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On Preconditioned MHSS Real-Valued Iteration Methods for a Class of Complex Symmetric Indefinite Linear Systems

Zhi-Ru Ren[1], Yang Cao[2,*] and Li-Li Zhang[3]

[1] *School of Statistics and Mathematics, Central University of Finance and Economics, Beijing 100081, P.R. China.*
[2] *School of Transportation, Nantong University, Nantong 226019, P.R. China.*
[3] *School of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou 450046, P.R. China*

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Abstract. A generalized preconditioned modified Hermitian and skew-Hermitian splitting (GPMHSS) real-valued iteration method is proposed for a class of complex symmetric indefinite linear systems. Convergence theory is established and the spectral properties of an associated preconditioned matrix are analyzed. We also give several variants of the GPMHSS preconditioner and consider the spectral properties of the preconditioned matrices. Numerical examples illustrate the effectiveness of our proposed method.

AMS subject classifications: 65F10, 65F50

Key words: Complex linear systems, PMHSS iteration, real-valued form, convergence, preconditioning.

1. Introduction

We consider the iterative solution of a large, sparse and nonsingular linear system of the form

$$Ax = b\,, \quad \text{with } A = W + \mathrm{i}T\,, \quad x = y + \mathrm{i}z \quad \text{and} \quad b = f + \mathrm{i}g\,, \tag{1.1}$$

where the $n \times n$ matrices W and T are real symmetric, the vectors y, z, f, g are all in $\mathbb{R}^n$, and $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. We assume that $W \neq 0$ and $T \neq 0$, which implies that the matrix A is neither Hermitian nor skew-Hermitian. Complex symmetric systems arise from many problems in scientific computing and engineering applications, including diffuse optical tomography [1], quantum mechanics [22], molecular scattering [20], structural dynamics [15], electrical power system modeling [17], lattice quantum chromodynamics [16], and so on; see also Ref. [10].

*Corresponding author. *Email address:* caoyangnt@ntu.edu.cn (Y. Cao)

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Many efficient iteration methods have been proposed to solve the complex system (1.1). When W and T are symmetric matrices with one positive definite and the other either positive semidefinite or positive definite, Bai *et al.* [5, 6] introduced the modified Hermitian and skew-Hermitian splitting (MHSS) and preconditioned MHSS (PMHSS) iteration methods, which were proved more efficient than the HSS iteration method [9] for solving the complex linear system (1.1). They also established convergence theory for these MHSS and PMHSS iteration methods [5, 6]. In addition, the PMHSS method naturally involves a preconditioner for the complex symmetric matrix A, and some properties of the preconditioned matrix are also presented in Ref. [6]. When W and T are symmetric indefinite matrices and satisfy certain assumptions, Xu [24] presented a generalization of the PMHSS iteration method for the complex linear system (1.1). Recently, Cao & Ren [13] proposed two variants of the PMHSS iteration methods for a class of complex symmetric indefinite linear systems (1.1), and proved the convergence of these variants. However, the application of these methods to the complex system (1.1) requires complex arithmetic, which one may wish to avoid for several reasons. For instance, if most of the entries of A are real (with complex entries localized in just a few positions), then it is wasteful to use complex arithmetic throughout the code [12]. As mentioned in Ref. [14], another difficulty is the scarcity of available preconditioning software supporting complex arithmetic, compared to the widespread availability of high-quality packages that could be used for an equivalent real formulation of the complex linear system (1.1).

In fact, the complex linear system (1.1) can be transformed into the $2n \times 2n$ equivalent real formulation

$$\begin{pmatrix} W & -T \\ T & W \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{1.2}$$

The block two-by-two linear system (1.2) may be solved efficiently in real arithmetic by Krylov subspace methods such as the preconditioned GMRES and QMR [21], by C-to-R [2], or by alternating splitting iteration methods such as PHSS and PMHSS [7], for which high-quality preconditioners are crucial to guarantee their accuracy, efficiency and robustness in practical computation as discussed in Refs. [3, 10] and references therein. The C-to-R and the PMHSS iteration methods are presented for solving the system (1.2) when W and T are symmetric matrices with one positive definite and the other either positive semidefinite or positive definite. Moreover, the PMHSS iteration method [7] is a skilful preconditioned modification of the HSS iteration method initially introduced to solve non-Hermitian positive definite linear systems [5, 9]. It can be accommodated to solve the real linear system (1.2) using only real arithmetic. PMHSS iteration methods naturally result in matrix splitting preconditioners called PMHSS preconditioners, for Krylov subspace iteration methods such as GMRES and QMR employed to solve the complex and real linear systems (1.1) and (1.2). However, there are few effective real-valued iterative methods to solve the equivalent system (1.2) when the matrix W or the matrix T in the system (1.1) is symmetric indefinite.

In this paper, we explore preconditioned real-valued iteration methods for the complex symmetric indefinite linear system (1.1) with W and T satisfying the case $-T \prec W \preceq T$. Here and in the sequel, for any matrices B and C, the notation $B \prec C$ means that $C - B$

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is symmetric positive definite and $B \preceq C$ means that $C - B$ is symmetric positive semidefinite. We present a generalization of the PMHSS iteration method to solve a $2n \times 2n$ real equivalent formulation of (1.1) and prove its convergence. Moreover, we obtain a quasi-optimal preconditioning matrix and a quasi-optimal parameter for the generalized PMHSS (GPMHSS) iteration method by analyzing an upper bound of the spectral radius of the iteration matrix. The GPMHSS iteration method results in a matrix splitting preconditioner called the GPMHSS preconditioner for the block two-by-two linear system. Based on the GPMHSS preconditioner, we construct a few of block diagonal and block tridiagonal preconditioners and analyze the spectral properties of these preconditioned matrices. Then these preconditioners are applied to accelerate the convergence rate of Krylov subspace methods when they are used to solve the block two-by-two real equivalent formulation. Numerical results show that the GPMHSS preconditioner and its several variants are effective for the real linear system (1.2).

The remainder of this paper is as follows. In Section 2, we establish a generalization of the PMHSS real-valued iteration method for solving the complex symmetric indefinite linear system (1.1) and prove its unconditional convergence. In Section 3, we introduce the GPMHSS preconditioner and analyze the spectral properties of the preconditioned matrix. In Section 4 we give several variants of the GPMHSS preconditioner. Numerical experiments are used in Section 5 to confirm the effectiveness of the preconditioned iteration methods. Finally, our brief concluding remarks are made in Section 6.

Throughout the paper, we adopt the following notations. The symbol $(\cdot)^T$ denotes the transpose of either a vector or a matrix, $\bar{\lambda}$ stands for the conjugate of a complex number λ, and $\lambda_{\min}(E)$ and $\lambda_{\max}(E)$ represent the smallest and the largest eigenvalues of a symmetric matrix E, respectively.

2. A Generalization of the PMHSS Real-Valued Iteration Method

Given $-T \prec W \preceq T$, we know that $W + T$ is a symmetric positive definite matrix and $T - W$ is a symmetric positive semidefinite matrix. The complex linear system (1.1) can be transformed into the following equivalent linear system

$$[W + T + \mathrm{i}(T - W)](y + \mathrm{i}z) = (1 - \mathrm{i})(f + \mathrm{i}g),$$

and hence the equivalent $2n \times 2n$ real formulation

$$\begin{pmatrix} W + T & -(T - W) \\ T - W & W + T \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} f + g \\ g - f \end{pmatrix} := \begin{pmatrix} p \\ q \end{pmatrix} \tag{2.1}$$

involving the positive definite coefficient matrix

$$\mathscr{A} := \begin{pmatrix} W + T & -(T - W) \\ T - W & W + T \end{pmatrix}. \tag{2.2}$$

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Let α be a positive real number and V be a symmetric positive definite matrix. Then the block two-by-two linear system (2.1) can be rewritten as a fixed point equation

$$\begin{pmatrix} \alpha V + W + T & 0 \\ 0 & \alpha V + W + T \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \alpha V & T - W \\ -(T - W) & \alpha V \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}.$$

Alternatively, it can be rewritten as another fixed point equation

$$\begin{pmatrix} \alpha V + T - W & 0 \\ 0 & \alpha V + T - W \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \alpha V & -(W + T) \\ W + T & \alpha V \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} q \\ -p \end{pmatrix}.$$

By alternating between these two splittings, we obtain the generalized PMHSS (GPMHSS) real-valued iteration method for the complex symmetric indefinite linear system (1.1) for the case $-T \prec W \preceq T$ as follows.

Method 2.1. (The GPMHSS real-valued iteration method)
Let $(y^{(0)^T}, z^{(0)^T})^T \in \mathbb{R}^{2n}$ be an arbitrary initial guess with $y^{(0)}, z^{(0)} \in \mathbb{R}^n$. For $k = 0, 1, 2, \cdots$, until the sequence of iterates $\{(y^{(k)^T}, z^{(k)^T})^T\}_{k=0}^{\infty} \in \mathbb{R}^{2n}$ converges, compute the next iterate $(y^{(k+1)^T}, z^{(k+1)^T})^T$ according to the following procedure:

$$\begin{cases} \begin{pmatrix} \alpha V + W + T & 0 \\ 0 & \alpha V + W + T \end{pmatrix}\begin{pmatrix} y^{(k+\frac{1}{2})} \\ z^{(k+\frac{1}{2})} \end{pmatrix} = \begin{pmatrix} \alpha V & T - W \\ -(T - W) & \alpha V \end{pmatrix}\begin{pmatrix} y^{(k)} \\ z^{(k)} \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}, \\ \begin{pmatrix} \alpha V + T - W & 0 \\ 0 & \alpha V + T - W \end{pmatrix}\begin{pmatrix} y^{(k+1)} \\ z^{(k+1)} \end{pmatrix} = \begin{pmatrix} \alpha V & -(W + T) \\ W + T & \alpha V \end{pmatrix}\begin{pmatrix} y^{(k+\frac{1}{2})} \\ z^{(k+\frac{1}{2})} \end{pmatrix} + \begin{pmatrix} q \\ -p \end{pmatrix}, \end{cases}$$

where α is a given positive constant and $V \in \mathbb{R}^{n \times n}$ is a prescribed symmetric positive definite matrix.

In particular, when the preconditioning matrix V is taken to be the identity matrix I, the Method 2.1 reduces to the generalized MHSS (GMHSS) iteration method. As V and $W + T$ are symmetric positive definite matrices and $T - W$ is a symmetric positive semidefinite matrix, we know that each iterate of the GPMHSS real-valued iteration alternates between two symmetric positive definite matrices $\alpha V + W + T$ and $\alpha V + T - W$. The two linear subsystems in Method 2.1 therefore can be solved efficiently using real arithmetic either exactly by a Cholesky factorization or inexactly by the conjugate gradient method. In fact, the roles of the matrices $W + T$ and $T - W$ can be reversed, so that we may first solve the linear system with the coefficient matrix $\alpha V + T - W$ and then solve the linear system with the coefficient matrix $\alpha V + W + T$. Further, if we write

$$u^{(k)} = \begin{pmatrix} y^{(k)} \\ z^{(k)} \end{pmatrix}, \quad u^{(k+\frac{1}{2})} = \begin{pmatrix} y^{(k+\frac{1}{2})} \\ z^{(k+\frac{1}{2})} \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} p \\ q \end{pmatrix},$$

then after straightforward computations we can re-express the GPMHSS iteration in the standard form

$$u^{(k+1)} = \mathscr{L}(V; \alpha) u^{(k)} + \mathscr{R}(V; \alpha) h, \quad k = 0, 1, 2, \cdots, \tag{2.3}$$

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where

$$\mathscr{L}(V;\alpha)=\begin{pmatrix} \alpha V+T-W & 0 \\ 0 & \alpha V+T-W \end{pmatrix}^{-1}\begin{pmatrix} \alpha V & -(W+T) \\ W+T & \alpha V \end{pmatrix} \cdot\begin{pmatrix} \alpha V+W+T & 0 \\ 0 & \alpha V+W+T \end{pmatrix}^{-1}\begin{pmatrix} \alpha V & T-W \\ -(T-W) & \alpha V \end{pmatrix}$$

and

$$\mathscr{R}(V;\alpha)=\begin{pmatrix} \alpha V+T-W & 0 \\ 0 & \alpha V+T-W \end{pmatrix}^{-1}\begin{pmatrix} \alpha V & 0 \\ 0 & \alpha V \end{pmatrix} \cdot\begin{pmatrix} \alpha V+W+T & 0 \\ 0 & \alpha V+W+T \end{pmatrix}^{-1}\begin{pmatrix} I & I \\ -I & I \end{pmatrix},$$

with $I\in\mathbb{R}^{n\times n}$ being the identity matrix. It is notable that $\mathscr{L}(V;\alpha)$ is the iteration matrix of the GPMHSS real-valued iteration method. The fixed point iteration (2.3) converges to the solution of the linear system (2.1) for an arbitrary initial guess if and only if $\rho(\mathscr{L}(V;\alpha))<1$, where $\rho(\mathscr{L}(V;\alpha))$ denotes the spectral radius of the iteration matrix.

The following theorem describes the convergence result of the GPMHSS real-valued iteration method.

Theorem 2.1. *Let $A=W+\mathrm{i}T\in\mathbb{C}^{n\times n}$ be a nonsingular matrix with W being symmetric indefinite, T being symmetric positive definite and $-T\prec W\preceq T$. Let V be a symmetric positive definite matrix and α be a positive constant. Then the spectral radius $\rho(\mathscr{L}(V;\alpha))$ of the GPMHSS iteration method satisfies*

$$\rho(\mathscr{L}(V;\alpha))\le\sigma(\alpha)\equiv\max_{\widetilde{\lambda}_j\in\mathrm{sp}(V^{-1}(W+T))}\frac{\sqrt{\alpha^2+\widetilde{\lambda}_j^2}}{\alpha+\widetilde{\lambda}_j},$$

where $\mathrm{sp}(V^{-1}(W+T))$ denotes the spectrum of the matrix $V^{-1}(W+T)$. Consequently,

$$\rho(\mathscr{L}(V;\alpha))\le\sigma(\alpha)<1,$$

so the GPMHSS iteration method converges to the unique solution of the linear system (2.1) *for any initial guess.*

Proof. We first introduce the notation

$$\widetilde{W}=V^{-1/2}WV^{-1/2}\quad\text{and}\quad\widetilde{T}=V^{-1/2}TV^{-1/2}.$$

From the premise, we know that $\widetilde{W}+\widetilde{T}$ is a symmetric positive definite matrix and $\widetilde{T}-\widetilde{W}$ is a symmetric positive semidefinite matrix. Then there exist orthogonal matrices $\widetilde{Q}_1$, $\widetilde{Q}_2\in\mathbb{R}^{n\times n}$, a positive diagonal matrix $\widetilde{\Lambda}_1=\mathrm{diag}(\widetilde{\lambda}_1,\widetilde{\lambda}_1,\cdots,\widetilde{\lambda}_n)\in\mathbb{R}^{n\times n}$ and a positive semidefinite diagonal matrix $\widetilde{\Lambda}_2=\mathrm{diag}(\widetilde{\mu}_1,\widetilde{\mu}_1,\cdots,\widetilde{\mu}_n)\in\mathbb{R}^{n\times n}$ such that

$$\widetilde{W}+\widetilde{T}=\widetilde{Q}_1\widetilde{\Lambda}_1\widetilde{Q}_1^T\quad\text{and}\quad\widetilde{T}-\widetilde{W}=\widetilde{Q}_2\widetilde{\Lambda}_2\widetilde{Q}_2^T.$$

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From block scaling and matrix similarity, we know that the iteration matrix $\mathscr{L}(V;\alpha)$ of the GPMHSS iteration method is similar to

$$\widetilde{\mathscr{L}}(\alpha) := \begin{pmatrix} \alpha I & -(\widetilde{W}+\widetilde{T}) \\ \widetilde{W}+\widetilde{T} & \alpha I \end{pmatrix} \begin{pmatrix} \alpha I+\widetilde{W}+\widetilde{T} & 0 \\ 0 & \alpha I+\widetilde{W}+\widetilde{T} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \alpha I & \widetilde{T}-\widetilde{W} \\ -(\widetilde{T}-\widetilde{W}) & \alpha I \end{pmatrix} \begin{pmatrix} \alpha I+\widetilde{T}-\widetilde{W} & 0 \\ 0 & \alpha I+\widetilde{T}-\widetilde{W} \end{pmatrix}^{-1},$$

so that $\rho(\mathscr{L}(V;\alpha)) = \rho(\widetilde{\mathscr{L}}(\alpha))$. Then from straightforward computation,

$$\begin{aligned}
\rho(\widetilde{\mathscr{L}}(\alpha)) &\le \left\| \begin{pmatrix} \alpha I & -(\widetilde{W}+\widetilde{T}) \\ \widetilde{W}+\widetilde{T} & \alpha I \end{pmatrix} \begin{pmatrix} \alpha I+\widetilde{W}+\widetilde{T} & 0 \\ 0 & \alpha I+\widetilde{W}+\widetilde{T} \end{pmatrix}^{-1} \right\|_2 \\
&\quad \cdot \left\| \begin{pmatrix} \alpha I & \widetilde{T}-\widetilde{W} \\ -(\widetilde{T}-\widetilde{W}) & \alpha I \end{pmatrix} \begin{pmatrix} \alpha I+\widetilde{T}-\widetilde{W} & 0 \\ 0 & \alpha I+\widetilde{T}-\widetilde{W} \end{pmatrix}^{-1} \right\|_2 \\
&= \left\| \begin{pmatrix} \alpha I & -\widetilde{\Lambda}_1 \\ \widetilde{\Lambda}_1 & \alpha I \end{pmatrix} \begin{pmatrix} \alpha I+\widetilde{\Lambda}_1 & 0 \\ 0 & \alpha I+\widetilde{\Lambda}_1 \end{pmatrix}^{-1} \right\|_2 \\
&\quad \cdot \left\| \begin{pmatrix} \alpha I & \widetilde{\Lambda}_2 \\ -\widetilde{\Lambda}_2 & \alpha I \end{pmatrix} \begin{pmatrix} \alpha I+\widetilde{\Lambda}_2 & 0 \\ 0 & \alpha I+\widetilde{\Lambda}_2 \end{pmatrix}^{-1} \right\|_2 \\
&= \max_{\widetilde{\lambda}_j \in \mathrm{sp}(\widetilde{W}+\widetilde{T})} \left| \frac{\alpha \pm \mathrm{i}\widetilde{\lambda}_j}{\alpha+\widetilde{\lambda}_j} \right| \cdot \max_{\widetilde{\mu}_j \in \mathrm{sp}(\widetilde{T}-\widetilde{W})} \left| \frac{\alpha \pm \mathrm{i}\widetilde{\mu}_j}{\alpha+\widetilde{\mu}_j} \right| \\
&= \max_{\widetilde{\lambda}_j \in \mathrm{sp}(V^{-1}(W+T))} \frac{\sqrt{\alpha^2+\widetilde{\lambda}_j^2}}{\alpha+\widetilde{\lambda}_j} \cdot \max_{\widetilde{\mu}_j \in \mathrm{sp}(V^{-1}(T-W))} \frac{\sqrt{\alpha^2+\widetilde{\mu}_j^2}}{\alpha+\widetilde{\mu}_j} \\
&\le \max_{\widetilde{\lambda}_j \in \mathrm{sp}(V^{-1}(W+T))} \frac{\sqrt{\alpha^2+\widetilde{\lambda}_j^2}}{\alpha+\widetilde{\lambda}_j} \equiv \sigma(\alpha) < 1
\end{aligned}$$

holds true for any $\alpha > 0$, where we have used $\mathrm{sp}(\widetilde{W}+\widetilde{T}) = \mathrm{sp}(V^{-1}(W+T))$ and $\mathrm{sp}(\widetilde{T}-\widetilde{W}) = \mathrm{sp}(V^{-1}(T-W))$. The GPMHSS iteration method therefore converges unconditionally to the unique solution of the block two-by-two linear system (2.1) for any initial guess. □

Theorem 2.1 gives an unconditional convergence property for the GPMHSS real-valued iteration method. As we know, the convergence rate depends on the spectral radius of the GPMHSS iteration matrix $\mathscr{L}(V;\alpha)$ and it is not an easy task to find the optimal spectral radius. Nevertheless, from Theorem 2.1 we have $\rho(\mathscr{L}(V;\alpha)) \le \sigma(\alpha)$, so we consider the bound $\sigma(\alpha)$ instead of the spectral radius $\rho(\mathscr{L}(V;\alpha))$. Thus analogous to Corollary 2.1 in Ref. [5], we obtain the quasi-optimal parameter α^* that minimizes the upper bound $\sigma(\alpha)$ and the corresponding optimal upper bound $\sigma(\alpha^*)$ in the following corollary.

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Corollary 2.1. *Let the conditions of Theorem* 2.1 *be satisfied, and* $\widetilde{\lambda}_{\min}$ *and* $\widetilde{\lambda}_{\max}$ *be the extreme eigenvalues of the matrix* $V^{-1}(W+T)$. *Then the quasi-optimal parameter* α^* *that minimizes the upper bound* $\sigma(\alpha)$ *of the spectral radius* $\rho(\mathscr{L}(V;\alpha))$ *is*

$$\alpha^* \equiv \arg\min_{\alpha}\left\{\max_{\widetilde{\lambda}_{\min}\le\widetilde{\lambda}_j\le\widetilde{\lambda}_{\max}}\frac{\sqrt{\alpha^2+\widetilde{\lambda}_j}}{\alpha+\widetilde{\lambda}_j}\right\}=\sqrt{\widetilde{\lambda}_{\min}\widetilde{\lambda}_{\max}}$$

and

$$\sigma(\alpha^*)=\frac{\sqrt{\widetilde{\lambda}_{\min}+\widetilde{\lambda}_{\max}}}{\sqrt{\widetilde{\lambda}_{\min}}+\sqrt{\widetilde{\lambda}_{\max}}}=\frac{\sqrt{\kappa(V^{-1}(W+T))+1}}{\sqrt{\kappa(V^{-1}(W+T))}+1},$$

where $\kappa(V^{-1}(W+T))$ *is the spectral condition number of the matrix* $V^{-1}(W+T)$.

From Corollary 2.1, we know that the smaller the condition number of the matrix $V^{-1}(W+T)$ is, the faster the asymptotic convergence rate of the GPMHSS iteration is. In actual computation, we can choose the preconditioning matrix $V=W+T$ to render $\kappa(V^{-1}(W+T))=1$. Then it holds that

$$\begin{aligned}\mathscr{L}(\alpha):=&\mathscr{L}(W+T;\alpha)\\ =&\frac{1}{\alpha+1}\begin{pmatrix}\alpha(W+T)+T-W & 0\\ 0 & \alpha(W+T)+T-W\end{pmatrix}^{-1}\\ &\cdot\begin{pmatrix}\alpha(W+T) & T-W\\ -(T-W) & \alpha(W+T)\end{pmatrix}\begin{pmatrix}\alpha I & -I\\ I & \alpha I\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathscr{R}(\alpha):=&\mathscr{R}(W+T;\alpha)\\ =&\frac{\alpha}{\alpha+1}\begin{pmatrix}\alpha(W+T)+T-W & 0\\ 0 & \alpha(W+T)+T-W\end{pmatrix}^{-1}\begin{pmatrix}I & I\\ -I & I\end{pmatrix}.\end{aligned}$$

Moreover, the spectral radius of the iteration matrix $\mathscr{L}(\alpha)$ satisfies

$$\rho(\mathscr{L}(\alpha))\le(\sqrt{\alpha^2+1})/(\alpha+1).$$

The quasi-optimal parameter is $\alpha^*=1$ and $\rho(\mathscr{L}(\alpha^*))\le\sigma(\alpha^*)=1/\sqrt{2}$.

3. The GPMHSS Preconditioner

The GPMHSS real-valued iteration method for solving the linear system (2.1) belongs to the class of stationary iterative methods. If we introduce the matrices

$$\begin{aligned}\mathscr{F}(V;\alpha)=P(\alpha)&\begin{pmatrix}\alpha V+W+T & 0\\ 0 & \alpha V+W+T\end{pmatrix}\begin{pmatrix}V^{-1} & 0\\ 0 & V^{-1}\end{pmatrix}\\ &\cdot\begin{pmatrix}\alpha V+T-W & 0\\ 0 & \alpha V+T-W\end{pmatrix}\end{aligned}\tag{3.1}$$

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and

$$\mathscr{G}(V;\alpha)=P(\alpha)\begin{pmatrix} \alpha V & -(W+T)\\ W+T & \alpha V\end{pmatrix}\begin{pmatrix} V^{-1} & 0\\ 0 & V^{-1}\end{pmatrix}\begin{pmatrix} \alpha V & T-W\\ -(T-W) & \alpha V\end{pmatrix}$$

with

$$P(\alpha)=\frac{1}{2\alpha}\begin{pmatrix} I & -I\\ I & I\end{pmatrix}, \tag{3.2}$$

then

$$\mathscr{A}=\mathscr{F}(V;\alpha)-\mathscr{G}(V;\alpha)\quad\text{and}\quad\mathscr{L}(V;\alpha)=\mathscr{F}(V;\alpha)^{-1}\mathscr{G}(V;\alpha). \tag{3.3}$$

Thus the GPMHSS real-valued iteration method is induced by the matrix splitting $\mathscr{A}=\mathscr{F}(V;\alpha)-\mathscr{G}(V;\alpha)$ defined in (3.3). The splitting matrix $\mathscr{F}(V;\alpha)$ given in (3.1) can also be used as a preconditioning matrix for the block two-by-two matrix $\mathscr{A}\in\mathbb{R}^{2n\times 2n}$ in (2.2), which is called the GPMHSS preconditioner.

When used to precondition Krylov subspace methods in actual implementations, the action of the preconditioning matrix $\mathscr{F}(V;\alpha)$ is often realized through solving a sequence of generalized residual equations of the form

$$\mathscr{F}(V;\alpha)v=r\,, \tag{3.4}$$

where $v=(v_1^T,v_2^T)^T\in\mathbb{R}^{2n}$ with $v_1,v_2\in\mathbb{R}^n$ denotes the generalized residual vector, and $r=(r_1^T,r_2^T)^T\in\mathbb{R}^{2n}$ with $r_1,r_2\in\mathbb{R}^n$ denotes the current residual vector. From the structure of the matrices $\mathscr{F}(V;\alpha)$ in (3.1) and $P(\alpha)$ in (3.2), we obtain the following algorithm for computing the vector v.

Algorithm 3.1. When the GPMHSS preconditioner $\mathscr{F}(V;\alpha)$ in (3.1) is used to precondition Krylov subspace methods, we can solve the generalized residual equations (3.4) by the following procedure:

(1) compute $\widetilde{v}_1,\widetilde{v}_2\in\mathbb{R}^n$ by solving the linear systems

$$\begin{cases}(\alpha V+W+T)\widetilde{v}_1=\alpha(r_1+r_2)\,,\\ (\alpha V+W+T)\widetilde{v}_2=\alpha(-r_1+r_2)\,;\end{cases}$$

(2) compute $\widehat{v}_1,\widehat{v}_2\in\mathbb{R}^n$ by

$$\widehat{v}_1=V\,\widetilde{v}_1\,,\quad\widehat{v}_2=V\,\widetilde{v}_2\,;$$

(3) compute $v_1,v_2\in\mathbb{R}^n$ by solving the linear systems

$$\begin{cases}(\alpha V+T-W)v_1=\widehat{v}_1\,,\\ (\alpha V+T-W)v_2=\widehat{v}_2\,.\end{cases}$$

Since both matrices $\alpha V+W+T$ and $\alpha V+T-W$ are symmetric positive definite, the above four linear systems can be solved efficiently either exactly by Cholesky factorization or inexactly by the conjugate gradient method.

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In particular, when $V = W + T$, the GPMHSS real-valued iteration scheme is induced by the matrix splitting

$$\mathscr{A} = \mathscr{F}(\alpha) - \mathscr{G}(\alpha)\,,$$

where

$$\mathscr{F}(\alpha) := \mathscr{F}(W+T,\alpha) = (\alpha+1)P(\alpha)\begin{pmatrix} \alpha(W+T)+T-W & 0 \\ 0 & \alpha(W+T)+T-W \end{pmatrix} \tag{3.5}$$

and

$$\mathscr{G}(\alpha) := \mathscr{G}(W+T,\alpha) = P(\alpha)\begin{pmatrix} \alpha I & -I \\ I & \alpha I \end{pmatrix}\begin{pmatrix} \alpha(W+T) & T-W \\ -(T-W) & \alpha(W+T) \end{pmatrix},$$

with $P(\alpha)$ being defined in (3.2). When the splitting matrix $\mathscr{F}(\alpha)$ is used as a preconditioner to accelerate Krylov subspace methods, the convergence rate relates closely to spectral properties of the preconditioned matrix $\mathscr{F}(\alpha)^{-1}\mathscr{A}$. Based on the structure of $\mathscr{A}$ in (2.2) and $\mathscr{F}(\alpha)$ in (3.5), we adapt Theorem 2.4 in Ref. [7] as follows.

Theorem 3.1. *Let $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$ be a nonsingular matrix with W being symmetric indefinite, T being symmetric positive definite and $-T \prec W \preceq T$. Let α be a positive constant. Define $Z^{(\alpha)} = [\alpha(W+T)+T-W]^{-\frac{1}{2}}[W+T-\alpha(T-W)][\alpha(W+T)+T-W]^{-\frac{1}{2}}$. Denote by $\mu_1^{(\alpha)}, \mu_2^{(\alpha)}, \cdots, \mu_n^{(\alpha)}$ the eigenvalues of the symmetric matrix $Z^{(\alpha)} \in \mathbb{R}^{n\times n}$, and by $q_1^{(\alpha)}, q_2^{(\alpha)}, \cdots, q_n^{(\alpha)}$ the corresponding $[\alpha(W+T)+T-W]^{-1}$-orthogonal eigenvectors. Then the eigenvalues of the matrix $\mathscr{F}(\alpha)^{-1}\mathscr{A}$ are*

$$\lambda_{\pm}^{\alpha,j} = \frac{\alpha}{\alpha^2+1}\left(1 \pm \mathrm{i}\frac{\alpha-1}{\alpha+1}\right)\left(1 \pm \mathrm{i}\mu_j^{(\alpha)}\right), \quad j = 1,2,\cdots,n\,,$$

and the corresponding eigenvectors are

$$\widetilde{\omega}_j^{(\alpha)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \omega_j^{(\alpha)} \\ \mathrm{i}\omega_j^{(\alpha)} \end{pmatrix}, \quad \widetilde{\omega}_{n+j}^{(\alpha)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \omega_j^{(\alpha)} \\ -\mathrm{i}\omega_j^{(\alpha)} \end{pmatrix}, \quad j = 1,2,\cdots,n\,,$$

with $\omega_j^{(\alpha)} = [\alpha(W+T)+T-W]^{-\frac{1}{2}}q_j^{(\alpha)}$. Therefore, we have $\mathscr{F}(\alpha)^{-1}\mathscr{A} = \widetilde{\Omega}^{(\alpha)}\Lambda^{(\alpha)}\widetilde{\Omega}^{(\alpha)^{-1}}$, where

$$\widetilde{\Omega}^{(\alpha)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \Omega^{(\alpha)} & \Omega^{(\alpha)} \\ \mathrm{i}\Omega^{(\alpha)} & -\mathrm{i}\Omega^{(\alpha)} \end{pmatrix}, \qquad \Omega^{(\alpha)} = \left(\omega_1^{(\alpha)}, \omega_2^{(\alpha)}, \cdots, \omega_n^{(\alpha)}\right) \in \mathbb{R}^{n\times n}\,,$$

and

$$\Lambda^{(\alpha)} = \begin{pmatrix} \Lambda_+^{(\alpha)} & 0 \\ 0 & \Lambda_-^{(\alpha)} \end{pmatrix}, \qquad \Lambda_{\pm}^{(\alpha)} = \mathrm{diag}(\lambda_{\pm}^{(\alpha,1)}, \lambda_{\pm}^{(\alpha,2)}, \cdots, \lambda_{\pm}^{(\alpha,n)}) \in \mathbb{C}^{n\times n}\,.$$

Moreover, $\Omega^{(\alpha)} \in \mathbb{R}^{n\times n}$ is orthogonal and $\widetilde{\Omega}^{(\alpha)} \in \mathbb{C}^{2n\times 2n}$ is unitary, so it holds that

$$\kappa_2(\widetilde{\Omega}^{(\alpha)}) = \kappa_2(\Omega^{(\alpha)}) = 1\,,$$

where $\kappa_2(\cdot)$ represents the condition number in the Euclidean norm.

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If α=1, then the GPMHSS preconditioner is

$$\mathscr{F} = 2\begin{pmatrix} I & -I \\ I & I \end{pmatrix}\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}.$$

When it is used to precondition the matrix $\mathscr{A} \in \mathbb{R}^{2n\times 2n}$, the eigenvalues of the preconditioned matrix $\mathscr{F}^{-1}\mathscr{A}$ are contained within the complex disc centred at $(1,0)$ with radius $\sqrt{2}/2$. Moreover, Theorem 3.1 indicates that the matrix $\mathscr{F}^{-1}\mathscr{A}$ is diagonalizable, where the matrix $\widetilde{\Omega}^{(1)}$ formed by its eigenvectors is unitary. Thus when employed to solve the block two-by-two linear system (2.1), the preconditioned Krylov subspace iteration methods can be expected to converge rapidly and independently of both problem data and mesh size. From Algorithm 3.1, we know that only the linear subsystem with the coefficient matrix T has to be solved when the matrix $\mathscr{F}$ is used as a preconditioner.

4. Several Variants of the GPMHSS Preconditioner

We have established that the eigenvalues of the preconditioned matrix $\mathscr{F}(\alpha)^{-1}\mathscr{A}$ are clustered within a complex disc centred at $(1,0)$ with radius $\sqrt{\alpha^2+1}/(\alpha+1)$ and the matrix of the corresponding eigenvectors is unitary. The GPMHSS preconditioner $\mathscr{F}(\alpha)$ in (3.5) is essentially a scaled product of the orthogonal matrix

$$G = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix} \tag{4.1}$$

and the symmetric positive definite matrix

$$\mathscr{B}(\alpha) = \begin{pmatrix} \alpha(W+T)+T-W & 0 \\ 0 & \alpha(W+T)+T-W \end{pmatrix}, \tag{4.2}$$

where $W+T$ is symmetric positive definite and $T-W$ is symmetric positive semidefinite. By further simplifying and modifying the GPMHSS preconditioner, we now construct a block diagonal preconditioner and two block triangular preconditioners for the block two-by-two matrix $\mathscr{A} \in \mathbb{R}^{2n\times 2n}$ in (2.2). The eigenvalues of these preconditioned matrices are also analyzed in this section.

4.1. Block diagonal preconditioner

Based on the structure of the GPMHSS preconditioner, we can use the block diagonal preconditioning matrix $\mathscr{B}(\alpha)$ defined in (4.2) as a preconditioner for the block two-by-two matrix $\mathscr{A}$ in (2.2), where the matrix $\mathscr{B}(\alpha)$ is obtained by directly dropping $P(\alpha)$ in the matrix $\mathscr{F}(\alpha)$. Since the matrix $\alpha(W+T)+T-W$ is symmetric positive definite, this block diagonal preconditioner $\mathscr{B}(\alpha)$ is symmetric positive definite. When $\mathscr{B}(\alpha)$ is used to accelerate the convergence rate of Krylov subspace methods, we have two linear subsystems with coefficient matrix $\alpha(W+T)+T-W$ to solve.

From Theorem 2.1 in Ref. [8] we have the spectral properties of the preconditioned matrix $\mathscr{B}(\alpha)^{-1}\mathscr{A}$ as follows.

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Theorem 4.1. *Let $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$ be a nonsingular matrix with W being symmetric indefinite, T being symmetric positive definite and $-T \prec W \preceq T$. Let α be a positive constant. Define $Z^{(\alpha)} = [\alpha(W+T)+T-W]^{-\frac{1}{2}}[W+T-\alpha(T-W)][\alpha(W+T)+T-W]^{-\frac{1}{2}}$. Denote by $\mu_1^{(\alpha)}, \mu_2^{(\alpha)}, \cdots, \mu_n^{(\alpha)}$ the eigenvalues of the symmetric matrix $Z^{(\alpha)} \in \mathbb{R}^{n\times n}$, and by $q_1^{(\alpha)}, q_2^{(\alpha)}, \cdots, q_n^{(\alpha)}$ the corresponding $[\alpha(W+T)+T-W]^{-1}$-orthogonal eigenvectors. Set $\omega_j^{(\alpha)} = [\alpha(W+T)+T-W]^{-\frac{1}{2}} q_j^{(\alpha)}$ $(j = 1, 2, \cdots, n)$ and $\Omega^{(\alpha)} = \left(\omega_1^{(\alpha)}, \omega_2^{(\alpha)}, \cdots, \omega_n^{(\alpha)}\right) \in \mathbb{R}^{n\times n}$. Then the eigenvalues of the matrix $\mathscr{B}(\alpha)^{-1}\mathscr{A}$ are given by*

$$\lambda_{\pm}^{\alpha,j} = \frac{(\alpha+\mathrm{i})\left(1 \mp \mathrm{i}\mu_j^{(\alpha)}\right)}{\alpha^2+1}, \quad j = 1, 2, \cdots, n,$$

and the corresponding eigenvectors are

$$\widetilde{\omega}_j^{(\alpha)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \omega_j^{(\alpha)} \\ \mathrm{i}\omega_j^{(\alpha)} \end{pmatrix}, \quad \widetilde{\omega}_{n+j}^{(\alpha)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \omega_j^{(\alpha)} \\ -\mathrm{i}\omega_j^{(\alpha)} \end{pmatrix}, \quad j = 1, 2, \cdots, n.$$

Therefore, we have $\mathscr{B}(\alpha)^{-1}\mathscr{A} = \widetilde{\Omega}^{(\alpha)}\Lambda^{(\alpha)}\widetilde{\Omega}^{(\alpha)^{-1}}$ with

$$\widetilde{\Omega}^{(\alpha)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \Omega^{(\alpha)} & \Omega^{(\alpha)} \\ \mathrm{i}\Omega^{(\alpha)} & -\mathrm{i}\Omega^{(\alpha)} \end{pmatrix}$$

a unitary matrix such that $\kappa_2(\widetilde{\Omega}^{(\alpha)}) = 1$, where $\kappa_2(\cdot)$ represents the condition number in the Euclidean norm and

$$\Lambda^{(\alpha)} = \begin{pmatrix} \Lambda_-^{(\alpha)} & 0 \\ 0 & \Lambda_+^{(\alpha)} \end{pmatrix}, \qquad \Lambda_{\pm}^{(\alpha)} = \mathrm{diag}(\lambda_{\pm}^{(\alpha,1)}, \lambda_{\pm}^{(\alpha,2)}, \cdots, \lambda_{\pm}^{(\alpha,n)}).$$

Denoting

$$\mathrm{sp}\left(Z^{(\alpha)}\right) \subseteq \left[\mu_{\min}^{(\alpha)}, \mu_{\max}^{(\alpha)}\right]$$

and

$$\eta^{(\alpha)} = \max\left\{\left|\alpha\mu_{\min}^{(\alpha)} - 1\right|, \left|\alpha\mu_{\max}^{(\alpha)} - 1\right|\right\},$$

from Theorem 4.1 we know that the eigenvalues of the matrix $\mathscr{B}(\alpha)^{-1}\mathscr{A}$ are bounded as

$$\mathrm{sp}\left(\mathscr{B}(\alpha)^{-1}\mathscr{A}\right) \subseteq \left[\frac{\alpha+\mu_{\min}^{(\alpha)}}{\alpha^2+1}, \frac{\alpha+\mu_{\max}^{(\alpha)}}{\alpha^2+1}\right] \times \left[-\frac{\eta^{(\alpha)}}{\alpha^2+1}, \frac{\eta^{(\alpha)}}{\alpha^2+1}\right].$$

When α=1, the block diagonal preconditioner is reduced to

$$\mathscr{B} = 2\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}.$$

Moreover, the eigenvalues of the preconditioned matrix $\mathscr{B}^{-1}\mathscr{A}$ are located within the complex rectangle $(0, 1] \times [-1, 1]$.

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4.2. Block triangular preconditioners

Since a block matrix may be approximated by its block triangular parts more accurately than by its block diagonal part and a block triangular matrix may have essentially the same computational cost as the corresponding block diagonal matrix, we construct the following block triangular preconditioners

$$\mathscr{C}_1(\alpha)=\frac{1}{\sqrt{\alpha^2+1}}G\begin{pmatrix}\alpha(W+T)+T-W & 0\\ \alpha(T-W)-W-T & \alpha(W+T)+T-W\end{pmatrix} \tag{4.3}$$

and

$$\mathscr{C}_2(\alpha)=\frac{1}{\sqrt{\alpha^2+1}}G\begin{pmatrix}\alpha(W+T)+T-W & W+T-\alpha(T-W)\\ 0 & \alpha(W+T)+T-W\end{pmatrix} \tag{4.4}$$

for the block two-by-two matrix $\mathscr{A}$ in (2.2), where the matrix G is defined in (4.1). Like the GPMHSS preconditioner $\mathscr{F}(\alpha)$ in (3.5), the block triangular preconditioners $\mathscr{C}_1(\alpha)$ and $\mathscr{C}_2(\alpha)$ are scaled products of the orthogonal matrix G and block triangular matrices. It is expected that $\mathscr{C}_1(\alpha)$ and $\mathscr{C}_2(\alpha)$ exhibit both richer theoretical properties and better numerical results than the GPMHSS preconditioner $\mathscr{F}(\alpha)$. The matrix $\alpha(W+T)+T-W$ is symmetric positive definite, so the preconditioners $\mathscr{C}_1(\alpha)$ and $\mathscr{C}_2(\alpha)$ are nonsingular. Owing to the multiplicative structures, the actions of these two preconditioning matrices can be accomplished through solving only linear subsystems of the same coefficient matrix $\alpha(W+T)+T-W$.

From Theorem 3.1 in Ref. [4], we immediately obtain the eigenvalue distributions of the preconditioned matrices $\mathscr{C}_1(\alpha)^{-1}\mathscr{A}$ and $\mathscr{C}_2(\alpha)^{-1}\mathscr{A}$ as follows.

Theorem 4.2. *Let $A=W+\mathrm{i}T\in\mathbb{C}^{n\times n}$ be a nonsingular matrix with W being symmetric indefinite, T being symmetric positive definite and $-T\prec W\preceq T$. Let α be a positive constant. Then the eigenvalues of the matrices $\mathscr{C}_1(\alpha)^{-1}\mathscr{A}$ and $\mathscr{C}_2(\alpha)^{-1}\mathscr{A}$ are respectively located within the same union of circles with centre $c_\pm(\alpha)$ and radius $\delta(\alpha)$ on the complex plane, where*

$$c_\pm(\alpha)=\frac{1}{\sqrt{2(\alpha^2+1)}}((\alpha+1)\pm\mathrm{i}(\alpha-1))$$

and

$$\delta(\alpha)=\|E(\alpha)\|\sqrt{1+\|E(\alpha)\|^2}$$

with $E(\alpha)=(\alpha(W+T)+(T-W))^{-1}(W+T-\alpha(T-W))$.

By selecting a suitable parameter α, we may guarantee that all the eigenvalues of these block triangular preconditioned matrices are located in the right half or left half of the complex plane, which ensures the convergence of Krylov subspace iteration methods such as GMRES and its restarted variants when they are preconditioned by the matrices $\mathscr{C}_1(\alpha)$ and $\mathscr{C}_2(\alpha)$, respectively. When $\alpha=1$, the block triangular preconditioners $\mathscr{C}_1(\alpha)$ and $\mathscr{C}_2(\alpha)$ are respectively reduced to

$$\mathscr{C}_1=\sqrt{2}G\begin{pmatrix}T & 0\\ -W & T\end{pmatrix}\quad\text{and}\quad\mathscr{C}_2=\sqrt{2}G\begin{pmatrix}T & W\\ 0 & T\end{pmatrix}.$$

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Moreover, from Theorem 4.2 we know that the eigenvalues of $\mathscr{C}_1^{-1}\mathscr{A}$ and $\mathscr{C}_2^{-1}\mathscr{A}$ are contained within the same union of two complex discs centred at $(1,0)$ with radius less than one.

Remark 4.1. When these block preconditioners are used to accelerate the convergence rate of Krylov subspace iteration methods, it is necessary to solve linear subsystems with the coefficient matrix $\alpha(W+T)+T-W$. In order to avoid inversion of the matrix $\alpha(W+T)+T-W$ exactly at each step of solving the linear subsystems, we can also construct the inexact block preconditioners for the block two-by-two matrix (2.2) and analyze the spectral properties of the preconditioned matrices [19].

Remark 4.2. When the matrices $\mathscr{C}_1(\alpha)$ and $\mathscr{C}_2(\alpha)$ are used as preconditioners to accelerate Krylov subspace methods such as GMRES, the pre-factor $1/\sqrt{\alpha^2+1}$ has no effect on the preconditioned systems and can be neglected. Here we add the pre-factor just for the purpose of eigenvalue analysis.

5. Numerical Results

In this section, we provide two examples to verify the effectiveness of the GPMHSS real-valued iteration method and the proposed preconditioners for the complex symmetric indefinite linear system (1.1). We first use the GMRES, HSS [9] and GPMHSS iteration methods to solve the equivalent linear system (2.1). Then the preconditioned GMRES methods incorporated with the HSS preconditioner $\mathscr{P}_{HSS}$ [9, 11], the preconditioner $\mathscr{P}_S$ [18], the GPMHSS preconditioner $\mathscr{F}(\alpha)$ in (3.5), the block diagonal preconditioner $\mathscr{B}(\alpha)$ in (4.2), and the block triangular preconditioners $\mathscr{C}_1(\alpha)$ and $\mathscr{C}_2(\alpha)$ in (4.3) and (4.4) respectively are also used to solve the linear system (2.1). Here, the preconditioners $\mathscr{P}_{HSS}$ and $\mathscr{P}_S$ are respectively defined as

$$\begin{aligned}\mathscr{P}_{HSS} &= \frac{1}{2\alpha}(\alpha\mathscr{I}+\mathscr{H})(\alpha\mathscr{I}+\mathscr{S}) \\ &= \frac{1}{2\alpha}\begin{pmatrix} \alpha I+W+T & 0 \\ 0 & \alpha I+W+T \end{pmatrix}\begin{pmatrix} \alpha I & -(T-W) \\ T-W & \alpha I \end{pmatrix}\end{aligned}$$

and

$$\mathscr{P}_S = \begin{pmatrix} W+T & 0 \\ T-W & \widetilde{S} \end{pmatrix},$$

where $\mathscr{H}$ and $\mathscr{S}$ in $\mathscr{P}_{HSS}$ are the Hermitian and skew-Hermitian parts of $\mathscr{A}$, and $\widetilde{S} = W+T+(T-W)\widetilde{M}^{-1}(T-W)$ in $\mathscr{P}_S$ is an approximation to the Schur complement matrix of $\mathscr{A}$ with $\widetilde{M} = \mathrm{diag}(W+T)$. To compare these iteration methods, we consider the number of iteration steps (denoted by "IT") and elapsed CPU times in seconds (denoted by "CPU").

In the implementation, the parameters α for the HSS iteration method and the HSS preconditioner $\mathscr{P}_{HSS}$ were chosen to be the optimal $\alpha^* = \sqrt{\lambda_{\min}(W+T)\lambda_{\max}(W+T)}$, and the parameters α for the GPMHSS iteration method and the preconditioners $\mathscr{F}(\alpha)$, $\mathscr{B}(\alpha)$,

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Table 1: Numerical results from different iteration methods for Example 5.1.

Methods		Grids			
		16×16	32×32	64×64	128×128
GMRES	IT	76	223	547	1339
	CPU	0.2600	0.9691	19.6478	635.8150
	RES	7.73e-7	8.64e-7	9.65e-7	9.94e-7
HSS	IT	51	105	197	358
	CPU	0.0206	0.1484	1.6353	18.6814
	RES	8.98e-7	9.97e-7	9.79e-7	9.88e-7
GPMHSS	IT	20	20	20	20
	CPU	0.0109	0.0226	0.1072	0.6185
	RES	9.62e-7	9.54e-7	9.54e-7	9.54e-7

$\mathscr{C}_1(\alpha)$ and $\mathscr{C}_2(\alpha)$ were set to be $\alpha^* = 1$. The initial guess was $x^{(0)} = (0,\ 0, \cdots,\ 0)^T \in \mathbb{R}^n$, and the iteration was terminated when the current iterate satisfied

$$\text{RES} \le \frac{\|b - Ax^{(k)}\|_2}{\|b\|_2} \le 10^{-6}$$

or the prescribed iteration number $k_{max} = 1500$ was exceeded. All runs were performed in MATLAB 2010 on an Intel Core (4G RAM) Windows 7 system.

Example 5.1. The first complex symmetric indefinite linear system (1.1) is of the form

$$\left[\left(K - (3-\sqrt{3})\omega^2 I\right) + \mathrm{i}\left(K + (3+\sqrt{3})\tau^2 I\right)\right]x = b\,, \tag{5.1}$$

where ω and τ are two positive parameters, and K is the five-point centred difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0,\ 1]\times[0,\ 1]$ with the mesh-size $h = 1/(m+1)$. The matrix $K \in \mathbb{R}^{n\times n}$ has the form $K = I \otimes V_m + V_m \otimes I$ with $V_m = h^{-2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$, so K is an $n\times n$ block tridiagonal matrix with $n = m^2$.

This problem is a technical modification of Example 4.1 in Ref. [5]. For this test problem, we set the right-hand side vector $b = (1+\mathrm{i})Ax^\star$, with $x^\star$ being the vector of all entries equal to 1, and normalized the system by multiplying both sides with h^2. In the computation, we took the grid size $m = 16,\ 32,\ 64,\ 128$ and chose $\omega = 20$ such that four test problems were complex symmetric indefinite linear systems. The parameter τ values chosen were 100, 400, 1600 and 7000 such that the assumption $-T \prec W \preceq T$ was satisfied.

Table 1 describes numerical results of the GMRES, HSS and GPMHSS iteration methods for Example 5.1. From this table, we see that the GMRES method converges very slowly, while the HSS and the GPMHSS iteration methods converge quite fast. Moreover, the number of iteration steps and the CPU times of GPMHSS iteration method are much less than those of the HSS iteration method.

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Table 2: Numerical results from different preconditioned GMRES methods for Example 5.1.

Preconditioners		Grids			
		16×16	32×32	64×64	128×128
$\mathscr{P}_{HSS}$	IT	24	41	66	104
	CPU	0.0143	0.0836	0.7797	7.5773
	RES	5.90e-7	7.28e-7	8.67e-7	9.76e-7
$\mathscr{P}_S$	IT	16	16	15	15
	CPU	0.0206	0.0525	0.1624	0.8518
	RES	5.14e-7	4.74e-7	7.72e-7	3.52e-7
$\mathscr{F}(\alpha)$	IT	6	5	4	4
	CPU	0.0077	0.0197	0.0502	0.1609
	RES	8.89e-8	1.15e-7	1.21e-7	1.67e-9
$\mathscr{B}(\alpha)$	IT	9	9	7	7
	CPU	0.0071	0.0255	0.0468	0.2327
	RES	6.26e-7	7.14e-9	5.63e-8	6.90e-10
$\mathscr{C}_1(\alpha)$	IT	4	4	3	3
	CPU	0.0067	0.0135	0.0425	0.1378
	RES	6.38e-8	2.97e-10	1.28e-7	1.74e-9
$\mathscr{C}_2(\alpha)$	IT	4	4	3	3
	CPU	0.0069	0.0111	0.0366	0.1355
	RES	6.44e-8	2.98e-10	1.30e-7	1.74e-7

In Table 2, we list numerical results of preconditioned GMRES methods with the preconditioners $\mathscr{P}_{HSS}$, $\mathscr{P}_S$, $\mathscr{F}(\alpha)$, $\mathscr{B}(\alpha)$, $\mathscr{C}_1(\alpha)$ and $\mathscr{C}_2(\alpha)$ for Example 5.1. From this table, we observe that all these preconditioned GMRES methods converge fast. The HSS preconditioned GMRES method requires the most iteration steps and CPU time, and our proposed $\mathscr{F}(\alpha)$, $\mathscr{B}(\alpha)$, $\mathscr{C}_1(\alpha)$ and $\mathscr{C}_2(\alpha)$ preconditioned GMRES methods use much less iteration steps and lower CPU time for all four grids than the existing $\mathscr{P}_{HSS}$ and $\mathscr{P}_S$ preconditioned GMRES methods, which shows that our proposed preconditioned GMRES methods are effective for Example 5.1. Further, from Tables 1 and 2 we observe that the $\mathscr{F}(\alpha)$, $\mathscr{B}(\alpha)$, $\mathscr{C}_1(\alpha)$ and $\mathscr{C}_2(\alpha)$ preconditioned GMRES methods are more effective than the GPMHSS iteration method. This shows that the GPMHSS iteration method is not competitive as a solver, but the splitting matrix and its variants are good preconditioners in accelerating the convergence rate of the GMRES method for Example 5.1.

Example 5.2. [5, 15, 23] The second complex symmetric indefinite linear system (1.1) arises from the direct frequency domain analysis of an n-degree-of-freedom (n-DOF) linear system

$$M\ddot{q}+(C_V+\frac{1}{\omega}C_H)\dot{q}+Kq=p\ ,$$

where q is the configuration vector, p is the vector of generalized components of dynamic forces, M is the inertia matrix, C_V and C_H are the viscous and the hysteretic damping

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matrices, respectively, and K is the stiffness matrix. Complex harmonic excitation at the driving circular frequency ω (i.e. of the type $p(t) = f e^{i\omega t}$) admits the steady-state solution $q(t) = \tilde{q}(\omega)e^{i\omega t}$, where $\tilde{q}$ solves the following complex symmetric indefinite linear system

$$E(\omega)\tilde{q}(\omega) \equiv \left[(-\omega^2 M + K) + \mathrm{i}(\omega C_V + \tau K)\right]\tilde{q}(\omega) = f, \tag{5.2}$$

where $E(\omega)$ is the dynamic impedance matrix and τ is a damping coefficient. Ref. [15] provides more details.

In the computation, we took $M = I$, $C_V = 10I$ and K the five-point centred difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions on a uniform mesh in the unit square $[0,1]\times[0,1]$ with the mesh-size $h = 1/(m+1)$. The matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $K = I \otimes V_m + V_m \otimes I$ with $V_m = h^{-2}\mathrm{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$, so K is an $n \times n$ block-tridiagonal matrix with $n = m^2$. Analogous to Example 5.1, we set the right-hand side vector $b = (1 + \mathrm{i})Ax^\star$ and normalized the linear system by multiplying both sides with h^2. Then from (5.2) we had

$$W = -\frac{\omega^2}{(m+1)^2} I + (I \otimes V_m + V_m \otimes I)$$

and

$$T = \frac{10\omega}{(m+1)^2} I + \tau(I \otimes V_m + V_m \otimes I)\,,$$

where we took $m = 16,\ 32,\ 64,\ 128$. We set the parameter $\omega = 10$ for all test problems and $\tau = 120,\ 500,\ 1800,\ 7000$ for the four test sizes, with the assumption $-T \prec W \preceq T$ satisfied.

In Table 3, we list numerical results of the GMRES, HSS and GPMHSS iteration methods for Example 5.2. Table 4 lists numerical results of the preconditioned GMRES methods with the preconditioners $\mathscr{P}_{HSS}$, $\mathscr{P}_S$, $\mathscr{F}(\alpha)$, $\mathscr{B}(\alpha)$, $\mathscr{C}_1(\alpha)$ and $\mathscr{C}_2(\alpha)$ for Example 5.2. Table 3 shows that the proposed GMPHSS iteration method costs much less iterations and CPU times than the GMRES and the HSS iteration methods. From Table 4, we see that the convergence of all these preconditioned GMRES methods is fast for Example 5.2, with the HSS preconditioned GMRES method requiring the most iteration steps and CPU time. Moreover, for all four grids our proposed $\mathscr{F}(\alpha)$, $\mathscr{B}(\alpha)$, $\mathscr{C}_1(\alpha)$ and $\mathscr{C}_2(\alpha)$ preconditioned GMRES methods use much less iteration steps and have lower CPU times than the existing $\mathscr{P}_{HSS}$ and $\mathscr{P}_S$ preconditioned GMRES methods. Further, when the grid number increases, the iteration steps of the $\mathscr{F}(\alpha)$, $\mathscr{B}(\alpha)$, $\mathscr{C}_1(\alpha)$ and $\mathscr{C}_2(\alpha)$ preconditioned GMRES methods are nearly constant and even decrease, while those of the $\mathscr{P}_{HSS}$ preconditioned GMRES method increase rapidly. Our proposed preconditioners are thus also effective in accelerating the convergence rate of the GMRES method for Example 5.2.

6. Conclusion

We have proposed a generalised PMHSS (GPMHSS) real-valued iteration method for the complex symmetric indefinite linear system (1.1) with $-T \prec W \preceq T$. It is proved that

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Table 3: Numerical results from different iteration methods for Example 5.2.

Methods		Grids			
		16×16	32×32	64×64	128×128
GMRES	IT	79	227	550	1341
	CPU	0.2715	0.9971	21.0832	662.4171
	RES	9.44e-7	8.70e-7	9.84e-7	9.73e-7
HSS	IT	63	112	200	359
	CPU	0.0238	0.1796	1.6515	18.3772
	RES	9.41e-7	9.71e-7	9.89e-7	9.99e-7
GPMHSS	IT	20	20	20	20
	CPU	0.0098	0.0364	0.1089	0.6903
	RES	9.54e-7	9.54e-7	9.54e-7	9.54e-7

Table 4: Numerical results from different preconditioned GMRES methods for Example 5.2.

Preconditioners		Grids			
		16×16	32×32	64×64	128×128
$\mathscr{P}_{HSS}$	IT	27	43	67	105
	CPU	0.0182	0.0851	0.8151	7.4936
	RES	8.50e-7	8.84e-7	9.58e-7	9.16e-7
$\mathscr{P}_S$	IT	15	15	15	15
	CPU	0.0157	0.0499	0.2815	0.8123
	RES	6.73e-7	3.98e-7	3.89e-7	5.04e-7
$\mathscr{F}(\alpha)$	IT	5	4	4	3
	CPU	0.0091	0.0177	0.0317	0.1221
	RES	3.37e-8	1.13e-7	2.57e-9	2.07e-7
$\mathscr{B}(\alpha)$	IT	7	7	7	5
	CPU	0.0114	0.0123	0.0570	0.1968
	RES	8.39e-7	1.46e-8	3.23e-10	9.96e-8
$\mathscr{C}_1(\alpha)$	IT	4	3	3	3
	CPU	0.0095	0.0090	0.0361	0.1525
	RES	6.83e-11	7.04e-8	1.57e-9	2.69e-11
$\mathscr{C}_2(\alpha)$	IT	4	3	3	3
	CPU	0.0096	0.0088	0.0349	0.1476
	RES	6.83e-11	7.02e-8	1.57e-9	2.69e-11

the GPMHSS iteration method is unconditionally convergent. The splitting matrix of the GPMHSS iteration method can be used as a preconditioner to accelerate the convergence rate of the GMRES method for solving the real equivalent form (2.1). In addition, we constructed several block diagonal and triangular preconditioners for the real equivalent form (2.1). Numerical results show that our proposed preconditioners outperform the existing

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ones. However, the design of an efficient method for general complex symmetric indefinite linear systems should be investigated further.

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