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On semi-Markov processes on arbitrary spaces

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1. *Introduction.* Let E be an arbitrary set, $\mathfrak{A}$ a σ-algebra of subsets of

$$E, F = (-\infty, +\infty), \quad F^- = (-\infty, 0), \quad F^+ = [0, \infty),$$

$\mathfrak{B}$ the Borel sets of F. We write $A \times B$ for the product of the sets A and B, and

$$\mathfrak{L} = \mathfrak{A} \otimes \mathfrak{B}$$

for the product σ-algebra of $\mathfrak{A}$ and $\mathfrak{B}$ (i.e. the σ-algebra generated by the rectangles $A \times B$ with $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$).

A mapping $Q: E \times \mathfrak{L} \Rightarrow [0, 1]$ will be called a semi-Markovian kernel if

(i) for fixed $x \in E$, $Q(x, \cdot)$ is a measure on $\mathfrak{L}$;

(ii) for fixed $C \in \mathfrak{L}$ the function $Q(\cdot, C)$ is $\mathfrak{A}$-measurable;

(iii) for any $x \in E$, $Q(x, E \times F^-) = 0$.

We note that if Q is a semi-Markovian kernel, then $K(x, A) = Q(x, A \times F)$ $(x \in E, A \in \mathfrak{A})$ is a (possibly sub-) Markovian kernel on the measurable space $(E, \mathfrak{A})$.

If a function f on $E \times F$ is $\mathfrak{A} \otimes \mathfrak{L}$-measurable, then every section $f(x, \cdot)$ is $\mathfrak{L}$-measurable and every section $f(\cdot, t)$ is $\mathfrak{A}$-measurable. Let L denote the Banach space of $\mathfrak{A} \otimes \mathfrak{B}$-measurable real-valued functions f on $E \times F$ with the usual norm

$$\|f\| = \sup_{(x,t) \in E \times F} |f(x, t)|.$$

A semi-Markovian kernel Q on $E \times \mathfrak{L}$ induces a linear transformation T on the space L by

$$Tf(x, t) = \int_{E \times F} Q(x, d(y \times u)) f(y, t - u). \tag{1}$$

It is clear that T is a positive contraction, i.e.

(i) $f \in L, f \geqslant 0 \Rightarrow Tf \geqslant 0$,

(ii) $f \in L \Rightarrow \|Tf\| \leqslant \|f\|$.

A non-negative function $f \in L$ is called regular if $f = Tf$, super-regular if $f \geqslant Tf$, and is called the potential of g if $f = \sum_{k=0}^{\infty} T^k g$. From the Riesz decomposition theorem we have that every super-regular function f can be written (in one and only one way) as the sum of a potential and a regular function, i.e. we can write

$$f = \sum_{k=0}^{\infty} T^k g + h \tag{2}$$

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where $h = Th$. The main topic of interest in this paper is formulated as follows. We start from the equation

$$f = g + Tf \quad (g \in L), \tag{3}$$

and consider the existence, uniqueness, and the limiting behaviour as $t \uparrow \infty$ of the solutions $f(x, t) \in L$ of this equation. Naturally this leads to questions concerning the super-regular and regular functions.

In the special case where E has countably many points, Q defines a semi-Markov process in the sense of Smith (11). The general case of arbitrary E is mentioned in Lévy (7). What we have then is a stochastic process on $(E, \mathfrak{A})$. The paths are piecewise constant (normally), and if jumps accumulate to a finite z then the process is left undefined for $t \geqslant z$. Just as in the completely discontinuous Markov processes, the epochs of jumps are regeneration points erasing the influence of the past. The only difference is that sojourn time at a point $x \in E$ has an arbitrary distribution which may depend on the terminal state. The mechanism of the process is determined by Q where $Q(x, A \times B)$ is the probability that a sojourn time at x has a duration $t \in B$ and ends by a jump into A. This rough description will be made precise in section 2. There we give a rigorous definition of a semi-Markov process by utilizing the formalism of Dynkin (2). It differs fundamentally from the definitions of Smith (11) and Pyke and Schaufele (10) by allowing the process to terminate; in this respect it is in the spirit of Dynkin (2) and Feller (4). On the other hand, being constructive, it is similar to the definition of Pyke (9).

Quite a few questions concerning the functions or additive functionals of such processes as well as certain questions concerning their sample paths reduce to questions about the existence, uniqueness, and the limiting behaviour of solutions to equation (3). An investigation of these problems in the countable state space case can be found in the review paper of Çinlar (1) which is based primarily on Feller (3, 4). Most of that theory about the existence and uniqueness of solutions to (3) goes through for the general case of arbitrary state spaces $(E, \mathfrak{A})$. In section 3 we will sketch the main results and give some extensions.

New problems arise, however, when one considers the limiting behaviour as $t \uparrow \infty$ of the solutions $f(x, t)$ of (3). In the case of countable E such questions are answered by the use of renewal theory, since then the epochs of successive entrances to a fixed state $x \in E$ form a renewal process. Due to the possible lack of such imbedded renewal processes, in the general case, renewal theory fails to apply directly. The theorems we give in section 4 show the equivalence of the generalized versions of the renewal theorems of Blackwell and Smith. The method of proof followed is that of Feller (5).

2. *Definition of a semi-Markov process.* Suppose we are given

(i) a function $\zeta(\omega)$ on some space Ω taking values in $\{1, 2, 3, \ldots, +\infty\}$;

(ii) for each $n = 0, 1, \ldots$ a σ-algebra $\mathfrak{M}_n$ on the space $\Omega_n = \{\omega : \zeta(\omega) > n\}$;

(iii) for each $x \in E$ a probability measure P_x on the σ-algebra $\mathfrak{M}$, on Ω, which contains all the $\mathfrak{M}_n$;

(iv) a pair of functions $\xi_n(\omega) = \xi(n, \omega), \tau_n(\omega) = \tau(n, \omega)$ defined for $\omega \in \Omega, 0 \leqslant n < \zeta(\omega)$; ξ takes values in the state space $(E, \mathfrak{A})$; τ takes values in the real line and is such that, for any $\omega \in \Omega, 0 = \tau_0(\omega) \leqslant \tau_1(\omega) \leqslant \ldots$.

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Suppose these elements define a Markov chain $((\xi_n, \tau_n), \zeta, \mathfrak{M}_n, P_x)$ in the sense of Dynkin (cf. Dynkin (2), p. 79) such that

$$P_x\{\xi_{n+1} \in A, \tau_{n+1} \in B | \mathfrak{M}_n\} = Q(\xi_n, A \times (B - \tau_n)) \quad \text{(a.e. } \Omega_n, P_x),$$

for any $A \in \mathfrak{A}$, $B \in \mathfrak{B}$; ($B - t$ is the set $\{u : u + t \in B\}$). Now define

$$z(\omega) = \sup_{0 \leqslant n < \zeta(\omega)} \tau_n(\omega) \quad (\omega \in \Omega);$$

$$\nu(t, \omega) = \sup\{n : \tau_n(\omega) \leqslant t\} \quad (t < z(\omega), \omega \in \Omega);$$

$$x_t(\omega) = \xi(\nu(t, \omega), \omega) \quad (t < z(\omega), \omega \in \Omega).$$

Then, we call $X = (x_t, z, P_x)$ a semi-Markov process with kernel Q.

The set Ω is the sample space, ζ is the total number of transitions, z is the termination time, $\nu(t)$ is the number of transitions in $[0, t]$, x_t is the state of the process at time epoch t. The epochs τ_n are instants of jumps; at such points the process X regenerates itself, namely as far as predicting the 'future' of x_t is concerned, all knowledge previous to the random time τ_n lose their prognostic value once ξ_n is known. Thus, what we have is a generalization of a Markov process where the independence of 'past' and 'future' need only hold for certain random 'present' instants τ_n.

Defining

$$P(t, x, A) = P_x\{x_t \in A\} \quad \text{if} \quad t \geqslant 0, \ A \in \mathfrak{A}$$
$$= 0 \quad \text{if} \quad t < 0, \ A \in \mathfrak{A}$$

we see that

$$P(t, x, A) = l_A(x)\, Q(x, E \times (t, \infty)) + \int_{E \times F} Q(x, d(y \times u))\, P(t - u, y, A), \tag{4}$$

where l_A is the indicator function of A. This is the generalized version of Kolmogorov's backward equations (needless to say, the semi-group property does not hold). We note that, for fixed $A \in \mathfrak{A}$, considered as a function of $E \times F$, P satisfies the equation (3) for a suitable g. Thus the study of that equation will give us the solution to (4) as a by-product.

We will call the process (x_t, z, P_x) conservative if $P_x\{z < \infty\} = 0$ for all $x \in E$. It is clear that whether the process is conservative or not depends largely on the semi-Markovian kernel Q which defines it. As in the case of Markov processes, we will see that a process is conservative if and only if the minimal solution of (4) gives

$$P(t, x, E) = 1.$$

Letting $K(x, A) = Q(x, A \times F)$ $(x \in E, A \in \mathfrak{A})$, we obtain a Markovian kernel on $(E, \mathfrak{A})$. This is the kernel corresponding to the Markov chain $(\xi_n, \zeta; \mathfrak{M}'_n, P_x)$ where $\mathfrak{M}'_n$ is the σ-algebra generated by $\xi_1, \ldots, \xi_n$. For fixed $x \in E$ and $B \in \mathfrak{B}$ we have

$$Q(x, A \times B) \leqslant K(x, A) \quad \text{for any} \quad A \in \mathfrak{A}.$$

Thus, by the Radon–Nikodym theorem, the derivative

$$G(x, y; B) = \frac{Q(x, dy \times B)}{K(x, dy)}$$

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exists. Let $r_0 < r_1 < r_2 < \ldots$ be the rational numbers of F^+ and let

$$f_n(x,y) = \frac{Q(x, dy \times (-\infty, r_n])}{K(x, dy)}$$

be selected so that $f_0 \leqslant f_1 \leqslant f_2 \leqslant \ldots$ and $\lim f_n = 1$ identically. For $t \in F^+$, if r_{n_k} is a sequence of rational numbers decreasing to t as $k \uparrow \infty$, set

$$G(x,y;(-\infty,t]) = g(x,y;t) = \lim_k f_{nk}(x,y).$$

Then, $g(x,\cdot,t)$ and $g(\cdot,y,t)$ are $\mathfrak{A}$-measurable and $g(x,y,\cdot)$ is a distribution function: the conditional distribution of a sojourn time given that the sojourn is at state x and will end by a jump to y if $g(\cdot,\cdot;t)$ is $\mathfrak{A}\otimes\mathfrak{A}$ measurable.

In the opposite direction, suppose we are given on a measurable space $(E, \mathfrak{A})$

(i) a Markovian kernel $K(x,A)$, i.e. a mapping from $E \times \mathfrak{A}$ into $[0,1]$ such that $K(\cdot,A)$ is an $\mathfrak{A}$-measurable function for fixed $A \in \mathfrak{A}$, and $K(x,\cdot)$ is a measure on $\mathfrak{A}$ for fixed $x \in E$;

(ii) a measure $G(x,y;\cdot)$ on $\mathfrak{L}$ defined for each $x,y \in E$ such that $G(x,y;F^-) = 0$, $G(x,y,F) = 1$, and $G(\cdot,y,B)$ is $\mathfrak{A}$-measurable for fixed $y \in E$, $B \in \mathfrak{B}$, and $G(x,\cdot,B)$ is $\mathfrak{A}$-measurable for fixed $x \in E$. $B \in \mathfrak{B}$.

Letting

$$Q(x, A \times B) = \int_A K(x,dy)\, G(x,y;B)$$

we obtain, for fixed $x \in E$, a set function on rectangles $A \times B$ of $\mathfrak{A} \times \mathfrak{B}$. This can be extended in a unique manner to a measure on $\mathfrak{A}\otimes\mathfrak{B}$ for each fixed $x \in E$. Then the resulting mapping $Q(x,C)$ of $E \times (\mathfrak{A}\otimes\mathfrak{B})$ is a semi-Markovian kernel if $Q(\cdot,C)$ is $\mathfrak{A}$-measurable for each fixed $C \in \mathfrak{A}\otimes\mathfrak{B}$.

A special case arises when we have the representation

$$Q(x, A \times B) = K(x,A)\, G(x,B) \tag{5}$$

for any $x \in E$, $A \in \mathfrak{A}$, $B \in \mathfrak{B}$ where K is a Markovian kernel on $(E, \mathfrak{A})$ and $G(x,\cdot)$ is a measure on $\mathfrak{B}$ for each fixed x whereas $G(\cdot,B)$ is $\mathfrak{A}$-measurable for each $B \in \mathfrak{B}$. In this case $G(x,\cdot)$ is the distribution of the sojourn time at $x \in E$ and that sojourn time is independent of the terminal state. In particular, all pure-jump Markov processes are of this type: in that case we have

$$G(x,(t,\infty)) = e^{-\lambda(x)t}$$

where λ is an $\mathfrak{A}$-measurable function.

Throughout the remainder of this paper we shall assume the representation (5) is possible. To see that there is no loss of generality in assuming this we note the following device due to Pyke and Schaufele(10). Let (ξ_n, τ_n) be the semi-Markov chain corresponding to $Q(x,C)$, and write $y_n(\omega) = (\xi_n(\omega), \xi_{n+1}(\omega))$ $(n = 0,1,\ldots)$. Then, the process (y_n, τ_n) is another semi-Markovian chain where the initial distribution is given by

$$P_{(x,y)}(W) = P_x\{W \mid \xi_1 = y\} \quad (W \in \mathfrak{M});$$

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and the transitions are regulated by

$$\begin{aligned} &P_x\{y_{n+1} \in A, \tau_{n+1} \in B | \sigma(y_0, \ldots, y_n; \tau_1, \ldots, \tau_n)\} \\ &\quad = P_x\{(\xi_{n+1}, \xi_{n+2}) \in A_1 \times A_2, \tau_{n+1} \in B | \sigma(\xi_0 = x, \xi_1, \ldots, \xi_{n+1}; \tau_1, \ldots, \tau_n)\} \\ &\quad = P_x\{\xi_{n+1} \in A_1, \xi_{n+1} \in A_2, \tau_{n+1} \in B | \xi_n, \xi_{n+1}, \tau_n\} \\ &\quad = K(\xi_{n+1}, A_2)\, l_{A_1}(\xi_{n+1})\, G(\xi_n, \xi_{n+1}; B - \tau_n) \end{aligned}$$

where $A = A_1 \times A_2, A_1, A_2 \in \mathfrak{A}, B \in \mathfrak{B}$. Now, letting

$$\tilde{K}((x,y), A_1 \times A_2) = l_{A_1}(x)\, K(x, A_2),$$

$$\tilde{G}((x,y), B) = G(x, y; B),$$

$$\tilde{Q}((x,y), (A_1 \times A_2) \times B) = \tilde{K}((x,y), A_1 \times A_2)\, \tilde{G}((x,y), B)$$

we see that $\tilde{Q}$ defines a semi-Markovian kernel

$$\tilde{Q}: \tilde{E} \times (\tilde{\mathfrak{A}} \otimes \mathfrak{B}) \Rightarrow [0, 1]$$

where $\tilde{E} = E \times E, \tilde{\mathfrak{A}} = \mathfrak{A} \otimes \mathfrak{A}$, and that $\tilde{Q}$ has the desired representation (5).

2. *The fundamental equation.* Assuming that Q has the representation (5), the integral

$$Tf(x,t) = \int_F \int_E Q(x, dy, du) f(y, t-u),$$

where $Q(x, dy, du) = Q(x, d(y \times u)) = K(x, dy)\, G(x, du)$, is well defined for functions f whose sections $f(x, \cdot)$ and $f(\cdot, t)$ and $\mathfrak{B}$- and $\mathfrak{A}$-measurable respectively, for each $x \in E$ and $t \in F$. Further, the resulting function Tf has the same property, i.e. all sections of Tf are measurable with respect to the appropriate σ-algebras. Thus, the iterates of T are also well defined.

Let L be the class of bounded real valued functions f on $E \times F$ that are $\mathfrak{A} \otimes \mathfrak{B}$-measurable. The class L is a Banach space if the norm is defined as usual by $\|f\| = \sup_{(x,t) \in E \times F} |f(x,t)|$.

Let $g \in L$ and consider the equation (3), i.e.

$$f = g + Tf. \tag{6}$$

By iteration we have

$$f = g + Tg + \ldots + T^n g + T^{n+1} f \quad (n = 0, 1, 2, \ldots).$$

If the pointwise limit of $\sum_{k=0}^{n} T^k g$ exists as $n \to \infty$, then it is a solution of the equation, but may or may not be a solution in L. On the other hand, if the strong limit

$$Rg = s \lim_{n \to \infty} \sum_{k=0}^{n} T^k g \tag{7}$$

exists, then Rg is a solution of (6) and is in L; that is, if it exists, Rg satisfies

$$TRg = RTg = Rg - g.$$

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If f is any other solution of (6), then we must have $T(f-Rg) = f-Rg$. On the other hand, since for any solution $f \in L$ of (6) we have $g = f-Tf$ and

$$\left\| \sum_{k=0}^{n} T^k g \right\| = \|f - T^{n+1} f\| \leqslant 2\|f\| \quad (n \geqslant 0),$$

Rg exists whenever there is a solution $f \in L$ of (6). Thus we have

THEOREM. *The equation* (6) *has a solution* $f \in L$ *if and only if the limit* Rg *defined by* (7) *exists. All solutions of* (6), *if there are any, are of the form*

$$f = Rg + h$$

where h *satisfies*

$$h = Th.$$

Therefore, Rg *is the unique solution (when it exists) if and only if* $h = Th$, $h \in L$ *implies* $h = 0$.

The following simple result is applicable to almost all the equations that are likely to arise in applications to semi-Markov processes. (8)

PROPOSITION. *Suppose* $g \geqslant 0$ *vanishes on* $E \times F^-$, *and* $g(x,t) \leqslant G(x,(t,\infty))$ *for* $t \geqslant 0$. *Then* Rg *exists and* $Rg \leqslant 1$. (9)

Proof. Let e be the indicator function of $E \times F^+$. Then, $g \leqslant e - Te$, and

$$0 \leqslant Rg \leqslant e - \lim_n T^n e \leqslant e. \qquad \blacksquare$$

COROLLARY. *Let* (x_t, z, P_x) *be a semi-Markov process with kernel* Q *satisfying* (5) *with* $Q(x, E \times F) = 1$ *and consider the equation* (4) *for* $P(t,x,A) = P_x\{x_t \in A\}, t \geqslant 0$. (10) *Then*

$$P(t,x,A) = R(l_A \cdot (e - Te))(x,t) \tag{11}$$

is a solution for (4). *This solution is the minimal solution; therefore, it is the unique solution if and only if* $P(t,x,E) = 1$ *for all* $t > 0$ *for any* $x \in E$.

Proof. Existence of the solution indicated by (11) follows from proposition (9). Since $R(e-Te) = e - s\lim T^n e \leqslant e$, $P(t,x,F) \leqslant 1$; further, the minimality follows from the maximality of $s\lim_n T^n e$ amongst the solutions of $h = Th$, $0 \leqslant h \leqslant e$. If $P(t,x,E) = 1$, then the solution is unique because of its minimal character. Otherwise, $\operatorname{slim} T^n e = h \neq 0$ and there exist other solutions. $\blacksquare$

We remark the following interpretation for the limit

$$h = s\lim_{n\to\infty} T^n e: \tag{12}$$

we have $h(x,t) = P_x\{z \leqslant t, \zeta = \infty\}$ where z is the termination epoch of the semi-Markov process. Since h is the maximal solution of

$$h = Th \quad (0 \leqslant h \leqslant e), \tag{13}$$

and since all bounded solutions of $h = Th$ that vanish on $E \times F^-$ are linear combinations of solutions of (13), we have that all such solutions vanish if and only if the limit in (12) vanishes. Further criteria on this point can be found in Feller (4) or in Çinlar (1).

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We shall extend the above result to the case of the regular functions which do not necessarily vanish on $E \times F^-$. For this purpose we will exclude the degenerate case $P_x\{\zeta < \infty\} > 0$ by taking $Q(x, E \times F) = 1$ for all x. Further, we shall assume non-periodicity: $X = (x_t, z, P_x)$ is non-periodic if, for any $r > 0$ and $A \in \mathfrak{A}$, the event

$$\{\{\tau_n : \xi_n \in A, n \geqslant 0\} \subset \{0, r, 2r, \ldots\}\}$$

has P_x measure zero for all x.

PROPOSITION. *Let f be a bounded non-negative regular function such that*

$$\sup_{x \in E} f(x,t) \to 0 \quad as \quad t \to -\infty,$$

and suppose X is non-periodic. Then, $f = 0$ if and only if $s\lim_{n\to\infty} T^n e = 0$. (14)

Proof. The necessity is obvious. To prove the sufficiency, consider the Markov chain $[(\xi_n, \tau_n), \mathfrak{M}_n]$. Since f is regular $(f(\xi_n, t-\tau_n), \mathfrak{M}_n)$ is a martingale. By the martingale convergence theorem

$$\lim_{n\to\infty} f(\xi_n(\omega), t-\tau_n(\omega)) = \gamma(t, \omega)$$

exists a.e. P_x, and $E_x\gamma(t) = E_x f(\xi_n, t-\tau_n) = f(x,t)$ for any $x \in E$.

Now, $s\lim_n T^n e = 0$ implies that $\tau_n \uparrow \infty$ a.e. P_x for any x. The sufficiency therefore follows from observing that $0 \leqslant f(\xi_n, t-\tau_n) \leqslant \sup_x f(x, t-\tau_n) \to 0$ with probability one as $n \uparrow \infty$. ∎

COROLLARY. *Suppose X is non-periodic and f is a bounded regular function so that*

$$\sup_{y \in E} |f(y, t+s) - f(y,t)| \to 0 \quad (t \to -\infty)$$

for all $s > 0$, then $f(x, \cdot)$ is a constant, say, $g(x)$. (15)

Proof. Note that, for fixed $s > 0$, $h(x,t) = f(x, t+s) - f(x,t)$ satisfies (14). The proof of (14) holds, with some simple modifications, to show that $h = 0$.

LEMMA. *Let $(E, \mathscr{A}, \mathfrak{A})$ be a topological state space, and suppose the Markov kernel $K(x, A)$ is such that*

$$P_x\{\xi_n \in A \text{ infinitely often}\} = 1$$

for any $x \in E$ and any open set $A \in \mathscr{A}$. Then, all bounded regular functions of ξ_n are constants. (16)

Proof. Suppose $f \geqslant 0$, bounded, and

$$f(x) = \int_E K(x, dy) f(y).$$

If $\mathfrak{M}_n'$ is the σ-algebra generated by $\xi_0, \ldots, \xi_n$, then $(f(\xi_n), \mathfrak{M}_n')$ is a martingale and

$$\lim_{n\to\infty} f(\xi_n(\omega)) = \gamma(\omega)$$

exists a.e. P_x. Let A_y be an open set in $\mathscr{A}$ such that for any $z \in A_y$

$$|f(z) - f(y)| < \eta.$$

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By the recurrence hypothesis, ξ_n enters A_y infinitely often. Thus,

$$P_x\{|f(\xi_n)-f(y)| < \eta \text{ infinitely often}\} = 1,$$

and, therefore,
$$P_x\{|\gamma(\omega)-f(y)| \leqslant \eta\} = 1.$$

Since η is arbitrary,
$$P_x\{\gamma(\omega) = f(y)\} = 1.$$

Similarly, we can show $P_x\{\gamma(\omega) = f(z)\} = 1$. Thus, $f(y) = f(z)$ for all $y, z \in E$.

THEOREM. *Suppose the conditions of* (16) *are fulfilled. If* $X = (\xi(t), z, P_x)$ *is a non-periodic semi-Markov process, then any bounded regular function* f *with*

$$\sup_x |f(x, t+s)-f(x,t)| \to 0 \quad as \quad t \to -\infty \tag{17}$$

for all $s > 0$ *is a constant.*

Proof. The recurrence condition (16) implies that $P_x\{z < \infty\} = 0$ for any x (cf. Theorem of Feller (4)). Thus, by corollary (15), $f(x,t) = g(x)$ independent of t. Regularity of f implies that g is a regular function for the kernel K, and thus by Lemma (16), g is constant.

4. *Limit theorems.* In this section we are interested in the limiting behaviour as $t \uparrow \infty$ of functions of the form $Rg(x,t)$ where g vanishes on $E \times F^-$.

We introduce the following class of functions on $E \times F^+$ (in case E consists of a single point these functions were called 'directly Riemann integrable' by Feller (5)). Let π be a measure on $\mathfrak{A}$. Let $\delta > 0$ be fixed and write $I_n = [n\delta - \delta, n\delta)$ $(n = 1, 2, \ldots)$. For any arbitrary $\mathfrak{A}\otimes\mathfrak{B}$-measurable function g from $E \times F^+$ into F, let

$$f_n'(x) = \inf_{t \in I_n} g(x,t), \quad f_n''(x) = \sup_{t \in I_n} g(x,t) \tag{18}$$

and formally define
$$\Gamma' = \delta \sum_{n=0}^{\infty} \int_E f_n' \, d\pi, \quad \Gamma'' = \delta \sum_{n=0}^{\infty} \int_E f_n'' \, d\pi. \tag{19}$$

Function g is said to belong to $\mathscr{D}_\pi$ if the two series Γ' and Γ'' converge absolutely and if, further, $|\Gamma' - \Gamma''| < \eta$ if δ is sufficiently small. If $g \in \mathscr{D}_\pi$ then g^+, g^-, $|g| \in \mathscr{D}_\pi$ also, and further, g is integrable in the ordinary sense with respect to the product measure $\pi \times \lambda$ where λ is the Lebesgue measure on the line.

We will assume a regularity condition essentially stating that there will be no instantaneous states: we assume, concerning the sojourn time at x, that there exists $\delta > 0$ such that

$$\sup_{x \in E} G(x, [0, \delta]) < 1. \tag{20}$$

Throughout we assume aperiodicity.

LEMMA. *For fixed* $A \in \mathfrak{A}$, $B \in \mathfrak{B}$ *the function* $Rl_{A\times B}$ *is bounded if* $\lambda(B) < \infty$.

Proof. Let $g(x,t) = G(x, (t, \infty))$; then $Rg = e$ where e is the indicator function of $E \times F^+$. By our assumption (20), there exists $\delta > 0$ such that $\sup_x G(x, [0, \delta]) < 1$, and thus, $\inf_x g(x, \delta) = \eta > 0$. Let $B = [0, \delta)$ and $A \in \mathfrak{A}$ be arbitrary. Then $g \geqslant \eta l_{A\times B}$ and since $R = \Sigma T^n$ is a positive operator

$$\eta R l_{A\times B} \leqslant Rg = e. \tag{21}$$

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Any finite interval $C \in \mathfrak{B}$ is covered by a finite number of intervals $I_n = [n\delta - \delta, \delta)$. Now the lemma follows upon observing that

$$Rl_{A\times C}(x,t) \leqslant \sum_\nu Rl_{A\times I_\nu}(x,t) = \sum_\nu Rl_{A\times B}(x, t-\nu\delta+\delta)$$

where the sum is over the set $\{\nu\}$ such that $\bigcup_\nu I_\nu$ covers C. ∎

THEOREM. *Suppose*

$$\lim_{t\to\infty} Rl_{A\times B}(x,t) = \mu(A,B)$$

exists independent of x for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. Then, (22)

$$\mu(A,B) = \pi(A)\,\lambda(B), \tag{23}$$

where λ is the Lebesgue measure on $\mathfrak{B}$ and π is an invariant measure for the kernel K, that is

$$\pi(A) = \int \pi(dx)\,K(x,A) \quad (A \in \mathfrak{A}). \tag{24}$$

Proof. Suppose $Rl_{A\times B}(x,t) \to \mu(A,B)$ as $t \to \infty$, and let f be a continuous function on F vanishing outside a finite interval. Then, with $g(x,t) = l_A(x)f(t)$,

$$Rg(x,t+u) \to \int \mu(A,dv) f(u-v) \equiv h_A(u) \quad (t\to\infty). \tag{25}$$

Since $Rg = g + TRg$, and since $g(x,y) \to 0$ as $t \to \infty$, we see that $TRg(x,t+u) \to h_A(u)$ as $t \to \infty$. But

$$\begin{aligned}\lim_{t\to\infty} TRg(x,t+u) &= \lim_t \int\int Q(x,dy,dv)\,Rg(y,t+u-v)\\ &= \int\int Q(x,dy,dv) \lim_t Rg(y,t+u-v)\\ &= \int G(x,dv)\,h_A(u-v),\end{aligned}$$

since $\lim_t Rg(y,t+z)$ is independent of y. We now have

$$h_A(u) = \int_F G(x,dv)\,h_A(u-v)$$

and h_A is continuous. By the Choquet–Deny theorem (Feller(5), p. 351), this implies that h_A is periodic with period equalling that of $G(x,\cdot)$ if $G(x,\cdot)$ is a lattice distribution. Since $G(x,\cdot)$ cannot have the same period for all $x \in E$ (by our assumption of aperiodicity), h_A must be constant. This in turn implies, by the definition (25) of h_A, that $\mu(A,\cdot)$ is equivalent to the Lebesgue measure, thus proving (23).

To show that π is an invariant measure let $B = [0,1]$ and $g = l_{A\times B}$. Since

$$Rg = g + RTg \quad \text{and since} \quad g(x,t) \to 0 \quad \text{as} \quad t \to \infty,$$

$$\lim_t RTg(x,t) = \lim_t Rg(x,t) = \pi(A)\,\lambda(B) = \pi(A).$$

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On the other hand,

$$\lim_t RTg(x,t) = \iint \mu(dy,du)\, Tg(y,u)$$

$$= \int \pi(dy) \int Tg(y,u)\, du$$

$$= \int \pi(dy)\, K(y,A) \int G(y,u-B)\, du$$

$$= \int \pi(dy)\, K(y,A),$$

since $$\int G(y,u-B)\, du = \int G(y,(u-1,u])\, du = 1. \quad \blacksquare$$

THEOREM. *For any* $A \in \mathfrak{A}$ *and* $B \in \mathfrak{B}$ (26)

$$Rl_{A\times B}(x,t) \to \pi(A)\,\lambda(B) \quad (t \to \infty) \tag{27}$$

if and only if $$Rg(x,t) \to \int_F du \int_E \pi(dy)\, g(y,u) \quad (t \to \infty)$$

for all $g \in \mathscr{D}_\pi$.

Proof. The sufficiency is obvious.

Let $\delta > 0$, $I_n = [n\delta - \delta, n\delta)$, and consider $g_{jk} \in \mathscr{D}_\pi$ with form

$$g_{jk}(x,t) = l_{I_\nu}(t) f_{jk}(x),$$

where f_{jk} is a non-negative simple function on $(E, \mathfrak{A}, \pi)$, that is, f_{jk} is a linear combination of a finite number of indicator functions of sets in $\mathfrak{A}$. Then, Rg_{jk} is a linear combination of a finite number of terms $Rl_{A_\nu \times I_j}$. Thus (27) implies that

$$Rg_{jk}(x,t) \to \iint g_{jk}\, d\pi\, d\lambda \quad (t \to \infty). \tag{28}$$

Next take $f_j \in L_1(E, \mathfrak{A}, \pi)$, $f_j \geqslant 0$ and put

$$g_j(x,t) = l_{I_j}(t) f_j(x).$$

There exists a non-decreasing sequence of simple functions f_{jk} which increase to f_j uniformly in E as $k \uparrow \infty$. Now,

$$Rg_j(x,t) - Rg_{jk}(x,t) \leqslant Rl_{E\times I_j}(x,t) \sup_y (f_j(y) - f_{jk}(y))$$

$$\leqslant M(x,\delta) \sup_y (f_j(y) - f_{jk}(y)), \tag{29}$$

where $M(x,\delta)$ is independent of t by Lemma (21). We can write

$$\left| Rg_j(x,t) - \delta \int f_j\, d\pi \right| \leqslant |Rg_j(x,t) - Rg_{jk}(x,t)|$$

$$+ \left| Rg_{jk}(x,t) - \delta \int f_{jk}\, d\pi \right| + \left| \delta \int f_{jk}\, d\pi - \delta \int f_j\, d\pi \right|. \tag{30}$$

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Let $\eta > 0$; pick N_1 so that $M(x,\delta)\sup_y (f_j(y) - f_{jk}(y)) < \frac{1}{3}\eta$ whenever $k \geqslant N_1$; next choose N_2 so that $\delta\left|\int f_{jk} - \int f_j\right| < \frac{1}{3}\eta$ whenever $k \geqslant N_2$; and set $N = \max(N_1, N_2)$. Now choose T large enough so that $\left|Rg_{jN}(x,t) - \delta\int f_{jN}\,d\pi\right| < \frac{1}{3}\eta$ whenever $t \geqslant T$. Then, putting $k = N$ and $t \geqslant T$ in (30) we obtain $\left|Rg_j(x,t) - \delta\int f_j\,d\pi\right| < \eta$. Since η is arbitrary this implies

$$Rg_j(x,t) \to \delta\int f_j\,d\pi = \int\int g_j\,d\pi\,d\lambda \quad (t \to \infty). \tag{31}$$

Now take $f_j \in L_1(E, \mathfrak{A}, \pi)$, $f_j \geqslant 0$, and $\Sigma\int f_j\,d\pi < \infty$, and let

$$g(x,t) = \sum_j g_j(x,t) = \sum_j f_j(x)\,l_{I_j}(t). \tag{32}$$

We then have, since R is positive, using an estimate through Lemma (21),

$$\sum_{j=0}^{n} Rg_j(x,t) \leqslant Rg(x,t) \leqslant \sum_{j=0}^{n} Rg_j(x,t) + M(\delta)\,\delta \sum_{j=n+1}^{\infty} f_j\,d\pi.$$

First taking limits as $t \uparrow \infty$, then letting $n \uparrow \infty$ gives from (31)

$$Rg(x,t) \to \delta \sum_{j=0}^{\infty} \int_E f_j\,d\pi \quad (t \to \infty). \tag{33}$$

Finally, let $g \in \mathscr{D}_\pi$ be arbitrary and define f_n' and f_n'' by (18). Let g' and g'' be defined by (32) as corresponding to f_n' and f_n'' respectively. Now the proof is completed by applying (33) to Rg' and Rg'', observing

$$Rg' \leqslant Rg \leqslant Rg'',$$

and using the definition of $\mathscr{D}_\pi$. ∎

Supposing that $\lim_t Rl_{A\times B}(x,t)$ exists, the identification of the limit is not difficult. Existence of the limit implies, by Theorem (22), the existence of an invariant measure for the underlying Markov chain induced by the kernel K. A large number of necessary and sufficient conditions for the existence of such measures are reported in Ito (6), Neveu (8) and their references. Supposing, then, that there exists a unique (up to a multiplication by a constant) positive invariant measure α, to find π we need only to compute the normalizing factor c since we must have $\pi = c\alpha$ for some $c \geqslant 0$.

Assumption (20) implies that the process is conservative (cf. Theorem (3.13) in Çinlar (1)). Thus, from Corollary (10),

$$Rg(x,t) = P_x\{x_t \in E\} = 1 \quad \text{where} \quad g(x,t) = G(x,(t,\infty)).$$

Let the mean sojourn time in state x be denoted by $m(x)$, i.e.

$$m(x) = \int_{F^+} tG(x,dt) = \int_{F^+} g(x,t)\,dt.$$

If $\int m\,d\alpha < \infty$, then $g \in \mathscr{D}_\alpha$ since $g(x,\cdot)$ is non-increasing. Then, the preceding theorem applies to give

$$1 = \lim_t Rg(x,t) = c\int_E\int_F g\,d\alpha\,d\lambda = c\int_E m\,d\alpha.$$

Thus, if $\int m\,d\alpha < \infty$, then $\pi = c\alpha$ is such that $\int m\,d\pi = 1$. Otherwise, if $\int m\,d\alpha$ diverges, we must have $c = 0$ and then $\pi = 0$.

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COROLLARY. *Let (x, z, P_x) be a non-periodic semi-Markov process satisfying* (20), *suppose* $\lim_t Rl_{A\times B}(x,t)$ *exists, and let* α *be the unique invariant measure for the underlying Markov chain. Then, if* $\int m\,d\alpha < \infty$, (34)

$$\lim_{t\to\infty} P_x\{x_t \in A\} = \frac{\int_A m\,d\alpha}{\int_E m\,d\alpha} \quad (A \in \mathfrak{A}). \tag{35}$$

Proof. From Corollary (10) we have that

$$P_x\{x_t \in A\} = R(l_{A\times F^+}\cdot g)\,(x,t)$$

where $g(x,t) = G(x,(t,\infty))$. Since $l_{A\times F^+}\cdot g(x,\cdot)$ is non-increasing,

$$\iint l_{A\times F}\cdot g\,d\alpha\,d\lambda = \int_A \alpha(dx)\int_{F^+} g(x,t)\,dt = \int_A m\,d\alpha \leqslant \int_E m\,d\alpha < \infty$$

implies that $l_{A\times F^+}\cdot g \in \mathscr{D}_\alpha$. Now Theorem (26) applies to give, in view of the remarks preceding the corollary, the result desired. ∎

Remark. Concerning the existence of the limit of $Rl_{A\times B}(x,t)$ as $t\to\infty$, we remark the need for additional structure on the state space $(E,\mathfrak{A})$. In the case of discrete E the limits exist by Blackwell's renewal theorem. For non-discrete case the problem remains to be solved.

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