+1})\}$, the point in $\left\{\Theta(j) \cap \left(C \setminus \bigcup_{i=1}^{t_{n+1}} \mathcal{U}_i(\varepsilon_{n+1})\right)\right\}$ nearest to $x$ will be denoted by $y_x^j$. Let $\mathcal{P}(1) = \left\{y_x^1 : x \in \bigcup_{i=1}^{t_1} \partial\{\mathcal{U}_i(\varepsilon_1)\}\right\}$. For all $n \in \mathbb{Z}^+ \setminus \{0\}$, we will define $\mathcal{P}(n+1)$ as being the union of $\mathcal{P}(n)$ with the set $\left\{y_x^j : x \in \bigcup_{i=1}^{t_{n+1}} \partial\{\mathcal{U}_i(\varepsilon_{n+1})\}$ and $1 \leq j \leq n+1\right\}$.

*Convention.* Given a finite partition $\mathcal{Q} = \{x_0, x_1, \dots, x_n\}$ of $C$, we will say that $A$ is an *interval determined by the partition* $\mathcal{Q}$ if, and only if, $A$ is an interval with endpoints $x_i, x_{i+1}$ belonging to $\mathcal{Q}$ such that $A \cap \mathcal{Q} = \{x_i, x_{i+1}\}$. Thus, a partition is characterized by the intervals that it determines.

Notice that for each $n \in \mathbb{Z}^+$, $\mathcal{P}(n+1)$ is a refinement of $\mathcal{P}(n)$. Moreover, if $A$ is an interval determined by the partition $\mathcal{P}(n+1)$ but is not determined by $\mathcal{P}(n)$ (therefore $A$ is determined by $\mathcal{P}(k)$ for all $k \geq n+1$), then necessarily $A$ is contained in some interval determined by $\mathcal{P}(n)$ which contains points where $T$ is not defined.

The *infinite partition* $\mathcal{P}$ in $C$ is defined as follows. A sub-interval of $C$ will be an interval determined by $\mathcal{P}$ if, and only if, for some $n_0$ it is an interval determined by the partition $\mathcal{P}(n_0)$ that does not contain points where $T$ is not defined.

Notice that each intervals determined by $\mathcal{P}$ is also an interval determined by $\mathcal{P}(n)$ for some integers $n$ sufficiently large, and its endpoints accumulates in the set $C \setminus \mathrm{Dom}(T)$. Moreover, we have the following.

CLAIM 5.3. *If $[a, b) \subset A$, where $A$ is an interval determined by the infinite partition $\mathcal{P}$ of $C$ and $n_0$ is the least positive integer such that $A$ is determined by $\mathcal{P}(n_0)$, then $\#\{\Theta(n) \cap [a, b)\} = \#\{\Theta(n) \cap T([a, b))\}$ for all $n \geq n_0$.*

In fact, let $n_0$ be the least positive integer such that $A$ is fixed by $\mathcal{P}(n_0)$. As $[a, b)$ is contained in the domain of continuity of $T$, we only consider the case in which $T$ preserves orientation on the component of $\mathrm{Dom}(T)$ containing $[a, b)$, thus $T([a, b)) = [T(a), T(b))$. Therefore, if $x \in \Theta(n) \cap [a, b)$ for $n \geq n_0$, then $T(x) \in \Theta(n) \cap [T(a), T(b))$ if, and only if, $x$ is none of the last elements of the truncate orbits of $T$ that define $\Theta(n)$. By the construction of $\mathcal{P}(n)$ and $\Theta(n)$, these elements do not belong to the intervals determined by $\mathcal{P}(n)$ included in a component of continuity of $T$. Therefore, as $\Theta(n)$ is made up of a finite number of truncate orbits of $T$, it follows from the condition on $[a, b)$ that $\#\{\Theta(n) \cap [a, b)\} \leq \#\{\Theta(n) \cap [T(a), T(b))\}$ for all $n \geq n_0$. Similarly, it is verified that $\#\{\Theta(n) \cap [T(a), T(b))\} \leq \#\{\Theta(n) \cap [a, b)\}$ for all $n \geq n_0$. Therefore, the claim is proved.

Given $n \in \mathbb{Z}^+$, let $\Theta(n)$ be as above. Let us now define a sequence $\mu_n$, $n = 1, 2, \dots$ of measures of probability on $C$ such that for each $n$, $\mu_n$ is supported on $\Theta(n)$.

- For each $p \in \Theta(1)$, define $\mu_1(\{p\}) = 1/m_1$, where $m_1$ is the number of elements of $\Theta(1)$.
- For $n \geq 2$, the measure of probability is defined as follows. For each $s \in \Theta(n)$, we define $\mu_n(\{s\}) = \mu_{n-1}([p, q))/m_n$, where $[p, q)$ is the smallest sub-interval of $C$ containing $s$ such that $p, q \in \Theta(n-1)$, and $m_n$ is the number of elements of $\Theta(n) \cap [p, q)$.

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As $C$ is compact, the sequence of probability $\{\mu_n : n = 1, 2, \dots\}$ admits a subsequence, which for simplicity will continue to be denoted as $\{\mu_n : n = 1, 2, \dots\}$, and which converges in the weak topology to a probability $\mu$; i.e.

$$\int_C \varphi \, d\mu_n \to \int_C \varphi \, d\mu, \quad \text{for all } \varphi \in \mathrm{C}^0(C).$$

Notice that, as $C$ is a space metric, $\mu$ and $\mu_n$ will be regular measures.

LEMMA 5.4. *Let $\{\mu_n\}_{n\geq 1}$ be the sequence of measures defined above. Then we have that:*

(1) *for each $p \in \bigcup_{n=1}^{\infty} \Theta(n)$, $\lim_{n\to\infty} \mu_n(\{p\}) = 0$.*

*If $[p, q)$ is a sub-interval of $C$ with $p, q \in \bigcup_{n=1}^{\infty} \Theta(n)$, then there exists $n_0 \in \mathbb{Z}^+$ such that:*

(2) *$\mu_n([p, q)) = \mu_{n_0}([p, q))$ for all $n \geq n_0$;*

(3) *$\mu([p, q)) = \mu_{n_0}([p, q))$, and $\mu$ is positive in every open set of $C$.*

*Proof.* Let $s_0$ be the least positive integer such that $p \in \Theta(s_0)$. An easy computation shows that

$$\mu_{s_0+k}(\{p\}) = \frac{\mu_{s_0+k-1}([p, q_k))}{m_{s_0+k}} = \frac{\mu_{s_0}(\{p\})}{(m_{s_0+1})(m_{s_0+2}) \cdots (m_{s_0+k})}$$

for all $k \geq 1$, where $q_k \in \Theta(s_0 + k - 1)$ is such that $[p, q_k) \cap \Theta(s_0 + k - 1) = \{p\}$. Here, $m_{s_0+k}$ denotes the number of elements of the set $[p, q_k) \cap \Theta(s_0 + k)$. Therefore, (1) follows since $m_{s_0+k} \geq 2$ for all $k \geq 1$. Now, let $n_0$ be the smallest positive integer such that $p, q \in \Theta(n_0)$. Assume that $p, q \in \Theta(n_0) \setminus \Theta(n_0 - 1)$ with $[p, q) \cap \Theta(j) = \emptyset$ for all $1 \leq j \leq n_0 - 1$. In this case, it is proved by induction on $n$ that $\mu_n([p, q)) = \mu_{n_0}([p, q)) = \#\{\Theta(n_0) \cap [p, q)\}\mu_{n_0}(\{p\})$, for all $n \geq n_0$. Now suppose a second case; i.e. $q \in \Theta(n_0)$, $[p, q) \cap \Theta(n_0) = \{p\}$ and $p \in \Theta(n_0 - 1)$ with $p \notin \Theta(j)$ for all $1 \leq j \leq n_0 - 2$. In this case, it is easy to see that it follows from the definition of $\mu_n$ that for all $n \geq n_0$, $\mu_n([p, q)) = \mu_{n_0}([p, r))/m_0$, where $[p, r)$ is the smallest interval containing $[p, q)$ with $r \in \Theta(n_0 - 1)$, and $m_0$ is the number of elements of $\Theta(n_0) \cap [p, r)$. In order to prove (2) in the general case, it is sufficient to split the interval $[p, q)$ into a finite union of sub-intervals that satisfies the previous two cases. Now we will prove (3). As $\mu_n$ converges to $\mu$ in the weak topology, for the open $(p, q)$ and closed $[a, b]$ sub-intervals of $C$, we have that $\mu((p, q)) \leq \liminf_{n\to\infty} \mu_n((p, q))$ and $\limsup_{n\to\infty} \mu_n([p, q]) \leq \mu([p, q])$. On the other hand, from the definition of $\mu_n$, it follows that the bounded sequences $\{\mu_n((p, q)) : n = 1, 2, \dots\}$ and $\{\mu_n([p, q]) : n = 1, 2, \dots\}$ are increasing and decreasing, respectively. Thus, we have that

$$\mu((p, q)) \leq \lim_{n\to\infty} \mu_n((p, q)) \quad \text{and} \quad \lim_{n\to\infty} \mu_n([p, q]) \leq \mu([p, q]). \tag{§}$$

On the other hand, let $\{(p_j, q_j) : j = 1, 2, \dots\}$ be a sequence of open sub-intervals of $C$ with endpoints in $\bigcup_{n=1} \Theta(n)$ such that $[p, q] \subset (p_{j+1}, q_{j+1}) \subset (p_j, q_j)$ for all $j$, and $\bigcap_{j=1}(p_j, q_j) = [p, q]$. As $\mu$ is a regular measure, we have that $\mu([p, q]) = \lim_{j\to\infty} \mu((p_j, q_j))$. An easy computation shows that this last equality, together with the first inequality of (§), items (1) and (2) of the lemma, and the fact that $\mu_n$ is also a regular measure, implies that there exists $n_0 \in \mathbb{Z}^+$ such that $\mu([p, q]) \leq \lim_{j,n\to\infty} \mu_n((p_j, q_j)) = \mu_{n_0}([p, q))$. On the other hand, from the second inequality

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of (§), and items (1) and (2) of the lemma, we have that $\mu_{n_0}([p,q)) \leq \mu([p,q])$. Thus, $\mu([p,q]) = \mu_{n_0}([p,q))$. Considering an increasing sequence $\{[p_j,q_j] : j = 1,2,\dots\}$ of closed sub-intervals in $C$ with endpoints in $\bigcup_{n=1}\Theta(n)$ in such a way that $[p_j,q_j] \subset [p_{j+1},q_{j+1}] \subset [p,q]$ for all $j$ and $\bigcup_{j=1}[p_j,q_j] = (p,q)$, a similar argument shows that for large enough $n_0$, $\mu((p,q)) = \mu_{n_0}([p,q))$. Therefore, this last equality and the previous one prove (3); thus the proof is complete. □

LEMMA 5.5. *Let $T : C \to C$ be as in Proposition 5.1; then on $C$ there exists an infinite partition $\mathcal{P}$ and a measure of probability $\mu$ positive in open sets such that for each interval $A$ determined by $\mathcal{P}$ there exists a constant $\lambda_A > 0$ with the following property: for each sub-interval $[p,q)$ of interval $A$ we have that*

$$\mu(T([p,q))) = \lambda_A \mu([p,q)).$$

*Proof.* Let $\mathcal{P}$ be the infinite partition on $C$ obtained above. As before, we only consider the case when $T|_A$ is increasing (the same reasoning applies to the other case). Let $j \in \mathbb{Z}^+$ be the smallest integer such that the interval $A$ is fixed by $\Theta(j)$. From the density of $\bigcup_{n=1}\Theta(n)$ in $C$ and continuity of $T$ in $A$, we can assume that $p,q \in \bigcup_{n=1}\Theta(n)$. First, the following particular cases will be considered.

*Particular cases.* $q \in \Theta(i)$, $p \in \Theta(i+1) \setminus \Theta(i)$ and $[p,q] \cap \Theta(i) = \{q\}$ or $p \in \Theta(i)$, $q \in \Theta(i+1) \setminus \Theta(i)$ and $[p,q] \cap \Theta(i) = \{p\}$.

We remark that, from construction, necessarily $i \geq j$. Now, let us prove the lemma in the first particular case. The other is similar.

(1) Let $I_{q,i}$ be the smallest interval in $C$ containing $[p,q)$ closed to the left and open to the right, such that it has $q$ and some element of $\Theta(i)$ as endpoints.

(2) Let $\alpha(q,i)$ (respectively $\tilde{\alpha}(\overline{pq},i)$) be the number of elements of $\Theta(i+1) \cap I_{q,i}$ (respectively of $\Theta(i+1) \cap [p,q)$).

From Claim 5.3, we have the following.

(3) The number of elements of $\Theta(i+1) \cap [T(p),T(q))$ is $\tilde{\alpha}(\overline{pq},i)$.

(4) Let $I_{T(q),i}$ be the smallest sub-interval of $C$ closed to the left and open to the right, containing the interval $[T(p),T(q))$, such that it has as endpoints $T(q)$ and an element of $\Theta(i)$.

(5) Let $\alpha(T(q),i)$ denote the number of elements of $\Theta(i+1) \cap I_{T(q),i}$. Let $\{\mu_k : k = 1,2,\dots\}$ be the sequence of probability in $C$ defined above such that it converges in the weak topology to the probability $\mu$. From (3) of Lemma 5.4, we have that there exists an integer $k_0$ (which can be assumed to be greater than $i+1$) such that $\mu(I_{q,i}) = \mu_k(I_{q,i}) = \mu_{k_0}(I_{q,i})$ and $\mu(I_{T(q),i}) = \mu_k(I_{T(q),i}) = \mu_{k_0}(I_{T(q),i})$, for all $k \geq k_0$.

From (1), construction of $\mu_k$ and the conditions on the particular case we have the following.

(6) $\mu([p,q)) = \mu_{k_0}([p,q)) = (\mu(I_{q,i})/\alpha(q,i))\tilde{\alpha}(\overline{pq},i)$.

Notice that $\mu(I_{q,i})/\alpha(q,i)$ is the $\mu_{k_0}$-measure of each element of $\Theta(i+1) \cap I_{q,i}$. From (3), (5) and similar considerations we have the following.

(7) $\mu([T(p),T(q))) = \mu_{k_0}([T(p),T(q))) = (\mu(I_{T(q),i})/\alpha(T(q),i))\tilde{\alpha}(\overline{pq},i)$.

On the other hand, in this particular case, there are only two possibilities: either $I_{q,i} \subset A$ or $I_{q,i} \supset A$. If $I_{q,i} \subset A$, then it can be proved that $\mu(I_{q,i}) = [\mu(A)/(\delta_j\delta_{j+1}\cdots\delta_i\delta_{i+1})]\eta_{i+1}$,

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$\mu(I_{T(q),i}) = [\mu(T(A))/(\delta_j\delta_{j+1}\cdots\delta_i\delta_{i+1})]\eta_{i+1}$, and $\alpha(q,i) = \alpha(T(q),i)$, where $\eta_{i+1} = \#\{\Theta(i+1)\cap[p,q)\}$, $\delta_j = \#\{\Theta(j)\cap A\}$ and for all $k \in \{j+1,\ldots,i+1\}$, $\delta_k$ denotes the number of elements of $\Theta(k)$ in the smallest interval containing $[p,q)$, closed to the left and open to the right, with endpoints belonging to $\Theta(k-1)$. On the other hand, it is easy to see that if $I_{q,i} \supset A$, then $\mu(A) = (\mu(I_{q,i})/\alpha(q,i))\#(\Theta(i+1)\cap A)$ and $\mu(T(A)) = (\mu(I_{T(q),i})/\alpha(T(q),i))\#(\Theta(i+1)\cap A)$. Therefore, in both cases, from (6) and (7) we have the following.

(8) For all intervals $[p,q)$, as in the particular case,

$$\mu([T(p),T(q))) = (\mu(T(A))/\mu(A))\mu([p,q)).$$

Now we will prove the lemma in the general case. As above, let $j$ be the smallest positive integer such that $A$ is fixed by $\Theta(j)$ and let $i$ be the smallest positive integer such that $p,q \in \Theta(i)$ (notice that by construction of $\Theta(n)$, necessarily $i \geq j$). It follows from the definition of the infinite partition $\mathcal{P}$ that the endpoints of $A$ belong to $\Theta(j)$ or one of them belongs to $\Theta(j-1)$ and the other one to $\Theta(j)$. For simplicity, suppose that the left endpoint belongs to $\Theta(j)$ and the right endpoint to $\Theta(j-1)$, the other cases being similar. Let $l_j$ be the smallest positive integer with $j \leq l_j \leq i$ such that $[p,q)\cap\Theta(l_j) \neq \emptyset$. Split the interval $[p,q)$ into the union of intervals determined by $\Theta(l_j)$; i.e. we have (9).

(9)
$$\mu([p,q)) = \mu([p,p_{l_j}^1)) + \sum_{k=1}^{\tilde{\alpha_1}(l_j)-1} \mu([p_{l_j}^k, p_{l_j}^{k+1})) + \mu([p_{l_j}^{\tilde{\alpha}_1(l_j)}, q)),$$

where $p_{l_j}^k \in \Theta(l_j)$, for all $k \in \{1,\ldots,\tilde{\alpha_1}(l_j)\}$ and

(10) $\tilde{\alpha_1}(l_j)$ is the number of elements of $\Theta(l_j)\cap[p,q)$.

From (9) and (10) we have the following.

(11)
$$\mu([T(p),T(q))) = \mu([T(p),T(p_{l_j}^1))) + \sum_{k=1}^{\tilde{\alpha}_1(l_j)-1} \mu([T(p_{l_j}^k),T(p_{l_j}^{k+1}))) + \mu([T(p_{l_j}^{\tilde{\alpha}_1(l_j)}),T(q))).$$

On the other hand, let us proceed as in the particular case.

(12) Let $I_{\overline{pq},l_j}$ (respectively $I_{T(\overline{pq}),l_j}$) be the smallest sub-interval of $C$ containing $[p,q)$ (respectively $[T(p),T(q))$) such that it is closed to the left and open to the right with endpoint elements in $\Theta(l_j)$.

(13) From Claim 5.3, the number of elements of $\Theta(l_j)\cap I_{\overline{pq},l_j}$ and $\Theta(l_j)\cap T(I_{\overline{pq},l_j})$ is $\tilde{\alpha}_1(l_j)+1$.

From Lemma 5.4 there exists $k_0 \in \mathbb{Z}^+$ (which can be assumed to be larger than $i+1$) so that for all $k \geq k_0$:

$\mu(I_{\overline{pq},l_j}) = \mu_k(I_{\overline{pq},l_j})$ and

$\mu(T(I_{\overline{pq},l_j})) = \mu_k(I_{T(\overline{pq}),l_j})$.

In a similar way (and with the same notations) as in (7)

$\mu(I_{\overline{pq},l_j}) = [\mu(A)/\delta_j\delta_{j+1}\cdots\delta_{l_j}](\tilde{\alpha}_1(l_j)+1)$ and

$\mu((I_{T(\overline{pq}),l_j})) = [\mu(T(A))/\delta_j\delta_{j+1}\cdots\delta_{l_j}](\tilde{\alpha}_1(l_j)+1)$.

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Considering $k_0$ large enough, from the construction of the probabilities $\mu_k$, for all $k \in \{1, \dots, \tilde{\alpha_1}(l_j)\}$,

$\mu([p_{l_j}^k, p_{l_j}^{k+1})) = \mu(I_{\overline{pq},l_j})/(\tilde{\alpha}_1(l_j)+1) = \mu(A)/\delta_j\delta_{j+1}\cdots\delta_{l_j}$,

$\mu([T(p_{l_j}^k), T(p_{l_j}^{k+1}))) = \mu((I_{T(\overline{pq}),l_j}))/(\tilde{\alpha}_1(l_j)+1) = \mu(T(A))/\delta_j\delta_{j+1}\cdots\delta_{l_j}$.

Replacing these latter equations in (9) and (11), we have that

$$\mu([p,q)) = \mu([p, p_{l_j+1}^1)) + (\mu(A)/\delta_j\delta_{j+1}\cdots\delta_i\delta_{l_j})(\tilde{\alpha_1}(l_j)-1) + \mu([p_{l_j}^{\tilde{\alpha}_1(l_j)}, q)),$$

$$\mu([T(p),T(q))) = \mu([T(p), T(p_{l_j+1}^1))) + (\mu(T(A))/\delta_j\delta_{j+1}\cdots\delta_{l_j})(\tilde{\alpha_1}(l_j)-1) + \mu([T(p_{l_j}^{\tilde{\alpha}_1(l_j)}), T(q))).$$

On the other hand, as $\Theta(n) \subset \Theta(n+1)$ for all $n \in \mathbb{Z}^+$, and each $\Theta(n)$ is made up of a finite number of elements, then the intervals $[p, p_{l_j+1}^1)$, $[p_{l_j}^{\tilde{\alpha}_1(i)}, q)$, and their respective images by $T$ can be split into a finite union of intervals that satisfies the particular case.

Therefore, in the general case it follows from (8) and the last two equalities that

$$\mu([T(p),T(q))) = (\mu(T(A))/\mu(A))\mu([p,q)),$$

thus proving the lemma. □

*Proof of Proposition 5.1.* Let $\mu$ be the probability in $C$ obtained in Lemma 5.5. For fixed $x_0 \in C$, define the map $g : C \to \mathbb{R}/\mathbb{Z}$ as $g(x) = \mu([x_0, x))$. As $\mu$ is positive in each open set, so $g$ is an increasing map. We claim that $g$ is a continuous map. Let us suppose the contrary. Let $x \in C$ be a discontinuity point of $g$ and let $\{r_n\}_{n=1}$ and $\{s_n\}_{n=1}$ be sequences of elements belonging to $\bigcup_n \Theta(n)$ converging to $x$ so that: (1) for each $n$, $r_n$ and $s_n$ belong to $\Theta(n) \setminus \Theta(n-1)$ (notice that this is always possible from the characteristics of the definition of $\Theta(n)$), (2) $[r_{n+1}, s_{n+1}) \subsetneq [r_n, s_n)$ with $\{x\} = \bigcap_n [r_n, s_n)$, and (3) $\Theta(n) \cap [r_n, s_n) = \{r_n\}$. It follows from the monotonicity of $g$ and the assumption of discontinuity in $x$ that given $\epsilon_0 > 0$, $|g(s_n) - g(r_n)| \geq \epsilon_0$ for all $n$ large enough. On the other hand, if $\{\mu_k : k = 1, 2, \dots\}$ is the sequence of probability that converges to $\mu$ (in the weak topology), we have that there are $k_n \in \mathbb{Z}^+$ so that $|g(s_n) - g(r_n)| = \mu([r_n, s_n)) = \mu_{k_n}([r_n, s_n))$. However, the choice of $r_n$ and $s_n$ implies that $\mu_{k_n+1}([r_{n+1}, s_{n+1})) = \mu_{k_n}([r_n, s_n))/m_n$, where $m_n > 2$ is the number of elements of $\Theta(n+1) \cap [r_n, s_n)$. It can be shown by induction on $n$ that there exists a positive integer $n_0$, such that $\mu_{k_n}([r_n, s_n)) = \mu_{k_{n_0}}([r_{n_0}, s_{n_0}))/(m_n m_{n-1}\cdots m_{n_0-1})$ for all integers $n \geq n_0$. But, in the limit, this equality contradicts the assertion $|g(s_n) - g(r_n)| \geq \epsilon_0$. Therefore, $g$ is a continuous and increasing map on $C$, so that it is a homeomorphism.

Now, let $[r, s)$ be an arbitrary interval in $C$ contained in an interval $A$ determined by the infinite partition $\mathcal{P}$. We only consider the case where map $T$ is increasing in the component of continuity that contains interval $A$ (the other case is similar). It follows from Lemma 5.5 that there is a constant $\lambda_A > 0$ such that $|(g \circ T)(s) - (g \circ T)(r)| = |\mu([T(r), T(s))| = \lambda_A \mu([r, s)) = \lambda_A |g(s) - g(r)|$. On the other hand, let $E = g \circ T \circ g^{-1} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, and let $g(\mathcal{P})$ be the infinite partition in $\mathbb{R}/\mathbb{Z}$ induced by $\mathcal{P}$ through map $g$. If $[\tilde{r}, \tilde{s})$ is an arbitrary interval contained in the interval $g(A)$ obtained from the partition $g(\mathcal{P})$, then it follows from the previous equality that $|E(\tilde{r}) - E(\tilde{s})| = \lambda_A |\tilde{r} - \tilde{s}|$. Therefore, the proof follows by deleting from $\mathbb{R}/\mathbb{Z}$ (respectively $C$) the points which determine the partition $g(\mathcal{P})$ (respectively $\mathcal{P}$) and its respective points of accumulation. □

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