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ON STOCHASTIC COMPARISONS OF ORDER STATISTICS FROM HETEROGENEOUS EXPONENTIAL SAMPLES

YAMING YU
Department of Statistics, University of California, Irvine, CA 92697, USA
E-mail: yamingy@uci.edu

We show that the kth order statistic from a heterogeneous sample of $n \geq k$ exponential random variables is larger than that from a homogeneous exponential sample in the sense of star ordering, as conjectured by Xu and Balakrishnan [14]. As a consequence, we establish hazard rate ordering for order statistics between heterogeneous and homogeneous exponential samples, resolving an open problem of Păltănea [11]. Extensions to general spacings are also presented.

Keywords: convolution, dispersive ordering, exponential distribution, majorization, stochastic order

1. INTRODUCTION AND MAIN RESULTS

There exists a large literature on stochastic comparisons between order statistics arising from possibly heterogeneous populations, see Balakrishnan and Zhao [1] for a review. In reliability theory, order statistics play a prominent role as the lifetime of a k-out-of-n system can be represented by the $n-k+1$th order statistic of the n component lifetimes. Because of the complexity of the distribution of order statistics arising from heterogeneous populations, stochastic comparisons with those from a homogeneous population are helpful, and can provide bounds on tail probabilities and hazard rates. Of particular interest is the following result of Bon and Păltănea [2]. Let $X_1, \ldots, X_n$ be independent exponential random variables with rates $\lambda_1, \ldots, \lambda_n$ respectively. Let $Y_1, \ldots, Y_n$ be another set of independent exponential random variables with a common rate γ. Then the kth order statistic of the heterogeneous sample is larger than that of the homogeneous sample in the usual stochastic order, that is, $Y_{k:n} \leq_{\mathrm{st}} X_{k:n}$, if and only if

$$\gamma \geq \left(\binom{n}{k}^{-1} s_k(\lambda_1, \ldots, \lambda_n) \right)^{1/k}, \tag{1}$$

where s_k denotes the kth elementary symmetric function. For the sample maximum, Khaledi and Kochar [7] showed that (1) with $k = n$ implies that $X_{n:n}$ is larger than $Y_{n:n}$ according

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to both the hazard rate order and the dispersive order. In the context of failsafe systems, Păltănea [11] showed that (1) with $k=2$ is equivalent to $Y_{2:n} \le_{\rm hr} X_{2:n}$. In the $k=2$ case, Zhao et al. [18] obtained related results for the likelihood ratio ordering.

In this paper we aim to extend such results to general k. Specifically, we have

THEOREM 1: *For each $1 \le k \le n$, (1) is equivalent to $Y_{k:n} \le_{\rm hr} X_{k:n}$, and also equivalent to $Y_{k:n} \le_{\rm disp} X_{k:n}$.*

The statement concerning $\le_{\rm hr}$ in Theorem 1 confirms a conjecture of Păltănea [11]. Theorem 1 is established by considering the star order between $X_{k:n}$ and $Y_{k:n}$. For random variables X and Y supported on $(0,\infty)$ with distribution functions F and G respectively, we say X is smaller than Y in the star order, denoted by $X \le_* Y$ (or $F \le_* G$), if $G^{-1}F(x)/x$ is increasing in $x>0$, where G^{-1} denotes the right continuous inverse of G. Equivalently, $X \le_* Y$ if and only if, for each $c>0$, $F(cx)$ crosses $G(x)$ at most once, and from below, as x increases on $(0,\infty)$. For definitions and properties of various stochastic orders including $\le_*$, see Shaked and Shanthikumar [13].

Kochar and Xu [8] showed $Y_{n:n} \le_* X_{n:n}$, a key result that allowed Xu and Balakrishnan [14] to obtain stochastic comparison results for the ranges of heterogeneous exponential samples; see Genest et al. [5] for related work. Our Theorem 2 confirms the conjecture of Xu and Balakrishnan [14] (see also Balakrishnan and Zhao [1], Open Problem 5). A special case of our Theorem 2 was obtained by Kochar and Xu [9] for multiple-outlier models.

THEOREM 2: *For each $1 \le k \le n$, we have $Y_{k:n} \le_* X_{k:n}$.*

The star ordering is a strong variability ordering. It implies the Lorenz ordering $\le_{\rm L}$, which in turn implies the ordering of the coefficients of variation. Our Theorem 2 is a strengthening of the main result of Da et al. [3] who showed $Y_{k:n} \le_{\rm L} X_{k:n}$.

Theorems 1 and 2 can be extended from order statistics to general spacings.

COROLLARY 1: *For $1 \le m < k \le n$ we have $Y_{k:n} - Y_{m:n} \le_* X_{k:n} - X_{m:n}$.*

PROPOSITION 1: *For $1 \le m < k \le n$ we have $Y_{k:n} - Y_{m:n} \le_{\rm order} X_{k:n} - X_{m:n}$ if and only if*

$$\binom{n-m}{k-m}\gamma^{k-m} \ge \sum_{\mathbf{r}} s_{k-m}^{[\mathbf{r}]}(\lambda) \prod_{j=1}^{m} \frac{\lambda_{r_j}}{\Lambda - \sum_{i=1}^{j-1}\lambda_{r_i}} \tag{2}$$

where $\mathbf{r}=(r_1,\ldots,r_m)$, the outer sum is over all permutations of $\{1,\ldots,n\}$ using m at a time, $\lambda=(\lambda_1,\ldots,\lambda_n)$, $\Lambda=\sum_{i=1}^n \lambda_i$, $s_{k-m}^{[\mathbf{r}]}(\lambda) = s_{k-m}(\lambda\backslash\{\lambda_{r_1},\ldots,\lambda_{r_m}\})$, and $\le_{\rm order}$ is any of $\le_{\rm st}, \le_{\rm hr}$ or $\le_{\rm disp}$.

In the special case of $m=1$ and $k=n$, which corresponds to comparing the sample ranges, the condition (2) reduces to

$$\gamma \ge \left(\frac{\prod_{i=1}^n \lambda_i}{\Lambda/n}\right)^{1/(n-1)},$$

and we recover Theorem 4.1 of Xu and Balakrishnan [14]. In the case of ordinary spacings, that is, $k=m+1$, Proposition 1 can also be derived using the log-concavity arguments of Yu [15]. As noted by Păltănea [12], however, it is difficult to implement such an argument in general.

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In Section 2 we prove Theorem 2, from which Theorem 1, Corollary 1 and Proposition 1 are deduced. We refer to Păltănea [11] for numerical illustrations and Da et al. [3] for potential applications of these results. It would be interesting to see if results similar to Theorem 1 and Proposition 1 can be obtained for the likelihood ratio order, for which Theorem 2, based on the star order, is not helpful.

2. DERIVATION OF MAIN RESULTS

We need the following closure property of the star order with respect to mixtures.

LEMMA 1 (Lemma 3.1 of Xu and Balakrishnan [14]): *Let F be a distribution function with density f supported on $(0,\infty)$ such that $f(e^x)$ is log concave in $x \in \mathbb{R}$. Let $G_1,\dots,G_n$ be distribution functions with densities $g_1,\dots,g_n$ supported on $(0,\infty)$ such that $F \le_* G_i$ for each $i = 1,\dots,n$. Then $F \le_* \sum_{i=1}^n p_i G_i$ for $p_i > 0$ such that $\sum_{i=1}^n p_i = 1$.*

We also need the following representation of order statistics from heterogeneous exponential samples.

LEMMA 2 (Păltănea (2011), Theorem 4.1): *Let $S = \{X_1,\dots,X_n\}$ be a set of $n > 1$ independent exponential random variables with rates $\lambda_1,\dots,\lambda_n$ respectively. Denote $S^{[i]} = S\backslash\{X_i\}$ and let $X^{[i]}_{j:(n-1)}$ be the jth order statistic from $S^{[i]}$ with distribution function $F^{[i]}_{j:(n-1)}$ for $j \le n-1$. Then for $k \le n-1$ the $(k+1)$th order statistic $X_{(k+1):n}$ from S is the sum of two independent random variables: $X_{1:n}$, which has an exponential distribution with rate $\Lambda = \sum_{j=1}^n \lambda_j$, and a mixture of order statistics with distribution function $\Lambda^{-1}\sum_{i=1}^n \lambda_i F^{[i]}_{k:(n-1)}$.*

Finally, we need some stochastic comparison results concerning convolutions of gamma variables. For independent random variables $Z_1,\dots,Z_n \sim \text{gamma}(\alpha)$ let $F_{\boldsymbol\theta}$ denote the distribution function of the weighted sum $\sum_{i=1}^n \theta_i Z_i$ where $\boldsymbol\theta = (\theta_1,\dots,\theta_n)$ is a vector of positive weights. We use $\prec_w$ to denote weak sub-majorization [10]. The following Lemma is a consequence of Theorem 1 of Yu [16].

LEMMA 3: *For $\alpha > 0$, if $\log\boldsymbol\eta \prec_w \log\boldsymbol\theta$ then $F_{\boldsymbol\eta} \le_{\text{st}} F_{\boldsymbol\theta}$.*

Lemma 4 asserts a unique crossing between the distribution functions of two weighted sums of iid gamma variables when the weights form a special configuration. Lemma 4 is an important step in Yu's [17] investigation of the unique crossing conjecture of Diaconis and Perlman [4]; it is also a key tool in deriving our main results (we only need the $\alpha = 1$ case).

LEMMA 4 (Theorem 1 of Yu (2017)): *Suppose $\alpha \ge 1$. Suppose $0 < \theta_1 \le \cdots \le \theta_n$ and $\eta_1 \le \cdots \le \eta_n$ and (a) there exists $2 \le k \le n$ such that $\theta_i < \eta_i$ for $i<k$ and $\theta_i > \eta_i$ for $i \ge k$; (b) $\prod_{i=1}^n \eta_i > \prod_{i=1}^n \theta_i$. Then there exists $x_0 \in (0,\infty)$ such that $F_\eta(x) < F_\theta(x)$ for $x \in (0,x_0)$ and the inequality is reversed for $x > x_0$.*

Proof of Theorem 2: Let us use induction. The $k = 1$ case is trivial. Suppose the claim holds for a certain $k \ge 1$ and all $n \ge k$. We shall prove that it holds for $k+1$ and all $n \ge k+1$. Assume $\lambda_i,\ i = 1,\dots,n$, are not all equal, and let $F^{(\tau)}_{k:n}$ denote the distribution function of the kth order statistic from a sample of n iid exponential variables with rate τ. In the notation of Lemma 2, the induction hypothesis yields $F^{(\tau)}_{k:(n-1)} \le_* F^{[i]}_{k:(n-1)}$ for $\tau > 0,\ n \ge k+1$, and

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$i = 1, \ldots, n$. As noted by Xu and Balakrishnan [14], $F^{(\tau)}_{k:(n-1)}$ has a density f such that $f(e^x)$ is log-concave. By Lemma 1, we have

$$F^{(\tau)}_{k:(n-1)} \leq_* \sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} F^{[i]}_{k:(n-1)}. \tag{3}$$

As noted by Da et al. [3], when $x \downarrow 0$ we have

$$F^{[i]}_{k:(n-1)}(x) = s^{[i]}_k(\lambda)x^k + o(x^k), \quad F^{(\tau)}_{k:(n-1)} = \binom{n-1}{k}\tau^k x^k + o(x^k),$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $s^{[i]}_k(\lambda) = s_k(\lambda \backslash \{\lambda_i\})$. Thus, if $\binom{n-1}{k}\tau^k \geq \sum_{i=1}^n \lambda_i s^{[i]}_k(\lambda)/\Lambda$, or equivalently $\tau \geq \tau^*$ with $\tau^* = (n s_{k+1}(\lambda)/\Lambda/\binom{n}{k+1})^{1/k}$, then

$$F^{(\tau)}_{k:(n-1)} \leq_{\mathrm{st}} \sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} F^{[i]}_{k:(n-1)}. \tag{4}$$

Analogous considerations as $x \to \infty$ [12] reveal that if $\tau \leq \tau_* \equiv \sum_{i=1}^{n-k} \lambda_{(i)}/(n-k)$, where $\lambda_{(1)}, \ldots, \lambda_{(n)}$ are λ arranged in increasing order, then (4) holds with the direction of $\leq_{\mathrm{st}}$ reversed. For $\tau \in (\tau_*, \tau^*)$, star ordering implies that there exists $x_0 > 0$ such that

$$F^{(\tau)}_{k:(n-1)}(x) \leq \sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} F^{[i]}_{k:(n-1)}(x), \quad x \in (0, x_0), \tag{5}$$

and the inequality is reversed if $x \in (x_0, \infty)$. One can show strict inequality, that is, the crossing point is unique. Indeed, if there exists another $\tilde{x} \in (0, x_0)$, for example, such that equality holds in (5) at $x = \tilde{x}$, then a slight increase in τ would produce at least two crossings, near x_0 and $\tilde{x}$, respectively (equality cannot hold for all $x \in [\tilde{x}, x_0]$ unless the two distributions are identical, because these distribution functions can be written as linear combinations of exponential functions and are therefore analytic).

In view of Lemma 2, we can convolve both sides of (3) with an exponential with rate Λ and obtain that $\mathrm{expo}(\Lambda) * F^{(\tau)}_{k:(n-1)}$ crosses $F_{(k+1):n}$ (the distribution function of $X_{(k+1):n}$) exactly once, and from below, for $\tau \in (\tau_*, \tau^*)$. That there is at most one crossing follows from variation diminishing properties of TP2 kernels [6]. Upon close inspection there is exactly one crossing at a unique point. In particular, because $\Lambda > (n-k)\tau > (n-k)\tau_*$, convolving with $\mathrm{expo}(\Lambda)$ cannot reverse the sign of $F^{(\tau)}_{k:(n-1)}(x) - \Lambda^{-1}\sum_{i=1}^n \lambda_i F^{[i]}_{k:(n-1)}(x)$ as $x \to \infty$. We then recognize the crossing point, denoted by x_*, as a decreasing, continuous function of τ, because $F^{(\tau)}_{k:(n-1)}$ stochastically decreases in τ. Furthermore, the above analysis at the end points τ_* and τ^* shows that $x_*(\tau) \uparrow \infty$ as $\tau \downarrow \tau_*$ and $x_*(\tau) \downarrow 0$ as $\tau \uparrow \tau^*$.

The distribution of $Y_{(k+1):n}$ is the convolution

$$F^{(\gamma)}_{(k+1):n} = \mathrm{expo}(n\gamma) * \mathrm{expo}((n-1)\gamma) * \cdots * \mathrm{expo}((n-k)\gamma).$$

Suppose $F^{(\gamma)}_{(k+1):n}$ crosses $F_{(k+1):n}$ at some $x^* > 0$. Then we can choose $\tau \in (\tau_*, \tau^*)$ such that $x_*(\tau) = x^*$, that is, $\mathrm{expo}(\Lambda) * F^{(\tau)}_{k:(n-1)}$ crosses $F_{(k+1):n}$ at exactly x^*, from below. If $\gamma \geq \Lambda/n$ then Maclaurin's inequality yields $\gamma \geq \tau^* > \tau$, which implies that

$$F^{(\gamma)}_{(k+1):n} \leq_{\mathrm{st}} \mathrm{expo}(\Lambda) * F^{(\tau)}_{k:(n-1)}, \tag{6}$$

contradicting the existence of x^*. If $\gamma \leq \tau$ then the inequality (6) is reversed, and there is again no crossing. Thus we must have $\gamma \in (\tau, \Lambda/n)$. If $\prod_{i=0}^k((n-i)\gamma) \geq \Lambda \prod_{i=1}^k((n-i)\tau)$

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then one can show (the log applies element-wise)

$$-\log(n\gamma,(n-1)\gamma,\ldots,(n-k)\gamma) \prec_w -\log(\Lambda,(n-1)\tau,\ldots,(n-k)\tau),$$

which again implies (6) by Lemma 3. For the remaining case, $\prod_{i=0}^{k}((n-i)\gamma) < \Lambda\prod_{i=1}^{k}((n-i)\tau)$, the conditions of Lemma 4 are satisfied. It follows that $F^{(\gamma)}_{(k+1):n}$ crosses $\mathrm{expo}(\Lambda) * F^{(\tau)}_{k:(n-1)}$ exactly once, from below, at the same point x^*. We deduce that $F^{(\gamma)}_{(k+1):n}$ crosses $F_{(k+1):n}$ exactly once, from below, at x^*. Since $\gamma > 0$ is arbitrary, we have $Y_{(k+1):n} \le_* X_{(k+1):n}$. ■

Proof of Theorem 1: Theorem 1 can be deduced from Theorem 2 as discussed by Xu and Balakrishnan [14]. Specifically, given $Y_{k:n} \le_* X_{k:n}$, we have the equivalence

$$Y_{k:n} \le_{\mathrm{st}} X_{k:n} \iff Y_{k:n} \le_{\mathrm{disp}} X_{k:n}.$$

Since $Y_{k:n}$ has increasing failure rate, dispersive ordering implies hazard rate ordering. ■

Proof of Corollary 1: Extending Lemma 2 we can write the distribution function of $X_{k:n} - X_{m:n}$ as

$$\sum_{\mathbf{r}}\prod_{j=1}^{m}\frac{\lambda_{r_j}}{\Lambda-\sum_{i=1}^{j-1}\lambda_{r_i}}F^{[\mathbf{r}]}_{(k-m):(n-m)}, \tag{7}$$

where the notation is the same as in the statement of Proposition 1, and $F^{[\mathbf{r}]}_{(k-m):(n-m)}$ denotes the distribution function of the $(k-m)$th order statistics of $\{X_1,\ldots,X_n\}\backslash\{X_{r_1},\ldots,X_{r_m}\}$. The distribution of $Y_{k:n} - Y_{m:n}$ is simply $F^{(\gamma)}_{(k-m):(n-m)}$. The claim therefore follows from Theorem 2 and Lemma 1. ■

Proof of Proposition 1: Given the star ordering, we can establish the characterization for $\le_{\mathrm{st}}$ by examining the distribution function (7) near $x = 0$ (we did this for $m = 1$ in the proof of Theorem 2). Characterizations for $\le_{\mathrm{hr}}$ and $\le_{\mathrm{disp}}$ follow as in the proof of Theorem 1. ■

References

1. Balakrishnan, N. & Zhao, P. (2013). Ordering properties of order statistics from heterogeneous populations: A review with an emphasis on some recent developments. *Probability in the Engineering and Informational Sciences* 27: 403–443.
2. Bon, J.-L. & Păltănea, E. (2006). Comparison of order statistics in a random sequence to the same statistics with iid variables. *ESAIM: Probability and Statistics* 10: 1–10.
3. Da, G., Xu, M., & Balakrishnan, N. (2014). On the Lorenz ordering of order statistics from exponential populations and some applications. *Journal of Multivariate Analysis* 127: 88–97.
4. Diaconis, P. & Perlman, M.D. (1990). Bounds for tail probabilities of weighted sums of independent gamma random variables. *IMS Lecture Notes–Monograph Series* 16: 147–166.
5. Genest, C., Kochar, S.C., & Xu, M. (2009). On the range of heterogeneous samples. *Journal of Multivariate Analysis* 100: 1587–1592.
6. Karlin, S. (1968). *Total positivity*. Stanford: Stanford University Press.
7. Khaledi, B.-E. & Kochar, S. (2000). Some new results on stochastic comparisons on parallel systems. *Journal of Applied Probability* 37: 1123–1128.
8. Kochar, S.C. & Xu, M. (2009). Comparisons of parallel systems according to the convex transform order. *Journal of Applied Probability* 46: 342–352.
9. Kochar, S.C. & Xu, M. (2011). On the skewness of order statistics in the multiple-outlier models. *Journal of Applied Probability* 48: 271–284.

https://doi.org/10.1017/S026996481900041X Published online by Cambridge University Press

10. Marshall, A.W., Olkin, I., & Arnold, B. (2009). *Inequalities: Theory of majorization and its applications* (2nd ed.). New York: Springer.
11. Păltănea, E. (2008). On the comparison in hazard rate ordering of fail-safe systems. *Journal of Statistical Planning and Inference* 138: 1993–1997.
12. Păltănea, E. (2011). Bounds for mixtures of order statistics from exponentials and applications. *Journal of Multivariate Analysis* 102: 896–907.
13. Shaked, M. & Shanthikumar, J.G. (2007). *Stochastic orders*. New York: Springer.
14. Xu, M. & Balakrishnan, N. (2012). On the sample ranges from heterogeneous exponential variables. *Journal of Multivariate Analysis* 109: 1–9.
15. Yu, Y. (2009). Stochastic ordering of exponential family distributions and their mixtures. *Journal of Applied Probability* 46: 244–254.
16. Yu, Y. (2011). Some stochastic inequalities for weighted sums. *Bernoulli* 17: 1044–1053.
17. Yu, Y. (2017). On the unique crossing conjecture of Diaconis and Perlman on convolutions of gamma random variables. *Annals of Applied Probability* 27: 3893–3910.
18. Zhao, P., Li, X., & Balakrishnan, N. (2009). Likelihood ratio order of the second order statistic from independent heterogeneous exponential random variables. *Journal of Multivariate Analysis* 100: 952–962.

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