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ARCHIMEDEAN COPULAS AND TEMPORAL DEPENDENCE

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We study the dependence properties of stationary Markov chains generated by Archimedean copulas. Under some simple regularity conditions, we show that regular variation of the Archimedean generator at zero and one implies geometric ergodicity of the associated Markov chain. We verify our assumptions for a range of Archimedean copulas used in applications.

1. INTRODUCTION

A central aspect of time series analysis is the modeling of dependence over time. Workhorse time series models such as the autoregressive moving average (ARMA) model popularized by Box and Jenkins (1970), the generalized autoregressive conditional heteroskedasticity (GARCH) model of Engle (1982) and Bollerslev (1986), or the autoregressive conditional duration (ACD) model of Engle and Russell (1998) impose explicit conditions on the way in which a process evolves stochastically over time. It is natural to wonder what kind of mixing or ergodic properties might be implied or excluded by such conditions. Conditions under which ARMA processes are geometrically ergodic have been provided by Pham and Tran (1985) and Mokkadem (1988), while conditions under which GARCH and ACD processes are geometrically ergodic have been provided by Carrasco and Chen (2002) and Meitz and Saikkonen (2008). Results such as these enhance our understanding of the models underlying much empirical work and can be used to justify the application of invariance principles to partial sums of functions of processes driven by those models.

During the last five years, a new class of time series models has emerged in which copula functions are used to model dependence over time in a stationary Markov chain. The allure of this approach is that it facilitates the separate consideration of the dependence structure of the chain, specified using a copula, and the invariant distribution of the chain. This advantage was first emphasized by Darsow, Nguyen, and Olsen (1992). Chen and Fan (2006) introduced copula-based Markov models to the econometric literature, proposing a semiparametric estimation procedure in which the copula is specified parametrically while the

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invariant distribution is left unspecified. Following that contribution, a number of related papers have appeared, including Fentaw and Naik-Nimbalkar (2008), Gagliardini and Gouriéroux (2008), Bouyé and Salmon (2009), Chen, Koenker, and Xiao (2009), Chen, Wu, and Yi (2009), Ibragimov (2009), Ibragimov and Lentzas (2009), and Beare (2010).

Just as it is natural to ask whether ARMA, GARCH, or ACD models satisfy weak dependence conditions such as geometric ergodicity, it is also natural to ask when such conditions will be satisfied by copula-based Markov models. Chen and Fan (2006) suggested that Foster-Lyapunov drift conditions of the kind discussed in detail by Meyn and Tweedie (1993) could be used to verify suitable mixing conditions. This approach was used by Gagliardini and Gouriéroux (2008) to obtain conditions under which Markov chains generated by proportional hazard copulas are geometrically ergodic, and by Chen, Wu, and Yi (2009) to prove that Markov chains generated by Clayton, Gumbel, and t-copulas are geometrically ergodic. Beare (2010) proved that Markov chains generated by copulas with positive symmetric square integrable densities are geometrically ergodic, and commented on the relationship between maximal correlation and ρ-mixing.

In this paper we consider Markov chains generated by copulas that are strictly Archimedean. We identify conditions on the Archimedean generator that ensure the associated Markov chain is geometrically ergodic. These conditions are sufficiently general to encompass 11 of the parametric families of Archimedean copulas listed in Table 4.1 of Nelsen (2006). The key requirement we place upon the Archimedean generator is that it is regularly varying at zero and one. We prove geometric ergodicity by using the theory of regularly varying functions to verify a Foster-Lyapunov drift condition.

In a related contribution that may be of some independent interest, we provide an example of a parametric family of Archimedean copulas that generates a Markov chain that is ergodic but not geometrically ergodic. The key feature of this family is that the Archimedean generator is rapidly varying at zero. Our example is thus suggestive of a link between rapidly varying Archimedean generators and subgeometric rates of ergodicity.

The remainder of the paper is structured as follows. In Section 2 we define the notion of an Archimedean copula and explain what it means for the generator of an Archimedean copula to vary regularly at zero and one. In Section 3 we present our geometric ergodic theorem for Archimedean copulas and give a list of 11 parametric copula families to which the theorem applies. In Section 4 we state and discuss our example of an Archimedean copula that generates a subgeometric rate of ergodicity. Section 5 concludes. The proof of our main theorem is contained in the Appendix.

2. REGULARLY VARYING ARCHIMEDEAN GENERATORS

A copula is a bivariate probability distribution function on the unit square that has uniform marginal distribution functions. An Archimedean copula is a copula that

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can be defined in terms of a generator function φ in a way to be made precise shortly. Given a continuous, strictly decreasing function $\varphi : [0,1] \to [0,\infty]$ with $\varphi(1) = 0$, let the pseudo-inverse of φ, denoted $\varphi^{[-1]} : [0,\infty] \to [0,1]$, be defined by

$$\begin{aligned}\varphi^{[-1]}(u) &= \varphi^{-1}(u) \quad \text{for } u \in [0, \varphi(0)],\\ &= 0 \quad \text{for } u \in [\varphi(0), \infty].\end{aligned}$$

An Archimedean copula is defined as follows.

DEFINITION 2.1. *A copula $C : [0,1]^2 \to [0,1]$ is said to be Archimedean if there exists a continuous, strictly decreasing, convex function $\varphi : [0,1] \to [0,\infty]$ with $\varphi(1) = 0$ such that $C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$ for all $(u,v) \in [0,1]^2$. The function φ is referred to as the Archimedean generator of C. When $\varphi(0) = \infty$, C is said to be strictly Archimedean, and φ is said to be a strict Archimedean generator.*

Many examples of and facts about Archimedean copulas may be found in Chapter 4 of Nelsen (2006). Some of those examples may also be found in Section 3 below.

Definition 2.1 states that a copula C is Archimedean if we can find a generator φ such that $C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$ for all $u, v \in [0,1]$. It can be shown (see, e.g., Theorem 4.14 in Nelsen, 2006) that if φ is any function satisfying the conditions placed on the Archimedean generator in Definition 2.1, then $(u,v) \mapsto \varphi^{[-1]}(\varphi(u) + \varphi(v))$ is a well-defined copula. This result goes some way toward explaining the apparent popularity of Archimedean copulas in applied work: Constructing an Archimedean copula is as simple as choosing a continuous, strictly decreasing, convex function on $[0,1]$ that vanishes at one.

We are concerned in this paper with copulas that are strictly Archimedean. For strict Archimedean generators φ, there is no distinction between the pseudo-inverse $\varphi^{[-1]}$ and ordinary inverse φ^{-1}. The behavior of a strict Archimedean generator φ near the origin turns out to be of critical importance in the study of various limiting phenomena. Juri and Wüthrich (2002) have shown that if φ varies regularly at zero, then the index of regular variation determines the lower-tail dependence coefficient of the copula C. More strikingly, they have shown that, under mild regularity conditions, the extreme lower-tail dependence copula associated with any Archimedean copula whose generator is regularly varying at zero is a member of the family of Clayton copulas, with the Clayton parameter determined by the index of regular variation of the generator. Complementary results pertaining to upper-tail dependence were shown by Juri and Wüthrich (2003) to depend critically on the behavior of φ near one. In this paper we link regular variation of φ at zero and one to the property of geometric ergodicity in Markov chains whose dependence is characterized by the copula C.

Before we define the notions of regular variation at zero and one, it will be helpful to define the more standard notion of regular variation at infinity. Let f denote a positive measurable real valued function defined on $(1,\infty)$.

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DEFINITION 2.2. *The function f is said to be regularly varying at infinity with index $\varsigma \in \mathbb{R}$, written $f \in \mathscr{R}_\varsigma(\infty)$, if $f(sx)/f(x) \to s^\varsigma$ as $x \to \infty$, for all $s \in (0,\infty)$. If $f \in \mathscr{R}_0(\infty)$, then f is said to be slowly varying at infinity.*

Our choice of $(1,\infty)$ as the domain of f is not, of course, entirely necessary; what matters is that f is defined in a neighborhood of infinity. The property of regular variation is determined solely by the behavior of $f(x)$ as $x \to \infty$. But the domain $(1,\infty)$ is convenient for our purposes.

An extensive treatment of the theory of regular variation has been provided by Bingham, Goldie, and Teugels (1987), henceforth referred to as BGT. Here we will require only the most basic elements of this theory. Intuitively, a function is regularly varying at infinity if it behaves like a polynomial in x for large x. More formally, any $f \in \mathscr{R}_\varsigma(\infty)$ satisfies the decomposition $f(x) = x^\varsigma \ell(x)$ for all x, for some $\ell \in \mathscr{R}_0(\infty)$. This decomposition may be proved by noting that $x^{-\varsigma} f(x)$ is a slowly varying function of x at infinity. Functions that are slowly varying at infinity may be viewed as asymptotically akin to a constant. Critically, the logarithm function falls into this category. Phillips (2007) discusses the properties of slowly varying functions in the context of time series regression on slowly varying trends, with particular attention devoted to semilogarithmic growth regressions and log periodogram analysis of long memory.

The definition of regular variation at zero is a simple adaptation of the definition of regular variation at infinity. Let φ denote a positive measurable real valued function defined on $(0,1)$. Alternatively, φ may be a nonnegative measurable extended real valued function defined on $[0,1]$, provided that it is positive and finite on $(0,1)$.

DEFINITION 2.3. *The function φ is said to be regularly varying at zero with index $\eta \in \mathbb{R}$, written $\varphi \in \mathscr{R}_\eta(0)$, if $\varphi(su)/\varphi(u) \to s^\eta$ as $u \downarrow 0$, for all $s \in (0,\infty)$. If $\varphi \in \mathscr{R}_0(0)$, then φ is said to be slowly varying at zero.*

We must also define what it means for φ to be regularly varying at one.

DEFINITION 2.4. *The function φ is said to be regularly varying at one with index $\zeta \in \mathbb{R}$, written $\varphi \in \mathscr{R}_\zeta(1)$, if $\varphi(1-su)/\varphi(1-u) \to s^\zeta$ as $u \downarrow 0$, for all $s \in (0,\infty)$. If $\varphi \in \mathscr{R}_0(1)$, then φ is said to be slowly varying at one.*

The definitions of regular variation at zero and one derive directly from the definition of regular variation at infinity. Specifically, $\varphi \in \mathscr{R}_\eta(0)$ if and only if the map $x \mapsto \varphi(1/x)$ is in $\mathscr{R}_{-\eta}(\infty)$, while $\varphi \in \mathscr{R}_\zeta(1)$ if and only if the map $x \mapsto \varphi(1-1/x)$ is in $\mathscr{R}_{-\zeta}(\infty)$. We may also decompose functions that are regularly varying at zero or one into the product of polynomials and slowly varying functions, as we did earlier: We have $\varphi \in \mathscr{R}_\eta(0)$ if and only if $\varphi(u) = u^\eta \ell(u)$ for all $u \in (0,1)$ and some $\ell \in \mathscr{R}_0(0)$, and similarly $\varphi \in \mathscr{R}_\zeta(1)$ if and only if $\varphi(u) = (1-u)^\zeta \ell(u)$ for all $u \in (0,1)$ and some $\ell \in \mathscr{R}_0(1)$. In this sense, functions in $\mathscr{R}_\eta(0)$ behave like $u \mapsto u^\eta$ near zero, while functions in $\mathscr{R}_\zeta(1)$ behave like $u \mapsto (1-u)^\zeta$ near one.

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If an Archimedean generator φ is regularly varying at zero and/or one, the indices of regular variation must fall within a specified range. When $\varphi \in \mathscr{R}_\eta(0)$, we must have $\eta \leq 0$, since otherwise φ would vanish at zero, violating the assumption of strict monotonicity. And when $\varphi \in \mathscr{R}_\zeta(1)$, we must have $\zeta \geq 1$, since otherwise φ would fail to be convex (if $0 < \zeta < 1$) or fail to vanish at one (if $\zeta < 0$), or fail at least one of these two conditions (if $\zeta = 0$). Theorem 4.4 of Juri and Wüthrich (2003) shows how the indices of regular variation η and ζ determine the upper- and lower-tail dependence coefficients of the Archimedean copula C generated by φ. When $\varphi \in \mathscr{R}_\eta(0)$, the lower-tail dependence coefficient of C is equal to $2^{1/\eta}$ (for $\eta < 0$) or equal to 0 (for $\eta = 0$). And when $\varphi \in \mathscr{R}_\zeta(1)$, the upper-tail dependence coefficient of C is equal to $2 - 2^{1/\zeta}$. For a definition and further discussion of tail dependence coefficients, see Section 5.4 in Nelsen (2006).

It turns out that many Archimedean copulas used in practice have generators that are regularly varying at zero and one. Examples are provided in the following section.

3. GEOMETRIC ERGODIC THEOREM FOR ARCHIMEDEAN COPULAS

Let $\{U_t : t \in \mathbb{Z}\}$ be a stationary Markov chain defined on a probability space $(\Omega, \mathscr{F}, P)$. We assume that the invariant distribution of the chain is uniform on $(0, 1)$; that is, $U_t \sim U(0, 1)$ for each $t \in \mathbb{Z}$. Let $C : [0, 1]^2 \to [0, 1]$ denote the joint distribution function of (U_0, U_1). Since $\{U_t\}$ is stationary with invariant distribution $U(0, 1)$, the joint distribution function C is the unique copula for each consecutive pair (U_t, U_{t+1}), $t \in \mathbb{Z}$. The Markov property ensures that the entire joint distribution of $\{U_t\}$ is uniquely determined by C.

We are concerned in this paper with identifying conditions on C that are sufficient for $\{U_t\}$ to be geometrically ergodic. Let $\mathscr{B}$ denote the σ-field of Borel subsets of $(0, 1)$.

DEFINITION 3.1. *The stationary Markov chain $\{U_t : t \in \mathbb{Z}\}$ is said to be geometrically ergodic if, for almost everywhere (a.e.) $u \in (0, 1)$, there exists a real number $r > 1$ such that*

$$\sum_{k=1}^{\infty} r^k \sup_{B \in \mathscr{B}} |P(U_k \in B | U_0 = u) - P(U_k \in B)| < \infty.$$

Remark 3.1. Definition 3.1 is a minor variation on Definition 15.7 of Meyn and Tweedie (1993). We have dropped those authors' requirement that $\{U_t\}$ be positive Harris recurrent, which is, in any case, implied by Assumption 3.1 below. We have also departed from Meyn and Tweedie's definition of geometric ergodicity by requiring that the summability condition in Definition 3.1 hold only for a.e. $u \in (0, 1)$, rather than all $u \in (0, 1)$. For our purposes, the distinction is immaterial.

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Remark 3.2. For a stationary real valued Markov chain, geometric ergodicity is equivalent to exponentially fast β-mixing. Davydov (1973) defined the β-mixing coefficients for Markov chains in a way that makes the connection to Definition 3.1 immediately apparent. Theorem 21.19 of Bradley (2007) states a number of equivalent dependence conditions for stationary Markov chains, including geometric ergodicity and exponentially fast β-mixing. In an earlier paper (Beare, 2010) we identified conditions on C under which $\{U_t\}$ is β-mixing at an exponential rate. A key condition was that C is absolutely continuous with square integrable density. Most Archimedean copulas used in applications do not satisfy this condition; indeed, no absolutely continuous copula exhibiting positive tail dependence admits a square integrable density, by Theorem 3.3 in Beare (2010), so our previous result is of limited applicability in the present context.

Remark 3.3. It is clear from Definition 3.1 that if $\{U_t\}$ is geometrically ergodic then, for any Borel measurable $h : (0,1) \to \mathbb{R}$, $\{h(U_t)\}$ is also geometrically ergodic. This property makes our assumption that $\{U_t\}$ has invariant distribution $U(0,1)$ innocuous. Suppose we have a stationary Markov chain $\{X_t\}$ with continuous invariant distribution function F, and with C the unique copula for (X_0, X_1); that is, C satisfying $P(X_0 \leq x_0, X_1 \leq x_1) = C(F(x_0), F(x_1))$ for all $x_0, x_1 \in \mathbb{R}$. The entire distribution of this chain is identical to the distribution of the chain $\{Q(U_t)\}$, where $Q : (0,1) \to \mathbb{R}$ is the quantile function corresponding to F. Thus, if C is such that $\{U_t\}$ is geometrically ergodic, then $\{X_t\}$ must also be geometrically ergodic. This conclusion remains true even if F is not continuous, though in this case C may be one of many copulas for (X_0, X_1)

Geometric ergodicity of $\{U_t\}$ will be established under the following assumption on C.

Assumption 3.1. The copula C is strictly Archimedean, with strict Archimedean generator φ satisfying the following conditions.

(i) $\varphi \in \mathscr{R}_\eta(0)$ for some $\eta \in (-\infty, 0]$, and $\varphi \in \mathscr{R}_\zeta(1)$ for some $\zeta \in [1, \infty)$.
(ii) φ is twice continuously differentiable on $(0,1)$.
(iii) φ'' is monotone in a right-neighborhood of zero and in a left-neighborhood of one.
(iv) φ'' is strictly positive on $(0,1)$.
(v) If $\eta = 0$, then
  (a) $-\varphi' \in \mathscr{R}_{-1}(0)$, and
  (b) $u\varphi'(u)$ is bounded away from zero for u in a right-neighborhood of zero.
(vi) If $\zeta = 1$, then φ' and φ'' are bounded away from zero in a left-neighborhood of one.

We shall shortly provide a number of examples of Archimedean copulas satisfying Assumption 3.1. First, we make the following remarks.

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Remark 3.4. Theorem 1 of Genest and MacKay (1986b) states that an Archimedean copula C with twice continuously differentiable generator φ is absolutely continuous if and only if $\lim_{u\downarrow 0}\varphi(u)/\varphi'(u) = 0$. Under Assumptions 3.1(i),(ii), since $\varphi \in \mathscr{R}_\eta(0)$ and φ' is monotone, the monotone density theorem (see Theorem 1.7.2 in BGT) implies that $-\varphi' \in \mathscr{R}_{\eta-1}(0)$, provided that $\eta < 0$. When $\eta = 0$, the same is true by Assumption 3.1(v)(a). With $\varphi \in \mathscr{R}_\eta(0)$ and $-\varphi' \in \mathscr{R}_{\eta-1}(0)$, it is easy to show that $-\varphi(\cdot)/\varphi'(\cdot) \in \mathscr{R}_1(0)$, from which it follows that $\lim_{u\downarrow 0}\varphi(u)/\varphi'(u) = 0$. Thus, under Assumption 3.1, C is absolutely continuous, and we may obtain its density c on $(0,1)^2$ by differentiation:

$$c(u,v) = -\frac{\varphi''(C(u,v))\varphi'(u)\varphi'(v)}{\varphi'(C(u,v))^3} \quad \text{for } (u,v) \in (0,1)^2. \tag{3.1}$$

As noted by Genest and MacKay (1986b), $c(u,v) > 0$ for all u, v such that $\varphi(u) + \varphi(v) < \varphi(0)$. C is strictly Archimedean under Assumption 3.1, so $\varphi(0) = \infty$, and $c > 0$ on $(0,1)^2$.

Remark 3.5. As noted in Remark 3.4, the monotone density theorem ensures that $-\varphi' \in \mathscr{R}_{\eta-1}(0)$ when $\eta < 0$. The point of Assumption 3.1(v)(a) is to ensure that this is also true when $\eta = 0$. In fact, we are unaware of any example of a strict Archimedean generator in $\mathscr{R}_0(0)$ that is twice continuously differentiable and that violates Assumption 3.1(v)(a), and must confess there is some possibility that Assumption 3.1(v)(a) is redundant. Charpentier and Segers (2007) provide an example of a continuously differentiable strict Archimedean generator φ such that $\varphi \in \mathscr{R}_0(0)$ and $-\varphi' \notin \mathscr{R}_{-1}(0)$; however, this generator is not twice differentiable and so would not satisfy Assumption 3.1(ii) above.

Remark 3.6. Assumptions 3.1(v)(b) and 3.1(vi) are not always satisfied. We shall provide an example of a copula that violates both of them, while satisfying the remaining parts of Assumption 3.1. Consider the strict Archimedean generator

$$\varphi(u) = \log(1 - \log u).$$

The Archimedean copula corresponding to this generator is a member of the so-called Gumbel-Barnett family of copulas, which forms the ninth entry in Table 4.1 of Nelsen (2006); we have set the parameter value θ equal to one. It is easily verified that φ satisfies Assumptions 3.1(i) through 3.1(v)(a) with $\eta = 0$ and $\zeta = 1$. Differentiating φ, we obtain

$$\varphi'(u) = \frac{-1}{u(1 - \log u)},$$

implying that $\lim_{u\downarrow 0} u\varphi'(u) = 0$. Thus, Assumption 3.1(v)(b) is violated. Differentiating φ again, we find that

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$$\varphi''(u) = \frac{-\log u}{u^2(1-\log u)^2}.$$

We can see that $\lim_{u\uparrow 1} \varphi''(u) = 0$, and so Assumption 3.1(vi) is also violated.

Remark 3.6 notwithstanding, there are many well-known families of Archimedean copulas that satisfy Assumption 3.1. We shall provide 11 examples of such families, each of which may be found in Table 4.1 of Nelsen (2006). Sometimes we must restrict the parameter space given by Nelsen to ensure that Assumption 3.1 is satisfied. To conserve space, we do not give details of precisely how Assumption 3.1 is verified in each example. Typically, verification can be achieved by differentiating φ twice and perhaps applying l'Hôpital's rule or a Taylor expansion where appropriate. Similar methods were presumably used to derive several of the tail dependence coefficients reported in Example 5.2.2 in Nelsen (2006).

Example 3.1
The family of Clayton copulas, listed as the first entry in Table 4.1 of Nelsen (2006), corresponds to the family of Archimedean generators

$$\varphi(u) = \frac{1}{\theta}\left(u^{-\theta} - 1\right), \quad \theta \in [-1, \infty) \setminus \{0\}.$$

This family satisfies Assumption 3.1 with $\eta = -\theta$ and $\zeta = 1$, provided that $\theta \in (0, \infty)$. When $\theta \in [-1, 0)$, $\varphi(0) = -1/\theta$, and so φ is not strict and Assumption 3.1 does not hold.

Example 3.2
The third entry in Table 4.1 of Nelsen (2006) is the family of Ali-Mikhail-Haq copulas, which have generators

$$\varphi(u) = \log \frac{1-\theta(1-u)}{u}, \quad \theta \in [-1, 1).$$

This family satisfies Assumption 3.1 with $\eta = 0$ and $\zeta = 1$, provided that $\theta \in (-1, 1)$. When $\theta = -1$, $\lim_{u\uparrow 1} \varphi''(u) = 0$, and so Assumption 3.1(vi) is violated.

Example 3.3
The well-known Gumbel or Gumbel-Hougaard family of copulas comprises the fourth entry in Table 4.1 of Nelsen (2006). The corresponding family of generators is

$$\varphi(u) = (-\log u)^{\theta}, \quad \theta \in [1, \infty).$$

This family satisfies Assumption 3.1 with $\eta = 0$ and $\zeta = \theta$.

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Example 3.4
The Frank family of copulas, listed fifth in Table 4.1 of Nelsen (2006), is defined by the generators

$$\varphi(u) = -\log \frac{e^{-\theta u} - 1}{e^{-\theta} - 1}, \quad \theta \in (-\infty, \infty) \setminus \{0\}.$$

This family satisfies Assumption 3.1 with $\eta = 0$ and $\zeta = 1$.

Example 3.5
The Joe family of copulas forms the sixth entry in Table 4.1 of Nelsen (2006), and has generators

$$\varphi(u) = -\log\left(1 - (1-u)^{\theta}\right), \quad \theta \in [1, \infty).$$

This family satisfies Assumption 3.1 with $\eta = 0$ and $\zeta = \theta$

Example 3.6
The family of copulas listed tenth in Table 4.1 of Nelsen (2006) has generators

$$\varphi(u) = \log\left(2u^{-\theta} - 1\right), \quad \theta \in (0, 1].$$

This family appears to be an invention of Nelsen, as do the families in our five subsequent examples. It satisfies Assumption 3.1 with $\eta = 0$ and $\zeta = 1$, provided that $\theta \in (0, 1)$. When $\theta = 1$, $\lim_{u \uparrow 1} \varphi''(u) = 0$, and so Assumption 3.1(vi) is violated.

Example 3.7
The twelfth family of copulas listed in Table 4.1 of Nelsen (2006) is defined by the generators

$$\varphi(u) = \left(u^{-1} - 1\right)^{\theta}, \quad \theta \in [1, \infty).$$

This family satisfies Assumption 3.1 with $\eta = -\theta$ and $\zeta = \theta$.

Example 3.8
Listed thirteenth in Table 4.1 of Nelsen (2006), we have the family of generators

$$\varphi(u) = (1 - \log u)^{\theta} - 1, \quad \theta \in (0, \infty).$$

The corresponding family of copulas satisfies Assumption 3.1 with $\eta = 0$ and $\zeta = 1$, provided that $\theta \in [1, \infty)$. When $\theta \in (0, 1)$, $\lim_{u \downarrow 0} u\varphi'(u) = 0$, and so Assumption 3.1(v)(b) is violated.

Example 3.9
The fourteenth family of copulas listed in Table 4.1 of Nelsen (2006) is defined by the generators

$$\varphi(u) = \left(u^{-1/\theta} - 1\right)^{\theta}, \quad \theta \in [1, \infty).$$

This family satisfies Assumption 3.1 with $\eta = -1$ and $\zeta = \theta$.

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Example 3.10
The sixteenth family of copulas listed in Table 4.1 of Nelsen (2006) is defined by the generators

$$\varphi(u) = (\theta u^{-1} + 1)(1-u), \quad \theta \in [0, \infty).$$

This family satisfies Assumption 3.1 with $\eta = -1$ and $\zeta = 1$, provided that $\theta \in (0, \infty)$. When $\theta = 0$, $\varphi(0) = 1$, and so φ is not strict and Assumption 3.1 does not hold.

Example 3.11
Our final example is provided by the seventeenth entry in Table 4.1 of Nelsen. This family of generators is given by

$$\varphi(u) = -\log \frac{(1+u)^{-\theta} - 1}{2^{-\theta} - 1}, \quad \theta \in (-\infty, \infty) \setminus \{0\},$$

and the corresponding family of copulas satisfies Assumption 3.1 with $\eta = 0$ and $\zeta = 1$.

We have seen that many families of Archimedean copulas satisfy Assumption 3.1 over much or all of their parameter space. The following theorem, which is the main result of the paper, states that Archimedean copulas satisfying Assumption 3.1 generate geometrically ergodic Markov chains. The proof is deferred to the Appendix.

THEOREM 3.1. *Suppose* $\{U_t : t \in \mathbb{Z}\}$ *is a stationary Markov chain whose invariant distribution is uniform on* $(0, 1)$. *Let* C *denote the joint distribution function of* (U_0, U_1). *If* C *satisfies Assumption 3.1, then* $\{U_t : t \in \mathbb{Z}\}$ *is geometrically ergodic.*

Remark 3.7. Theorem 3.1 shows that the 11 families of Archimedean copulas listed in Examples 3.1–3.11 generate geometrically ergodic Markov chains over the stated parameter ranges. For three of those families, this result was known already. Theorem 2.1 of Chen, Wu, and Yi (2009) established geometric ergodicity for the Clayton and Gumbel families, and Theorem 3.1 of Beare (2010) established geometric ergodicity for the Frank family. To the best of our knowledge, geometric ergodicity for the remaining eight families has not been previously established.

Remark 3.8. Theorem 3.1 is an application of the geometric ergodic theorem, discussed in detail by Meyn and Tweedie (1993). The proof involves verifying that the one-step dependence characterized by C satisfies a Foster-Lyapunov drift condition. Chen, Wu, and Yi (2009) used precisely this approach to prove geometric ergodicity for the Clayton and Gumbel families. Our proof is based loosely on theirs, though the conditions we impose on C are much weaker.

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Remark 3.9. It should perhaps be emphasized that Theorem 3.1 applies to specific copula functions, not to families of copulas functions. When we say that geometric ergodicity is satisfied by, for instance, the family of Clayton copulas with $\theta \in (0, \infty)$ (recall Example 3.1), we mean that each member of that family generates a geometrically ergodic Markov chain. We do not mean that geometric ergodicity is in some sense uniform over the family of Clayton copulas. The rate of β-mixing may be very slow (albeit geometric) for specific members of the family. This is similar to, for instance, the autoregressive model, where geometric decay of the autocovariances is satisfied when the autoregressive coefficient is less than one in absolute value, but with a rate that becomes slower as the absolute value of the autoregressive coefficient approaches one.

Remark 3.10. One might wonder if it is necessarily the case that higher tail dependence values will lead to slower rates of β-mixing. This is an interesting question to which we do not have an answer. Ideally, we would like to obtain a generalization of Theorem 3.1 in which the rate of geometric decay of β-mixing coefficients is specified in terms of the indices of regular variation ζ and η. Our proof of Theorem 3.1 involves an application of Theorem 16.0.1 of Meyn and Tweedie (1993), which delivers us geometric ergodicity but does not indicate the precise β-mixing rate. We would therefore need to take a different approach to the proof in order to determine whether the rate of β-mixing varies with tail dependence in a systematic way.

4. AN EXAMPLE OF SUBGEOMETRIC ERGODICITY

In the previous section we saw that many families of Archimedean copulas can be used to generate Markov chains that are geometrically ergodic. In this section we identify a family of Archimedean copulas for which the associated rate of ergodicity is subgeometric.

Example 4.1
Consider the family of Archimedean generators

$$\varphi(u) = \exp\left(u^{-\theta}\right) - \mathrm{e}, \quad \theta \in (0, \infty).$$

The corresponding family of copulas forms the twentieth entry in Table 4.1 of Nelsen (2006). Clearly, φ is not regularly varying at zero, and so Theorem 3.1 cannot be applied. In fact, $\log \varphi \in \mathscr{R}_{-\theta}(0)$, and φ is said to be rapidly varying at zero; see Section 2.4 in BGT for a formal definition of rapid variation and further discussion. We will show that, when $\theta > 1$, a Markov chain generated by φ is not geometrically ergodic.

Let $\{U_t : t \in \mathbb{Z}\}$ be a stationary Markov chain with the joint distribution of U_0 and U_1 given by C, the Archimedean copula generated by φ. Geometric ergodicity of $\{U_t\}$ is equivalent to exponential decay of the β-mixing coefficients associated with $\{U_t\}$; see, e.g., Theorem 21.19 in Bradley (2007). The β-mixing coefficients for $\{U_t\}$ are bounded from below by the corresponding α-mixing coefficients.

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To disprove geometric ergodicity, it therefore suffices to demonstrate that the α-mixing coefficients for $\{U_t\}$ do not decay to zero at an exponential rate. In fact, we will show that $\liminf_{k\to\infty} k\alpha_k \geq 1$, demonstrating that the decay rate of α_k is no faster than k^{-1}.

The kth α-mixing coefficient α_k for $\{U_t\}$ is defined as the supremum of $|P(A \cap B) - P(A)P(B)|$ over all $A \in \sigma(U_t : t \leq 0)$ and $B \in \sigma(U_t : t \geq k)$. Therefore, for $k \in \mathbb{N}$, we have

$$\alpha_k \geq \left| P\left(U_0 \leq k^{-1}, U_k \leq k^{-1}\right) - P\left(U_0 \leq k^{-1}\right) \right.$$
$$\left. \times P\left(U_k \leq k^{-1}\right) \right| = \left| C_k\left(k^{-1}, k^{-1}\right) - k^{-2} \right|, \tag{4.1}$$

where C_k denotes the joint distribution function of U_0 and U_k. By elementary arguments,

$$\begin{aligned} C_k\left(k^{-1}, k^{-1}\right) &\geq P\left(U_0 \leq k^{-1}, U_{k-1} \leq k^{-1}, U_k \leq k^{-1}\right) \\ &\geq P\left(U_0 \leq k^{-1}, U_{k-1} \leq k^{-1}\right) \\ &\quad + P\left(U_{k-1} \leq k^{-1}, U_k \leq k^{-1}\right) - P\left(U_{k-1} \leq k^{-1}\right) \\ &= C_{k-1}\left(k^{-1}, k^{-1}\right) + C\left(k^{-1}, k^{-1}\right) - k^{-1}. \end{aligned}$$

On recursion, we obtain

$$C_k\left(k^{-1}, k^{-1}\right) \geq kC\left(k^{-1}, k^{-1}\right) - 1 + k^{-1}. \tag{4.2}$$

Convexity of φ implies that

$$\varphi\left(k^{-1} + \frac{\varphi\left(k^{-1}\right)}{\varphi'\left(k^{-1}\right)}\right) \geq \varphi\left(k^{-1}\right) + \varphi'\left(k^{-1}\right) \cdot \frac{\varphi\left(k^{-1}\right)}{\varphi'\left(k^{-1}\right)} = 2\varphi\left(k^{-1}\right), \tag{4.3}$$

provided of course that $k^{-1} + \varphi(k^{-1})/\varphi'(k^{-1}) > 0$. Since $\log\varphi \in \mathscr{R}_{-\theta}(0)$ and $\log\varphi$ is convex, the monotone density theorem implies that $-\varphi'(\cdot)/\varphi(\cdot) \in \mathscr{R}_{-\theta-1}(0)$. It follows that $k^{\gamma}\varphi(k^{-1})/\varphi'(k^{-1}) \to 0$ as $k \to \infty$ for any $\gamma < \theta + 1$, and so we have $k^{-1} + \varphi(k^{-1})/\varphi'(k^{-1}) > 0$ for all k sufficiently large. From (4.3) we obtain

$$C\left(k^{-1}, k^{-1}\right) = \varphi^{-1}\left(2\varphi\left(k^{-1}\right)\right) \geq k^{-1} + \frac{\varphi\left(k^{-1}\right)}{\varphi'\left(k^{-1}\right)} \tag{4.4}$$

for all k sufficiently large. Combining (4.4) with (4.2) yields

$$C_k\left(k^{-1}, k^{-1}\right) \geq k^{-1} + \frac{k\varphi\left(k^{-1}\right)}{\varphi'\left(k^{-1}\right)}$$

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for all k sufficiently large. Recalling that $k^{\gamma}\varphi\left(k^{-1}\right)/\varphi'(k^{-1}) = o(1)$ for any $\gamma < \theta + 1$, and our assumption that $\theta > 1$, we deduce that $C_k(k^{-1}, k^{-1}) \geq k^{-1} + o(k^{-1})$. In view of (4.1), this proves that $\liminf_{k\to\infty} k\alpha_k \geq 1$.

We conclude this section with some remarks on Example 4.1.

Remark 4.1. Inspection of our demonstration that $\{U_t\}$ is not geometrically ergodic reveals that only three features of the generator φ were essential to our argument. They are: (1) φ is differentiable; (2) $\log\varphi \in \mathscr{R}_{\eta}(0)$ for some $\eta < -1$; (3) $\log\varphi$ is convex. In fact, property (3) was used only to justify the application of the monotone density theorem, and thus need hold only locally to zero. Our argument thus demonstrates that any reasonably behaved Archimedean generator that diverges sufficiently rapidly at zero will generate a Markov chain that fails to be geometrically ergodic.

Remark 4.2. Perhaps the best-known nontrivial example of a stationary Markov chain that is not geometrically ergodic is the stationary linear process $X_t = \sum_{j=0}^{\infty} 2^{-j}\varepsilon_{t-j}$, $t \in \mathbb{Z}$, formed from independent innovations ε_t, $t \in \mathbb{Z}$, that are each equal to 0 with probability $1/2$ and $1/2$ with probability $1/2$. This process was studied in detail by Andrews (1984); see also Example 2.15 in Bradley (2007) and the references given there. It is known that $\{X_t\}$ is not α-mixing, and in fact satisfies $\alpha_k = 1/4$ for all k. Further, the marginal distribution of X_0 is $U(0, 1)$, so that the unique copula characterizing the dependence structure of $\{X_t\}$ is simply the joint distribution function of X_0 and X_1. As noted in Remark 4.2 in Beare (2010), this copula is absolutely singular with respect to Lebesgue measure on the unit square. In contrast, the Archimedean copula in Example 4.1 is absolutely continuous with respect to Lebesgue measure on the unit square, and admits a density that is positive almost everywhere. Using, for instance, Theorems 21.3 and 21.5 in Bradley (2007), one may show that this property implies that $\{U_t\}$ is ergodic and β-mixing. The rates of ergodicity and β-mixing are, however, subexponential.

The example just given of a stationary autoregressive process that fails to be α-mixing brings to mind the limitations of mixing conditions as characterizations of weak dependence. Linear processes are known to satisfy laws of large numbers and invariance principles under suitable summability and moment conditions — see, e.g., Phillips and Solo (1992) — but these conditions are not sufficient for the satisfaction of strong mixing conditions such as α-mixing. Geometric ergodicity is an even more powerful dependence condition that may be more than sufficient for whatever purpose is at hand.

Remark 4.3. It is not clear from our discussion whether geometric ergodicity obtains in Example 4.1 when $\theta \in (0, 1)$. What we do know in this case is that $\{U_t\}$ fails to be ρ-mixing, and has $\rho_k = 1$ for all k. See Beare (2010) for further discussion of ρ-mixing in copula-based Markov models. The failure of ρ-mixing is a consequence of the fact that the copula in Example 4.1 exhibits perfect (i.e.,

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unit) lower-tail dependence, which is itself a consequence of the rapid variation of φ at zero; see Theorem 3.9 of Juri and Wüthrich (2002). Chen, Hansen, and Carrasco (2010) identify conditions under which stationary diffusions that fail to be ρ-mixing must necessarily exhibit subgeometric rates of α- and β-mixing.

Remark 4.4. Given that the rate of α-mixing in Example 4.1 has been shown to be no faster than k^{-1}, it is tempting to describe $\{U_t\}$ as exhibiting long memory of some form. Ibragimov and Lentzas (2009) considered the possibility that copulas may be used to generate "long memory-like" behavior in Markov chains. Nevertheless, the traditional definition of long memory concerns the summability of autocovariances, and it is not clear to us that the nonsummability of α-mixing coefficients implies that the autocovariances of $\{U_t\}$, or indeed of $\{U_t^p\}$ for some power p, are themselves nonsummable. We therefore refrain from suggesting a connection between long memory and rapid variation of φ at zero.

5. CONCLUSION

In this paper we have identified conditions under which a Markov chain whose dependence is characterized by an Archimedean copula will be geometrically ergodic. These conditions are sufficiently general to encompass 11 families of Archimedean copulas described in the monograph of Nelsen (2006), over a range of possible parameter values. We hope that our results will prove useful to researchers developing statistical methods for models of this kind, by making invariance principles or moment inequalities for weakly dependent processes more readily applicable.

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APPENDIX: Proof of Theorem 3.1

In our proof of Theorem 3.1 we shall employ five supplementary lemmas. Proving these lemmas requires multiple applications of the monotone density theorem and Potter's theorem. For a statement of these results, refer to Theorem 1.7.2 and Theorem 1.5.6 in BGT.

LEMMA A.1. *Under Assumption 3.1, for* $p \in [0, 1-1/\eta)$, *with* $1/\eta := -\infty$ *when* $\eta = 0$, *we have*

$$\lim_{u \downarrow 0} \int_0^1 \frac{\varphi''(su)}{\varphi''(u)} \left(\frac{\varphi'(su)}{\varphi'(u)} \right)^{-2} \left(\frac{\varphi(su)}{\varphi(u)} - 1 \right)^p ds = \int_0^1 s^{-\eta} (s^\eta - 1)^p ds.$$

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Proof. The integrand on the left-hand side of the equation to be proved is written as the product of three terms. Since $\varphi \in \mathscr{R}_\eta(0)$, the third term satisfies

$$\lim_{u \downarrow 0} \left(\frac{\varphi(su)}{\varphi(u)} - 1 \right)^p = (s^\eta - 1)^p$$

pointwise in s. We know from the monotone density theorem (when $\eta < 0$) or by Assumption 3.1(v)(a) (when $\eta = 0$) that $-\varphi' \in \mathscr{R}_{\eta-1}(0)$, so the second term satisfies

$$\lim_{u \downarrow 0} \left(\frac{\varphi'(su)}{\varphi'(u)} \right)^{-2} = s^{2-2\eta}$$

pointwise in s. Since $-\varphi' \in \mathscr{R}_{\eta-1}(0)$ and $\eta - 1 < 0$, the monotone density theorem also implies that $\varphi''(u) \in \mathscr{R}_{\eta-2}(0)$, and so the first term satisfies

$$\lim_{u \downarrow 0} \frac{\varphi''(su)}{\varphi''(u)} = s^{\eta-2}$$

pointwise in s. Consequently, our integrand converges pointwise to $s^{-\eta}(s^\eta - 1)^p$ as $u \downarrow 0$. Using Potter's theorem, we can show that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that our integrand is bounded by $2s^{(p-1)\eta-\varepsilon}$ for all $u \in (0, \delta)$. Since $(p-1)\eta > -1$, we may choose ε small enough to make this bound integrable on $(0, 1)$. The dominated convergence theorem now delivers our desired result. ■

LEMMA A.2. *Under Assumption 3.1, for $p \in (0, 1)$ we have*

$$\lim_{u \downarrow 0} \varphi(u)^{-p} \int_0^1 \frac{\varphi''(su)}{\varphi''(u)} \left(\frac{\varphi'(su)}{\varphi'(u)} \right)^{-2} (\varphi(su) - \varphi(u))^{-p} \, ds = 0.$$

Proof. Convexity of φ implies the inequality $\varphi(su) - \varphi(u) \geq -(1-s)u\varphi'(u)$, valid for $s \in (0, 1)$. We know from the monotone density theorem (when $\eta < 0$) or by Assumption 3.1(v)(b) (when $\eta = 0$) that $u\varphi'(u)$ is bounded away from zero in a neighborhood of zero. Since $\lim_{u \downarrow 0} \varphi(u) = \infty$, it remains only to show that

$$\limsup_{u \downarrow 0} \int_0^1 \frac{\varphi''(su)}{\varphi''(u)} \left(\frac{\varphi'(su)}{\varphi'(u)} \right)^{-2} (1-s)^{-p} \, ds < \infty.$$

Using the monotone density theorem, we can show that the above integrand converges pointwise to $s^{-\eta}(1-s)^{-p}$ as $u \downarrow 0$. And using Potter's theorem, we can show that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that our integrand is bounded by $2s^{-\eta-\varepsilon}(1-s)^{-p}$ for all $u \in (0, \delta)$. Since $p < 1$, we may choose $\varepsilon < 1 - \eta$ to make this bound integrable in s. The dominated convergence theorem thus yields

$$\lim_{u \downarrow 0} \int_0^1 \frac{\varphi''(su)}{\varphi''(u)} \left(\frac{\varphi'(su)}{\varphi'(u)} \right)^{-2} (1-s)^{-p} \, ds = \int_0^1 s^{-\eta}(1-s)^{-p} ds.$$

Since $p < 1$, the limiting integral is finite, and we are done. ■

LEMMA A.3. *Under Assumption 3.1, if $\zeta > 1$ then for $p < 1 - 1/\zeta$ we have*

$$\lim_{u \downarrow 0} \int_1^{\frac{1}{u}} \frac{\varphi''(1-su)}{\varphi''(1-u)} \left(\frac{\varphi'(1-su)}{\varphi'(1-u)} \right)^{-2} \left(\frac{\varphi(1-su)}{\varphi(1-u)} - 1 \right)^p ds = \int_1^\infty s^{-\zeta}(s^\zeta - 1)^p ds.$$

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Proof. Since $\varphi \in \mathscr{R}_{\zeta}(1)$ and $\zeta > 1$, we know from the monotone density theorem that $-\varphi' \in \mathscr{R}_{\zeta-1}(1)$ and $\varphi'' \in \mathscr{R}_{\zeta-2}(1)$. Consequently, as $u \downarrow 0$, the integrand on the left-hand side of the equation to be proved converges to $s^{-\zeta}(s^{\zeta}-1)^{p}$ pointwise on $(1,\infty)$. Using Potter's theorem, we can show that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that our integrand is bounded by $2s^{(p-1)\zeta+\varepsilon}$ for all $u \in (0,\delta)$. Since $(p-1)\zeta < -1$, we may choose ε small enough to make this bound integrable on $(1,\infty)$. The dominated convergence theorem now delivers our desired result. ■

LEMMA A.4. *Fix $u_0 \in (0,1)$. Under Assumption 3.1, for $u \in [u_0,1)$ and $p \in [0, 1-1/\eta)$ we have*

$$\int_0^1 \left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(u)}{w}\right)\right)-\varphi(u)\right)^{p} dw \le -\varphi'(u_0)\int_0^{u_0}\frac{\varphi(v)^{p}\varphi''(v)}{\varphi'(v)^{2}}dv < \infty.$$

Proof. Since φ is decreasing and strictly convex, $\varphi(\varphi'^{-1}(\varphi'(\cdot)/w))$ is decreasing for each $w \in (0,1)$. Combined with the nonnegativity of φ, we find that

$$\int_0^1 \left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(u)}{w}\right)\right)-\varphi(u)\right)^{p} dw \le \int_0^1 \varphi\left(\varphi'^{-1}\left(\frac{\varphi'(u_0)}{w}\right)\right)^{p} dw.$$

The first inequality to be proved follows easily using the change of variables $w = \varphi'(u_0)/\varphi'(v)$. It remains to show that $\varphi(\cdot)^{p}\varphi'(\cdot)^{-2}\varphi''(\cdot)$ is integrable on $(0,u_0)$. Twice continuous differentiability of φ on $(0,1)$ ensures integrability provided that our integrand does not diverge too rapidly at the origin. In fact, Assumption 3.1(v)(a) and the monotone density theorem imply that $\varphi \in \mathscr{R}_{\eta}(0)$, $-\varphi' \in \mathscr{R}_{\eta-1}(0)$ and $\varphi'' \in \mathscr{R}_{\eta-2}(0)$, and so we have $\varphi(\cdot)^{p}\varphi'(\cdot)^{-2}\varphi''(\cdot) \in \mathscr{R}_{\eta(p-1)}(0)$. Since $\eta(p-1) > -1$, integrability holds. ■

LEMMA A.5. *Fix $u_0, u_1 \in (0,1)$ with $u_0 < u_1$. Under Assumption 3.1, for $u \in [u_0,u_1]$ and $p \in [0,1)$ we have*

$$\int_0^1 \left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(u)}{w}\right)\right)-\varphi(u)\right)^{-p} dw$$
$$\le (1-\sqrt{p})^{-\sqrt{p}}u_1^{\sqrt{p}-p}(-\varphi'(u_0))^{1-p}\left(\int_0^{u_1}\left(\frac{\varphi''(v)}{\varphi'(v)^{2}}\right)^{1/(1-\sqrt{p})}dv\right)^{1-\sqrt{p}} < \infty.$$

Proof. Combining the change of variables $w = \varphi'(u)/\varphi'(su)$ with the inequality $\varphi(su)-\varphi(u) \ge -(u-su)\varphi'(u)$, valid for $s \in (0,1)$ due to the convexity of φ, we obtain

$$\int_0^1 \left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(u)}{w}\right)\right)-\varphi(u)\right)^{-p} dw \le u^{1-p}(-\varphi'(u))^{1-p}\int_0^1 (1-s)^{-p}\frac{\varphi''(su)}{\varphi'(su)^{2}}ds.$$

Applying Hölder's inequality and the change of variables $v = su$,

$$\int_0^1 (1-s)^{-p}\frac{\varphi''(su)}{\varphi'(su)^{2}}ds \le \left(\int_0^1 (1-s)^{-\sqrt{p}}ds\right)^{\sqrt{p}}\left(\int_0^1\left(\frac{\varphi''(su)}{\varphi'(su)^{2}}\right)^{1/(1-\sqrt{p})}ds\right)^{1-\sqrt{p}}$$
$$= (1-\sqrt{p})^{-\sqrt{p}}u^{\sqrt{p}-1}\left(\int_0^{u}\left(\frac{\varphi''(v)}{\varphi'(v)^{2}}\right)^{1/(1-\sqrt{p})}dv\right)^{1-\sqrt{p}}.$$

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The first inequality to be proved now follows from the inequalities $u_0 \le u \le u_1$ and $-\varphi'(u) \le -\varphi'(u_0)$. It remains to show that $(\varphi'(\cdot)^{-2}\varphi''(\cdot))^{1/(1-\sqrt{p})}$ is integrable on $(0, u_1)$. Twice continuous differentiability of φ on $(0, 1)$ ensures integrability provided that our integrand does not diverge too rapidly at the origin. In fact, Assumption 3.1(v)(a) and the monotone density theorem imply that $-\varphi' \in \mathscr{R}_{\eta-1}(0)$ and $\varphi'' \in \mathscr{R}_{\eta-2}(0)$, and so we have $(\varphi'(\cdot)^{-2}\varphi''(\cdot))^{1/(1-\sqrt{p})} \in \mathscr{R}_{-\eta/(1-\sqrt{p})}(0)$. Since $-\eta/(1-\sqrt{p}) > -1$, integrability holds. ■

The proof of Theorem 3.1 involves an application of the geometric ergodic theorem. This result is presented in many ways and discussed in great detail by Meyn and Tweedie (1993), which we shall henceforth refer to as MT. A version of the geometric ergodic theorem is given below as Theorem A.1. First, we require an additional definition.

DEFINITION A.1. *A set $S \in \mathscr{B}$ is said to be small if there exists a nontrivial measure ν on $\mathscr{B}$ such that $P(U_1 \in B | U_0 = u) \ge \nu(B)$ for a.e. $u \in S$ and all $B \in \mathscr{B}$.*

The above definition of a small set differs somewhat from the definition given by MT. Aside from the a.e. qualifier, our definition is more narrow than theirs. But it is sufficient for our purposes.

The statement of Theorem A.1 employs the notions of irreducibility and aperiodicity. For definitions, we refer the reader to MT. Here, we note only that our Markov chain $\{U_t : t \in \mathbb{Z}\}$ is irreducible and aperiodic whenever C admits a density c that is positive on $(0, 1)^2$.

THEOREM A.1. *Suppose $\{U_t : t \in \mathbb{Z}\}$ is irreducible and aperiodic, and there exists a function $V : (0, 1) \to [1, \infty)$, a small set $S \in \mathscr{B}$, and constants $a < 1$, $b < \infty$ such that*

$$\mathbb{E}(V(U_1) | U_0 = u) \le aV(u) + b1_S(u) \tag{A.1}$$

for a.e. $u \in (0, 1)$. Then $\{U_t : t \in \mathbb{Z}\}$ is geometrically ergodic.

Proof of Theorem A.1. By Proposition 5.5.3 in MT, every small set is petite (defined on p. 124 in MT), and so the assumptions of Theorem A.1 are stronger than those of Theorem 16.0.1 of MT (aside from the a.e. qualifier in Definition A.1, which we may safely ignore). Thus, the equivalence of (ii) and (iv) in Theorem 16.0.1 of MT implies that

$$|P(U_k \in B | U_0 = u) - P(U_k \in B)| \le V(u) A e^{-\gamma k}$$

for a.e. $u \in (0, 1)$, all $B \in \mathscr{B}$, all $k \in \mathbb{N}$, and some $A < \infty$ and $\gamma > 0$. It follows immediately that $\{U_t\}$ is geometrically ergodic. ■

We are now in a position to provide a proof of Theorem 3.1

Proof of Theorem 3.1. Our proof consists of verifying the conditions of Theorem A.1. As noted above, irreducibility and aperiodicity of $\{U_t\}$ hold if C admits a density c that is positive on $(0, 1)^2$. Recalling Remark 3.4, this is indeed the case under Assumption 3.1. It remains for us to verify the drift condition (A.1) for suitably chosen V, S, a, and b. Our choice of these objects will depend critically on whether $\zeta = 1$ or $\zeta > 1$. We therefore separate the remainder of the proof into two parts.

Case 1. $\zeta = 1$. Fix a number $p \in (0, -1/\eta)$; here and in what follows, $1/\eta$ should be interpreted as $-\infty$ when $\eta = 0$. For our drift function V we choose $V(\cdot) = \varphi(\cdot)^p + 1$. As

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a first step toward verifying (A.1) for this choice of V we shall investigate the behavior of $\mathbb{E}(V(U_1)|U_0 = u)$ as $u \downarrow 0$. Following the construction on p. 157 of Genest and MacKay (1986a), we may express the relationship between U_0 and U_1 in the nonlinear autoregressive form

$$U_1 = \varphi^{-1}\left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(U_0)}{W}\right)\right) - \varphi(U_0)\right), \tag{A.2}$$

where W is a $U(0,1)$ random variable distributed independently of U_0. Using (A.2), for $u \in (0,1)$ we may write

$$\mathbb{E}\left(\varphi(U_1)^p|U_0 = u\right) = \int_0^1 \left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(u)}{w}\right)\right) - \varphi(u)\right)^p dw. \tag{A.3}$$

Applying the change of variables $w = \varphi'(u)/\varphi'(su)$ to the integral in (A.3) and rearranging terms, we obtain

$$\frac{\mathbb{E}\left(\varphi(U_1)^p|U_0 = u\right)}{\varphi(u)^p} = -\frac{u\varphi''(u)}{\varphi'(u)}\int_0^1 \frac{\varphi''(su)}{\varphi''(u)}\left(\frac{\varphi'(su)}{\varphi'(u)}\right)^{-2}\left(\frac{\varphi(su)}{\varphi(u)} - 1\right)^p ds. \tag{A.4}$$

Since $-\varphi' \in \mathscr{R}_{\eta-1}(0)$, the monotone density theorem implies that $\lim_{u\downarrow 0} u\varphi''(u)/\varphi'(u) = \eta - 1$. Combining this result with Lemma A.1, we obtain

$$\lim_{u\downarrow 0}\frac{\mathbb{E}\left(\varphi(U_1)^p|U_0 = u\right)}{\varphi(u)^p} = (1-\eta)\int_0^1 s^{-\eta}\left(s^\eta - 1\right)^p ds = \int_0^1 \left(r^{\frac{\eta}{1-\eta}} - 1\right)^p dr =: \xi_0, \tag{A.5}$$

where we have used the change of variables $s = r^{1/(1-\eta)}$. Clearly $\xi_0 \geq 0$, with $\xi_0 = 0$ when $\eta = 0$. When $\eta < 0$, since $p \in (0, -1/\eta)$, Hölder's inequality implies that

$$\xi_0 < \left(\int_0^1 \left(r^{\frac{\eta}{1-\eta}} - 1\right)^{-1/\eta} dr\right)^{-p\eta} = \left(\frac{\eta - 1}{\eta}\int_0^1 q^{-1/\eta} dq\right)^{-p\eta} = 1,$$

where we have used the change of variables $r = (1-q)^{(\eta-1)/\eta}$. Hence $\xi_0 \in [0,1)$.

We have shown that $\mathbb{E}\left(\varphi(U_1)^p|U_0 = u\right)/\varphi(u)^p \to \xi_0 \in [0,1)$ as $u \downarrow 0$. Since $\varphi(u)^p \to \infty$ as $u \downarrow 0$, it follows easily that $\mathbb{E}(V(U_1)|U_0 = u)/V(u) \to \xi_0$ as $u \downarrow 0$. Consequently, for any arbitrary constant $a \in (\xi_0, 1)$, there must exist $u_0 \in (0,1)$ such that

$$\mathbb{E}(V(U_1)|U_0 = u) \leq aV(u) \quad \text{for all } u \in (0, u_0). \tag{A.6}$$

Lemma A.4 and (A.3) ensure the existence of $b < \infty$ such that

$$\mathbb{E}(V(U_1)|U_0 = u) \leq b \quad \text{for all } u \in [u_0, 1). \tag{A.7}$$

Combining (A.6) and (A.7), we obtain

$$\mathbb{E}(V(U_1)|U_0 = u) \leq aV(u) + b1_{[u_0,1)}(u) \quad \text{for all } u \in (0,1).$$

To verify the drift condition (A.1), it remains only to show that $[u_0, 1)$ is a small set. Consider the expression for the copula density c given in (3.1). Since $C > 0$ on $[u_0, 1)^2$,

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the denominator on the right-hand side of (3.1) is bounded away from $-\infty$ on $[u_0, 1)^2$. Further, since $\zeta = 1$, Assumptions 3.1(vi), 3.1(v)(a) and 3.1(vi) jointly imply that φ' and φ'' are bounded away from zero, implying that the numerator on the right-hand side of (3.1) is also bounded away from zero. Hence, c is bounded away from zero on $[u_0, 1)^2$. Let $\kappa = \inf_{(u,v)\in[u_0,1)^2} c(u,v) > 0$, and for $B \in \mathscr{B}$ let $\nu B = \kappa \int_B 1_{[u_0,1)}(v)\mathrm{d}v$. Clearly ν is a nontrivial measure on $\mathscr{B}$. For any $u \in [u_0, 1)$ and any $B \in \mathscr{B}$ we have

$$P(U_1 \in B | U_0 = u) = \int_B c(u,v)dv \geq \int_B c(u,v)1_{[u_0,1)}(v)dv \geq \kappa \int_B 1_{[u_0,1)}(v)dv = \nu B,$$

implying that $[u_0, 1)$ is small. Our desired result now follows from Theorem A.1 for the case where $\zeta = 1$.

Case 2. $\zeta > 1$. This time we fix $p \in (0, \min\{-1/\eta, 1/\zeta, 1 - 1/\zeta\})$, and for our drift function V we choose $V(u) = \varphi(u)^p + \varphi(u)^{-p}$. We will investigate the behavior of $\mathbb{E}(V(U_1)|U_0 = u)$ as $u \downarrow 0$ and as $u \uparrow 1$, beginning with the former scenario. The proof that $\lim_{u\downarrow 0} \mathbb{E}\left(\varphi(U_1)^p | U_0 = u\right)/\varphi(u)^p = \xi_0 \in [0, 1)$ given for Case 1 continues to apply here. Trivially modifying (A.4), we have

$$\frac{\mathbb{E}\left(\varphi(U_1)^{-p} | U_0 = u\right)}{\varphi(u)^p} = -\frac{u\varphi''(u)}{\varphi'(u)}\varphi(u)^{-p}\int_0^1 \frac{\varphi''(su)}{\varphi''(u)}\left(\frac{\varphi'(su)}{\varphi'(u)}\right)^{-2}(\varphi(su) - \varphi(u))^{-p}ds.$$

As noted in the proof for Case 1, $\lim_{u\downarrow 0} u\varphi''(u)/\varphi'(u) = \eta - 1$, and so Lemma A.2 implies that $\lim_{u\downarrow 0} \mathbb{E}\left(\varphi(U_1)^{-p} | U_0 = u\right)/\varphi(u)^p = 0$. We have now established that

$$\begin{aligned}\lim_{u\downarrow 0}\frac{\mathbb{E}(V(U_1)|U_0 = u)}{V(u)} &= \lim_{u\downarrow 0}\frac{\varphi(u)^p}{\varphi(u)^p + \varphi(u)^{-p}} \\ &\quad \times \left(\frac{\mathbb{E}\left(\varphi(U_1)^p | U_0 = u\right)}{\varphi(u)^p} + \frac{\mathbb{E}\left(\varphi(U_1)^{-p} | U_0 = u\right)}{\varphi(u)^p}\right) \\ &= 1 \cdot (\xi_0 + 0) = \xi_0 \in [0, 1).\end{aligned}$$

Next consider the behavior of $\mathbb{E}(V(U_1)|U_0 = u)$ as $u \uparrow 1$. Applying the change of variables $w = \varphi'(1-u)/\varphi'(1-su)$ to the integral in (A.3) with $1-u$ in place of u, and rearranging terms, we obtain

$$\begin{aligned}&\frac{\mathbb{E}\left(\varphi(U_1)^p | U_0 = 1-u\right)}{\varphi(1-u)^p} \\ &\quad = -\frac{u\varphi''(1-u)}{\varphi'(1-u)}\int_1^{\frac{1}{u}} \frac{\varphi''(1-su)}{\varphi''(1-u)}\left(\frac{\varphi'(1-su)}{\varphi'(1-u)}\right)^{-2}\left(\frac{\varphi(1-su)}{\varphi(1-u)} - 1\right)^p ds. \qquad \textbf{(A.8)}\end{aligned}$$

The monotone density Theorem implies that $\lim_{u\downarrow 0} u\varphi''(1-u)/\varphi'(1-u) = 1-\zeta$, while Lemma A.3 implies that the integral in (A.8) converges to $\int_1^\infty s^{-\zeta}(s^\zeta - 1)^p ds$ as $u \downarrow 0$. Since $p < 1 - 1/\zeta$, this integral is finite. We have thus shown that

$$\lim_{u\uparrow 1}\frac{\mathbb{E}\left(\varphi(U_1)^p | U_0 = u\right)}{\varphi(u)^p} = (\zeta - 1)\int_1^\infty s^{-\zeta}\left(s^\zeta - 1\right)^p ds. \qquad \textbf{(A.9)}$$

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In fact, by an identical argument, (A.9) remains true with $-p$ in place of p. Consequently,

$$\begin{aligned}\lim_{u\uparrow 1}\frac{\mathbb{E}(V(U_1)|U_0=u)}{V(u)} &= \lim_{u\uparrow 1}\left(\frac{\varphi(u)^p}{\varphi(u)^p+\varphi(u)^{-p}}\cdot\frac{\mathbb{E}\left(\varphi(U_1)^p|U_0=u\right)}{\varphi(u)^p}\right)\\ &\quad+\lim_{u\uparrow 1}\left(\frac{\varphi(u)^{-p}}{\varphi(u)^p+\varphi(u)^{-p}}\cdot\frac{\mathbb{E}\left(\varphi(U_1)^{-p}|U_0=u\right)}{\varphi(u)^{-p}}\right)\\ &= (\zeta-1)\int_1^\infty s^{-\zeta}\left(s^\zeta-1\right)^{-p}ds =: \xi_1. \end{aligned} \tag{A.10}$$

Applying the change of variables $s = r^{1/(1-\zeta)}$ to the integral defining ξ_1 in (A.10), we obtain

$$\xi_1 = \int_0^1\left(r^{\frac{\zeta}{1-\zeta}}-1\right)^{-p}dr,$$

which is well defined since $\zeta > 1$. Clearly $\xi_1 > 0$. And since $p \in (0, 1/\zeta)$, Hölder's inequality implies that

$$\xi_1 < \left(\int_0^1\left(r^{\frac{\zeta}{1-\zeta}}-1\right)^{-1/\zeta}dr\right)^{p\zeta} = \left(\frac{\zeta-1}{\zeta}\int_0^1 q^{-1/\zeta}dq\right)^{p\zeta} = 1,$$

where we have used the change of variables $r = (1-q)^{(\zeta-1)/\zeta}$. Hence $\xi_1 \in (0,1)$.

We have now shown that $\mathbb{E}(V(U_1)|U_0=u)/V(u) \to \xi_0 \in [0,1)$ as $u \downarrow 0$, and that $\mathbb{E}(V(U_1)|U_0=u)/V(u) \to \xi_1 \in (0,1)$ as $u \uparrow 1$. Consequently, there exists $a \in (\max\{\xi_0,\xi_1\},1)$ and $u_0, u_1 \in (0,1)$ with $u_0 < u_1$ such that

$$\mathbb{E}(V(U_1)|U_0=u) \le aV(u) \quad \text{for all } u \in (0,u_0)\cup(u_1,1). \tag{A.11}$$

Lemma A.4, Lemma A.5, and (A.3) ensure the existence of $b < \infty$ such that

$$\mathbb{E}(V(U_1)|U_0=u) \le b \quad \text{for all } u \in [u_0,u_1]. \tag{A.12}$$

Combining (A.11) and (A.12), we obtain

$$\mathbb{E}(V(U_1)|U_0=u) \le aV(u) + b1_{[u_0,u_1]}(u) \quad \text{for all } u \in (0,1).$$

To verify the drift condition (A.1), it remains only to show that $[u_0,u_1]$ is a small set. Recalling the proof that $[u_0,1)$ was small in Case 1, it should be clear that we need only show that c is bounded away from zero on $[u_0,u_1]^2$. But this is obvious from (3.1) in view of the fact that $-\varphi'$ and φ'' are continuous and strictly positive on $[u_0,u_1]$ (recall Assumptions 3.1(ii) and 3.1(iv)), while $-\varphi'\circ C$ is continuous and therefore bounded on $[u_0,u_1]^2$. We may therefore apply Theorem A.1 to obtain our desired result for the case where $\zeta > 1$ also. ■

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