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Describing quadratic Cremer point polynomials by parabolic perturbations

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Abstract. We describe two infinite-order parabolic perturbation procedures yielding quadratic polynomials having a Cremer fixed point. The main idea is to obtain the polynomial as the limit of repeated parabolic perturbations. The basic tool at each step is to control the behaviour of certain external rays.

Polynomials of the Cremer type correspond to parameters at the boundary of a hyperbolic component of the Mandelbrot set. In this paper we concentrate on the main cardioid component. We investigate the differences between two-sided (i.e. alternating) and one-sided parabolic perturbations.

In the *two-sided case*, we prove the existence of polynomials having an explicitly given external ray accumulating both at the Cremer point and at its non-periodic preimage. We think of the Julia set as containing a 'topologist's double comb'.

In the *one-sided case* we prove a weaker result: the existence of polynomials having an explicitly given external ray accumulating at the Cremer point, but having in the impression of the ray both the Cremer point and its other preimage. We think of the Julia set as containing a 'topologist's single comb'.

By tuning, similar results hold on the boundary of any hyperbolic component of the Mandelbrot set.

1. *Introduction*

This paper is concerned with the question of producing and describing some connected non-locally connected quadratic Julia sets of the Cremer type.

We do not yet have a complete understanding of the geometry and topology of quadratic Cremer type Julia sets. Primarily this is because we need local connectivity to be able to use the standard topological models for the filled Julia sets, e.g. the pinched disk model or the disked tree model; cf. [**Do**].

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Recently, however, the subject has been the object of a lot of work and partial results can be found, for instance, in [**GMO**], [**Ki**], [**PM1**], [**PM2**] and [**Sø2**]. This paper is a continuation of [**Sø2**] and is based on results first presented in [**Sø1**].

2. *Preliminaries*

In this section we quickly recall the tools and concepts introduced in [**Sø2**], and introduce the notation used throughout the paper. The tools and concepts are standard for the subject of 'holomorphic dynamics'. The basic references are [**DH**], [**Mi**], and [**Po**]. For further details see [**Sø2**].

2.1. *The Riemann map and external rays.* Let K be a connected compact full (i.e. $\mathbb{C} - K$ is connected) subset of $\mathbb{C}$ containing at the least 2 points. By the Riemann mapping theorem, there is a *unique* biholomorphic map $\varphi_K : \overline{\mathbb{C}} - K \xrightarrow{\simeq} \overline{\mathbb{C}} - \overline{\mathbb{D}}$, normalised by the conditions

$$\varphi_K(\infty) = \infty \quad \text{and} \quad \lim_{z\to\infty} \frac{\varphi_K(z)}{z} \in \mathbb{R}^+.$$

We denote its inverse by ψ_K. For each $\theta \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ we define the map

$$\psi_{K,\theta} : \mathbb{R}^+ \to \mathbb{C} \quad \text{by} \quad \psi_{K,\theta}(s) := \psi_K(e^{s+2\pi i\theta}),$$

where s will be called the *potential*. By an *external ray of external argument* θ we understand the curve

$$\mathcal{R}_K(\theta) := \psi_{K,\theta}(\mathbb{R}^+).$$

Note that we count angles in turns. If the limit

$$\lim_{s\to 0^+} \psi_{K,\theta}(s)$$

exists, then the ray is said to *land* at the limit point, which is then called the *landing point* and is denoted by $\gamma_K(\theta)$. Whenever it is well-defined, γ_K is called the *Carathéodory Loop*.

The set $\overline{\mathcal{R}_K(\theta)} - \mathcal{R}_K(\theta)$ is known as the *principal impression of the external ray* $\mathcal{R}_K(\theta)$, and is the natural generalisation of the notion of a 'landing point'. Clearly, the notions coincide whenever the external ray lands.

Generalising these notions further, we also consider the *impression of the external ray* $\mathcal{R}_K(\theta)$:

$$\bigcap_{\varepsilon>0} \left(\overline{\mathcal{R}_K\left(\overline{\omega_\varepsilon(\theta)}\right)} - \mathcal{R}_K\left(\overline{\omega_\varepsilon(\theta)}\right) \right),$$

where $\omega_\varepsilon(\theta) :=]\theta - \varepsilon, \theta + \varepsilon[/\mathbb{Z}$. A ray that has trivial impression (i.e. one point) is said to be of the *first kind*.

The classical Carathéodory theorem can be expressed so that it links the notion of impression to the question of local connectivity.

THEOREM 2.1. (Carathéodory) *K is locally connected if and only if all external rays are of the first kind.*

See, for example, [**Po**, §2]. However, the impression may be non-trivial, even when the external rays land, as the following example shows.

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Example 2.2. The double comb—a classical example. For $r \in]0,1]$ consider the union of the following line-segments in $\mathbb{C}$:

- $[0,1]$, $[0,i]$, $[i,1+i]$;
- $\left[\frac{1}{2^{2n-1}}, \frac{1}{2^{2n-1}} + i \cdot r\left(1 - \frac{1}{2^{2n}}\right)\right]$ for $n \in \mathbb{N}$; and
- $\left[\frac{1}{2^{2n}} + i, \frac{1}{2^{2n}} + i \cdot \left(1 - r\left(1 - \frac{1}{2^{2n+1}}\right)\right)\right]$ for $n \in \mathbb{N}_o$.

$r = 1$. The set is non-locally connected on $[0,i]$ and locally connected elsewhere. The sketched ray is the only one not being of the first kind. Its impression and principal impression coincides: $[0,i]$. Such a ray is said to be of the *third kind*.

$1/2 < r < 1$. The set is non-locally connected on the open segment $]0,i[$ and locally connected elsewhere. There is one ray which is not of the first kind. It is the sketched ray, which has as principal impression $[i \cdot (1-r), i \cdot r]$ and impression $[0,i]$ (see Figure 2.1). Rays like this are said to be of the *fourth kind*.

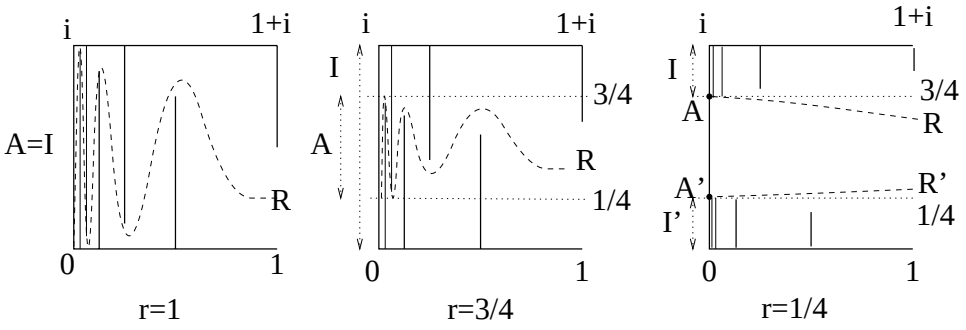


FIGURE 2.1. Sketches of a double comb for three different values of r. For simplicity, the external rays are denoted R and R', the corresponding principal impressions A and A', and the ditto impressions I and I'.

$0 < r \leq 1/2$. The set is non-locally connected on $]0, i \cdot r]$ and $[i \cdot (1-r), i[$; and locally connected elsewhere. There are two rays which are not of the first kind. These are the rays landing at $i \cdot r$ and $i \cdot (1-r)$. They have impressions $[0, i \cdot r]$ and $[i \cdot (1-r), i]$ respectively. Such rays are said to be of the *second kind*. The set can be considered as the union of *two topologist's combs*. For $r = 1/2$ the two second kind rays coalesce to a single ray.

Remark. It is classical and simple to check that all rays are of one of the four kinds just described; again see [**Po**].

2.2. *Quadratic polynomials with an indifferent fixed point.* The Mandelbrot set M is the set of parameters c such that the Julia set of the quadratic polynomial $z \mapsto z^2 + c$ is connected.

In the sequel we consider the family of quadratic polynomials having an indifferent

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fixed point:

$$P_t(z) := z^2 + \gamma_{H_o}(t), \quad t \in \mathbb{T},$$

where

$$\gamma_{H_o}(t) := \frac{e^{2\pi it}}{2}\left(1 - \frac{e^{2\pi it}}{2}\right), \quad t \in \mathbb{T},$$

is the parametrization by internal arguments of the *main cardioid* (the boundary of the main component H_o) of the Mandelbrot set.

Note that 0 is the *critical point* of P_t and that $c_t := \gamma_{H_o}(t)$ is the *critical value*. The least repelling fixed point is denoted by α_t, the other by β_t. The multiplier of α_t is $e^{2\pi it}$. So α_t is a parabolic fixed point whenever t is rational. All other periodic orbits are repelling.

The *Julia set* of P_t is denoted by J_t and the *filled Julia set* by K_t. The normalised Riemann map of $\mathbb{C} - K_t$ is denoted by φ_t and its inverse by ψ_t. Accordingly, we take as a convention that $\mathcal{R}_t$, respectively γ_t, are abbreviations for $\mathcal{R}_{K_t}$, respectively γ_{K_t}. It is well-known that K_t is of capacity radius 1, and that φ_t conjugates the action of P_t to the action of $z \mapsto z^2$. Furthermore, in the appropriate domain of definition the mapping $(z, P) \mapsto \psi_P(z)$ is continuous. This immediately yields the following.

PROPOSITION 2.3. *Let $s > 0$. Then $\Psi_s : [0,1]^2 \to \mathbb{C}$, given by*

$$\Psi_s : (t, \theta) \mapsto \psi_t(e^{s + i \cdot 2\pi\theta}),$$

is well-defined and continuous in both t and θ.

2.3. *Landing properties.* Much of what we do in the sequel is based on the following major theorem due to Douady and Hubbard which was introduced and discussed in detail in [**Sø2**].

THEOREM 2.4. (The cardioid landing theorem.) *For all rational $t \in]0,1[$, we have:*

(1) $\gamma_t(\theta^-(t)) = \gamma_t(\theta^+(t)) = \alpha_t (\in J_t)$;

(2) $\gamma_M(\theta^-(t)) = \gamma_M(\theta^+(t)) = \gamma_{H_o}(t)$ $(= c_t)$;

(3)

$$\theta^-(t) = \sum_{0<p/q<t} \frac{1}{2^q} < \theta^+(t) = \sum_{0<p/q\le t} \frac{1}{2^q}.$$

Note that we sum over *all pairs* (p, q), such that $0 < p/q < t$ (respectively $0 < p/q \le t$), whether (p, q) are relatively prime or not.

We let these formulas also serve as definitions for irrational $t \in [0,1]$, in which case the functions coincide allowing us to denote the argument simply by $\theta(t)$. In [**Sø2**] we proved the following elementary results.

PROPOSITION 2.5. *Consider $\theta^\pm : [0,1] \to [0,1]$.*

(1) *$\theta^\pm$ are strictly increasing functions.*

(2) *If $t < t'$ then $\theta^+(t) < \theta^-(t')$.*

(3) *Let $t_o \in [0,1]$ be any point. Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that:*

$$\forall t \in]t_o - \delta, t_o + \delta[\cap [0,1] : \theta^-(t_o) - \varepsilon < \theta^-(t) \le \theta^+(t) < \theta^+(t_o) + \varepsilon.$$

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PROPOSITION 2.6. *Let* $((t_n, \theta_n))_{n\in\mathbb{N}}$ *be any sequence in*

$$\{(t, \theta) \in [0, 1] \times [0, 1] \mid \theta = \theta^-(t) \text{ or } \theta = \theta^+(t)\}.$$

(1) *If* $t_n \nearrow t_\infty$ *strictly as* $n \to \infty$ *then* $\lim_{n\to\infty}(t_n, \theta_n) = (t_\infty, \theta^-(t_\infty))$.
(2) *If* $t_n \searrow t_\infty$ *strictly as* $n \to \infty$ *then* $\lim_{n\to\infty}(t_n, \theta_n) = (t_\infty, \theta^+(t_\infty))$.

LEMMA 2.7. *For* $t \in \mathbb{Q} \cap]0, 1[$ *:*

$$\gamma_t\left(\frac{\theta^-(t)}{2}\right) = -\alpha_t \quad \textit{and} \quad \gamma_t\left(\frac{\theta^+(t)}{2}\right) = \alpha_t.$$

Figure 2.2 illustrates the lemma.

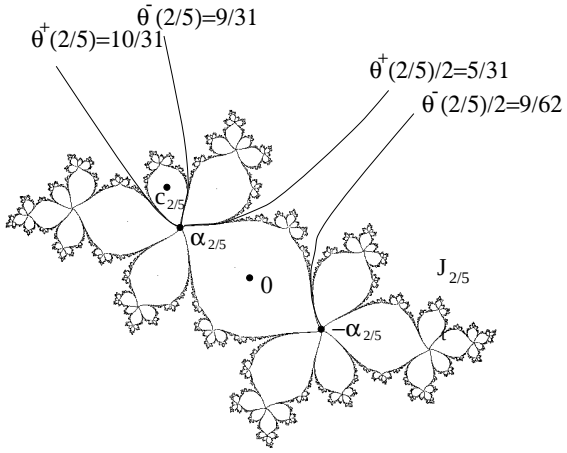


FIGURE 2.2. The external ray of argument $\theta^-(2/5)/2$, respectively $\theta^+(2/5)/2$, lands at $-\alpha_{2/5}$, respectively $\alpha_{2/5}$, in $J_{2/5}$.

2.4. *Continued fraction expansions.* Recall that a real number $x \in]0, 1]$ can be represented by its *(simple) continued fraction expansion*:

$$x = [a_1, a_2, a_3, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

where simple means that the *coefficients* are positive integers $a_i \in \mathbb{N}$. The number of coefficients is called its *length*. It is *infinite* if and only if x is *irrational*, in which case the expansion is *unique*. The number $x_n := [a_1, a_2, a_3, \ldots, a_n]$ is called the nth *convergent*. The convergents tend faster than exponentially to x, and they are ordered as follows:

$$0 < x_2 < x_4 < \cdots < x_{2m} < \cdots < x < \cdots < x_{2m+1} < \cdots < x_3 < x_1 \le 1. \qquad (2.1)$$

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We say that they form an (alternating) *two-sided* approximation to x. If we only consider the convergents on one side of x, e.g. $(x_{2n})_n$, we say that they form a *one-sided* approximation.

2.5. *Parabolic perturbations—definitions.*

First-order parabolic perturbations. Consider any parameter $\gamma_{H_o}(p/q)$ at the main cardioid ∂H_o with rational internal argument. By a *first-order parabolic perturbation* of $\gamma_{H_o}(p/q)$ we will understand a perturbation of this parameter to a parameter of one of the four kinds marked (a)–(d) on Figure 2.3. We do not treat perturbations of kinds (c) and (d) in this paper.

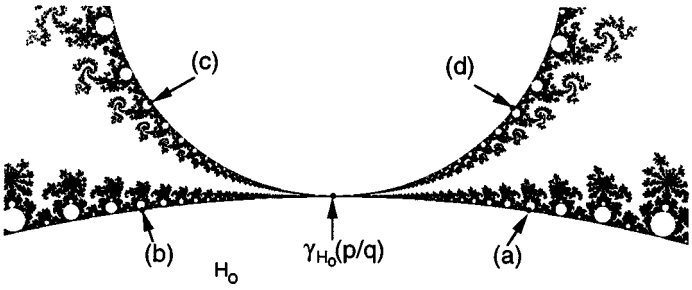


FIGURE 2.3. First-order parabolic perturbations.

Definition 2.8. Represent p/q by a continued fraction expansion $[a_1, a_2, \ldots, a_n]$. By a *first-order parabolic perturbation* of $\gamma_{H_o}(p/q)$ we understand a parameter $\gamma_{H_o}(t(x))$ on ∂H_o having internal argument $t(x) = [a_1, a_2, \ldots, a_n, x]$, where $x \in \mathbb{Z} \setminus \{0, -1\}$ is the *type* of the perturbation.

Note that $t(x) \neq p/q$ and it tends to p/q monotonously as $|x|$ tends to infinity.

Repeated parabolic perturbations. Recursively using the definition above we define the following.

Definition 2.9. A parameter c' in M is said to be a *parabolic perturbation of order n* of another parameter c if it can be reached from c by n repeated first-order perturbations. By the *type* of the perturbation we understand the sequence of types for the involved first-order perturbations.

This evidently extends to the infinite order case.

Definition 2.10. A parameter c' in M is said to be a *parabolic perturbation of infinite order* of another parameter c if it can be reached from c as the limit of infinitely (but countable) many repeated first-order perturbations. By the *type* of the perturbation we understand the sequence of types for the involved first-order perturbations.

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We shall be interested in large values of the $|a_i|$'s, and in infinite-order parabolic perturbations with $|a_i|$ tending fast to ∞. If c' has been obtained by an infinite-order parabolic perturbation of a parameter c on ∂H_o, clearly c' is also a point on ∂H_o. Of course, the type can consist of any sequence but we are in particular interested in two cases:

Two-sided parabolic perturbation. We alternate each time between perturbations to the left and to the right, such that the limit is approximated (alternating) two-sided. Hence, the type is a sequence of positive integers, and we consider *every* convergent subsequence of the simple continued fraction expansion of the limit as an approximation of the limit.

One-sided parabolic perturbation. We take perturbations to one side only, such that the limit is approximated one-sided. Hence, the type is a sequence of negative integers, and we consider *every second* convergent subsequence of the simple continued fraction expansion of the limit as an approximation of the limit.

Remark. We want to obtain control at each step by choosing the coefficients very large. So it is clear that a two-sided perturbation gives more control than a one-sided peturbation. For a one-sided perturbation, when using the simple continued fraction representation, every second coefficient becomes 1—the worst possible for us! This means that in some cases we can relax the condition to use any positive integer instead of 1, without loosing further control (actually we might gain some).

3. *Two-sided infinite-order parabolic perturbations*

3.1. *Setting up a scheme.* In the following, by more or less heuristical arguments, we are going to present a scheme for producing connected non-locally connected quadratic Julia sets by infinite-order parabolic perturbations. We have in mind to produce a Julia set containing a double comb; see Figure 2.1 ($r = 1$).

Initialisation. We start from any rational t_m, corresponding to a point at the main cardioid, having a simple continued fraction expansion $[a_1, a_2, \ldots, a_m]$ of, say, *odd length*. In Figure 3.1 we think of t_m as being $1/3$; thus $m = 1$ and $a_1 = 3$, but the actual numbers are of minor importance.

Representing t_m by its convergent p_m/q_m with $(p_m, q_m) = 1$ and $0 < p_m < q_m$, it follows that the polynomial P_{t_m} has a parabolic fixed point at α_{t_m} of combinatorial rotation number p_m/q_m. Thus the Fatou–Leau flower has exactly q_m petals. One of these petals contains the critical point. So the critical point Fatou component extends from α_{t_m} to $-\alpha_{t_m}$. This is the part of the Julia set which is important for us. In Figure 3.1 we schematically indicate its position neglecting the remaining part of the Julia set.

In particular, we are interested in $\mathcal{R}_{t_m}(\theta^-(t_m)/2)$ and $\mathcal{R}_{t_m}(\theta^+(t_m)/2)$. By Lemma 2.7 they land at $-\alpha_{t_m}$ and α_{t_m} respectively.

It is of importance for the following, that given any $\varepsilon > 0$ we can find a neighbourhood of $\theta^-(t_m)/2$ and a neighbourhood of t_m so that all rays $\mathcal{R}_t(\theta)$ with argument θ in the

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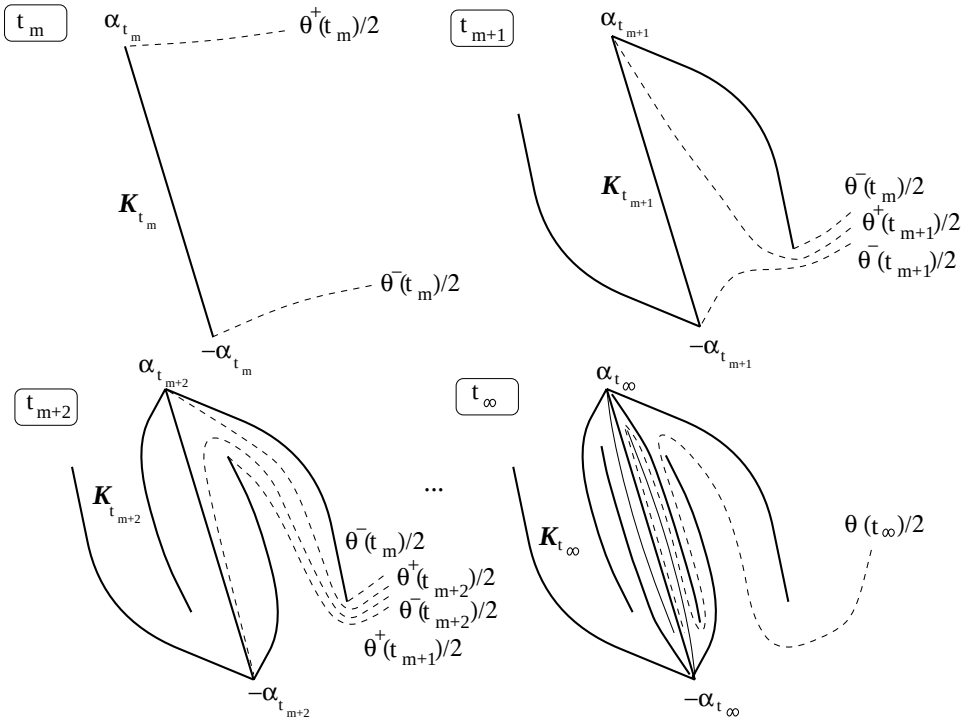


FIGURE 3.1. Our expectations to the central part of the Julia set when repeatedly performing right and left first-order perturbations. The type of the infinite-order parabolic perturbation has positive terms only. We move at the main cardioid, starting with a parameter represented by a simple continued fraction expansion of odd length.

former neighbourhood and parameter t in the latter get at least ε-close to $-\alpha_t$. For a precise statement see Lemma 3.2.

First perturbation. We now perform a type $a_{m+1} > 0$ first-order parabolic perturbation of t_m this way giving us t_{m+1} which is slightly smaller than t_m.

If the perturbation is small enough, we expect to get a *tooth* in the Julia set extending from $\alpha_{t_{m+1}}$ towards its negative $-\alpha_{t_{m+1}}$, since the ray $\mathcal{R}_{t_{m+1}}(\theta^-(t_m)/2)$ must still land, and by continuity at a point close to $-\alpha_{t_{m+1}}$. Of course by symmetry we also expect to get a tooth in the Julia set extending from $-\alpha_{t_{m+1}}$ towards its negative $\alpha_{t_{m+1}}$.

By the discussion above, if the perturbation is sufficiently small, not only does the ray $\mathcal{R}_{t_{m+1}}(\theta^-(t_m)/2)$ get close to $-\alpha_{t_{m+1}}$ but so do the two new important rays $\mathcal{R}_{t_{m+1}}(\theta^-(t_{m+1})/2)$ and $\mathcal{R}_{t_{m+1}}(\theta^+(t_{m+1})/2)$. Note that the first of these according to Lemma 2.7 has to land at $-\alpha_{t_{m+1}}$, while the second has to land at $\alpha_{t_{m+1}}$ *but only after* getting close to $-\alpha_{t_{m+1}}$. This observation is what is going to lead us to the construction of a non-locally connected Julia set with rays having non-trivial principal impression or impression.

Again, given any $\varepsilon > 0$ we can find a neighbourhood of $\theta^-(t_{m+1})/2$ and

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a neighbourhood of t_{m+1} so that all rays $\mathcal{R}_t(\theta)$ with argument θ in the former neighbourhood and parameter t in the latter get at least ε-close to $-\alpha_t$. Furthermore, we can choose these neighbourhoods to have common points with those from the initial step.

The ray $\mathcal{R}_t(\theta^+(t_m)/2)$ has to land at a periodic point of period dividing q_m, since the ray itself is periodic of eventual period q_m. In all of $\overline{H}_o$ the landing point of this ray depends continuously on the parameter; cf. [**DH**, part II, Théorème p. 77]. On $\overline{H}_o \setminus \{\gamma_{H_o}(t_m)\}$ the landing point remains repelling because there already is an attracting fixed point for parameters inside H_o; for parameters at the boundary if that was not the case it would have to land at an eventual parabolic cycle of rotation number t_m—the only such parameter being $\gamma_{H_o}(t_m)$. This implies that off this point the landing point has constant period. If this period was lower than q_m, say $\tilde{q}$, then α_{t_m} would be a multiple solution to

$$P_c^{\circ\tilde{q}}(z) = z$$

which violates its multiplier being exactly a q_mth root of unity. In particular, this holds on the cardioid. The landing of $\mathcal{R}_t(\theta^-(t_m)/2)$ can be analysed the same way. Let us state the result explicitly

LEMMA 3.1. *For $t \in [0,1] - \{t_m\}$, and q_m the period of $\theta^{\pm}(t_m)$:*

(1) *The ray $\mathcal{R}_t(\theta^+(t_m)/2)$ has to land at a repelling periodic point of period exactly q_m. When t tends to t_m, this point tends to α_{t_m}.*
(2) *The ray $\mathcal{R}_t(\theta^-(t_m)/2)$ has to land at a strictly preperiodic point which is eventually repelling of period exactly q_m. When t tends to t_m, this point tends to $-\alpha_{t_m}$.*

Actually, we do not use the lemma in the proof we are going to present, but it is very precise and descriptive itself, and we later use part of it to gain a more precise description of the scenario.

Second perturbation. We now do a type $a_{m+2} > 0$ perturbation of t_{m+1} giving us t_{m+2} which is slightly bigger than t_{m+1}.

We expect to get a tooth in the Julia set extending from $-\alpha_{t_{m+2}}$ towards the new landing point of $\mathcal{R}_{t_{m+2}}(\theta^+(t_{m+1})/2)$ which will be close to $\alpha_{t_{m+2}}$; see the schematic drawing on Figure 3.1

Again, if the perturbation is sufficiently small not only does the ray $\mathcal{R}_{t_{m+2}}(\theta^+(t_{m+1})/2)$ get close to $\alpha_{t_{m+2}}$, but so do the two new important rays $\mathcal{R}_{t_{m+2}}(\theta^-(t_{m+2})/2)$ and $\mathcal{R}_{t_{m+2}}(\theta^+(t_{m+2})/2)$. Now the rays start oscillating:

(1) $\mathcal{R}_{t_{m+2}}(\theta^-(t_m)/2)$ still has to get close to $-\alpha_{t_{m+2}}$;
(2) $\mathcal{R}_{t_{m+2}}(\theta^+(t_{m+1})/2)$ has to get close to $-\alpha_{t_{m+2}}$ before it lands close to $\alpha_{t_{m+2}}$;
(3) $\mathcal{R}_{t_{m+2}}(\theta^+(t_{m+2})/2)$ has to get close to $-\alpha_{t_{m+2}}$ before it lands at $\alpha_{t_{m+2}}$;
(4) $\mathcal{R}_{t_{m+2}}(\theta^-(t_{m+2})/2)$ has to get close to $-\alpha_{t_{m+2}}$ and then close to $\alpha_{t_{m+2}}$ before it lands at $-\alpha_{t_{m+2}}$.

Lemma 3.1 yields that the ray $\mathcal{R}_t(\theta^+(t_{m+1})/2)$ lands at a periodic point of period q_{m+1}, whose distance to α_t could be controlled.

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In the limit. Now the process can be iterated. At each step we get a new tooth, so that, in the limit, we expect to get a Julia set containing a topologist's double comb.

In fact, if there is one such comb, there are infinitely many preimages of it also. This makes the proof or even the formulation of a precise statement difficult. So instead of formulating a statement directly on the Julia set, we will formulate and prove a theorem on the rays.

3.2. *Statement of Theorem I.*

THEOREM I. *Consider any rational number t_m represented by the continued fraction expansion $[a_1, a_2, \dots, a_m]$. Then*

$$\exists N_{m+1}\ \forall a_{m+1} > N_{m+1},\ \exists N_{m+2}\ \forall a_{m+2} > N_{m+2}, \dots, \exists N_{m+n}\ \forall a_{m+n} > N_{m+n}, \dots$$

such that if $t_\infty = [a_1, a_2, \dots, a_m, a_{m+1}, \dots]$ then the external ray $\mathcal{R}_{t_\infty}(\theta(t_\infty)/2)$ accumulates at both α_{t_∞} and $-\alpha_{t_\infty}$.

Remarks.

(1) The conclusion may be rephrased: *... then the principal impression of the external ray $\mathcal{R}_{t_\infty}(\theta(t_\infty)/2)$ contains both α_{t_∞} and $-\alpha_{t_\infty}$.*

(2) Note that N_{m+n} depends not only on $a_1, a_2, \dots, a_m$ but also on the choices of a_{m+1}, $a_{m+2}, \dots, a_{m+n-1}$.

(3) If we represent the convergents t_n of t_∞ by p_n/q_n, where $(p_n, q_n) = 1$ and $0 < p_n < q_n$, then Lemma 3.1 immediately gives the following complement to the theorem: *For any n, the external ray $\mathcal{R}_{t_\infty}(\theta^+(t_n)/2)$ lands at a (q_n)-periodic point, and the external ray $\mathcal{R}_{t_\infty}(\theta^-(t_n)/2)$ lands at a point which is not periodic but whose first image is periodic of period q_n.*

(4) The critical point 0 cannot be accessible, since a ray landing at 0 together with its symmetric copy would form a curve separating the two points α_t and $-\alpha_t$, implying that the impression of any other ray, in particular the $(\theta(t)/2)$-ray, would be disjoint from either the connected component containing α_t or the other. Recently, by a similar argument, Kiwi [**Ki**] has proved that for a polynomial of any degree, to any Cremer point approximated by small cycles there is a corresponding non-accessible critical point.

(5) By tuning with obvious modifications the theorem and remarks hold on the boundary of any hyperbolic component of M.

3.3. *Two descriptive lemmas.* Before we proceed to the proof of Theorem I, we will prove two lemmas which are going to be the basic tools for the proof.

Conventions. In the remainder of this paper

$$\forall n: \quad a_n, N_n \in \mathbb{N}$$

and we take as a convention

$$-^n := \begin{cases} +, & \text{for } n \text{ even}, \\ -, & \text{for } n \text{ odd}. \end{cases}$$

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LEMMA 3.2. *Consider any rational number t_n represented by the continued fraction expansion $[a_1, a_2, \ldots, a_n]$ and any $\varepsilon_{n+1} > 0$. Then there is a $\rho_{n+1} > 0$ such that if we let*

$$\Omega_{n+1} := \omega_{\rho_{n+1}}(t_n) \times \omega_{\rho_{n+1}}(\theta^{-^n}(t_n))$$

then for all $(t, \theta) \in \Omega_{n+1}$,

$$d(\mathcal{R}_t(\theta/2), (-1)^n \alpha_t) < \varepsilon_{n+1}.$$

Proof of Lemma 3.2. We only treat the case of n being *even*; the odd case can be dealt with similarly.

The ray $\mathcal{R}_{t_n}(\theta^+(t_n)/2)$ lands at α_{t_n}, so let z be a point on the ray such that

$$d(z, \alpha_{t_n}) < \frac{\varepsilon_{n+1}}{2}. \tag{3.1}$$

Set $s = \log|\varphi_{t_n}(z)|$; then Proposition 2.3 assures that Ψ_s is continuous in t and θ. Considering the function

$$\delta_s : [0,1]^2 \to [0, +\infty[\quad \text{given by} \quad \delta_s : (t, \theta) \mapsto d(\Psi_s(t, \theta/2), \alpha_t),$$

we see that δ_s is also continuous since it is the composite of continuous mappings.

Note that by 3.1

$$\delta_s(t_n, \theta^+(t_n)) = d(z, \alpha_{t_n}) < \frac{\varepsilon_{n+1}}{2}.$$

The continuity of δ_s ensures the existence of $\rho > 0$ so that

$$\forall (t, \theta) \in \omega_\rho(t_n) \times \omega_\rho(\theta^+(t_n)) : \quad \delta_s(t, \theta) < \varepsilon_{n+1},$$

and since

$$d(\mathcal{R}_t(\theta/2), \alpha_t) \le \delta_s(t, \theta)$$

any $\rho_{n+1} \le \rho$ satisfies the demands of the lemma. □

LEMMA 3.3. *Consider any rational number t_n represented by the continued fraction expansion $[a_1, a_2, \ldots, a_n]$ and any $\varepsilon_{n+1} > 0$. Let Ω_{n+1} be the set provided by Lemma 3.2. Then there is an $N_{n+1} > 0$, so that for any integer $a_{n+1} \ge N_{n+1}$ we have that if $t_{n+1} := [a_1, a_2, \ldots, a_n, a_{n+1}]$ then*

$$(t_{n+1}, \theta^{-^{n+1}}(t_{n+1})) \in \Omega_{n+1}.$$

Proof of Lemma 3.3. Again, we only treat the case of n being *even*, as the odd case can be dealt with similarly.

Note that t_{n+1} is a rational number and that

$$t_{n+1} \searrow t_n \quad \text{as} \quad a_{n+1} \to +\infty,$$

so by Proposition 2.6(2)

$$\theta^-(t_{n+1}) \searrow \theta^+(t_n) \quad \text{as} \quad a_{n+1} \to +\infty.$$

In particular, there is an N_{n+1} satisfying the requirements. □

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3.4. *Proof of Theorem I.* In the sequel let t_m be given by the continued fraction $[a_1, a_2, \ldots, a_m]$. Take any sequence $(\varepsilon_n)_{n>m}$, with $\varepsilon_n > 0$, tending to 0.

Recursively using Lemmas 3.2 and 3.3.

Step 0. Lemma 3.2 applies with $n = m$ and provides us with ρ_{m+1} and Ω_{m+1}. From Lemma 3.3 we get a corresponding value for N_{m+1}.

Choose any $a_{m+1} \geq N_{m+1}$, and consider a_{m+1}, and therefore also t_{m+1}, as fixed in the sequel.

Step (i). The process can be repeated by setting $n = m + 1$, and in general at step (i) with $n = m + i$. At each step the process provides us with ρ_{m+i+1}, Ω_{m+i+1} and N_{m+i+1}, so that we can choose $a_{m+i+1} \geq N_{m+i+1}$ at each step assuring the properties of the two lemmas.

Deriving the theorem. We get a recursively defined sequence of sets $(\Omega_n)_{n>m}$. By Lemma 3.2, any common point (t, θ) for all these sets

$$(t, \theta) \in \Omega^\infty := \bigcap_{n>m} \Omega_n,$$

defines a parameter t for which the external ray of external argument $\theta/2$ gets arbitrarily close to the two distinct points α_t and $-\alpha_t$ (note that $t \neq 0$) thus accumulates these two points. Since t_∞ has an infinite continued fraction expansion it is irrational, so to prove the theorem it suffices to prove that

$$(t_\infty, \theta(t_\infty)) \in \Omega^\infty.$$

Indeed, let $n \geq m$ be any *odd* number, then since (t_n) is the sequence of convergents for the simple continued fraction expansion of t_∞,

$$t_{n+1} < t_\infty < t_n \tag{3.2}$$

and by Lemma 3.3

$$t_n - \rho_{n+1} < t_{n+1} < t_n + \rho_{n+1}. \tag{3.3}$$

Combining (3.2) and (3.3) we get

$$t_\infty \in \omega_{\rho_{n+1}}(t_n). \tag{3.4}$$

From (3.2) and Proposition 2.5 we have

$$\theta^+(t_{n+1}) < \theta(t_\infty) < \theta^-(t_n). \tag{3.5}$$

By Lemma 3.3

$$\theta^{-^n}(t_n) - \rho_{n+1} < \theta^{-^{n+1}}(t_{n+1}) < \theta^{-^n}(t_n) + \rho_{n+1}. \tag{3.6}$$

Hence (3.5) and (3.6) yield

$$\theta(t_\infty) \in \omega_{\rho_{n+1}}(\theta^{-^n}(t_n)).$$

Combined with (3.4) this is what was required. The case of n being even is treated similarly. □

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Remark. Inspecting the proof we see that initially we choose a sequence $(\varepsilon_n)_n$ of positive real numbers tending to 0. The choice of this sequence indicates how fast we require the ray of interest to accumulate the fixed point and its non-periodic preimage, this directly affects the actual size and growth of the N_n's: *'The slower accumulation required the weaker are the demands imposed on the N_n's.'*

3.5. *Weakening the control.* In Theorem I we set up demands for all of the coefficients a_n for $n > m$. In fact, by re-examining the proof it is clear that if we infinitely often choose a_n's with n being even and infinitely often choose a_n's with n odd honouring the demands of Lemma 3.3, then we will still end up with a parameter t_∞ for which the $(\theta(t_\infty)/2)$-ray accumulates both α_{t_∞} and $-\alpha_{t_\infty}$. Details are left to the reader.

3.6. *Strengthening the control.* Theorem I was proved by showing that we could find a sequence $(s_n)_n$ of potentials (tending to 0) so that the points $\psi_{t_\infty}(e^{s_n+i\cdot\pi\theta(t_\infty)})$ on the ray $\mathcal{R}_{t_\infty}(\theta(t_\infty)/2)$ were ε_n-close to $(-1)^{n+1}\alpha_{t_\infty}$.

This actually also follows directly from the definition of the principal impression together with the statement of the theorem. In fact, after extracting a subsequence we can assume that the sequence of potentials is *strictly decreasing*. This way we get information on the order in which the neighbourhoods are visited when moving on the ray.

This means that the $(\theta(t_\infty)/2)$-ray somehow oscillates between the fixed point and its non-periodic preimage. The reader should think of the ray as a topologist's sine as indicated in the scheme in Figure 3.1, although we have not been able to formulate or prove a rigourous statement about this. Indeed, we cannot even rule out the possibility of the ray spiralling instead!

We could have incorporated the property of the decreasing potentials in the theorem and in the construction proving it. This of course imposes further demands on the N_n's, but it also yields further information on the behaviour of the rays $\mathcal{R}_{t_\infty}(\theta^{\pm}(t_n)/2)$ for $n > m$, as well as opening some possibilities for generalising the theorem. In fact, what we want to use is already hidden in the proof of Lemma 3.2. Examining this proof, a slightly stronger version of the lemma, which takes account of the potentials, it becomes immediate.

LEMMA 3.4. *Consider any rational number t_n represented by the continued fraction expansion $[a_1, a_2, \ldots, a_n]$, any potential $s_n > 0$ and any $\varepsilon_{n+1} > 0$. Then there is a $\rho_{n+1} > 0$ and a potential $s_{n+1} \in]0, s_n[$ such that if we let*

$$\Omega_{n+1} := \omega_{\rho_{n+1}}(t_n) \times \omega_{\rho_{n+1}}(\theta^{-n}(t_n))$$

then for all $(t, \theta) \in \Omega_{n+1}$,

$$d(\psi_t(e^{s_{n+1}+i\cdot\pi\theta}), (-1)^n\alpha_t) < \varepsilon_{n+1}.$$

In Lemma 3.3 the reference to Lemma 3.2 is replaced by a reference to this lemma, and in this way uses the new definition of the set Ω_{n+1}.

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Alternative proof of Theorem I. With this in mind the rest of the proof of Theorem I, which was presented in §3.4, goes through with the modification that the recursive use of Lemma 3.2 is replaced by a recursive use of Lemma 3.4, and the addition that this way we also define the sequence of potentials $(s_n)_{n>m}$ at which the 'ray-bundle' visits the appropriate neighbourhoods. □

In fact, from this proof we can extract a bit more information than stated in the theorem.

For simplicity we assume m and n to be *odd*; the general formulation is left to the reader!

- First, at potential s_{m+1}, both $\mathcal{R}_{t_\infty}(\theta^-(t_n)/2)$ and $\mathcal{R}_{t_\infty}(\theta^+(t_n)/2)$ get ε_{m+1}-close to $-\alpha_{t_\infty}$.
- Then, at potential s_{m+2}, they get ε_{m+2}-close to α_{t_∞}.
- At potential s_{m+3}, they get ε_{m+3}-close to $-\alpha_{t_\infty}$.
- ...
- Then, at potential s_n, they get ε_n-close to α_{t_∞}.
- Finally, the ray $\mathcal{R}_{t_\infty}(\theta^+(t_n)/2)$ lands at a q_n-periodic point, which we think of as being in a small cycle near α_{t_∞}, although we might need more control to be sure of this cycle actually being small.

  The other ray $\mathcal{R}_{t_\infty}(\theta^-(t_n)/2)$ gets ε_{n+1}-close to $-\alpha_{t_\infty}$ and lands at a strictly preperiodic point of eventual period q_n, which we analogously think of as being part of a 'small pre-cycle' near $-\alpha_{t_\infty}$.

3.7. *Corollary: realising non-landing rays.* The following proposition is an immediate corollary of Theorem I.

PROPOSITION 3.5. *Consider any rational number t_m represented by the continued fraction expansion $[a_1, a_2, \ldots, a_m]$. Then*

$$\exists N_{m+1}\ \forall a_{m+1} > N_{m+1},\ \exists N_{m+2}\ \forall a_{m+2} > N_{m+2}, \ldots, \exists N_{m+n}\ \forall a_{m+n} > N_{m+n}, \ldots$$

such that if $t_\infty = [a_1, a_2, a_3, \ldots, a_m, a_{m+1}, \ldots]$ then $\mathcal{R}_{t_\infty}(\theta(t_\infty)/2)$ does not land.

Note that as in Theorem I, N_{m+n} depends not only on $a_1, a_2, \ldots, a_m$ but also on the choices of $a_{m+1}, a_{m+2}, \ldots, a_{m+n-1}$.

However, by using a variation of the alternative proof of Theorem I we get another proof of this proposition, which might cover cases not covered by Theorem I.

Proof of Proposition 3.5. Let $t'_m := [a_1, a_2, a_3, \ldots, a_m, 1]$ and

$$I := [\min\{t_m, t'_m\}, \max\{t_m, t'_m\}].$$

Then any t_∞ that can possibly be reached by a two-sided infinite-order parabolic perturbation of t_m is contained in I.

For all $n > m$, we set

$$\varepsilon_n \equiv \varepsilon := k \cdot \inf_{t\in I} |\alpha_t|, \quad \text{for any } 0 < k < 1,$$

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and we repeat the rest of the construction of the alternative proof of Theorem I. Then we are provided with a parameter t_∞, for which the ray $\mathcal{R}_{t_\infty}(\theta(t_\infty)/2)$ infinitely often and alternating enters the separated balls $B_\varepsilon(\alpha_{t_\infty})$ and $B_\varepsilon(-\alpha_{t_\infty})$, thus the ray cannot land. □

Remarks.

(1) It is clear that in this way we obtain quadratic Julia sets for which the external ray $\mathcal{A}_{t_\infty}(\theta(t_\infty)/2)$ accumulates at a set (its principal impression) which has diameter at least $2(1-k)\varepsilon$.

(2) It does not follow from this proposition that the impression of the $(\theta(t_\infty)/2)$-ray contains the points $\pm\alpha_{t_\infty}$; in fact, it does not have to be the case: α_{t_∞} could be the centre of a Siegel disk of the non-locally connected Herman type (i.e. the critical point is not contained in the boundary of the Siegel disk).

(3) We think of these Julia sets as containing a double comb of the type described in §2.2 with $1/2 < r < 1$. This corresponds to controlling the property that the tooth created at every step extends only some of the way from the fixed point to its negative; compare with the illustration of the scheme—Figure 3.1.

4. *One-sided infinite-order parabolic perturbations*

If, in the construction in the previous section, we make no restrictions on, say, the a_n's with n odd, we are now going to see that we can still choose the a_n's with n even so that we get an external ray with non-trivial impression.

Vaguely, think of this as the Julia set containing at the least a 'topologist's single comb'; see Figure 2.1 ($r < \frac{1}{2}$). Looking at the one-sided scheme, as in §3.1, we think of this as only controlling the length of the teeth in one of the combs.

The extreme would be to approximate from one side only, which is accomplished by setting, for example, $a_{m+2i} = 1, i > 0$, and by only considering t_{m+2i} as the approximants. To conform with §3, we will set up a scheme for the construction in this particular case, while the statement and the proof that follows are in the general case.

4.1. *Setting up a scheme.*

Initialisation and first perturbation. The considerations and discussion concerning the initialisation and first perturbation presented in §3.1 also apply here, since the first steps are identical.

The only difference is that after the first perturbation in §3.1, we had the second approximant given by $[a_1, a_2, \ldots, a_m, a_{m+1}]$, which is of even length. Here we let $t_{m+1} := [a_1, a_2, \ldots, a_m, (a_{m+1} - 1)]$ and consider it as an intermediate step towards

$$t_{m+2} := [a_1, a_2, \ldots, a_m, (a_{m+1} - 1), 1].$$

This way the parameter t_{m+2} is considered as being the *second* approximation to the parameter which we want to get in the limit.

Note that t_{m+2} is slightly smaller than t_m and that a 'new' tooth extends from $\alpha_{t_{m+2}}$ towards the landing point of the $(\theta^-(t_m)/2)$-ray. If the perturbation is sufficiently small not only does $\mathcal{R}_{t_{m+2}}(\theta^-(t_m)/2)$ get close to $-\alpha_{t_{m+2}}$ but so do the two 'new' important

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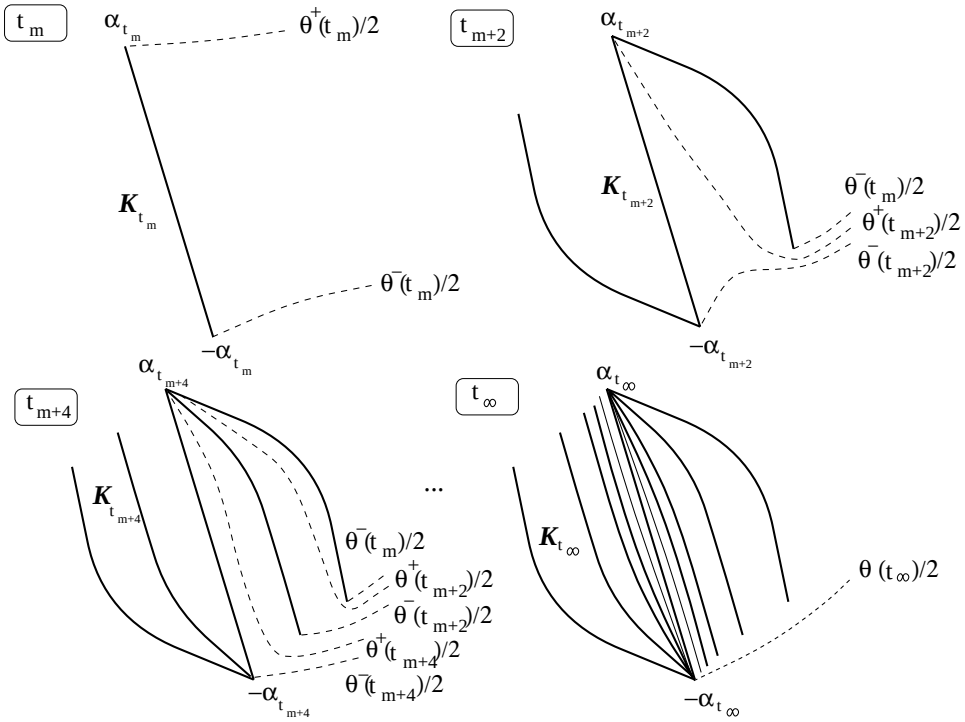


FIGURE 4.1. Our expectations to the central part of the Julia set when repeatedly performing first-order perturbations to the right. The type of the infinite-order parabolic perturbation has negative terms only. We move at the main cardioid, starting with a parameter represented by a simple continued fraction expansion of odd length.

rays $\mathcal{R}_{t_{m+2}}(\theta^-(t_{m+2})/2)$ and $\mathcal{R}_{t_{m+2}}(\theta^+(t_{m+2})/2)$. The latter of these lands at $\alpha_{t_{m+2}}$. Under further perturbations the former properties are preserved, while the latter one of landing replaced by one stating that the ray $\mathcal{R}_t(\theta^+(t_{m+2})/2)$ gets close to α_t, see Figure 4.1.

Second perturbation. We now do another perturbation to the right, but this time of t_{m+2} yielding a t_{m+4} being slightly smaller than t_{m+2}. This perturbation, of course, is expected to exhibit exactly the same features as the first. Thus a new tooth is born as well as two new rays of arguments $\theta^{\pm}(t_{m+4})/2$ close to $\theta^-(t_{m+2})/2$, etc.

In the limit. The process can be iterated always doing first-order parabolic perturbations to the right: at every step we get a new tooth, so that in the limit we expect to get a Julia set containing a topologist's comb.

As before, if there is one such comb then there are infinitely many preimages of it, too. Once again we formulate and prove a theorem on the rays only.

Since we never used the fact that every second coefficient in the continued fraction expansion we were constructing was 1, we will suppress this demand in the statement.

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4.2. *Statement of Theorem II.*

THEOREM II. *Given any rational number t_m represented by the continued fraction expansion $[a_1, a_2, \ldots, a_m]$. Then*

$$\exists N_{m+1}\ \forall a_{m+1} > N_{m+1}\ \forall a_{m+2}\ \exists N_{m+3}\ \forall a_{m+3} > N_{m+3}\ \forall a_{m+4}$$
$$\ldots \exists N_{m+2n+1}\ \forall a_{m+2n+1} > N_{m+2n+1}\ \forall a_{m+2n+2} \ldots$$

such that if $t_\infty = [a_1, a_2, a_3, \ldots, a_m, a_{m+1}, \ldots]$ then the sequence of external rays $(\mathcal{R}_{t_\infty}(\theta^{-m+1}(t_{m+2n})/2))_n$ accumulates at both α_{t_∞} and $-\alpha_{t_\infty}$.

Furthermore, the limit external ray $\mathcal{R}_{t_\infty}(\theta(t_\infty)/2)$ accumulates at $(-1)^m\alpha_{t_\infty}$.

Remarks.

(1) The formulation of the first conclusion of the theorem is just a descriptive way of stating '*the impression of the ray $\mathcal{R}_{t_\infty}(\theta(t_\infty)/2)$ contains both α_{t_∞} and $-\alpha_{t_\infty}$*'.

(2) Initially, by choosing the parity of m, one has the possibility of choosing between one-sided infinite-order parabolic perturbations of $\gamma_{H_o}(t_m)$ to the right or to the left; *odd* m gives the former, *even* the latter.

(3) Note that as in §3.5, the statement can be relaxed from demanding control at every second step to only demanding control infinitely often.

(4) Remarks (2)–(5) to Theorem I remain valid here with obvious modifications.

4.3. *Two descriptive lemmas.* To prove Theorem II we need an extension of Lemma 3.2.

LEMMA 4.1. *Consider any rational number t_n represented by the continued fraction expansion $[a_1, a_2, \ldots, a_n]$ and any $\varepsilon_{n+1} > 0$. Then there is a $\rho_{n+1} > 0$ such that:*

(1) *for all $t \in \omega_{\rho_{n+1}}(t_n)$,*

$$d(\mathcal{R}_t(\theta^{-n+1}(t_n)/2), (-1)^{n+1}\alpha_t) < \varepsilon_{n+1};$$

(2) *and if we set*

$$\Omega_{n+1} := \omega_{\rho_{n+1}}(t_n) \times \omega_{\rho_{n+1}}(\theta^{-n}(t_n)),$$

then for all $(t, \theta) \in \Omega_{n+1}$,

$$d(\mathcal{R}_t(\theta/2), (-1)^n\alpha_t) < \varepsilon_{n+1}.$$

Proof of Lemma 4.1.

Part (1). This is immediate by using the same method as in the proof of Lemma 3.2. Here we use that for any fixed potential $s > 0$ and any fixed external argument $\theta^{-n+1}(t_n)/2$ the mapping from $\mathbb{T}$ to $\mathbb{R}^+$ given by

$$t \mapsto d(\Psi_s(t, \theta^{-n+1}(t_n)/2), (-1)^{n+1}\alpha_t)$$

is continuous, and vanishes in t_n. This supplies us with ρ'_{n+1}.

Part (2). This is exactly Lemma 3.2, and thus supplies us with ρ''_{n+1}.

Now, any

$$\rho_{n+1} \le \min(\rho'_{n+1}, \rho''_{n+1})$$

satisfies both demands of the lemma. □

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We also need a replacement for Lemma 3.3.

LEMMA 4.2. *Consider any rational number t_n represented by the continued fraction expansion $[a_1, a_2, \ldots, a_n]$ and any $\varepsilon_{n+1} > 0$. Let Ω_{n+1} be the set provided by Lemma 4.1. Then there is a $N_{n+1} > 0$, so that for any integer $a_{n+1} \geq N_{n+1}$ and any $a_{n+2} \in \mathbb{N}$ we have that if $t_{n+2} := [a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}]$ then*

$$(t_{n+2}, \theta^{\pm}(t_{n+2})) \in \Omega_{n+1}.$$

Proof of Lemma 4.2. Lemma 3.3 applies with the reference to Lemma 3.2 replaced by a reference to the just proven Lemma 4.1, and thus supplies us with N_{n+1} so that for any $a_{n+1} \geq N_{n+1}$,

$$(t_{n+1}, \theta^{-^{n+1}}(t_{n+1})) \in \Omega_{n+1}. \tag{4.1}$$

Let us assume n to be *odd*; the other case can be dealt with similarly.

If we now choose a_{n+2} as any positive integer, then by (4.1) and the fact that the t_i's are the convergents of a simple continued fraction expansion,

$$t_n - \rho_{n+1} < t_{n+1} < t_{n+2} < t_n. \tag{4.2}$$

By Proposition 2.5 and (4.1) again we also have

$$\theta^-(t_n) - \rho_{n+1} < \theta^+(t_{n+1}) < \theta^-(t_{n+2}) < \theta^+(t_{n+2}) < \theta^-(t_n). \tag{4.3}$$

This inequality combined with the inequality (4.2) is exactly the property required by the lemma. □

4.4. *Proof of Theorem II.* The proof is more or less a repetition of the proof of Theorem I.

Let $t_m := [a_1, a_2, \ldots, a_m]$ with m of any parity. Take any sequence $(\varepsilon_{m+2i+1})_{i \geq 0}$, with $\varepsilon_{m+2i+1} > 0$ tending to 0.

Recursively using Lemmas 4.1 and 4.2.

Step (0). Lemma 4.1 applies with $n = m$ and provides us with a ρ_{m+1} and Ω_{m+1}, and Lemma 4.2 provides us with a corresponding value for N_{m+1}.

Choose any $a_{m+1} \geq N_{m+1}$, and any $a_{m+2} \in \mathbb{N}$ and think of them, and therefore of t_{m+2}, as fixed in the sequel.

Step (i). The process can be repeated by setting $n = m + 2$, and in general at step (i) with $n = m + 2i$. At each step the process provides us with ρ_{m+2i+1}, Ω_{m+2i+1} and N_{m+2i+1}, so that we can choose $a_{m+2i+1} \geq N_{m+2i+1}$ at each step assuring the properties of the two lemmas for any choice of the integer a_{m+2i+2}.

Deriving the theorem. We get a recursively defined sequence of sets $(\Omega_{m+2i+1})_{i \geq 0}$, and exactly as in the proof of Theorem I it is easily seen that the point $(t_\infty, \theta(t_\infty))$ is in all of these sets.

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In particular, the ray $\mathcal{R}_{t_\infty}(\theta^{-m+1}(t_{m+2i+2})/2)$ is ε_{m+2i+3}-close to $(-1)^{m+1}\alpha_{t_\infty}$ by part (1) of Lemma 4.1 and ε_{m+2i+1}-close to $(-1)^m\alpha_{t_\infty}$ by Lemma 4.2. From this, the first part of the theorem directly follows.

By part (2) of Lemma 4.1, t_∞ is a parameter for which the external ray of external argument $\theta(t_\infty)/2$ gets arbitrarily close to the point $(-1)^m\alpha_{t_\infty}$, thus it accumulates this point, as stated in the second part of the theorem. □

5. *Concluding remarks*

We hope in the future to obtain stronger and more explicit versions of Theorems I and II. In particular:

(1) To find estimates on the numbers N_{m+n} in each of the cases. To do so, one probably would have to perform a very careful parabolic implosion analysis.

(2) To do the estimates in (1) independently of the choices of a_{m+1} to a_{m+n-1}. This would probably lead to the theorems being true under some Diophantine conditions. Perhaps, we could then derive exactly which Cremer point polynomials are covered by Theorems I and II.

Theorem I gives examples of quadratic polynomials having a remarkable ray either of the third or of the fourth kind. In fact, the construction in the scheme as well as the proof of the theorem suggest that here we have a ray of the third kind, thus the following conjecture.

CONJECTURE 5.A. *For the polynomials given by Theorem I, the $(\theta(t_\infty)/2)$-ray is of the third kind.*

The scheme and proof of Theorem II (one-sided version) suggest the following conjecture.

CONJECTURE 5.B. *There are quadratic Julia sets with an external ray of the second kind.*

Furthermore, in the general version of Theorem II we should be able to control the uncontrolled coefficients so much that we obtain cases in between those described by Theorem I and the strictly one-sided version of II. So, the following seems reasonable.

CONJECTURE 5.C. *Theorem II covers cases where the $(\theta(t)/2)$-ray is of the fourth kind.*

In conclusion, this means that we believe that we can find examples of *all four kinds of external rays* in the quadratic family.

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