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LOGICS FOR THE RELATIONAL SYLLOGISTIC

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Abstract. The Aristotelian syllogistic cannot account for the validity of certain inferences involving relational facts. In this paper, we investigate the prospects for providing a relational syllogistic. We identify several fragments based on (a) whether negation is permitted on all nouns, including those in the subject of a sentence; and (b) whether the subject noun phrase may contain a relative clause. The logics we present are extensions of the classical syllogistic, and we pay special attention to the question of whether *reductio ad absurdum* is needed. Thus our main goal is to derive results on the existence (or nonexistence) of syllogistic proof systems for relational fragments. We also determine the computational complexity of all our fragments.

§1. Introduction. Augustus De Morgan famously observed that the Aristotelian syllogistic cannot account for the validity of even the most elementary inferences involving relational facts, for example (De Morgan, 1847, p. 114):

$$\frac{\text{Every man is an animal}}{\text{He who kills a man kills an animal.}} \tag{1}$$

De Morgan was certainly not the first to notice the problem of relational inference: for example, it is given prominent treatment in the Port Royal Logic (Arnauld, 1662, Ch. III). But whereas the Port-Royalists took such inferences to demonstrate at most the occasional need for ingenious reformulation, De Morgan saw in them clear evidence that binary relations must be governed by logical principles lying outside the scope of the traditional syllogistic, and made one of the first serious attempts to extend syllogism-like principles to relational judgments (see De Morgan, 1860). Ultimately, of course, De Morgan's attempt was to be without issue: the methods he employed were rendered obsolete at the end of the nineteenth century by the rise of predicate logic, which provided an expressive yet easily understood apparatus for formalizing relational facts.

From a computational point of view, however, expressive power is a double-edged sword: roughly speaking, the more expressive a language is, the harder it is to compute with. In the last decade, this trade-off has led to renewed interest in *inexpressive* logics, in which the problem of determining entailments is algorithmically decidable with (in ideal cases) low complexity. The logical fragments subjected to this sort of complexity-theoretic analysis have naturally enough tended to be those which owe their salience to the syntax of first-order logic, for example: the two-variable fragment, the guarded fragment, and various quantifier-prefix fragments. But of course it is equally reasonable to consider instead logics

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defined in terms of the syntax of *natural languages*. Perhaps, therefore, it is time to return to where De Morgan and his contemporaries left off, but with a computational agenda in mind. As we shall see, this leads to a rather different view of the syllogistic than that defended by certain latter-day advocates of term logic.

This paper investigates the following six logics: (i) $\mathcal{S}$, which corresponds to the traditional syllogistic; (ii) $\mathcal{S}^\dagger$, which extends $\mathcal{S}$ with negated nouns such as "non-man" or "non-animal"; (iii) $\mathcal{R}$, which extends $\mathcal{S}$ with transitive verbs such as "kills"; (iv) $\mathcal{R}^\dagger$, which extends $\mathcal{S}$ with both these constructions; (v) $\mathcal{R}^*$, which extends $\mathcal{R}$ by allowing subject noun phrases to contain relative clauses as in the conclusion of Argument (1); and (vi) $\mathcal{R}^{*\dagger}$, which extends $\mathcal{R}^*$ with negated nouns. We derive results on the existence (or nonexistence) of syllogistic proof systems for these logics, paying particular regard to the need for the rule of *reductio ad absurdum*. Specifically, we present sound and complete syllogistic systems for both $\mathcal{S}$ and $\mathcal{S}^\dagger$ that do not employ *reductio* (we call such systems *direct syllogistic systems*). We then show that, by contrast, there is no sound and complete direct syllogistic system for the fragment $\mathcal{R}$; however, we do present a direct syllogistic system for this fragment that is sound and *refutation-complete* (i.e., becomes complete if *reductio ad absurdum* is allowed as a single, final step). We then consider *indirect syllogistic systems*—that is, those in which the rule of *reductio ad absurdum* may be employed at any point during the derivation. We show that—unless PTIME=NPTIME—there is no direct syllogistic system that is sound and refutation-complete for the fragment $\mathcal{R}^*$; however, we do provide an indirect syllogistic system that is sound and complete for this fragment. Finally, we show that neither $\mathcal{R}^\dagger$ nor $\mathcal{R}^{*\dagger}$ has even an indirect syllogistic system that is sound and complete. We also obtain results on the computational complexity of determining validity in the above fragments, and show how these results are related to the existence of sound and complete proof systems (of various kinds) for the fragments in question. Specifically, we show that the problem of determining the validity of sequents in any of the fragments $\mathcal{S}$, $\mathcal{S}^\dagger$, or $\mathcal{R}$ is NLOGSPACE-complete; the problem of determining the validity of sequents in the fragment $\mathcal{R}^*$ is co-NPTIME-complete; and the problem of determining the validity of sequents in either of the fragments $\mathcal{R}^\dagger$ or $\mathcal{R}^{*\dagger}$ is EXPTIME-complete.

The contribution of the present paper—with its emphasis on computational complexity and limited expressive power—is thus a product of its own time, and differs from De Morgan's work in both motivation and outcome. (For a modern reconstruction of De Morgan's work, see Merrill, 1990.) Nevertheless, we hope that the technical results below will be of interest to historians of logic, not only in respect of the relational syllogistic itself, but also—as one of our anonymous referees pointed out—in respect of the study of mediæval (and later) analyses of multiple quantification more generally.

§2. Preliminaries.

In this section, we define the logics studied in this paper and establish the formal apparatus needed to analyze them.

2.1. The fragments $\mathcal{S}$ and $\mathcal{S}^\dagger$.

Fix a countably infinite set **P**. We may assume **P** to contain various English common count nouns such as "man", "animal", etc. A *unary atom* is an element of **P**; a *unary literal* is an expression of either of the forms p or $\bar{p}$, where p is a unary atom. A unary literal is called *positive* if it is a unary atom; otherwise, *negative*. We use the (possibly decorated) variables o, p, q to range over unary atoms, and l, m, n to range over unary literals. With these conventions, an $\mathcal{S}$*-formula* is an expression of any of the forms

$$\exists(p, l), \quad \exists(l, p), \quad \forall(p, l), \quad \forall(l, \bar{p}). \tag{2}$$

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We provide English glosses for $\mathcal{S}$-formulas as follows:

$$\begin{array}{llll} \forall(p,q) & \text{Every } p \text{ is a } q & \forall(p,\bar{q}) & \text{No } p \text{ is a } q \\ \exists(p,q) & \text{Some } p \text{ is a } q & \exists(p,\bar{q}) & \text{Some } p \text{ is not a } q. \end{array} \tag{3}$$

Note that $\mathcal{S}$ contains the formulas $\exists(\bar{p},q)$ and $\forall(\bar{p},\bar{q})$, which are not glossed in (3); however, according to the semantics below, these formulas are logically equivalent to $\exists(q,\bar{p})$ and $\forall(q,p)$, respectively. We may regard $\mathcal{S}$ as the language of the traditional syllogistic.

The syntax for $\mathcal{S}$ given above suggests a natural generalization. An $\mathcal{S}^{\dagger}$-*formula* is an expression of either of the forms

$$\exists(l,m), \quad \forall(l,m). \tag{4}$$

We provide English glosses for $\mathcal{S}^{\dagger}$-formulas in the same way as for $\mathcal{S}$, except that we sometimes require *negated* subjects:

$$\exists(\bar{p},\bar{q}) \quad \text{Some non-}p \text{ is not a } q \qquad \forall(\bar{p},q) \quad \text{Every non-}p \text{ is a } q.$$

We may regard $\mathcal{S}^{\dagger}$ as the extension of the traditional syllogistic with noun-level negation.

2.2. The fragments $\mathcal{R}$ and $\mathcal{R}^{\dagger}$. Now fix a countably infinite set $\mathbf{R}$, disjoint from $\mathbf{P}$. We may assume $\mathbf{R}$ to contain various English transitive verbs such as "kill", "admire", etc. A *binary atom* is an element of $\mathbf{R}$; a *binary literal* is an expression of either of the forms r or $\bar{r}$, where r is a binary atom. A binary literal is called *positive* if it is a binary atom; otherwise, *negative*. We use the (possibly decorated) variables r, s to range over binary atoms, and t to range over binary literals. With these conventions, a *c-term* is an expression of any of the forms

$$l, \quad \exists(p,t), \quad \forall(p,t). \tag{5}$$

Thus, all literals are, by definition, c-terms. We use the variables c, d to range over c-terms. With this convention, an $\mathcal{R}$-*formula* is an expression of any of the forms

$$\exists(p,c), \quad \exists(c,p), \quad \forall(p,c), \quad \forall(c,\bar{p}); \tag{6}$$

Thus, all $\mathcal{S}$-formulas are, by definition, $\mathcal{R}$-formulas.

We gloss (non-literal) c-terms using complex noun phrases, as follows:

$$\begin{array}{llll} \exists(q,r) & \text{thing which } r\text{s some } q & \forall(q,r) & \text{thing which } r\text{s every } q \\ \exists(q,\bar{r}) & \text{thing which does not } r \text{ every } q & \forall(q,\bar{r}) & \text{thing which } r\text{s no } q. \end{array} \tag{7}$$

And we gloss $\mathcal{R}$-formulas involving such c-terms accordingly, thus:

$$\begin{array}{llll} \forall(p,\exists(q,r)) & \text{Every } p \ r\text{s some } q & \forall(p,\exists(q,\bar{r})) & \text{No } p \ r\text{s every } q \\ \exists(p,\exists(q,r)) & \text{Some } p \ r\text{s some } q & \exists(p,\exists(q,\bar{r})) & \text{Some } p \text{ does not } r \text{ every } q \\ \forall(p,\forall(q,r)) & \text{Every } p \ r\text{s every } q & \forall(p,\forall(q,\bar{r})) & \text{No } p \ r\text{s any } q \\ \exists(p,\forall(q,r)) & \text{Some } p \ r\text{s every } q & \exists(p,\forall(q,\bar{r})) & \text{Some } p \ r\text{s no } q. \end{array}$$

In these glosses, quantifiers in subjects are assumed to have wide scope; those in objects, narrow scope. Also, the "not" in "Some p does not r every q" is assumed to scope over the direct object. (Equivalently: there is some p such that there is some q which it doesn't r.) Again, formulas of the forms $\exists(c,p)$ and $\forall(c,\bar{p})$, which are not glossed here, will turn out to be equivalent to formulas which are. We may regard $\mathcal{R}$ as the language of the *relational syllogistic*.

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Let us now extend $\mathcal{R}$ with noun-level negation just as we did with $\mathcal{S}$. An *e-term* is an expression of any of the forms

$$l, \quad \exists(l, t), \quad \forall(l, t). \tag{8}$$

We gloss nonliteral e-terms along the lines of (7), but with (possibly) negated direct objects, for example:

$\forall(\bar{q}, r)$ thing which rs every non-q $\quad$ $\forall(\bar{q}, \bar{r})$ thing which rs no non-q.

(Another way to gloss the latter term would be "thing which rs only qs".) Thus, all c-terms are, by definition, e-terms. We use the variables e, f to range over e-terms. With this convention, an $\mathcal{R}^\dagger$*-formula* is an expression of any of the forms

$$\exists(l, e), \quad \exists(e, l), \quad \forall(l, e), \quad \forall(e, l), \tag{9}$$

with the obvious English glosses. Note however that noun-level negation may be required, both in subjects and direct objects:

$\exists(\bar{p}, \forall(\bar{q}, r))$ Some non-p rs every non-q.

We may regard $\mathcal{R}^\dagger$ as the extension of the relational syllogistic $\mathcal{R}$ with noun-level negation.

2.3. The fragments $\mathcal{R}^*$ and $\mathcal{R}^{*\dagger}$. The reader may have noticed that neither $\mathcal{R}$ nor $\mathcal{R}^\dagger$ can express the conclusion of Argument (1). We now rectify this matter by introducing two additional fragments which can.

It will be convenient to extend the 'bar'-notation (as in $\bar{p}$ and $\bar{r}$) to all e-terms. If $l = \bar{p}$ is a negative unary literal, then we take $\bar{l}$ to denote p; and similarly for binary literals. If e is an e-term of the form $\forall(l, t)$, we denote by $\bar{e}$ the corresponding e-term $\exists(l, \bar{t})$; if e is an e-term of the form $\exists(l, t)$, we denote by $\bar{e}$ the corresponding e-term $\forall(l, \bar{t})$. Thus, for all e-terms e, we have $\bar{\bar{e}} = e$.

Recalling that a c-term c is an expression of any of the forms l, $\exists(p, t)$ or $\forall(p, t)$, we call c *positive* if the literal l or t is positive; otherwise, negative. Evidently, if c is a positive c-term, then $\bar{c}$ is a negative c-term, and vice versa. We now define an $\mathcal{R}^*$*-formula* to be an expression of any of the forms

$$\exists(c^+, d), \quad \exists(d, c^+), \quad \forall(c^+, d), \quad \forall(d, \overline{c^+}), \tag{10}$$

where c^+ ranges over positive c-terms and d over all c-terms (positive or negative). These forms may then be glossed by combining the glosses (3) and (7) in the obvious way. Thus, for example, the conclusion of Argument (1) may be formalized in $\mathcal{R}^*$ as

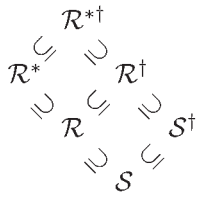
$\forall(\exists(\text{man}, \text{kill}), \exists(\text{animal}, \text{kill}))$

Everything which kills a man (is a thing which) kills an animal.

It is straightforward to check that, in providing English glosses for $\mathcal{R}^*$-formulas, noun-level negation (i.e., expressions such as "non-man") is not required. The study of $\mathcal{R}^*$ is motivated in part by its close connection to the system described in McAllester & Givan (1992), and in part by the fact that, from both a proof-theoretic and complexity-theoretic point of view, it occupies an intermediate position, in a sense that will be made precise below.

We come finally to the most general of the fragments studied here. Recalling that the variables e and f range over e-terms, an $\mathcal{R}^{*\dagger}$*-formula* is an expression of either of the forms

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$\mathcal{S}$	syllogistic
$\mathcal{S}^{\dagger}$	syllogistic with noun-negation
$\mathcal{R}$	relational syllogistic
$\mathcal{R}^{\dagger}$	relational syllogistic with noun-negation
$\mathcal{R}^{*}$	relational syllogistic with complex subjects
$\mathcal{R}^{*\dagger}$	relational syllogistic with complex subjects and noun-negation

Fig. 1. The six syllogistic fragments: $\mathcal{S}$, $\mathcal{S}^{\dagger}$, $\mathcal{R}$, $\mathcal{R}^{\dagger}$, $\mathcal{R}^{*}$ and $\mathcal{R}^{*\dagger}$.

$$\exists(e, f), \quad \forall(e, f). \tag{11}$$

These formulas receive the obvious glosses, for example:

$\forall(\exists(\overline{\text{animal}}, \text{kill}), \exists(\overline{\text{man}}, \text{kill}))$

Everything which kills a non-animal (is a thing which) kills a non-man.

We may regard $\mathcal{R}^{*\dagger}$ as the result of extending $\mathcal{R}^{*}$ with noun-level negation.

We denote the set of $\mathcal{S}$-formulas simply by $\mathcal{S}$, and similarly for $\mathcal{S}^{\dagger}$, $\mathcal{R}$, $\mathcal{R}^{\dagger}$, $\mathcal{R}^{*}$ and $\mathcal{R}^{*\dagger}$. Many of the results established below apply to all of these fragments. Accordingly, we say that a *syllogistic fragment* is any of the fragments $\mathcal{S}$, $\mathcal{S}^{\dagger}$, $\mathcal{R}$, $\mathcal{R}^{\dagger}$, $\mathcal{R}^{*}$, or $\mathcal{R}^{*\dagger}$, and we use the variable $\mathcal{F}$ to range over the syllogistic fragments. Figure 1 gives an overview of the syllogistic fragments, and Figure 2 a quick reference guide to their syntax.

It will be convenient to extend the bar notation to formulas. If φ is an $\mathcal{R}^{*\dagger}$-formula of the form $\forall(e, f)$, we denote by $\bar{\varphi}$ the $\mathcal{R}^{*\dagger}$-formula $\exists(e, \bar{f})$; if φ is an $\mathcal{R}^{*\dagger}$-formula of the form $\exists(e, f)$, we denote by $\bar{\varphi}$ the $\mathcal{R}^{*\dagger}$-formula $\forall(e, \bar{f})$. It is easy to verify that $\bar{\bar{\varphi}} = \varphi$, and that, for any syllogistic fragment $\mathcal{F}$, $\varphi \in \mathcal{F}$ implies $\bar{\varphi} \in \mathcal{F}$.

2.4. Semantics. We now provide semantics for the fragment $\mathcal{R}^{*\dagger}$ (and hence for any syllogistic fragment). A *structure* $\mathfrak{A}$ is a triple $\langle A, \{p^{\mathfrak{A}}\}_{p\in\mathbf{P}}, \{r^{\mathfrak{A}}\}_{r\in\mathbf{R}}\rangle$, where A is a nonempty set, $p^{\mathfrak{A}} \subseteq A$, for every $p \in \mathbf{P}$, and $r^{\mathfrak{A}} \subseteq A^2$, for every $r \in \mathbf{R}$. The set A is called the *domain* of $\mathfrak{A}$. Given a structure $\mathfrak{A}$, we extend the maps $p \mapsto p^{\mathfrak{A}}$ and $r \mapsto r^{\mathfrak{A}}$ to all e-terms by setting

$$\begin{aligned}
\bar{p}^{\mathfrak{A}} &= A \setminus p^{\mathfrak{A}} \\
\bar{r}^{\mathfrak{A}} &= A^2 \setminus r^{\mathfrak{A}} \\
\exists(l, t)^{\mathfrak{A}} &= \{a \in A \mid \langle a, b\rangle \in t^{\mathfrak{A}} \text{ for some } b \in l^{\mathfrak{A}}\} \\
\forall(l, t)^{\mathfrak{A}} &= \{a \in A \mid \langle a, b\rangle \in t^{\mathfrak{A}} \text{ for all } b \in l^{\mathfrak{A}}\}.
\end{aligned}$$

We define the truth-relation $\models$ between structures and $\mathcal{R}^{*\dagger}$-formulas by declaring $\mathfrak{A} \models \forall(e, f)$ if and only if $e^{\mathfrak{A}} \subseteq f^{\mathfrak{A}}$, and $\mathfrak{A} \models \exists(e, f)$ if and only if $e^{\mathfrak{A}} \cap f^{\mathfrak{A}} \neq \emptyset$. If Θ is a set of formulas, we write $\mathfrak{A} \models \Theta$ if, for all $\theta \in \Theta$, $\mathfrak{A} \models \theta$. Of course, this defines the truth relation for formulas of any syllogistic fragment $\mathcal{F}$.

A formula θ is *satisfiable* if there exists $\mathfrak{A}$ such that $\mathfrak{A} \models \theta$; a set of formulas Θ is *satisfiable* if there exists $\mathfrak{A}$ such that $\mathfrak{A} \models \Theta$. If, for all structures $\mathfrak{A}$, $\mathfrak{A} \models \Theta$ implies $\mathfrak{A} \models \theta$, we write $\Theta \models \theta$. We take it as uncontroversial that $\Theta \models \theta$ constitutes a rational reconstruction of the pre-theoretic judgment that a conclusion θ may be validly inferred

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Expression	Variables	Syntax
unary atom	o, p, q	
binary atom	r, s	
unary literal	l, m, n	$p \mid \bar{p}$
binary literal	t	$r \mid \bar{r}$
positive c-term	b^+, c^+	$p \mid \exists(p,r) \mid \forall(p,r)$
c-term	c, d	$l \mid \exists(p,t) \mid \forall(p,t)$
e-term	e, f	$l \mid \exists(l,t) \mid \forall(l,t)$
$\mathcal{S}$-formula		$\exists(p,l) \mid \exists(l,p) \mid \forall(p,l) \mid \forall(l,\bar{p})$
$\mathcal{S}^\dagger$-formula		$\exists(l,m) \mid \forall(l,m)$
$\mathcal{R}$-formula		$\exists(p,c) \mid \exists(c,p) \mid \forall(p,c) \mid \forall(c,\bar{p})$
$\mathcal{R}^\dagger$-formula		$\exists(l,e) \mid \exists(e,l) \mid \forall(l,e) \mid \forall(e,l)$
$\mathcal{R}^*$-formula		$\exists(c^+,d) \mid \exists(d,c^+) \mid \forall(c^+,d) \mid \forall(d,\overline{c^+})$
$\mathcal{R}^{*\dagger}$-formula		$\exists(e,f) \mid \forall(e,f)$

Fig. 2. Syntax of syllogistic fragments: quick reference guide.

from premises Θ. For example, the valid argument

$$\frac{\begin{array}{l}\text{Some artist is a beekeeper}\\ \text{Every artist is a carpenter}\\ \text{No beekeeper is a dentist}\end{array}}{\text{Some carpenter is not a dentist}}$$

corresponds to the valid sequent of $\mathcal{S}$-formulas:

$$\{\exists(\text{artist}, \text{beekeeper}), \forall(\text{artist}, \text{carpenter}), \forall(\text{beekeeper}, \overline{\text{dentist}})\} \models \exists(\text{carpenter}, \overline{\text{dentist}}). \tag{12}$$

Likewise, the valid argument

$$\frac{\begin{array}{l}\text{Every artist hates every artist}\\ \text{No beekeeper hates any beekeeper}\end{array}}{\text{No artist is a beekeeper}}$$

corresponds to the valid sequent of $\mathcal{R}$-formulas:

$$\{\forall(\text{artist}, \forall(\text{artist}, \text{hate})), \forall(\text{beekeeper}, \forall(\text{beekeeper}, \overline{\text{hate}}))\} \models \forall(\text{artist}, \overline{\text{beekeeper}}). \tag{13}$$

We follow modern practice in taking universal quantification not to carry existential commitment: thus $\{\forall(p,q)\} \not\models \exists(p,q)$.

2.5. Absurdity, negation, and identifications. A simple check shows that, for any structure $\mathfrak{A}$ and any e-term e, $\bar{e}^{\mathfrak{A}} = A \setminus e^{\mathfrak{A}}$; hence, $\mathfrak{A} \not\models \exists(e, \bar{e})$. We refer to a formula of this form as an *absurdity*. Even the smallest syllogistic fragment, $\mathcal{S}$, contains absurdities, for example, those of the form $\exists(p, \bar{p})$. Where the fragment $\mathcal{F}$ is clear from context, we write $\bot$ indifferently to denote any absurdity in $\mathcal{F}$.

For any structure $\mathfrak{A}$ and any $\mathcal{R}^{*\dagger}$-formula φ, $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A} \not\models \bar{\varphi}$. Also, φ and $\bar{\varphi}$ belong to the same syllogistic fragments. Thus, if $\mathcal{F}$ is a syllogistic fragment and φ an $\mathcal{F}$ formula, we may regard $\bar{\varphi}$ as the negation of φ.

For any structure $\mathfrak{A}$ and any e-terms e and f, $\mathfrak{A} \models \exists(e, f)$ if and only if $\mathfrak{A} \models \exists(f, e)$. Also, these formulas belong to the same syllogistic fragments. In the sequel, therefore,

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we identify such pairs of formulas, silently transforming one to the other as necessary. Similarly, for the pair of formulas $\forall(e, f)$ and $\forall(\bar{f}, \bar{e})$. These identifications make no essential difference to the results derived below on syllogistic proof systems; however, they greatly simplify their presentation and analysis.

***2.6. Syllogistic rules and* reductio ad absurdum.** Let $\mathcal{F}$ be a syllogistic fragment. A *derivation relation* $\mathrel{|\!\sim}$ in $\mathcal{F}$ is a subset of $\mathbb{P}(\mathcal{F}) \times \mathcal{F}$, where $\mathbb{P}(\mathcal{F})$ is the power set of $\mathcal{F}$. For readability, we write $\Theta \mathrel{|\!\sim} \theta$ instead of $\langle \Theta, \theta \rangle \in \mathrel{|\!\sim}$. We say that $\mathrel{|\!\sim}$ is *sound* if $\Theta \mathrel{|\!\sim} \theta$ implies $\Theta \models \theta$, and *complete* (*for* $\mathcal{F}$) if $\Theta \models \theta$ implies $\Theta \mathrel{|\!\sim} \theta$. A set Θ of $\mathcal{F}$-formulas is *inconsistent* (*with respect to* $\mathrel{|\!\sim}$) if $\Theta \mathrel{|\!\sim} \perp$ for some absurdity $\perp \in \mathcal{F}$; otherwise, *consistent*. A weakening of completeness called *refutation-completeness* will prove important in the sequel: $\mathrel{|\!\sim}$ is *refutation-complete* if all unsatisfiable sets Θ are inconsistent. Completeness trivially implies refutation-completeness, but not conversely. We are primarily interested in derivation relations induced by two different sorts of deductive system: direct syllogistic systems and indirect syllogistic systems. These we now proceed to define.

Let $\mathcal{F}$ be a syllogistic fragment. We employ the following terminology. A *syllogistic rule* (sometimes, simply: *rule*) in $\mathcal{F}$ is a pair Θ/θ, where Θ is a finite set (possibly empty) of $\mathcal{F}$-formulas, and θ an $\mathcal{F}$-formula. We call Θ the *antecedents* of the rule, and θ its *consequent*. The rule Θ/θ is *sound* if $\Theta \models \theta$. We generally display rules in 'natural-deduction' style. For example,

$$\frac{\forall(q, o) \quad \exists(p, q)}{\exists(p, \bar{o})} \qquad \frac{\forall(q, \bar{o}) \quad \exists(p, q)}{\exists(p, o)} \tag{14}$$

where p, q, and o are unary atoms, are syllogistic rules in $\mathcal{S}$, corresponding to the traditional syllogisms *Darii* and *Ferio*, respectively. A *substitution* is a function $g = g_1 \cup g_2$, where $g_1 : \mathbf{P} \to \mathbf{P}$ and $g_2 : \mathbf{R} \to \mathbf{R}$. If θ is an $\mathcal{F}$-formula, denote by $g(\theta)$ the $\mathcal{F}$-formula which results by replacing any atom (unary or binary) in θ by its image under g, and similarly for sets of formulas. An *instance* of a syllogistic rule Θ/θ is the syllogistic rule $g(\Theta)/g(\theta)$, where g is a substitution.

Syllogistic rules which differ only with respect to renaming of unary or binary atoms will be informally regarded as identical, because they have the same instances. Thus, the letters p, q, and o in (14) function, in effect, as *variables* ranging over unary atoms. It is often convenient to display syllogistic rules using variables ranging over other types of expressions, understanding that these are just more compact ways of writing finite collections of syllogistic rules in the official sense. For example, the two rules (14) may be more compactly written

$$\frac{\forall(q, l) \quad \exists(p, q)}{\exists(p, l)} \text{(D1)}$$

where p and q range over unary atoms, but l ranges over unary *literals*. Note that (D1) is simply the *name* of the rule, used for reference only.

Fix a syllogistic fragment $\mathcal{F}$, and let X be a set of syllogistic rules in $\mathcal{F}$. Define $\vdash_{\mathsf{X}}$ to be the smallest derivation relation in $\mathcal{F}$ satisfying:

1. if $\theta \in \Theta$, then $\Theta \vdash_{\mathsf{X}} \theta$;
2. if $\{\theta_1, \ldots, \theta_n\}/\theta$ is a rule in X, g a substitution, $\Theta = \Theta_1 \cup \cdots \cup \Theta_n$, and $\Theta_i \vdash_{\mathsf{X}} g(\theta_i)$ for all i $(1 \leq i \leq n)$, then $\Theta \vdash_{\mathsf{X}} g(\theta)$.

It is simple to show that the derivation relation $\vdash_{\mathsf{X}}$ is sound if and only if each rule in X is sound.

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Informally, we imagine chaining together instances of the rules in X to construct *derivations*, in the obvious way; and we refer to the resulting proof system as the *direct syllogistic system defined by* X. We generally display derivations in natural-deduction style. Thus, for example, if X is any rule set containing (D1), the derivation

$$\dfrac{\forall(\text{beekeeper}, \overline{\text{dentist}}) \qquad \dfrac{\forall(\text{artist}, \text{carpenter}) \qquad \exists(\text{artist}, \text{beekeeper})}{\exists(\text{carpenter}, \text{beekeeper})}\,(\text{D1})}{\exists(\text{carpenter}, \overline{\text{dentist}})}\,(\text{D1})$$

establishes that

$$\{\exists(\text{artist}, \text{beekeeper}), \forall(\text{artist}, \text{carpenter}), \forall(\text{beekeeper}, \overline{\text{dentist}})\} \vdash_{\mathsf{X}} \exists(\text{carpenter}, \overline{\text{dentist}}),$$

which, given the soundness of (D1), entails the validity (12). Notice, incidentally, that the first application of (D1) in the above derivation depends on the silent identification of $\exists(p, q)$ and $\exists(q, p)$. In the sequel, we reason freely about derivations in order to establish properties of derivation relations.

Derivation relations defined by direct proof systems are easily seen to have polynomial-time complexity.

LEMMA 2.1. *Let $\mathcal{F}$ be a syllogistic fragment, and X a finite set of syllogistic rules in $\mathcal{F}$. The problem of determining whether $\Theta \vdash_{\mathsf{X}} \theta$, for a given set of $\mathcal{F}$-formulas Θ and $\mathcal{F}$-formula θ, is in* PTIME.

Proof. Let Σ be the set of all atoms (unary or binary) occurring in $\Theta \cup \{\theta\}$, together with one additional binary atom r. We first observe that, if there is a derivation of θ from Θ using the rules X, then there is such a derivation involving only the atoms occurring in Σ. For, given any derivation of θ from Θ, uniformly replace any unary atom that does not occur in $\Theta \cup \{\theta\}$ with one that does. Similarly, uniformly replace any binary atom which does not occur in $\Theta \cup \{\theta\}$ with one which does (or with r in case $\Theta \cup \{\theta\}$ contains no binary atoms). This process obviously leaves us with a derivation of θ from Θ, using the rules X.

To prove the lemma, let the the total number of symbols occurring in $\Theta \cup \{\theta\}$ be n. Certainly, $|\Sigma| \leq n$. Let X comprise k_1 proof rules, each of which contains at most k_2 atoms (unary or binary). The number of rule instances involving only atoms in Σ is bounded by $p(n) = k_1 n^{k_2}$. Hence, we need never consider derivations with 'depth' greater than $p(n)$. Let Θ_i be the set of formulas involving only the atoms in Σ, and derivable from Θ using a derivation of depth i or less ($0 \leq i \leq p(n)$). Evidently, $|\Theta_i| \leq |\Theta| + p(n)$. It is then straightforward to compute the successive Θ_i in total time bounded by a polynomial function of n. □

In addition to syllogistic rules, we consider the traditional rule of *reductio ad absurdum* (RAA), which allows us to derive a proposition φ (in some syllogistic fragment) by first assuming $\bar{\varphi}$ and deriving an absurdity. Again, we display this rule, in natural-deduction style, as

$$\dfrac{\begin{array}{c}\cdots\,[\bar{\varphi}]^i\,\cdots\,[\bar{\varphi}]^i\,\cdots \\ \vdots \\ \bot\end{array}}{\varphi}\,(\text{RAA})^i.$$

The interpretation is as follows: if an absurdity has been derived from the set of premises Φ together with the premise $\bar{\varphi}$, then φ may be derived from the premises Φ alone. The premise

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$\bar{\varphi}$ may be used several times (including 0) in the derivation of the absurdity; moreover, Φ is allowed to contain $\bar{\varphi}$. We say that the (0 or more) bracketed instances of the premise $\bar{\varphi}$ have been *discharged* by application of (RAA). The numerical superscript i simply allows us to keep track of which application of (RAA) was responsible for the discharge of which (instances of a) premise.

Fix a syllogistic fragment $\mathcal{F}$, and let X be a set of syllogistic rules in $\mathcal{F}$. Define $\Vdash_{\mathsf{X}}$ to be the smallest derivation relation in $\mathcal{F}$ satisfying:

1. if $\theta \in \Theta$, then $\Theta \Vdash_{\mathsf{X}} \theta$;
2. if $\{\theta_1, \ldots, \theta_n\}/\theta$ is a rule in X, g a substitution, $\Theta = \Theta_1 \cup \cdots \cup \Theta_n$, and $\Theta_i \Vdash_{\mathsf{X}} g(\theta_i)$ for all i $(1 \leq i \leq n)$, then $\Theta \Vdash_{\mathsf{X}} g(\theta)$;
3. if $\Theta \cup \{\bar{\theta}\} \Vdash_{\mathsf{X}} \bot$, then $\Theta \Vdash_{\mathsf{X}} \theta$.

It is again simple to show that $\Vdash_{\mathsf{X}}$ is sound if and only if each rule in X is sound.

Informally, we imagine chaining together instances of the rules in X and the rule (RAA) to form *indirect derivations* in the obvious way; and we refer to the resulting proof system as the *indirect syllogistic system defined by* X. We generally display indirect derivations in natural-deduction style, as the following example illustrates. First, consider the (evidently sound) syllogistic rules:

$$\frac{\forall(q, \bar{c}) \quad \exists(p, c)}{\exists(p, \bar{q})}\ (\text{D3}) \qquad \frac{\forall(p, \forall(o, t)) \quad \exists(q, o)}{\forall(p, \exists(q, t))}\ (\forall\forall).$$

Here, as usual, o, p, and q range over unary atoms, t over binary literals and c over c-terms. In addition, we generalize the rule (D1) given above so that the variable l (ranging over literals) may be replaced by the variable c (ranging over c-terms). Then we have the following indirect derivation.

$$\dfrac{\dfrac{\forall(\text{bkpr}, \overline{\exists(\text{bkpr}, \text{hate})}) \qquad \dfrac{\dfrac{\forall(\text{art}, \forall(\text{art}, \text{hate})) \quad [\exists(\text{art}, \text{bkpr})]^1}{\forall(\text{art}, \exists(\text{bkpr}, \text{hate}))}\ (\forall\forall) \qquad [\exists(\text{art}, \text{bkpr})]^1}{\exists(\text{bkpr}, \exists(\text{bkpr}, \text{hate}))}\ (\text{D1})}{\exists(\text{bkpr}, \overline{\text{bkpr}})}\ (\text{D3})}{\forall(\text{art}, \overline{\text{bkpr}})}\ (\text{RAA})^1.$$

Here, two instances of the premise $\exists$(artist, beekeeper) have been discharged by the final application of (RAA). This derivation establishes that, if X is any set of rules containing (D3), $(\forall\forall)$ and (D1) (generalized as indicated), then

$$\{\forall(\text{artist}, \forall(\text{artist}, \text{hate})),\ \forall(\text{beekeeper}, \overline{\exists(\text{beekeeper}, \text{hate})})\} \Vdash_{\mathsf{X}} \forall(\text{artist}, \overline{\text{beekeeper}}).$$

Bearing in mind that $\overline{\exists(\text{beekeeper}, \text{hate})} = \forall(\text{beekeeper}, \overline{\text{hate}})$, and given the soundness of the syllogistic rules employed, this entails the validity (13).

It is important to realize that (RAA) is not itself a syllogistic rule. In particular, an application of (RAA) in general *decreases* the set of premises of the derivation in which it features. As we shall see below, the special status of (RAA) is essential: indirect syllogistic systems are in general more powerful than direct syllogistic systems.

If $\vdash_{\mathsf{X}}$ is refutation-complete, then $\Vdash_{\mathsf{X}}$ is complete. For suppose $\Theta \models \theta$. Then $\Theta \cup \{\bar{\theta}\}$ is unsatisfiable; hence, by the refutation-completeness of $\vdash_{\mathsf{X}}$, $\Theta \cup \{\bar{\theta}\} \vdash_{\mathsf{X}} \bot$; hence, using a single, final application of (RAA), $\Theta \Vdash_{\mathsf{X}} \theta$. We stress, however, that (RAA) is not in general restricted to the final step in an $\Vdash_{\mathsf{X}}$ derivation; rather, it may be employed at

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any point, and any number of times. We remark that, for indirect derivation relations, the reasoning of Lemma 2.1 fails: in Section 5 we present a finite set of syllogistic rules R^* such that $\Vdash_{\mathsf{R}^*}$ is co-NPTIME hard.

We conclude this discussion by showing that indirect proof systems yield a version of Lindenbaum's Lemma. Let $\mathcal{F}$ be any syllogistic fragment. We say that a set of $\mathcal{F}$-formulas Θ is *$\mathcal{F}$-complete* if, for every $\mathcal{F}$-formula θ, either $\theta \in \Theta$ or $\bar{\theta} \in \Theta$. If the fragment $\mathcal{F}$ is clear from context, we say *complete* instead of *$\mathcal{F}$-complete*; notice however that $\mathcal{F}$-completeness of a set of formulas has nothing to do with the completeness of a derivation relation.

LEMMA 2.2. *Let $\mathcal{F}$ be a syllogistic fragment, Θ a set of $\mathcal{F}$-formulas and X a set of syllogistic rules in $\mathcal{F}$. If Θ is consistent with respect to $\Vdash_{\mathsf{X}}$, then there exists a complete set of $\mathcal{F}$-formulas $\Delta \supseteq \Theta$ consistent with respect to $\Vdash_{\mathsf{X}}$.*

Proof. Let $(\varphi_n)_{n\in\mathbb{N}}$ enumerate the formulas of $\mathcal{F}$. We define a sequence of consistent sets Δ_n as follows. Let $\Delta_0 = \Theta$. For $n \geq 0$, let

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n\} \text{ if } \Delta_n \cup \{\varphi_n\} \text{ is consistent} \\ \Delta_n \cup \{\bar{\varphi}_n\} \text{ otherwise.} \end{cases}$$

and let $\Delta = \bigcup_n \Delta_n$. We show by induction that each Δ_n is consistent. For suppose Δ_n is consistent, but Δ_{n+1} inconsistent. Then $\Delta_n \cup \{\varphi_n\} \Vdash_{\mathsf{X}} \bot$, so that $\Delta_n \Vdash_{\mathsf{X}} \bar{\varphi}_n$ using (RAA). In addition, $\Delta_n \cup \{\bar{\varphi}_n\} \Vdash_{\mathsf{X}} \bot'$ (where $\bot'$ is some absurdity). We take a derivation establishing the latter assertion and replace each (undischarged) premise $\bar{\varphi}$ with a derivation of $\bar{\varphi}$ from Δ_n. This shows Δ_n to be inconsistent, a contradiction. Thus, Δ_n is consistent, for all n.

Since derivations are finite, Δ is consistent. Obviously, Δ is $\mathcal{F}$-complete. □

§3. $\mathcal{S}$ and $\mathcal{S}^\dagger$: direct systems. In the previous section, we defined our general notions, including the fragments $\mathcal{S}$ and $\mathcal{S}^\dagger$. These are the simplest in our paper, coming without verbs. This section provides sound and complete syllogistic logics for them. Let S be the following set of rules, where p and q range over unary atoms, l over unary literals, and φ, ψ over $\mathcal{S}$-formulas.

$$\frac{\forall(q,l) \quad \exists(p,q)}{\exists(p,l)}\ (\text{D1}) \qquad \frac{\forall(p,q) \quad \forall(q,l)}{\forall(p,l)}\ (\text{B}) \qquad \frac{\forall(p,\bar{p})}{\forall(p,l)}\ (\text{A})$$

$$\frac{\exists(p,l) \quad \forall(p,q)}{\exists(q,l)}\ (\text{D2}) \qquad \frac{\psi \quad \bar{\psi}}{\varphi}\ (\text{X})$$

$$\frac{\forall(q,\bar{l}) \quad \exists(p,l)}{\exists(p,\bar{q})}\ (\text{D3}) \qquad \frac{}{\forall(p,p)}\ (\text{T}) \qquad \frac{\exists(p,l)}{\exists(p,p)}\ (\text{I})$$

Rules (D1) and (D3) we have met already (the latter in a rather more general form); Rules (D2) and (B) are new, but evidently versions of classical syllogisms. Rule (X) is the classical rule of *ex falso quodlibet* (also called *ex contradictione quodlibet*): from a contradiction, the reasoner may have anything he pleases [quodlibet]. This rule is not to be confused with (RAA): (X) is a syllogistic rule, in the technical sense of this paper; (RAA) is not. Rules (A), (T), and (I) have no classical counterparts. Rule (A) stems from the fact that if all p are non-p, then there are no p whatsoever; vacuously, then, all p are l. To see that (T) is needed, note that without it there would be no way to derive $\forall(p,p)$ from the empty set of premises. Rule (I) is self-explanatory. We remark that Rule (D1) is actually

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redundant in this context. For consider any instance $\{\forall(q', l'), \exists(p', q')\}/\exists(p', l')$ of (D1). If $l' = o$ is a positive literal, then, using the identification $\exists(e, f) = \exists(f, e)$, we may rewrite this instance as $\{\forall(q', o), \exists(q', p')\}/\exists(o, p')$, which is an instance of (D2). On the other hand, if $l' = \bar{o}$, then, using the identification $\forall(e, f) = \forall(\bar{f}, \bar{e})$, we may rewrite the instance as $\{\forall(o, \bar{q}'), \exists(p', q')\}/\exists(p', \bar{o})$, which is an instance of (D3). We retain Rule (D1) for ease of use, and to make the relationship to the system R introduced in Section 4.2 more transparent.

Turning now to the fragment $\mathcal{S}^\dagger$, let $\mathsf{S}^\dagger$ comprise the following syllogistic rules, where l, m and n range over unary literals, and φ, ψ over $\mathcal{S}^\dagger$-formulas.

$$\frac{\exists(l, n) \quad \forall(l, m)}{\exists(m, n)}\ (\mathrm{D}) \qquad \frac{\forall(l, m) \quad \forall(m, n)}{\forall(l, n)}\ (\mathrm{B}) \qquad \frac{\forall(l, \bar{l})}{\forall(l, m)}\ (\mathrm{A})$$

$$\frac{}{\forall(l, l)}\ (\mathrm{T}) \qquad \frac{\exists(l, m)}{\exists(l, l)}\ (\mathrm{I}) \qquad \frac{\psi \quad \bar{\psi}}{\varphi}\ (\mathrm{X}) \qquad \frac{\forall(\bar{l}, l)}{\exists(l, l)}\ (\mathrm{N})$$

The rules in $\mathsf{S}^\dagger$ are the natural generalizations of those in S to reflect the more liberal syntax of $\mathcal{S}^\dagger$, but with two small changes: first, rules (D1)–(D3) have merged into a single rule (D); second, an extra rule (N) has been added. The soundness of (N) is due to the requirement that our structures have nonempty domains.

That $\mathsf{S}^\dagger$ really is an extension of S is given by the following lemma.

LEMMA 3.1. *Any instance of a rule in* $\mathsf{S}^\dagger$ *which involves only formulas in the fragment* $\mathcal{S}$ *is an instance of a rule of* S*, and conversely.*

Proof. We prove the first statement by considering the rules of $\mathsf{S}^\dagger$ one at a time. We illustrate here with Rule (D). Suppose that all formulas in some instance $\{\exists(l, n), \forall(l, m)\}/\exists(m, n)$ of this rule are in $\mathcal{S}$. If m is positive, then so is l (since $\forall(l, m) \in \mathcal{S}$). In this case, our instance matches Rule (D2) in the system S. On the other hand, if m is negative, then n is positive (since $\exists(m, n) \in \mathcal{S}$). We then use the identifications $\forall(l, m) = \forall(\bar{m}, \bar{l})$, $\exists(l, n) = \exists(n, l)$, and $\exists(m, n) = \exists(n, m)$, and rewrite the rule instance as $\{\exists(n, l), \forall(\bar{m}, \bar{l})\}/\exists(n, m)$, which (bearing in mind that n and $\bar{m}$ are unary atoms) matches Rule (D3) in the system S. The other rules are dealt with similarly. Note in particular that all instances of Rule (N) involve a formula lying outside the fragment $\mathcal{S}$, so that the statement holds trivially for this rule. The second statement of the lemma is completely routine. □

In view of Lemma 3.1, we have taken the liberty of using the same names—(B), (A), (T), (I) and (X)—for corresponding rules in S and $\mathsf{S}^\dagger$.

It is obvious that $\vdash_{\mathsf{S}}$ and $\vdash_{\mathsf{S}^\dagger}$ are sound. We now prove their completeness (for the fragments for $\mathcal{S}$ and $\mathcal{S}^\dagger$, respectively). It is convenient to prove the completeness of $\vdash_{\mathsf{S}^\dagger}$, and then to derive the result for $\vdash_{\mathsf{S}}$ as a special case.

Starting on the proof. In the remainder of this section, then, a *formula* is an $\mathcal{S}^\dagger$-formula unless otherwise stated. A *universal* formula is one of the form $\forall(l, m)$; an *existential* formula is one of the form $\exists(l, m)$. If $\mathfrak{A}$ and $\mathfrak{B}$ are structures with disjoint domains A and B, respectively, denote by $\mathfrak{A} \cup \mathfrak{B}$ the structure with domain $A \cup B$ and interpretations $p^{\mathfrak{A}\cup\mathfrak{B}} = p^{\mathfrak{A}} \cup p^{\mathfrak{B}}$ for any unary atom p. (Note that $\mathcal{S}^\dagger$ features only unary atoms.)

LEMMA 3.2. *Suppose* $\Phi \cup \Psi \models \theta$*, where* Φ *is a set of universal formulas,* Ψ *a set of existential formulas, and* θ *a formula.*

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1. *If* $\Phi \cup \Psi$ *is satisfiable and* θ *is universal, then* $\Phi \models \theta$.
2. *If* $\Psi \neq \emptyset$ *and* θ *is existential, then there exists* $\psi \in \Psi$ *such that* $\Phi \cup \{\psi\} \models \theta$.
3. *If* $\Psi = \emptyset$ *and* $\theta = \exists(l, m)$ *is existential, then* $\Phi \models \forall(\bar{l}, l)$ *and* $\Phi \models \forall(\bar{m}, m)$.

Proof. In each case, assume the contrary.

1. There exist structures $\mathfrak{A} \models \Phi \cup \Psi$ and $\mathfrak{B} \models \Phi \cup \bar{\theta}$. We may assume that $A \cap B = \emptyset$. But then $\mathfrak{A} \cup \mathfrak{B} \models \Phi \cup \Psi \cup \{\bar{\theta}\}$, a contradiction.
2. For every $\psi \in \Psi$, there exists a structure $\mathfrak{A}_\psi$ such that $\mathfrak{A}_\psi \models \Phi \cup \{\psi, \bar{\theta}\}$. Again, we assume that the domains are pairwise disjoint. But then $\bigcup_{\psi \in \Psi} \mathfrak{A}_\psi \models \Phi \cup \Psi \cup \{\bar{\theta}\}$, a contradiction.
3. If $\Phi \not\models \forall(\bar{l}, l)$, there exists a structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Phi \cup \{\exists(\bar{l}, \bar{l})\}$. Choose $a \in \bar{l}^{\mathfrak{A}}$, and let $\mathfrak{B}$ be the structure obtained by restricting $\mathfrak{A}$ to the singleton domain $\{a\}$. Then $\mathfrak{B} \models \Phi \cup \{\bar{\theta}\}$, a contradiction. A similar argument applies if $\Phi \not\models \forall(\bar{m}, m)$. □

We write $\Phi \vdash_{\mathrm{BTA}} \varphi$ if there is a derivation in $\vdash_{\mathsf{S}^\dagger}$ of φ from Φ employing only the rules (B), (T) and (A). We call a set V of literals *consistent* if $l \in V$ implies $\bar{l} \notin V$ (otherwise *inconsistent*); we call V *complete* if $l \notin V$ implies $\bar{l} \in V$. Let Φ be a set of universal formulas and V a set of literals. Define V^Φ to be the set of literals:

$$\{m : \Phi \vdash_{\mathrm{BTA}} \forall(l, m) \text{ for some } l \in V\}.$$

We say that V is Φ-*closed* if $V^\Phi \subseteq V$.

LEMMA 3.3. *Let* Φ *be a set of universal formulas and* V *a set of literals. Then* V^Φ *is* Φ*-closed, and* $V \subseteq V^\Phi$.

Proof. Immediate, by rules (B) and (T). □

Evidently, the union of any collection of Φ-closed sets of literals is Φ-closed.

LEMMA 3.4. *Let* Φ *be a set of universal formulas and* V *a set of literals. If* V^Φ *is inconsistent, then there exist literals* $l, l' \in V$ *such that* $\Phi \vdash_{\mathrm{BTA}} \forall(l, \bar{l}')$.

Proof. If $m, \bar{m} \in V^\Phi$, pick $l, l' \in V$ with $\Phi \vdash_{\mathrm{BTA}} \forall(l, m)$ and $\Phi \vdash_{\mathrm{BTA}} \forall(l', \bar{m})$. Rewriting $\forall(l', \bar{m})$ as $\forall(m, \bar{l}')$, we have a derivation from Φ

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \forall(l, m) & \forall(m, \bar{l}')\end{array}}{\forall(l, \bar{l}')}\ (\mathrm{B}),$$

as claimed. □

LEMMA 3.5. *Let* Φ *be a set of universal formulas. Any nonempty,* Φ*-closed, consistent set of literals has a* Φ*-closed, consistent, complete extension.*

Proof. Let V_0 be a nonempty, Φ-closed, consistent set of literals. Enumerate the set of all literals as $l_1, l_2, \ldots$. For all $i \geq 0$, define

$$V_{i+1} = \begin{cases} V_i \text{ if } l_{i+1} \in V_i \\ \left(V_i \cup \{\bar{l}_{i+1}\}\right)^\Phi \quad \text{otherwise.} \end{cases}$$

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and define $V = \bigcup_i V_i$. Thus, V is Φ-closed, complete and includes V_0. We remark that, since $V_0 \subseteq V_i$, V_i is nonempty for all $i \geq 0$. To show that V is consistent, suppose otherwise. Let i be the least natural number such that V_{i+1} is inconsistent. Hence, $V_{i+1} = \left(V_i \cup \{\bar{l}_{i+1}\}\right)^{\Phi}$, and so by Lemma 3.4 there exist literals $l, l' \in V_i \cup \{\bar{l}_{i+1}\}$ such that $\Phi \vdash_{\mathrm{BTA}} \forall(l, \bar{l}')$. Since V_i is Φ-closed and consistent, l and l' cannot both be in V_i, and so we may assume without loss of generality that $l' = \bar{l}_{i+1}$, whence $\Phi \vdash_{\mathrm{BTA}} \forall(l, l_{i+1})$. Now, either $l \in V_i$ or $l = \bar{l}_{i+1}$. In the former case, we have $l_{i+1} \in V_i$, because V_i is Φ-closed; in the latter, let l'' be any literal in V_i (which we know to be nonempty). Then the inference

$$\frac{\forall(\bar{l}_{i+1}, l_{i+1})}{\forall(l'', l_{i+1})}\ (\mathrm{A})$$

guarantees that, again, $l_{i+1} \in V_i$. Either way, $V_{i+1} = V_i$, a contradiction. □

THEOREM 3.6. *The derivation relation $\vdash_{\mathsf{S}^\dagger}$ is sound and complete for $\mathcal{S}^\dagger$.*

Proof. Soundness is routine. To prove completeness, let Θ be a set of $\mathcal{S}^\dagger$-formulas and θ an $\mathcal{S}^\dagger$-formula, and suppose $\Theta \models \theta$. By the compactness theorem for first-order logic, we may safely assume that Θ is finite. Suppose for the moment that Θ is satisfiable, and write $\Theta = \Phi \cup \Psi$, where Φ is a set of universal formulas and Ψ a set of existential formulas. We consider three cases: (1) θ is universal; (2) θ is existential and Ψ is nonempty; and (3) θ is existential and Ψ is empty.

In the remainder of this proof, we simplify our notation to write $\vdash$ for $\vdash_{\mathsf{S}^\dagger}$.
Case (1): Write $\theta = \forall(l_0, m_0)$. By Lemma 3.2, Part 1, $\Phi \models \theta$. Let

$$V_0 = \{l_0, \bar{m}_0\}^{\Phi}. \tag{15}$$

Then V_0 is nonempty and Φ-closed, by Lemma 3.3. We claim that V_0 is inconsistent. For suppose otherwise. By Lemma 3.5, let V be a consistent complete extension of V_0, and define $\mathfrak{A}$ to be the structure with singleton domain $\{a\}$ given by

$$p^{\mathfrak{A}} = \begin{cases} \{a\} \text{ if } p \in V \\ \emptyset \text{ otherwise,} \end{cases}$$

for every atom p. It is easily seen that $\mathfrak{A} \models \Phi \cup \bar{\theta}$, a contradiction. So by Lemma 3.4, there exist literals $l, l' \in \{l_0, \bar{m}_0\}$ such that

$$\Phi \vdash_{\mathrm{BTA}} \forall(l, \bar{l}'). \tag{16}$$

By exchanging l and l' if necessary, we have two subcases: (i) $l = l_0$ and $l' = \bar{m}_0$; (ii) $l = l' \in \{l_0, \bar{m}_0\}$. In subcase (i), (16) simply asserts that $\Phi \vdash \theta$. In subcase (ii), we have one of the derivations from Φ

$$\frac{\begin{array}{c}\vdots\\ \forall(l_0, \bar{l}_0)\end{array}}{\forall(\bar{m}_0, \bar{l}_0)}\ (\mathrm{A}) \qquad \frac{\begin{array}{c}\vdots\\ \forall(\bar{m}_0, m_0)\end{array}}{\forall(l_0, m_0)}\ (\mathrm{A}),$$

and so $\Phi \vdash \theta$.
Case (2): Write $\theta = \exists(l, m)$. By Lemma 3.2, Part 2, there exists $\psi = \exists(l_0, m_0) \in \Psi$ such that $\Phi \cup \{\psi\} \models \theta$. Set

$$V_0 = \{l_0, m_0, \bar{l}\}^{\Phi}.$$

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The set V_0 must be inconsistent. For otherwise, we can easily construct, using a parallel argument to that employed in Case (1), a structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Phi \cup \{\psi, \bar{\theta}\}$, contradicting the fact that $\Phi \cup \{\psi\} \models \theta$. Hence, there exist literals $l_1, l_2 \in \{l_0, m_0, \bar{l}\}$ such that $\Phi \vdash_{\mathrm{BTA}} \forall(l_1, \bar{l}_2)$. If l_1 and l_2 are both in $\{l_0, m_0\}$, then Θ is unsatisfiable, contrary to hypothesis. So assume, without loss of generality, that $l_2 = \bar{l}$. Thus, $\Phi \vdash_{\mathrm{BTA}} \forall(l_1, l)$, and we have the following possibilities: (i) $l_1 = l_0$; (ii) $l_1 = m_0$; (iii) $l_1 = \bar{l}$. Possibility (i) yields $\Phi \vdash \forall(l_0, l)$. Possibility (ii) yields $\Phi \vdash \forall(m_0, l)$. Possibility (iii) also yields $\Phi \vdash \forall(l_0, l)$, via the derivation

$$\dfrac{\begin{array}{c}\vdots\\ \forall(\bar{l}, l)\end{array}}{\forall(l_0, l)}\ (\mathrm{A}).$$

In other words, we have proved:

$$\text{either } \Phi \vdash \forall(l_0, l) \text{ or } \Phi \vdash \forall(m_0, l). \tag{17}$$

Replacing l by m in the above argument yields, in exactly the same way:

$$\text{either } \Phi \vdash \forall(l_0, m) \text{ or } \Phi \vdash \forall(m_0, m). \tag{18}$$

Considering (17), we may assume, by transposing l_0 and m_0 if necessary, that $\Phi \vdash \forall(l_0, l)$. This leaves us with the two possibilities in (18). If $\Phi \vdash \forall(m_0, m)$, we have

$$\dfrac{\dfrac{\exists(l_0, m_0) \qquad \begin{array}{c}\vdots\\ \forall(l_0, l)\end{array}}{\exists(l, m_0)}\ (\mathrm{D}) \qquad \begin{array}{c}\vdots\\ \forall(m_0, m)\end{array}}{\exists(l, m)}\ (\mathrm{D});$$

if, on the other hand, $\Phi \vdash \forall(l_0, m)$, we have

$$\dfrac{\dfrac{\dfrac{\exists(l_0, m_0)}{\exists(l_0, l_0)}\ (\mathrm{I}) \qquad \begin{array}{c}\vdots\\ \forall(l_0, l)\end{array}}{\exists(l, l_0)}\ (\mathrm{D}) \qquad \begin{array}{c}\vdots\\ \forall(l_0, m)\end{array}}{\exists(l, m)}\ (\mathrm{D}). \tag{19}$$

Either way, $\Theta \vdash \theta$, as required.

Case (3): Write $\theta = \exists(l, m)$. Since $\Theta = \Phi \models \theta$, by Lemma 3.2, Part 3, $\Phi \models \forall(\bar{l}, l)$ and $\Phi \models \forall(\bar{m}, m)$. By Case (1), $\Phi \vdash \forall(\bar{l}, l)$ and $\Phi \vdash \forall(\bar{m}, m)$. Therefore, we have the derivation from Φ

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall(\bar{l}, l)\end{array}}{\exists(l, l)}\ (\mathrm{N}) \qquad \dfrac{\begin{array}{c}\vdots\\ \forall(\bar{m}, m)\end{array}}{\forall(l, m)}\ (\mathrm{A})}{\exists(l, m)}\ (\mathrm{D}),$$

and $\Theta \vdash \theta$.

We have now shown that, for Θ satisfiable, $\Theta \models \theta$ implies $\Theta \vdash \theta$. It remains only to consider the case where Θ is unsatisfiable. If so, let $\Theta' \cup \{\theta'\}$ be a minimal unsatisfiable subset of Θ. (Remember, we are allowed to assume that Θ is finite.) Hence, Θ' is satisfiable,

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with $\Theta' \models \bar{\theta}'$. By the previous argument, $\Theta' \vdash \bar{\theta}'$. Thus, we have the derivation from $\Theta' \cup \{\theta'\}$

$$\dfrac{\begin{array}{c}\vdots\\ \bar{\theta}' \quad \theta'\end{array}}{\theta}\ (\mathrm{X}),$$

and $\Theta \vdash \theta$. □

THEOREM 3.7. *The derivation relation* $\vdash_{\mathsf{S}}$ *is sound and complete for* $\mathcal{S}$.

Proof. Soundness is obvious. Suppose that Θ is a satisfiable set of $\mathcal{S}$-formulas and θ an $\mathcal{S}$-formula such that $\Theta \models \theta$. By Theorem 3.6, $\Theta \vdash_{\mathsf{S}^\dagger} \theta$. We must show that $\Theta \vdash_{\mathsf{S}} \theta$. Since derivations (in $\vdash_{\mathsf{S}^\dagger}$) are finite, we may assume without loss of generality that Θ is finite. For the moment, let us further assume that Θ is satisfiable.

Let p, q be any unary atoms. We claim that $\Theta \not\models \forall(\bar{p}, q)$. For otherwise, by Lemma 3.2 Part 1, there exists a subset $\Phi \subseteq \Theta$ of *universal* formulas such that $\Phi \models \forall(\bar{p}, q)$. Now let $\mathfrak{A}$ be any structure such that $o^{\mathfrak{A}} = \emptyset$ for every unary atom o. Since $\Phi \subseteq \mathcal{S}$, $\mathfrak{A} \models \Phi$; but $\mathfrak{A} \not\models \forall(\bar{p}, q)$, a contradiction. Furthermore, since $\vdash_{\mathsf{S}^\dagger}$ is sound, it follows of course that $\Theta \not\vdash_{\mathsf{S}^\dagger} \forall(\bar{p}, q)$.

Consider any derivation in $\vdash_{\mathsf{S}^\dagger}$ with premises Θ. We claim that, if this derivation contains any formula of the form $\exists(\bar{p}, \bar{q})$, then the final conclusion of this derivation is also of that form. For, since Θ is, by assumption, satisfiable, (X) cannot be used in the derivation; and the only other rules in $\mathsf{S}^\dagger$ with any premise of the form $\exists(\bar{p}, \bar{q})$ are (I) and (D), thus:

$$\dfrac{\exists(\bar{p}, \bar{q})}{\exists(\bar{p}, \bar{p})}\ (\mathrm{I}) \qquad \dfrac{\exists(\bar{p}, \bar{q}) \quad \forall(\bar{p}, l)}{\exists(\bar{q}, l)}\ (\mathrm{D}).$$

By the observation of the previous paragraph, the literal l in this instance of (D) must be negative. In either case, the consequent of the rule is of the form $\exists(\bar{p}, \bar{q})$, as claimed.

Now take any derivation of θ from Θ in $\vdash_{\mathsf{S}^\dagger}$. (We know that one exists.) Since $\theta \in \mathcal{S}$ is definitely not of the form $\exists(\bar{p}, \bar{q})$, it follows from the previous two paragraphs that this derivation cannot involve any formula of either of the forms $\forall(\bar{p}, q)$ or $\exists(\bar{p}, \bar{q})$. That is, all the formulas are in $\mathcal{S}$. By Lemma 3.1, $\Theta \vdash_{\mathsf{S}} \theta$.

This proves the theorem in the case where the (finite) set of $\mathcal{S}$-formulas Θ is satisfiable. If Θ is unsatisfiable, let $\Theta' \cup \{\theta'\}$ be a minimal unsatisfiable subset of Θ. Hence, Θ' is satisfiable, with $\Theta' \models \bar{\theta}'$. By the result just established, $\Theta' \vdash_{\mathsf{S}} \bar{\theta}'$. By a single application of (X), $\Theta \vdash_{\mathsf{S}} \theta$. □

§4. $\mathcal{R}$: a refutation-complete system. We turn from $\mathcal{S}$ to $\mathcal{R}$. We exhibit a set R of syllogistic rules in $\mathcal{R}$, and prove that $\vdash_{\mathsf{R}}$ is a sound and *refutation*-complete derivation relation for $\mathcal{R}$; we also prove that there is no finite set of rules X in $\mathcal{R}$ such that X is sound and *complete* for $\mathcal{R}$. This highlights the importance of (RAA) in obtaining complete logics. Our refutation-completeness proof also implies that the problem of determining whether a given $\mathcal{R}$-sequent is valid is in NLOGSPACE, a fact which is otherwise not obvious.

4.1. There are no sound and complete syllogistic systems for $\mathcal{R}$.

THEOREM 4.1. *There exists no finite set* X *of syllogistic rules in* $\mathcal{R}$ *such that* $\vdash_{\mathsf{X}}$ *is both sound and complete.*

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Proof. Let X be any finite set of syllogistic rules for $\mathcal{R}$, and suppose $\vdash_{\mathsf{X}}$ is sound. We show that it is not complete. Since X is finite, fix $n \in n$ greater than the number of antecedents in any rule in X.

Let $p_1, \ldots, p_n$ be distinct unary atoms and r a binary atom. Let Γ be the following set of $\mathcal{R}$-formulas:

$$\forall(p_i, \exists(p_{i+1}, r)) \quad (1 \leq i < n) \tag{20}$$

$$\forall(p_1, \forall(p_n, r)) \tag{21}$$

$$\forall(p, p) \qquad (p \in \mathbf{P}) \tag{22}$$

$$\forall(p_i, \bar{p}_j) \qquad (1 \leq i < j \leq n) \tag{23}$$

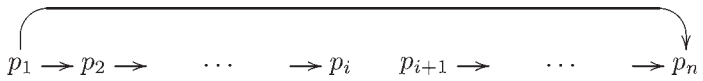
and let γ be the $\mathcal{R}$-formula $\forall(p_1, \exists(p_n, r))$. Observe that $\Gamma \models \gamma$. To see this, let $\mathfrak{A} \models \Gamma$. If $p_1^{\mathfrak{A}} = \emptyset$, then trivially $\mathfrak{A} \models \gamma$; on the other hand, if $p_1^{\mathfrak{A}} \neq \emptyset$, a simple induction using formulas (20) shows that $p_i^{\mathfrak{A}} \neq \emptyset$ for all i $(1 \leq i \leq n)$, whence $\mathfrak{A} \models \gamma$ by (21).

For $1 \leq i < n$, let $\Delta_i = \Gamma \setminus \{\forall(p_i, \exists(p_{i+1}, r))\}$.

CLAIM 4.2. *If $\varphi \in \mathcal{R}$ and $\Delta_i \models \varphi$, then $\varphi \in \Gamma$.*

It follows from this claim that $\Gamma \not\vdash_{\mathsf{X}} \gamma$. For, since no rule of X has more than $n - 1$ antecedents, any instance of those antecedents contained in Γ must be contained in Δ_i for some i. Let δ be the corresponding instance of the consequent of that rule. Since $\vdash_{\mathsf{X}}$ is sound, $\Delta_i \models \delta$. By Claim 4.2, $\delta \in \Gamma$. By induction on the number of steps in derivations, we see that no derivation from Γ leads to a formula not in Γ. But $\gamma \notin \Gamma$.

Proof of Claim. Certainly, Δ_i has a model, for instance the model $\mathfrak{A}_i$ given by:



(24)

Here, $A = \{p_1, \ldots, p_n\}$, $p_j^{\mathfrak{A}_i} = \{p_j\}$ for all j $(1 \leq j \leq n)$, and $r^{\mathfrak{A}_i}$ is indicated by the arrows. All other atoms (unary or binary) are assumed to have empty extensions. Note that there is no arrow from p_i to p_{i+1}.

We consider the various possibilities for φ in turn and check that either $\varphi \in \Gamma$ or there is a model of Δ_i in which φ is false.

(i) φ is of the form $\forall(p, p)$. Then $\varphi \in \Gamma$ by (22).

(ii) φ is not of the form $\forall(p, p)$, and involves at least one unary or binary atom other than $p_1, \ldots, p_n, r$. In this case, it is straightforward to modify $\mathfrak{A}_i$ so as to obtain a model $\mathfrak{A}_i'$ of Δ_i such that $\mathfrak{A}_i' \not\models \varphi$. Henceforth, then, we may assume that φ involves no atoms other than $p_1, \ldots, p_n, r$.

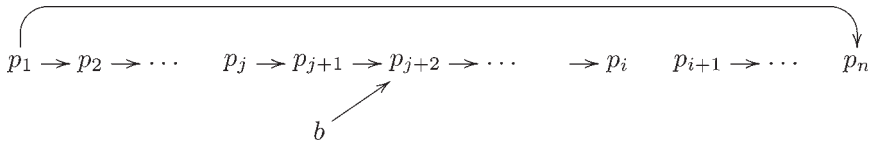
(iii) φ is of the form $\forall(p_j, p_k)$. If $j = k$, then $\varphi \in \Gamma$, by (22). If $j \neq k$, then $\mathfrak{A}_i \not\models \varphi$ by inspection.

(iv) φ is of the form $\forall(p_j, \bar{p}_k)$. If $j = k$, then $\mathfrak{A}_i \not\models \varphi$, since $p_j^{\mathfrak{A}_i} \neq \emptyset$. If $j \neq k$, then $\varphi \in \Gamma$, by (23) and the identification $\forall(p_j, \bar{p}_k) = \forall(p_k, \bar{p}_j)$.

(v) φ is of the form $\forall(p_j, \forall(p_k, r))$. If $j = 1$ and $k = n$, then $\varphi \in \Gamma$, by (21). So we may assume that either $j > 1$ or $k < n$, in which case, $k \neq j + 1$ implies $\mathfrak{A}_i \not\models \varphi$, by inspection. Hence, we may assume that $\varphi = \forall(p_j, \forall(p_{j+1}, r))$, with $j < n$. Let $\mathfrak{B}_{i,j}$ be the structure obtained from $\mathfrak{A}_i$ by adding a second point b to the interpretation of

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p_{j+1}, and to which p_j is not related by r. In pictures:



(This picture shows $j + 2 < i$. Similar pictures are possible in all other cases.) By inspection, $\mathfrak{B}_{i,j} \models \Delta_i$, but $\mathfrak{B}_{i,j} \not\models \varphi$.

(vi) φ is of the form $\forall(p_j, \exists(p_k, r))$. If $k = j + 1$, then $\varphi \in \Gamma$, by (20). Moreover, if $k \neq j + 1$, then, unless $j = 1$ and $k = n$, $\mathfrak{A}_i \not\models \varphi$, by inspection. Hence we may assume $\varphi = \forall(p_1, \exists(p_n, r))$. Let $\mathfrak{C}_i$ be the structure:

$$p_1 \to p_2 \to \cdots \to p_i,$$

with $p_j^{\mathfrak{C}_i} = \emptyset$ for all j $(i < j \leq n)$. Then $\mathfrak{C}_i \models \Delta_i$, but $\mathfrak{C}_i \not\models \varphi$.

(vii) φ is of either of the forms $\forall(p_j, \forall(p_k, \bar{r}))$, $\forall(p_j, \exists(p_k, \bar{r}))$. Define $\mathfrak{A}_i''$ to be like $\mathfrak{A}_i$ except that $r^{\mathfrak{A}_i''}$ additionally contains the pair of points $\langle p_j, p_k \rangle$. By inspection, $\mathfrak{A}_i'' \models \Delta_i$, but $\mathfrak{A}_i'' \not\models \varphi$.

(viii) φ is of the form $\exists(p, c)$. Let $\mathfrak{A}_0$ be a structure over any domain in which every atom has empty extension. Then $\mathfrak{A}_0 \models \Delta_i$, but $\mathfrak{A}_0 \not\models \varphi$. □

This also completes the proof of Theorem 4.1. □

4.2. A refutation-complete system for $\mathcal{R}$. Theorem 4.1 notwithstanding, we exhibit below a finite set R of rules in $\mathcal{R}$, such that $\vdash_{\mathsf{R}}$ is sound and *refutation*-complete. We remind the reader that p and q range over unary atoms, c over c-terms, and t over binary literals.

$$\frac{\exists(p,q) \qquad \forall(q,c)}{\exists(p,c)}\ (\mathrm{D1}) \qquad \frac{\forall(p,q) \qquad \forall(q,c)}{\forall(p,c)}\ (\mathrm{B})$$

$$\frac{\forall(p,q) \qquad \exists(p,c)}{\exists(q,c)}\ (\mathrm{D2}) \qquad \frac{}{\forall(p,p)}\ (\mathrm{T}) \qquad \frac{\exists(p,c)}{\exists(p,p)}\ (\mathrm{I})$$

$$\frac{\forall(q,\bar{c}) \qquad \exists(p,c)}{\exists(p,\bar{q})}\ (\mathrm{D3}) \qquad \frac{\forall(p,\bar{p})}{\forall(p,c)}\ (\mathrm{A}) \qquad \frac{\exists(p,\exists(q,t))}{\exists(q,q)}\ (\mathrm{II})$$

$$\frac{\forall(p,\forall(q',t)) \qquad \exists(q,q')}{\forall(p,\exists(q,t))}\ (\forall\forall) \qquad \frac{\exists(p,\exists(q,t)) \qquad \forall(q,q')}{\exists(p,\exists(q',t))}\ (\exists\exists)$$

$$\frac{\forall(p,\exists(q,t)) \qquad \forall(q,q')}{\forall(p,\exists(q',t))}\ (\forall\exists)$$

Rules (D1), (D2), (D3), (B), (A), (T) and (I) are natural generalizations of their namesakes in S. In contrast, $(\forall\forall)$, $(\exists\exists)$, $(\forall\exists)$ and (II) express genuinely relational logical

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principles. In some settings, these last rules are called *monotonicity principles*. Because we seek only refutation-completeness for $\vdash_{\mathsf{R}}$, we do not need a version of the rule (X).

To illustrate these rules, let n be any integer greater than 1, let

$$\Gamma^* = \{\forall(p_i, \exists(p_{i+1}, r)) \mid 1 \leq i < n\} \cup \{\forall(p_1, \forall(p_n, r))\},$$

and let $\gamma = \forall(p_1, \exists(p_n, r))$. Noting that $\bar{\gamma} = \exists(p_1, \forall(p_n, \bar{r}))$, we have the derivation (shown here for $n > 3$)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\exists(p_1, \forall(p_n, \bar{r}))}{\exists(p_1, p_1)}\,(\mathrm{I}) \qquad \forall(p_1, \exists(p_2, r))}{\exists(p_1, \exists(p_2, r))}\,(\mathrm{D1})}{\exists(p_2, p_2)}\,(\mathrm{II}) \qquad \forall(p_2, \exists(p_3, r))}{\exists(p_2, \exists(p_3, r))}\,(\mathrm{D1})}{\exists(p_3, p_3)}\,(\mathrm{II})}{\begin{array}{c}\\ \ddots\end{array}}$$

$$\dfrac{\dfrac{\forall(p_1, \forall(p_n, r)) \qquad \exists(p_n, p_n)}{\forall(p_1, \exists(p_n, r))}\,(\forall\forall) \qquad \exists(p_1, \forall(p_n, \bar{r}))}{\exists(p_1, \bar{p}_1)}\,(\mathrm{D3}),$$

showing that $\Gamma^* \cup \{\bar{\gamma}\} \vdash_{\mathsf{R}} \bot$. By contrast, since $\Gamma^* \subseteq \Gamma$, where Γ is the set of formulas used in the proof of Theorem 4.1, we know that, for *any* finite set X of syllogistic rules, n can be made sufficiently large that $\Gamma^* \nvdash_{\mathsf{X}} \gamma$.

Starting on the proof. For the remainder of this section, fix a finite, nonempty set Γ of $\mathcal{R}$-formulas. As usual, we take the (possibly decorated) variables p, q to range over unary atoms, r over binary atoms, t over binary literals, and c, d over c-terms. We write $c \Rightarrow d$ if $c = d$ or there exists a sequence of unary atoms $p_0, \ldots, p_k$ such that $c = p_0, \forall(p_k, d) \in \Gamma$, and $\forall(p_i, p_{i+1}) \in \Gamma$ for all i $(0 \leq i < k)$. If V is a set of c-terms, write $V \Rightarrow d$ if $c \Rightarrow d$ for some $c \in V$.

LEMMA 4.3. *Let V be a set of c-terms.*

1. *If $V \Rightarrow c$, then either $c \in V$ or there exists $p \in V$ such that $\Gamma \vdash_{\mathsf{R}} \forall(p, c)$;*
2. *if $V \Rightarrow p$, then there exists $p_0 \in V$ such that $\Gamma \vdash_{\mathsf{R}} \forall(p_0, p)$;*
3. *if $p \Rightarrow c$, then $\Gamma \vdash_{\mathsf{R}} \forall(p, c)$.*

Proof. Immediate, noting that R contains the rules (B) and (T). □

In the ensuing lemmas, we show that, if Γ is consistent (with respect to $\vdash_{\mathsf{R}}$), then Γ is satisfiable. As a first step, we create plenty of objects from which to construct a potential model of Γ. If $0 \leq i \leq 2$ and V is a set of c-terms with $1 \leq |V| \leq 2$, let $b_{V,i}$ denote some object or other, and assume that the various $b_{V,i}$ are distinct. Now set

$$B_0 = \{b_{\{p,c\},0} \mid \exists(p, c) \in \Gamma\}.$$

LEMMA 4.4. *Let $b_{V,0} \in B_0$, and let $p, c \in V$. Then $\Gamma \vdash_{\mathsf{R}} \exists(p, c)$.*

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Proof. If $p \neq c$, then $V = \{p, c\}$ and $\exists(p, c) \in \Gamma$ by construction. If $p = c$, then $V = \{p, d\}$ for some d, and we have the derivation

$$\frac{\exists(p, d)}{\exists(p, p)}\ \text{(I)}.$$

□

We now define sets $B_1, B_2, \ldots$ inductively as follows. Suppose B_k has been defined. Let

$$B_{k+1} = B_k \cup \{b_{\{p\},i} \mid 1 \leq i \leq 2 \text{ and, for some } b_{V,j} \in B_k \text{ and some } t,\ V \Rightarrow \exists(p, t)\}.$$

Let $B = \bigcup_{0 \leq k} B_k$. Evidently, B is finite. (Indeed, $|B|$ is bounded by a linear function of $|\Gamma|$.) It is immediate from the construction of B that, if $b_{V,i} \in B_0$, then $1 \leq |V| \leq 2$, V contains at least one unary atom p, and $i = 0$. On the other hand, if $b_{V,i} \in B_k$ for $k > 0$, then $V = \{p\}$ for some unary atom p, and i is either 1 or 2. The intuition here is that the elements of B_0 are witnesses for the existential formulas of Γ, while the elements of B_{k+1} are witnesses for the existential c-terms satisfied by elements of B_k.

LEMMA 4.5. *If $b_{V,i} \in B$, $V \Rightarrow p$ and $V \Rightarrow c$, then $\Gamma \vdash_{\mathsf{R}} \exists(p, c)$.*

Proof. Let k be the smallest number such that $b_{V,i} \in B_k$. We proceed by induction on k.

For the case $k = 0$, we have $b_{V,i} = b_{V,0} \in B_0$. By Lemma 4.3 Part 2, there exists $q_1 \in V$ such that $\Gamma \vdash_{\mathsf{R}} \forall(q_1, p)$. By Lemma 4.3 Part 1, either $c \in V$ or there exists $q_2 \in V$ such that $\Gamma \vdash_{\mathsf{R}} \forall(q_2, c)$. In the former case, Lemma 4.4 yields $\Gamma \vdash_{\mathsf{R}} \exists(q_1, c)$, so that we have the derivation

$$\frac{\begin{array}{cc}\vdots & \vdots \\ \exists(q_1, c) & \forall(q_1, p)\end{array}}{\exists(p, c)}\ \text{(D2)}$$

In the latter case, Lemma 4.4 yields $\Gamma \vdash_{\mathsf{R}} \exists(q_1, q_2)$, so that we have the derivation

$$\frac{\dfrac{\begin{array}{cc}\vdots & \vdots \\ \exists(q_1, q_2) & \forall(q_2, c)\end{array}}{\exists(q_1, c)}\ \text{(D1)} \qquad \begin{array}{c}\vdots \\ \forall(q_1, p)\end{array}}{\exists(p, c)}\ \text{(D2)}.$$

For the case $k > 0$, $b_{V,i} \in B_k$ implies $V = \{p_k\}$ for some p_k, and $1 \leq i \leq 2$. By construction of B_k, there exist $b_{W,j} \in B_{k-1}$, $p_{k-1} \in W$ and a binary literal t, such that $W \Rightarrow \exists(p_k, t)$. By inductive hypothesis, $\Gamma \vdash_{\mathsf{R}} \exists(p_{k-1}, \exists(p_k, t))$, and by Lemma 4.3 Part 3, $\Gamma \vdash_{\mathsf{R}} \forall(p_k, p)$, and $\Gamma \vdash_{\mathsf{R}} \forall(p_k, c)$. Therefore, we have the derivation

$$\frac{\dfrac{\dfrac{\begin{array}{c}\vdots \\ \exists(p_{k-1}, \exists(p_k, r))\end{array}}{\exists(p_k, p_k)}\ \text{(II)} \qquad \begin{array}{c}\vdots \\ \forall(p_k, p)\end{array}}{\exists(p_k, p)}\ \text{(D1)} \qquad \begin{array}{c}\vdots \\ \forall(p_k, c)\end{array}}{\exists(p, c)}\ \text{(D1)}.$$

□

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LEMMA 4.6. *If $b_{V,i} \in B$, $V \Rightarrow c$, $V \Rightarrow d$, and $c \neq d$, then there exists a unary atom p such that either:* (i) $\Gamma \vdash_{\mathsf{R}} \exists(p, c)$ *and* $\Gamma \vdash_{\mathsf{R}} \forall(p, d)$*; or* (ii) $\Gamma \vdash_{\mathsf{R}} \exists(p, d)$ *and* $\Gamma \vdash_{\mathsf{R}} \forall(p, c)$.

Proof. Suppose first that $c = q$ for some q. By Lemma 4.5, $\Gamma \vdash_{\mathsf{R}} \exists(q, d)$, and by rule (T), $\Gamma \vdash_{\mathsf{R}} \forall(q, q)$. Putting $p = q$ then satisfies Condition (ii). On the other hand, if $d = q$ for some q, then Condition (i) is satisfied, by a similar argument. Hence we may assume that neither c nor d is a unary atom. Since $c \neq d$, we have either $c \notin V$ or $d \notin V$, by construction of B. If the latter, then, by Lemma 4.3 Part 1, there exists $p \in V$ such that $\Gamma \vdash_{\mathsf{R}} \forall(p, d)$. But now we have $V \Rightarrow p$ and $V \Rightarrow c$, so that, by Lemma 4.5, $\Gamma \vdash_{\mathsf{R}} \exists(p, c)$, and Condition (i) is satisfied. If, on the other hand, $c \notin V$, Condition (ii) is satisfied, by a similar argument. □

The set B will form the domain of a structure $\mathfrak{B}$, defined as follows. If p is a unary atom, set

$$p^{\mathfrak{B}} = \{b_{V,i} \in B \mid V \Rightarrow p\};$$

and if r is a binary atom, set

$$\begin{aligned} r^{\mathfrak{B}} \;=\; & \{\langle b_{V,i}, b_{\{p\},1}\rangle \in B^2 \mid V \Rightarrow \exists(p, r)\} \cup \\ & \{\langle b_{V,i}, b_{W,j}\rangle \in B^2 \mid \text{ for some } q,\ V \Rightarrow \forall(q, r) \text{ and } W \Rightarrow q\}. \end{aligned}$$

The intuition is that the elements $b_{\{p\},1}$ are witnesses for the existential quantifiers in c-terms of the form $\exists(p, r)$, while the elements $b_{\{p\},2}$ are witnesses for the existential quantifiers in c-terms of the form $\exists(p, \bar{r})$.

LEMMA 4.7. *If Γ is unsatisfiable, then there exist an element $b_{V,i}$ of B, a unary atom p and a c-term c, such that $V \Rightarrow p$, $V \Rightarrow c$, $b_{V,i} \in p^{\mathfrak{B}}$ and $b_{V,i} \notin c^{\mathfrak{B}}$.*

Proof. Since Γ is unsatisfiable, let $\varphi \in \Gamma$ be such that $\mathfrak{B} \not\models \varphi$. If $\varphi = \exists(p, c)$, let $V = \{p, c\}$. Trivially, $V \Rightarrow p$ and $V \Rightarrow c$. By construction of B, $b_{V,0} \in B$, and by construction of $\mathfrak{B}$, $b_{V,0} \in p^{\mathfrak{B}}$, whence (since $\mathfrak{B} \not\models \varphi$) $b_{V,0} \notin c^{\mathfrak{B}}$. If, on the other hand, $\varphi = \forall(p, c)$, there exists $b_{V,i} \in B$ such that $b_{V,i} \in p^{\mathfrak{B}}$ and $b_{V,i} \notin c^{\mathfrak{B}}$. By construction of $\mathfrak{B}$, $V \Rightarrow p$; and since $\forall(p, c) \in \Gamma$, we have $V \Rightarrow c$, as required. □

We now prove the main Lemma, from which both the complexity and the refutation-completeness results follow.

LEMMA 4.8. *If Γ is unsatisfiable, then the following condition holds.*

(C) *There exist elements $b_{V,i}, b_{W,j}$ of B, unary atoms q, o, and binary atom r, such that one of the following is true:*

1. *$V \Rightarrow q$ and $V \Rightarrow \bar{q}$;*
2. *$V \Rightarrow \exists(q, \bar{r})$, $V \Rightarrow \forall(o, r)$, and $q \Rightarrow o$;*
3. *$V \Rightarrow \forall(q, \bar{r})$, $V \Rightarrow \exists(o, r)$, and $o \Rightarrow q$;*
4. *$V \Rightarrow \forall(q, \bar{r})$, $V \Rightarrow \forall(o, r)$, $W \Rightarrow q$, and $W \Rightarrow o$.*

Proof. Let V, i, p, and c be as in Lemma 4.7. We claim first that c cannot be of the form q, $\exists(q, r)$ or $\forall(q, r)$. For consider each possibility in turn.

(i) If $c = q$, then, by the construction of $\mathfrak{B}$, $V \Rightarrow c$ implies $b_{V,i} \in q^{\mathfrak{B}}$, contradicting $b_{V,i} \notin c^{\mathfrak{B}}$.

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(ii) If $c = \exists(q, r)$, then, by the construction of $\mathfrak{B}$, $V \Rightarrow c$ implies $b_{\{q\},1} \in B$, $b_{\{q\},1} \in q^{\mathfrak{B}}$, and $\langle b_{V,i}, b_{\{q\},1}\rangle \in r^{\mathfrak{B}}$, whence $b_{V,i} \in \exists(q, r)^{\mathfrak{B}}$, contradicting $b \notin c^{\mathfrak{B}}$.

(iii) If $c = \forall(q, r)$, then, since $V \Rightarrow c$, the construction of $\mathfrak{B}$ ensures that, for any $b_{W,j} \in q^{\mathfrak{B}}$, we have $W \Rightarrow q$ and hence $\langle b_{V,i}, b_{W,j}\rangle \in r^{\mathfrak{B}}$. That is, $b_{V,i} \in \forall(q, r)^{\mathfrak{B}}$, contradicting $b \notin c^{\mathfrak{B}}$.

Therefore, c is of one of the forms $\bar{q}$, $\exists(q, \bar{r})$, or $\forall(q, \bar{r})$. We consider each possibility in turn, and show that one of the four cases of Condition (**C**) holds.

(i) If $c = \bar{q}$, then $b_{V,i} \notin c^{\mathfrak{B}}$ means that $b_{V,i} \in q^{\mathfrak{B}}$, so that, by construction of $\mathfrak{B}$, $V \Rightarrow q$. But by assumption, $V \Rightarrow c$, and we have Case 1 of Condition (**C**).

(ii) If $c = \exists(q, \bar{r})$, then, since $V \Rightarrow c$, we have, by construction of $\mathfrak{B}$, $b_{\{q\},2} \in B$, and in fact $b_{\{q\},2} \in q^{\mathfrak{B}}$. Since $b_{V,i} \notin c^{\mathfrak{B}}$, we have $\langle b_{V,i}, b_{\{q\},2}\rangle \in r^{\mathfrak{B}}$. The construction of $\mathfrak{B}$ then guarantees that for some unary atom o, $V \Rightarrow \forall(o, r)$ and $q \Rightarrow o$; thus we have Case 2 of Condition (**C**).

(iii) If $c = \forall(q, \bar{r})$, then $b_{V,i} \notin c^{\mathfrak{B}}$ implies that there exists $b_{W,j} \in B$ such that $b_{W,j} \in q^{\mathfrak{B}}$ and $\langle b_{V,i}, b_{W,j}\rangle \in r^{\mathfrak{B}}$. By construction of $\mathfrak{B}$, $W \Rightarrow q$, and, for some unary atom o, either

(a) $V \Rightarrow \exists(o, r)$, $W = \{o\}$ and $j = 1$, or
(b) $V \Rightarrow \forall(o, r)$ and $W \Rightarrow o$.

But these yield Cases 3 and 4 of Condition (**C**), respectively. □

LEMMA 4.9. *The following are equivalent:*

1. $\Gamma \vdash_{\mathsf{R}} \bot$;
2. Γ *is unsatisfiable;*
3. *Condition* (**C**) *of Lemma 4.8 holds.*

Proof. For the implication $1 \Rightarrow 2$, we observe that $\vdash_{\mathsf{R}}$ is obviously sound. The implication $2 \Rightarrow 3$ is Lemma 4.8. For the implication $3 \Rightarrow 1$, suppose Condition (**C**) of Lemma 4.8 holds. This condition has four cases: we consider each in turn, showing that $\Gamma \vdash_{\mathsf{R}} \exists(p, \bar{p})$ for some unary atom p.

(i) $V \Rightarrow q$ and $V \Rightarrow \bar{q}$. By Lemma 4.5 we immediately have $\Gamma \vdash_{\mathsf{R}} \exists(q, \bar{q})$.

(ii) $V \Rightarrow \exists(q, \bar{r})$, $V \Rightarrow \forall(o, r)$, $q \Rightarrow o$. By Lemma 4.3 Part 2 (or Part 3), $\Gamma \vdash_{\mathsf{R}} \forall(q, o)$, and by Lemma 4.6, there exists p such that either:

(a) $\Gamma \vdash_{\mathsf{R}} \exists(p, \exists(q, \bar{r}))$ and $\Gamma \vdash_{\mathsf{R}} \forall(p, \forall(o, r))$; or
(b) $\Gamma \vdash_{\mathsf{R}} \exists(p, \forall(o, r))$ and $\Gamma \vdash_{\mathsf{R}} \forall(p, \exists(q, \bar{r}))$.

In Case (a), we then have

$$\dfrac{\dfrac{\exists(p, \exists(q, \bar{r})) \qquad \forall(q, o)}{\exists(p, \exists(o, \bar{r}))}\ (\exists\exists) \qquad \forall(p, \forall(o, r))}{\exists(p, \bar{p})}\ (\mathrm{D3}),$$

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while in Case (b), we have

$$\dfrac{\exists(p,\forall(o,r)) \quad \dfrac{\forall(p,\exists(q,\bar{r})) \qquad \forall(q,o)}{\forall(p,\exists(o,\bar{r}))}\ (\forall\exists)}{\exists(p,\bar{p})}\ (\mathrm{D3}).$$

(iii) $V \Rightarrow \forall(q,\bar{r})$, $V \Rightarrow \exists(o,r)$, $o \Rightarrow q$. By Lemma 4.3 Part 2 (or Part 3), $\Gamma \vdash_{\mathsf{R}} \forall(o,q)$, and by Lemma 4.6, there exists p such that either: (a) $\Gamma \vdash_{\mathsf{R}} \exists(p,\exists(o,r))$ and $\Gamma \vdash_{\mathsf{R}} \forall(p,\forall(q,\bar{r}))$; or (b) $\Gamma \vdash_{\mathsf{R}} \exists(p,\forall(q,\bar{r}))$ and $\Gamma \vdash_{\mathsf{R}} \forall(p,\exists(o,r))$. But then we can employ exactly the same derivation patterns as for Cases (ii)(a) and (ii)(b), respectively.

(iv) $V \Rightarrow \forall(q,\bar{r})$, $V \Rightarrow \forall(o,r)$, $W \Rightarrow q$, $W \Rightarrow o$. By Lemma 4.6, there exists p such that $\Gamma \vdash_{\mathsf{R}} \exists(p,\forall(p_1,u))$ and $\Gamma \vdash_{\mathsf{R}} \forall(p,\forall(p_2,\bar{u}))$, where u is either r or $\bar{r}$, and p_1 and p_2 are q and o in some order. By Lemma 4.5, $\Gamma \vdash_{\mathsf{R}} \exists(o,q)$, that is, $\Gamma \vdash_{\mathsf{R}} \exists(p_1,p_2)$. Thus we have the derivation

$$\dfrac{\dfrac{\forall(p,\forall(p_2,\bar{u})) \qquad \exists(p_1,p_2)}{\forall(p,\exists(p_1,\bar{u}))}\ (\forall\forall) \qquad \exists(p,\forall(p_1,u))}{\exists(p,\bar{p})}\ (\mathrm{D3}).$$

□

THEOREM 4.10. *The derivation relation $\vdash_{\mathsf{R}}$ is sound and refutation-complete for $\mathcal{R}$.*

Proof. Soundness is obvious. Refutation-completeness is the implication from 2 to 1 in Lemma 4.9. □

THEOREM 4.11. *The problem of determining the validity of a sequent in any of the fragments $\mathcal{S}$, $\mathcal{S}^{\dagger}$ and $\mathcal{R}$ is* NLOGSPACE*-complete.*

Proof. The lower bound is obtained by reduction of the reachability problem for directed graphs to the validity problem for $\mathcal{S}$. If $G = (V, E)$ is a directed graph, take the nodes V to be unary atoms. Let $\Theta_G = \{\forall(u,v) \mid (u,v) \in E\}$. It is easy to check that $\Theta_G \models \forall(u,v)$ if and only if v is reachable from u.

Recall that, by the Immerman-Szelepcsényi Theorem, NLOGSPACE is closed under complementation. The upper bound for $\mathcal{S}^{\dagger}$ is then immediate, since the problem of determining the satisfiability of a given set of $\mathcal{S}^{\dagger}$-formulas is (almost trivially) reducible to the problem 2SAT, which is well known to be NLOGSPACE-complete. (See, for example, Papadimitriou, 1994, pp. 185–398.)

It remains only to establish the upper bound for $\mathcal{R}$. For this, it suffices to show that the problem of determining the *un*satisfiability of a given set Γ of $\mathcal{R}$-formulas is in NLOGSPACE. Let Γ be a set of $\mathcal{R}$-formulas. Let B and $\mathfrak{B}$ be as defined for Lemmas 4.4–4.9. Lemma 4.9 guarantees that Γ is unsatisfiable if and only if there exist $b_{V,i}, b_{W,j} \in B$, unary atoms q, o, and a binary atom r satisfying one of the four cases in Condition (**C**) of Lemma 4.8. Nondeterministically guess these V, W, i, j, q, o, and r. This requires only logarithmic space, because only the indices of the relevant atoms need to be encoded, and the size of B is linear in the number of formulas in Γ. To check that $b_{V,i}, b_{W,j} \in B$ is essentially a

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graph-reachability problem, as are all the requirements in the four cases of Condition (**C**). Since graph reachability is in NLOGSPACE, this proves the theorem. □

§5. $\mathcal{R}^*$: an indirect system. Next, we consider the fragment $\mathcal{R}^*$. We refer the reader to Figure 2 for a review of the syntax of this fragment. The complexity-theoretic analysis of $\mathcal{R}^*$ has been done for us. It is contained in the more expressive logic (also inspired by natural language syntax) investigated in McAllester & Givan (1992), whose satisfiability problem is shown to be NPTIME-complete. Inspection of McAllester and Givan's NPTIME-hardness proof (*op. cit.* pp. 12–14) reveals that it employs only formulas in our fragment $\mathcal{R}^*$; hence the lower bound applies to $\mathcal{R}^*$ too. In other words:

PROPOSITION 5.1 (McAllester and Givan). *The problem of determining the validity of a sequent in the fragment $\mathcal{R}^*$ is* co-NPTIME*-complete.*

Proposition 5.1 is significant not least because of what we know about the complexity of searching for derivations in direct syllogistic systems.

COROLLARY 5.2. *If* PTIME≠NPTIME*, then there exists no finite set* X *of syllogistic rules in $\mathcal{R}^*$ such that $\vdash_{\mathsf{X}}$ is both sound and refutation-complete.*

Proof. Proposition 5.1 and Lemma 2.1. □

Let R^* be the following set of syllogistic rules in $\mathcal{R}^*$. Our result is that the *indirect* derivation relation $\Vdash_{\mathsf{R}^*}$ is complete. In the following, the variables b^+, c^+ range over positive c-terms, and d over c-terms. (As usual, p, q range over unary atoms and r over binary atoms.)

$$\frac{}{\forall(c^+, c^+)}\ (\mathrm{T}) \qquad \frac{\exists(c^+, d)}{\exists(c^+, c^+)}\ (\mathrm{I}) \qquad \frac{\forall(b^+, c^+) \qquad \forall(c^+, d)}{\forall(b^+, d)}\ (\mathrm{B})$$

$$\frac{\exists(b^+, c^+) \qquad \forall(c^+, d)}{\exists(b^+, d)}\ (\mathrm{D1}) \qquad \frac{\forall(b^+, c^+) \qquad \exists(b^+, d)}{\exists(c^+, d)}\ (\mathrm{D2})$$

$$\frac{\forall(p, q)}{\forall(\forall(q, r), \forall(p, r))}\ (\mathrm{J}) \qquad \frac{\forall(p, q)}{\forall(\exists(p, r), \exists(q, r))}\ (\mathrm{K}) \qquad \frac{\exists(p, q)}{\forall(\forall(p, r), \exists(q, r))}\ (\mathrm{L})$$

$$\frac{\exists(q, \exists(p, r))}{\exists(p, p)}\ (\mathrm{II}) \qquad \frac{\forall(p, \bar{p})}{\forall(c^+, \forall(p, r))}\ (\mathrm{Z}) \qquad \frac{\forall(p, \bar{p})}{\exists((\forall(p, r), \forall(p, r))}\ (\mathrm{W})$$

Rules (D1), (D2), (B), (T), and (I) are natural generalizations of their namesakes in R. Rules (J), (K), and (L) embody logical principles that are intuitively clear, yet not familiar when taken as single steps. If all porcupines are brown animals, then everything which attacks all brown animals attacks all porcupines (J), and everything which photographs some porcupine photographs some brown animal (K). And if some porcupines are brown animals, then everything which caresses all porcupines caresses some brown animals (L). We have already seen (II). Rule (Z) tells us that if there are no porcupines (say), then all farmers love all porcupines. Rule (W) tells us that under this same assumption, there is something which loves all porcupines (simply because we assume the universe is nonempty, and everything in it vacuously loves all porcupines). If we did not assume that the universe

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of a model is nonempty, then we would drop (W), and the completeness of the resulting system would be proved the same way.

All the rules of R are derivable in $\Vdash_{\mathsf{R}^*}$. For example, here is a proof of (A):

$$\dfrac{\dfrac{\dfrac{[\exists(p,\bar{d})]^1}{\exists(p,p)}(\mathrm{I}) \qquad \forall(p,\bar{p})}{\exists(p,\bar{p})}(\mathrm{D1})}{\forall(p,d)}(\mathrm{RAA})^1$$

The rest of this section is devoted to a proof that $\Vdash_{\mathsf{R}^*}$ is sound and complete for $\mathcal{R}^*$.

Starting the proof. To prove completeness, we need only show that every consistent set Γ in the fragment $\mathcal{R}^*$ is satisfiable. Also, by Lemma 2.2, we may assume that Γ is $\mathcal{R}^*$-complete. For the remainder of this section, fix some $\mathcal{R}^*$-complete set of formulas Γ which is consistent with respect to $\Vdash_{\mathsf{R}^*}$. We simplify our notation to write $\vdash$ for $\Vdash_{\mathsf{R}^*}$.

We shall construct a structure $\mathfrak{A}$ and prove that it satisfies Γ. First, let $\mathbf{C}^+$ be the set of positive c-terms. Then we define $\mathfrak{A}$ by setting:

$$\begin{aligned} A &= \{\langle c_1, c_2, Q\rangle \in \mathbf{C}^+ \times \mathbf{C}^+ \times \{\forall,\exists\} : \Gamma \vdash \exists(c_1, c_2)\} \\ p^{\mathfrak{A}} &= \{\langle c_1, c_2, Q\rangle \in A : \Gamma \vdash \forall(c_1, p) \text{ or } \Gamma \vdash \forall(c_2, p)\} \end{aligned}$$

and $\langle c_1, c_2, Q_1\rangle r^{\mathfrak{A}} \langle d_1, d_2, Q_2\rangle$ if and only if either:

(a) for some i, j, and $q \in \mathbf{P}$, $\Gamma \vdash \forall(c_i, \forall(q, r))$ and $\Gamma \vdash \forall(d_j, q)$; or
(b) $Q_2 = \exists$, and for some i and $q \in \mathbf{P}$, $d_1 = d_2 = q$, and $\Gamma \vdash \forall(c_i, \exists(q, r))$.

Note that the set A is nonempty. For let $p \in \mathbf{P}$. If $\Gamma \vdash \exists(p, p)$, then $\langle p, p, \forall\rangle \in A$. Otherwise, $\Gamma \vdash \forall(p, \bar{p})$, and so for all binary atoms r, $\Gamma \vdash \exists(\forall(p, r), \forall(p, r))$ by (W). Thus $\langle c, c, \forall\rangle \in A$, where c is $\forall(p, r)$.

LEMMA 5.3. *For all* $c \in \mathbf{C}^+$,

$$c^{\mathfrak{A}} = \{\langle d_1, d_2, Q\rangle \in A : \textit{ either } \Gamma \vdash \forall(d_1, c) \quad \textit{or} \quad \Gamma \vdash \forall(d_2, c)\}.$$

Proof. The result for c a unary atom is immediate. We shall often use the resulting fact that if $\Gamma \vdash \exists(p, p)$, then $p^{\mathfrak{A}} \neq \emptyset$; it contains both $\langle p, p, \forall\rangle$ and $\langle p, p, \exists\rangle$. The main work concerns c-terms of the form $\forall(p, r)$ and $\exists(p, r)$. We remark that all c-terms referred to in this proof are *positive*.

We begin with $c = \forall(p, r)$. Suppose $\langle d_1, d_2, Q\rangle \in \forall(p, r)^{\mathfrak{A}}$. Now, either $\Gamma \vdash \exists(p, p)$ or $\Gamma \nvdash \exists(p, p)$. If the former, then $\langle p, p, \forall\rangle \in p^{\mathfrak{A}}$. By the semantics of our fragment, $\langle d_1, d_2, Q\rangle r^{\mathfrak{A}} \langle p, p, \forall\rangle$. By the structure of $\mathfrak{A}$, there are i and q giving the derivation from Γ as in the tree on the left below:

$$\dfrac{\dfrac{\vdots}{\forall(d_i, \forall(q, r))} \qquad \dfrac{\dfrac{\vdots}{\forall(p, q)}}{\forall(\forall(q, r), \forall(p, r))}(\mathrm{J})}{\forall(d_i, \forall(p, r))}(\mathrm{B}) \qquad \dfrac{\vdots}{\forall(d_j, \forall(p, r))}(\mathrm{Z}).$$

This shows that $\Gamma \vdash \forall(d_i, \forall(p, r))$. On the other hand, if $\Gamma \nvdash \exists(p, p)$, we use the assumption that Γ is complete to assert that $\Gamma \vdash \forall(p, \bar{p})$. And then we have the derivation from Γ on the right above, for both j.

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Conversely, fix i and suppose that $\Gamma \vdash \forall(d_i, \forall(p, r))$. We claim that $\langle d_1, d_2, Q\rangle$ belongs to $\forall(p, r)^{\mathfrak{A}}$. For this, take any $\langle b_1, b_2, Q'\rangle \in p^{\mathfrak{A}}$ so that $\Gamma \vdash \forall(b_j, p)$ for some j. (We are thus using b_1, and b_2 to range over positive c-terms, just as the c's and d's do.) Then p, i, and j show that $\langle d_1, d_2, Q\rangle r^{\mathfrak{A}} \langle b_1, b_2, Q'\rangle$. This for all elements of $p^{\mathfrak{A}}$ shows that $\langle d_1, d_2, Q\rangle \in \forall(p, r)^{\mathfrak{A}}$.

We next prove the statement of our lemma for $c = \exists(p, r)$.

Suppose $\langle d_1, d_2, Q\rangle \in \exists(p, r)^{\mathfrak{A}}$. Thus we have $\langle d_1, d_2, Q\rangle r^{\mathfrak{A}} \langle b_1, b_2, Q'\rangle$ for some $\langle b_1, b_2, Q'\rangle \in p^{\mathfrak{A}}$. We first consider case (a) in the definition of our structure $\mathfrak{A}$: there are i, j, and q such that $\Gamma \vdash \forall(d_i, \forall(q, r))$ and $\Gamma \vdash \forall(b_j, q)$. We have $\Gamma \vdash \exists(b_1, b_2)$, since $\langle b_1, b_2, Q'\rangle \in A$. Further, let k be such that $\Gamma \vdash \forall(b_k, p)$. We show the desired conclusion using a derivation from Γ:

$$\dfrac{\forall(d_i, \forall(q, r)) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\exists(b_1, b_2)}}{\exists(b_k, b_j)} \qquad \vdots}{\exists(b_k, q)}\ (\mathrm{D1}) \qquad \vdots}{\dfrac{\dfrac{\exists(q, p)}{\forall(\forall(q, r), \exists(p, r))}\ (\mathrm{L})}{\forall(d_i, \exists(p, r))}\ (\mathrm{B})} \qquad \forall(b_k, p)}{\exists(b_k, q)}\ (\mathrm{D1})$$

This concludes the work in case (a). In case (b), $Q' = \exists$, there is some $q \in \mathbf{P}$ such that $b_1 = b_2 = q$, and for some i, $\Gamma \vdash \forall(d_i, \exists(q, r))$. Again we have $\Gamma \vdash \forall(q, p)$. So we have a derivation from Γ as follows:

$$\dfrac{\forall(d_i, \exists(q, r)) \qquad \dfrac{\dfrac{\vdots}{\forall(q, p)}}{\forall(\exists(q, r), \exists(p, r))}\ (\mathrm{K})}{\forall(d_i, \exists(p, r))}\ (\mathrm{B})$$

At this point, we know that if $\langle d_1, d_2, Q\rangle \in \exists(p, r)^{\mathfrak{A}}$, then $\Gamma \vdash \forall(d_i, \exists(p, r))$ for some i. We now verify the converse. Suppose $\langle d_1, d_2, Q\rangle \in A$, with i ($1 \leq i \leq 2$) such that $\Gamma \vdash \forall(d_i, \exists(p, r))$. Then $\Gamma \vdash \exists(d_1, d_2)$. We thus have a derivation from Γ:

$$\dfrac{\dfrac{\dfrac{\vdots}{\exists(d_1, d_2)}}{\exists(d_i, d_i)}\ (\mathrm{I}) \qquad \vdots}{\dfrac{\exists(d_i, \exists(p, r))}{\exists(p, p)}\ (\mathrm{II})} \qquad \forall(d_i, \exists(p, r))}\ (\mathrm{D1})$$

This goes to show that $\langle p, p, \exists\rangle \in A$. By the construction of $\mathfrak{A}$, $\langle d_1, d_2, Q\rangle r^{\mathfrak{A}} \langle p, p, \exists\rangle$, and $\langle p, p, \exists\rangle \in p^{\mathfrak{A}}$. So $\langle d_1, d_2, Q\rangle \in \exists(p, r)^{\mathfrak{A}}$. This completes the proof. □

LEMMA 5.4. $\mathfrak{A} \models \Gamma$.

Proof. The proof is by cases on the various formula types in $\mathcal{R}^*$. Using the fact that formulas $\exists(e, f)$ and $\exists(f, e)$ are identified, and similarly for $\forall(\bar{e}, \bar{f})$ and $\forall(f, e)$, we may take all $\mathcal{R}^*$-formulas to have one of the forms:

$$\forall(c^+, d^+), \quad \forall(c^+, \overline{d^+}), \quad \exists(c^+, d^+), \quad \exists(c^+, \overline{d^+}),$$

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where c^+ and d^+ range over *positive* c-terms. In the remainder of the proof, we omit the $^+$-superscripts for clarity: that is, c and d range over positive c-terms.

Let $\varphi \in \Gamma$ be $\forall(c, d)$. Using (B) and Lemma 5.3, we see that $c^{\mathfrak{A}} \subseteq d^{\mathfrak{A}}$.

Let $\varphi \in \Gamma$ be $\forall(c, \bar{d})$. Suppose toward a contradiction that $\mathfrak{A} \not\models \varphi$. Let $\langle b_1, b_2, Q\rangle \in c^{\mathfrak{A}} \cap d^{\mathfrak{A}}$. Let i and j be such that $\Gamma \vdash \forall(b_i, c)$ and $\Gamma \vdash \forall(b_j, d)$. Then using (B), $\Gamma \vdash \forall(b_i, \bar{d})$. And since $\Gamma \vdash \exists(b_i, b_j)$, we use (D1) to see that $\Gamma \vdash \exists(d, \bar{d})$. So Γ is inconsistent, a contradiction.

If $\varphi \in \Gamma$ is $\exists(c, d)$, then $(c, d, \exists) \in A$. Indeed, $(c, d, \exists) \in c^{\mathfrak{A}} \cap d^{\mathfrak{A}}$, by Rule (T) and Lemma 5.3.

Finally, consider the case when $\varphi \in \Gamma$ is of the form $\exists(c, \bar{d})$. Then, using (I), $\Gamma \vdash \exists(c, c)$, so $\langle c, c, \forall\rangle \in A$. Suppose toward a contradiction that $\mathfrak{A} \models \forall(c, d)$. Then $\langle c, c, \forall\rangle \in d^{\mathfrak{A}}$. But then we have $\Gamma \vdash \forall(c, d)$, by Lemma 5.3 again. One application of (D2) now shows that $\Gamma \vdash \exists(d, \bar{d})$. Thus we have a contradiction to the consistency of Γ. □

Hence, we have shown that any consistent set Γ of $\mathcal{R}^*$-formulas has a model. Therefore:

THEOREM 5.5. *The derivation relation* $\Vdash_{\mathsf{R}^*}$ *is sound and complete for* $\mathcal{R}^*$.

§6. $\mathcal{R}^\dagger$ and $\mathcal{R}^{*\dagger}$: no indirect systems. Our last fragments are $\mathcal{R}^\dagger$ and $\mathcal{R}^{*\dagger}$. We open with some simple complexity results, showing that these fragments have no direct syllogistic proof system which is both sound and refutation-complete. After this, we strengthen the result to show that there are no *in*direct syllogistic proof systems, either. The proof of this negative result is similar to what we saw earlier in Theorem 4.1 for $\mathcal{R}$, but the argument here is more intricate.

LEMMA 6.1. *The problem of determining the validity of a sequent in* $\mathcal{R}^\dagger$ *is* EXPTIME*-hard.*

Proof. The logic K^U is the basic modal logic K together with an additional modality U (for 'universal'), under the standard relational (Kripke) semantics together with the rule

$$\models_w U\varphi \quad \text{if and only if} \quad \models_{w'} \varphi \text{ for all worlds } w'.$$

The satisfiability problem for K^U is EXPTIME-hard. (The proof is an easy adaptation of the corresponding result for propositional dynamic logic; see, e.g., Harel *et al.*, 2000, 216 ff.) It suffices, therefore, to reduce this problem to satisfiability in $\mathcal{R}^\dagger$. Let φ be a formula of K^U.

We first transform φ into an equisatisfiable set of formulas $T_\varphi \cup S_\varphi$ of first-order logic; then we translate the formulas of $T_\varphi \cup S_\varphi$ into an equisatisfiable set of $\mathcal{R}^\dagger$-formulas. To simplify the notation, we shall take unary atoms (in $\mathcal{R}^\dagger$) to be unary predicates (in first-order logic); similarly, we take binary atoms to do double duty as binary predicates. Let r and e be binary atoms. For any K^U-formula ψ, let p_ψ be a unary atom, and define the set of first-order formulas T_ψ inductively as follows:

$$T_p = \emptyset \quad \text{(where } p \text{ is a proposition letter)}$$

$$T_{\psi\wedge\pi} = T_\psi \cup T_\pi \cup \big\{\forall x\big(p_\psi(x) \wedge p_\pi(x) \to p_{\psi\wedge\pi}(x)\big), \; \forall x\big(p_{\psi\wedge\pi}(x) \to p_\psi(x)\big), \; \forall x\big(p_{\psi\wedge\pi}(x) \to p_\pi(x)\big)\big\}$$

$$T_{\neg\psi} = T_\psi \cup \big\{\forall x\big(p_{\neg\psi}(x) \to \neg p_\psi(x)\big), \; \forall x\big(\neg p_{\neg\psi}(x) \to p_\psi(x)\big)\big\}$$

$$T_{\Box\psi} = T_\psi \cup \big\{\forall x\big(p_{\Box\psi}(x) \to \forall y\big(\neg p_\psi(y) \to \neg r(x, y)\big)\big),$$

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$$\forall x\big(\neg p_{\Box\psi}(x) \to \exists y\big(\neg p_\psi(y) \wedge r(x,y)\big)\big)\big\}$$

$$T_{U\psi} = T_\psi \cup \big\{\forall x\big(p_{U\psi}(x) \to \forall y(\neg p_\psi(y) \to \neg e(x,y))\big),$$

$$\forall x\big(\neg p_{U\psi}(x) \to \exists y\big(\neg p_\psi(y) \wedge e(x,y)\big)\big)\big\}.$$

Now let S_φ be the collection of five first-order formulas

$$\exists x\big(p_\varphi(x) \wedge p_\varphi(x)\big), \qquad \forall x\big(\pm p_\varphi(x) \to \forall y(\pm p_\varphi(y) \to e(x,y))\big).$$

(Although the first formula looks like it has a redundant conjunct, we state it in this way to make our work below a little easier.) We claim that the modal formula φ is satisfiable if and only if the set of first-order formulas $T_\varphi \cup S_\varphi$ is satisfiable. For let $\mathfrak{M}$ be any (Kripke) model of φ over a frame (W, R). Define the first-order structure $\mathfrak{A}$ with domain W, by setting $r^{\mathfrak{A}} = R$, $e^{\mathfrak{A}} = W^2$, and $p_\psi^{\mathfrak{A}} = \{w \mid \mathfrak{M} \models_w \psi\}$, for any subformula ψ of φ. It is then easy to check that $\mathfrak{A} \models T_\varphi \cup S_\varphi$. Conversely, suppose $\mathfrak{A} \models T_\varphi \cup S_\varphi$. We build a Kripke structure $\mathfrak{M}$ over the frame $(A, r^{\mathfrak{A}})$ by setting, for any proposition letter o mentioned in φ, $\mathfrak{M} \models_a o$ if and only if $a \in p_o^{\mathfrak{A}}$. A straightforward structural induction establishes that for any subformula ψ of φ, $\mathfrak{M} \models_a \psi$ if and only if $a \in p_\psi^{\mathfrak{A}}$. The formula $\exists x(p_\varphi(x) \wedge p_\varphi(x)) \in S_\varphi$ then ensures that φ is satisfied in $\mathfrak{M}$.

Now, all of the formulas in $T_\varphi \cup S_\varphi$ are of one of the forms

$$\forall x(\pm p(x) \to \pm q(x)) \qquad \forall x(\pm p(x) \to \forall y(\pm q(y) \to \pm r(x,y))) \tag{25}$$

$$\exists x(p(x) \wedge p(x)) \qquad \forall x(\pm p(x) \to \exists y(\pm q(y) \wedge r(x,y))) \tag{26}$$

$$\forall x(p(x) \wedge q(x) \to o(x)). \tag{27}$$

Notice that formulas of the forms (25) and (26) translate (in the obvious sense) directly into the fragment $\mathcal{R}^\dagger$; those of form (27), by contrast, do not. The next step is to eliminate formulas of this last type.

Let o^* be a new unary relation symbol. For $\theta \in T_\varphi \cup S_\varphi$ of the form (27), let r_θ be a new binary atom, and define R_θ to be the set of formulas

$$\forall x(\neg o(x) \to \exists z(o^*(z) \wedge r_\theta(x,z))) \tag{28}$$

$$\forall x(p(x) \to \forall z(\neg p(z) \to \neg r_\theta(x,z))) \tag{29}$$

$$\forall x(q(x) \to \forall z(p(z) \to \neg r_\theta(x,z))), \tag{30}$$

which are all of the forms in (25) or (26). It is easy to check that $R_\theta \models \theta$. For suppose (for contradiction) that $\mathfrak{A} \models R_\theta$ and a satisfies p and q but not o in $\mathfrak{A}$. By (28), there exists b such that $\mathfrak{A} \models r_\theta[a,b]$. If $\mathfrak{A} \not\models p[b]$, then (29) is false in $\mathfrak{A}$; on the other hand, if $\mathfrak{A} \models p[b]$, then (30) is false in $\mathfrak{A}$. Thus, $R_\theta \models \theta$ as claimed. Conversely, if $\mathfrak{A} \models \theta$, expand $\mathfrak{A}$ to a structure $\mathfrak{A}'$ by interpreting o^* and r_θ as follows:

$$(o^*)^{\mathfrak{A}} = A$$

$$r_\theta^{\mathfrak{A}} = \{\langle a, a\rangle \mid \mathfrak{A} \not\models o[a]\}.$$

We check that $\mathfrak{A}' \models R_\theta$. Formula (28) is true, because $\mathfrak{A}' \not\models o[a]$ implies $\mathfrak{A}' \models r_\theta[a,a]$. Formula (29) is true, because $\mathfrak{A}' \models r_\theta[a,b]$ implies $a = b$. To see that Formula (30) is true, suppose $\mathfrak{A}' \models q[a]$ and $\mathfrak{A}' \models p[b]$. If $a = b$, then $\mathfrak{A} \models o[a]$ (since $\mathfrak{A}' \models \theta$); that is, either $a \neq b$ or $\mathfrak{A} \models o[a]$. By construction, then, $\mathfrak{A}' \not\models r_\theta[a,b]$.

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Now let T^*_φ be the result of replacing all formulas θ in T_φ of form (27) with the corresponding trio R_θ. (The binary atoms r_θ for the various θ are assumed to be distinct; however, the same unary atom o^* can be used for all θ.) By the previous paragraph, $T^*_\varphi \cup S_\varphi$ is satisfiable if and only if $T_\varphi \cup S_\varphi$ is satisfiable, and hence if and only if φ is satisfiable. But $T^*_\varphi \cup S_\varphi$ is a set of formulas of the forms (25) and (26), and can evidently be translated into a set of $\mathcal{R}^\dagger$-formulas satisfied in exactly the same structures. Moreover, this set can be computed in time bounded by a polynomial function of $\|\varphi\|$. This completes the reduction. □

We note the following fact. (We omit a detailed proof, since subsequent developments do not hinge on this result.)

LEMMA 6.2. *The problem of determining the validity of a sequent in $\mathcal{R}^{*\dagger}$ is in* EXPTIME.

Proof. Trivial adaptation of Pratt-Hartmann (2004), Theorem 3, which considers a fragment obtained by adding relative clauses to the relational syllogistic. □

THEOREM 6.3. *The problem of determining the validity of a sequent in either of the fragments $\mathcal{R}^\dagger$ and $\mathcal{R}^{*\dagger}$ is* EXPTIME*-complete.*

Proof. Lemmas 6.1 and 6.2. □

COROLLARY 6.4. *There exists no finite set X of syllogistic rules in either $\mathcal{R}^\dagger$ or $\mathcal{R}^{*\dagger}$ such that $\vdash_{\mathsf{X}}$ is both sound and refutation-complete.*

Proof. It is a standard result that PTIME≠EXPTIME. The result is then immediate by Lemmas 2.1 and 6.1. □

Of course, Corollary 6.4 leaves open the possibility that there exist *indirect* syllogistic systems that are sound and complete for $\mathcal{R}^\dagger$ and $\mathcal{R}^{*\dagger}$. To show that there do not, stronger methods are required. That is the task of the remainder of this section.

Starting on the proof. Let $n \geq 2$; let $o_1, o_2, o_3, q_1, q_2, p_1, \ldots, p_n$ be unary atoms, and r, s binary atoms; and let $A^{(n)}$ be the set $\{a_1, \ldots, a_n, a'_1, \ldots, a'_n, u_0, u_1, u_2, u_3, u_4, v_1, v_2\}$. We take the structure $\mathfrak{A}^{(n)}$, with domain $A^{(n)}$, to be as depicted in Figure 3. Here, membership of an element in $p^{\mathfrak{A}^{(n)}}$ (where p is any unary atom) is indicated by writing p in a box next to that element. The extensions of the binary atoms r and s are depicted similarly: we indicate that a pair of elements is in $r^{\mathfrak{A}^{(n)}}$ by writing r next to an arrow between those elements (and likewise for s). The r-labeled arrow from the dotted box to the element u_2 is to be interpreted as follows: for any element a inside the dotted box, $\langle a, u_2\rangle \in r^{\mathfrak{A}^{(n)}}$.

Thus, $\mathfrak{A}^{(n)}$ contains two 'r-chains' of elements satisfying, successively, $p_1, \ldots, p_n$: the elements of the first chain additionally satisfy q_1 but not q_2; those of the second chain additionally satisfy q_2 but not q_1. The terminal elements of the two r-chains are related, by s, to the elements v_1 and v_2. There are also some additional 'outlier' elements, which ensure the truth of various existential formulas in $\mathfrak{A}$: for example, u_1 ensures (together with u_0) the truth of $\exists(o_1, \exists(p_1, \bar{r}))$; u_2 ensures the truth of $\exists(q_1, q_2)$, and so on. In the sequel, we define structures containing additional elements, which make *almost* the same $\mathcal{R}^{*\dagger}$-formulas true as $\mathfrak{A}^{(n)}$. These latter structures will then be used to show that there can be no sound and complete indirect syllogistic system for $\mathcal{R}^*$ or $\mathcal{R}^{*\dagger}$.

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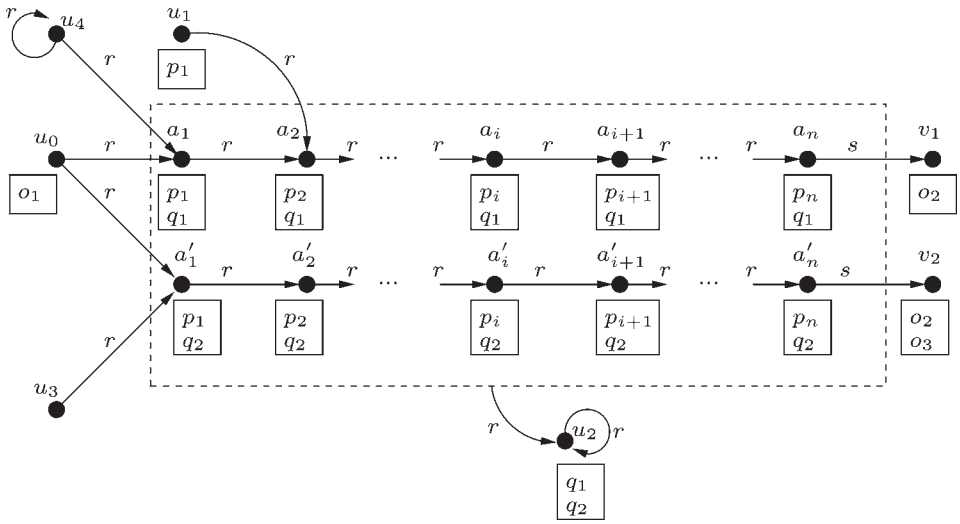


Fig. 3. The structure $\mathfrak{A}^{(n)}$. Every element inside the dotted box is related by r to u_2.

To avoid notational clutter, we drop the superscripts in $\mathfrak{A}^{(n)}$ (and in related constructions) where the value of n does not matter. By inspection, we have

$$\mathfrak{A} \models \forall(o_1, \exists(q_1, r)) \tag{31}$$

$$\mathfrak{A} \models \forall(o_1, \forall(\bar{p}_1, \bar{r})) \tag{32}$$

$$\mathfrak{A} \models \forall(q_1, \forall(\bar{q}_1, \bar{r})) \tag{33}$$

$$\mathfrak{A} \models \forall(q_2, \forall(\bar{q}_2, \bar{r})) \tag{34}$$

$$\mathfrak{A} \models \forall(p_i, \exists(p_{i+1}, r)) \quad (1 \leq i < n) \tag{35}$$

$$\mathfrak{A} \models \forall(q_1, \forall(o_3, \bar{s})) \tag{36}$$

$$\mathfrak{A} \models \forall(q_2, \forall(\bar{o}_3, \bar{s})) \tag{37}$$

$$\mathfrak{A} \models \forall(p_n, \exists(o_2, s)). \tag{38}$$

For any structure $\mathfrak{C}$, denote by $\mathrm{Th}(\mathfrak{C})$ the set of $\mathcal{R}^{\dagger}$-formulas true in $\mathfrak{C}$, and denote by $\mathrm{Th}^*(\mathfrak{C})$ the set of $\mathcal{R}^{*\dagger}$-formulas true in $\mathfrak{C}$. Recall that a set of formulas Φ is $\mathcal{R}^{\dagger}$-complete (or $\mathcal{R}^{*\dagger}$-complete) if, for every $\mathcal{R}^{\dagger}$-formula (respectively, $\mathcal{R}^{*\dagger}$-formula) φ, either $\varphi \in \Phi$ or $\bar{\varphi} \in \Phi$. Trivially, for any $\mathfrak{C}$, $\mathrm{Th}(\mathfrak{C})$ is $\mathcal{R}^{\dagger}$-complete, and $\mathrm{Th}^*(\mathfrak{C})$ is $\mathcal{R}^{*\dagger}$-complete.

Let γ be the $\mathcal{R}^{\dagger}$-formula given by

$$\gamma = \forall(o_1, \exists(\bar{q}_2, r)).$$

Noting that $o_1^{\mathfrak{A}} = \{u_0\}$, $\langle u_0, a_1 \rangle \in r^{\mathfrak{A}}$, and $a_1 \notin q_2^{\mathfrak{A}}$, we have $\mathfrak{A} \models \gamma$ (i.e., $\gamma \in \mathrm{Th}(\mathfrak{A})$). Let $\Gamma^{(n)}$ be obtained from $\mathrm{Th}(\mathfrak{A}^{(n)})$ by reversing the truth value of γ, and similarly for $\Gamma^{*(n)}$. That is:

$$\Gamma^{(n)} = \left(\mathrm{Th}\left(\mathfrak{A}^{(n)}\right) \setminus \{\gamma\}\right) \cup \{\bar{\gamma}\}$$

$$\Gamma^{*(n)} = \left(\mathrm{Th}^*\left(\mathfrak{A}^{(n)}\right) \setminus \{\gamma\}\right) \cup \{\bar{\gamma}\}.$$

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Again, we drop the (n) superscript when the value of n does not matter. Thus, Γ is $\mathcal{R}^\dagger$-complete and Γ^* is $\mathcal{R}^{*\dagger}$-complete.

LEMMA 6.5. Γ (*and therefore* Γ^*) *is unsatisfiable.*

Proof. Noting that $\bar{\gamma}$ is the formula $\exists(o_1, \forall(\bar{q}_2, \bar{r}))$, from (31) and (32), we see that, in any model $\mathfrak{B} \models \Gamma$, there exists $b \in p_1^{\mathfrak{B}} \cap q_1^{\mathfrak{B}} \cap q_2^{\mathfrak{B}}$. From (33) to (35), it then follows that, in any model $\mathfrak{B} \models \Gamma$, there exists $b \in p_n^{\mathfrak{B}} \cap q_1^{\mathfrak{B}} \cap q_2^{\mathfrak{B}}$. But, from (36) to (38), no model $\mathfrak{B} \models \Gamma$ can have any element $b \in p_n^{\mathfrak{B}} \cap q_1^{\mathfrak{B}} \cap q_2^{\mathfrak{B}}$. This proves the lemma. □

Fixing $n \geq 2$, for any i $(1 \leq i < n)$, let $B_i^{(n)} = A^{(n)} \cup \{b_1, \ldots, b_i, u_5\}$, and consider the structure $\mathfrak{B}_i^{(n)}$, with domain $B_i^{(n)}$, depicted in Figure 4. We employ the same notational conventions as in Figure 3. In particular, the r-labeled arrows from the dotted boxes are to be interpreted as follows: for any element a inside either of the dotted boxes, $\langle a, u_2 \rangle \in r^{\mathfrak{B}_i^{(n)}}$. Again, we drop superscripts where the value of n does not matter. The structure $\mathfrak{B}_i$ contains a copy of $\mathfrak{A}$, but has an additional r-chain whose elements satisfy both q_1 and q_2; notice, however, that this additional r-chain stops at the ith element.

We employ the following terminology. An *existential* formula is one of the form $\exists(e, f)$. If $\varphi = \exists(e, f)$ and $\mathfrak{C}$ is a structure, then a *witness for* $\mathfrak{C} \models \varphi$ is any $a \in C$ such that $a \in e^{\mathfrak{C}}$ and $a \in f^{\mathfrak{C}}$. Now, we saw above that $\mathfrak{A} \models \gamma$; by contrast $\mathfrak{B}_i \models \bar{\gamma}$ ($\bar{\gamma}$ is an existential formula, and u_5 is a witness). Let δ_i be the $\mathcal{R}^\dagger$-formula given by

$$\delta_i = \forall\big(p_i, \exists(p_{i+1}, r)\big).$$

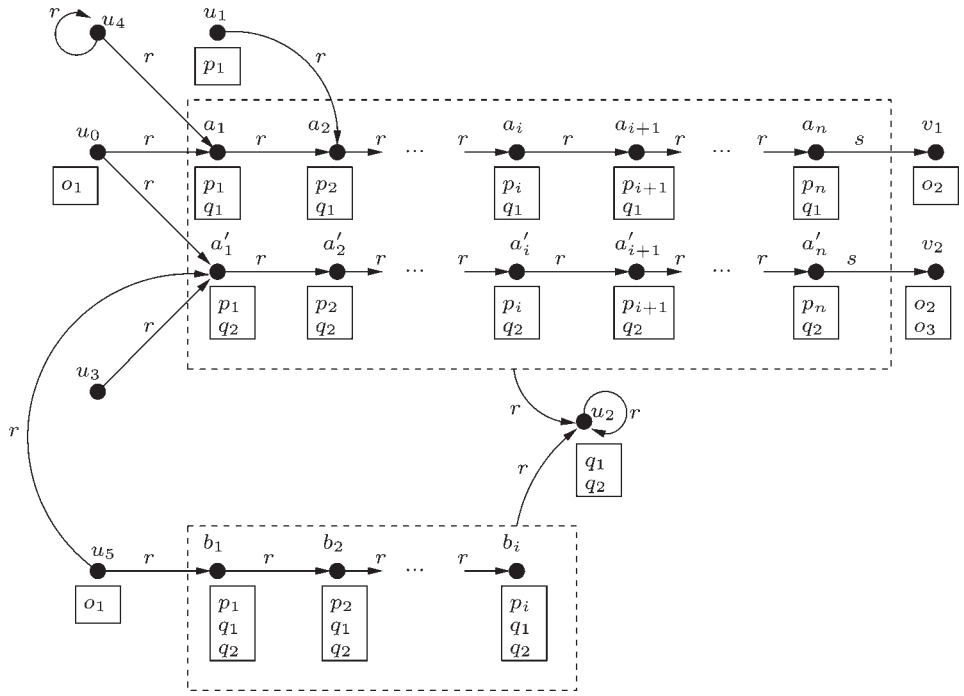


Fig. 4. The structure $\mathfrak{B}_i$. Every element inside either of the dotted boxes is related by r to u_2.

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We observed in (35) that $\mathfrak{A} \models \delta_i$. By contrast, $\mathfrak{B}_i \models \bar{\delta}_i$ ($\bar{\delta}_i$ is an existential formula, and b_i is a witness). However, it turns out that γ and δ_i are the *only* differences between $\mathfrak{A}$ and $\mathfrak{B}_i$ as far as $\mathcal{R}^{*\dagger}$ is concerned:

LEMMA 6.6. *For all i ($1 \leq i < n$):*

$$\mathrm{Th}\left(\mathfrak{B}_i^{(n)}\right) = \left(\mathrm{Th}\left(\mathfrak{A}^{(n)}\right) \setminus \{\gamma, \delta_i\}\right) \cup \{\bar{\gamma}, \bar{\delta}_i\};$$

similarly,

$$\mathrm{Th}^*\left(\mathfrak{B}_i^{(n)}\right) = \left(\mathrm{Th}^*\left(\mathfrak{A}^{(n)}\right) \setminus \{\gamma, \delta_i\}\right) \cup \{\bar{\gamma}, \bar{\delta}_i\}.$$

Proof. Since $\gamma, \delta_i, \bar{\gamma}, \bar{\delta}_i$ are all in $\mathcal{R}^\dagger$, the first of these statements follows instantly from the second. We proceed, therefore, to establish the second statement.

We first prove the statement for the case $i = 1$. Notice that here, the third r-'chain' in $\mathfrak{B}_1^{(n)}$ contains just one element, b_1. Our initial goal, then, is to show that $\mathrm{Th}^*(\mathfrak{B}_1^{(n)}) = (\mathrm{Th}^*(\mathfrak{A}^{(n)}) \setminus \{\gamma, \delta_1\}) \cup \{\bar{\gamma}, \bar{\delta}_1\}$. The basic intuition is simple: on the one hand, we may check the cases $n = 2$ and $n = 3$ by brute force; on the other, we see from Figure 4 that, once the ends of the first two r-chains are sufficiently distant from the elements u_5 and b_1, extending those r-chains further will have no effect on the differences between the formulas made true in $\mathfrak{A}^{(n)}$ and $\mathfrak{B}_1^{(n)}$.

To make this idea rigorous, we first establish three simple claims. We employ the following terminology. An *existential* e-term is one of the form $\exists(l, t)$. (As usual, l ranges over unary literals, and t over binary literals.) If $e = \exists(l, t)$, $\mathfrak{C}$ is a structure, and $a \in C$, then a *witness for* $a \in e^{\mathfrak{C}}$ is any $b \in C$ such that $b \in l^{\mathfrak{C}}$ and $\langle a, b\rangle \in t^{\mathfrak{C}}$.

CLAIM 6.7. *Let a be any element of $A^{(n)}$ ($n \geq 2$), and e any e-term. Then $a \in e^{\mathfrak{B}_1^{(n)}}$ if and only if $a \in e^{\mathfrak{A}^{(n)}}$.*

Proof. We prove the only-if direction by induction on n. The if-direction then follows by considering the e-term $\bar{e}$. The case $n = 2$ is checked by brute force. (We used a computer.) Assume the claim is true for $n = m \geq 2$; we show that it is true for $n = m + 1$. Let $a \in A^{(m+1)}$. For contradiction, suppose $a \in e^{\mathfrak{B}_1^{(m+1)}}$, but $a \notin e^{\mathfrak{A}^{(m+1)}}$. Then e is existential, and $a \in e^{\mathfrak{B}_1^{(m+1)}}$ has a witness in $B_1^{(m+1)} \setminus A^{(m+1)}$. Writing $e = \exists(l, t)$, there exists $b \in \{u_5, b_1\}$ such that $b \in l^{\mathfrak{B}_1^{(m+1)}}$ and $\langle a, b\rangle \in t^{\mathfrak{B}_1^{(m+1)}}$. In that case, if $a \in A^{(m)}$, we obviously have $a \in e^{\mathfrak{B}_1^{(m)}}$, whence $a \in e^{\mathfrak{A}^{(m)}}$ by inductive hypothesis, whence $a \in e^{\mathfrak{A}^{(m+1)}}$, since $\mathfrak{A}^{(m)}$ is a submodel of $\mathfrak{A}^{(m+1)}$—a contradiction. Thus, a is either a_{m+1} or a'_{m+1}. But, by inspection of Figure 4 (bearing in mind $i = 1$), it is easy to see that, for $a \in \{a_{m+1}, a'_{m+1}\}$, if $a \in e^{\mathfrak{B}_1^{(m+1)}}$ with a witness in $\{u_5, b_1\}$, then $a \in e^{\mathfrak{A}^{(m+1)}}$ with a witness in $\{u_0, u_2, a_1, a'_1\}$— again, a contradiction. □

CLAIM 6.8. *Let φ be any existential formula other than $\bar{\delta}_1$, and let $A_0 = \{a_1, a'_1, u_2\}$. If b_1 is a witness for $\mathfrak{B}_1^{(n)} \models \varphi$, then there exists $a \in A_0$ such that a is a witness for $\mathfrak{A}^{(n)} \models \varphi$.*

Proof. By induction on n. The cases $n = 2$ and $n = 3$ are checked by brute force. (We used a computer.) Note in passing that the condition that φ is not $\bar{\delta}_1$ is required for these cases. Suppose now the claim holds for $n = m \geq 3$; we show that it holds for $n = m + 1$.

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If e is any e-term, let $\hat{e}$ be the result of replacing any occurrence of the unary atom p_{m+1} by the unary atom p_m. Writing $\varphi = \exists(e, f)$, let $\hat{\varphi} = \exists(\hat{e}, \hat{f})$. Note that, since $m \geq 3$ and $\varphi \neq \bar{\delta}_1$, we have $\hat{\varphi} \neq \bar{\delta}_1$. The following facts are obvious (see Figures 3 and 4):

$$a \in e^{\mathfrak{A}^{(m+1)}} \quad \text{iff} \quad a \in \hat{e}^{\mathfrak{A}^{(m)}} \qquad \text{for all } a \in A_0$$

$$b_1 \in e^{\mathfrak{B}_1^{(m+1)}} \quad \text{iff} \quad b_1 \in \hat{e}^{\mathfrak{B}_1^{(m)}}.$$

It follows that, if b_1 is a witness for $\mathfrak{B}_1^{(m+1)} \models \varphi$, then b_1 is a witness for $\mathfrak{B}_1^{(m)} \models \hat{\varphi}$, whence, by inductive hypothesis, there exists $a \in A_0$ such that a is a witness for $\mathfrak{A}^{(m)} \models \hat{\varphi}$, whence there exists $a \in A_0$ such that a is a witness for $\mathfrak{A}^{(m+1)} \models \varphi$. This completes the induction. □

CLAIM 6.9. *Let φ be any existential formula other than $\bar{\gamma}$, and let $A_0 = \{u_0, u_2, u_3\}$. If u_5 is a witness for $\mathfrak{B}_1^{(n)} \models \varphi$, then there exists $a \in A_0$ such that a is a witness for $\mathfrak{A}^{(n)} \models \varphi$.*

Proof. Identical in structure to the proof of Claim 6.8. □

We can now prove Lemma 6.6 in the case $i = 1$. Claim 6.7 shows that any existential formula true in $\mathfrak{A}^{(n)}$ is true in $\mathfrak{B}_1^{(n)}$, and moreover, that if φ is an existential formula true in $\mathfrak{B}_1^{(n)}$ but false in $\mathfrak{A}^{(n)}$, then either u_5 or b_1 is a witness. But these possibilities are ruled out—except in the cases $\varphi = \bar{\gamma}$ and $\varphi = \bar{\delta}_1$—by Claims 6.8 and 6.9, respectively. Finally, if φ is universal (and not equal to γ or δ_1), then $\bar{\varphi}$ is existential (and not equal to $\bar{\gamma}$ or $\bar{\delta}_1$). The foregoing analysis shows that $\mathfrak{A}^{(n)}$ and $\mathfrak{B}_1^{(n)}$ agree on the truth value of $\bar{\varphi}$, hence on the truth value of φ.

We have thus proved the lemma for $i = 1$ and all values of n. Fix $n \geq 3$, and consider now the structure $\mathfrak{B}_2^{(n)}$, which differs from $\mathfrak{B}_1^{(n)}$ only in that the third r-chain has been extended from one to two elements. By inspection of Figure 4, the only effect on the set of sentences made true is to restore the truth of δ_1 and to falsify δ_2. That is:

$$\begin{aligned} \mathrm{Th}^*\left(\mathfrak{B}_2^{(n)}\right) &= \left(\mathrm{Th}^*\left(\mathfrak{B}_1^{(n)}\right) \setminus \{\bar{\delta}_1, \delta_2\}\right) \cup \{\delta_1, \bar{\delta}_2\} \\ &= \left(\mathrm{Th}^*\left(\mathfrak{A}^{(n)}\right) \setminus \{\gamma, \delta_2\}\right) \cup \{\bar{\gamma}, \bar{\delta}_2\}. \end{aligned}$$

Proceeding in the same way, we have, for all i $(1 \leq i < n - 1)$,

$$\begin{aligned} \mathrm{Th}^*\left(\mathfrak{B}_{i+1}^{(n)}\right) &= \left(\mathrm{Th}^*\left(\mathfrak{B}_i^{(n)}\right) \setminus \{\bar{\delta}_i, \delta_{i+1}\}\right) \cup \{\delta_i, \bar{\delta}_{i+1}\} \\ &= \left(\mathrm{Th}^*\left(\mathfrak{A}^{(n)}\right) \setminus \{\gamma, \delta_{i+1}\}\right) \cup \{\bar{\gamma}, \bar{\delta}_{i+1}\}. \end{aligned}$$

This proves the lemma. □

Fixing $n > 3$, for all i and j $(1 < i < j < n)$ define

$$\begin{aligned} \Delta_{i,j}^{(n)} &= \Gamma^{(n)} \setminus \{\delta_i, \delta_j\} \\ \Delta_{i,j}^{*(n)} &= \Gamma^{*(n)} \setminus \{\delta_i, \delta_j\}. \end{aligned}$$

Again (n)-superscripts are omitted where possible for clarity.

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LEMMA 6.10. *Let θ be an $\mathcal{R}^\dagger$-formula and θ^* an $\mathcal{R}^{*\dagger}$-formula. For all i, j $(1 < i < j < n)$, if $\Delta_{i,j} \models \theta$, then $\theta \in \Delta_{i,j}$. Likewise, if $\Delta^*_{i,j} \models \theta^*$, then $\theta^* \in \Delta^*_{i,j}$.*

Proof. We prove the first statement only; the proof of the second is similar. Given the equations $\Gamma^{(n)} = \left(\mathrm{Th}\left(\mathfrak{A}^{(n)}\right) \setminus \{\gamma\}\right) \cup \{\bar{\gamma}\}$ and $\Delta^{(n)}_{i,j} = \Gamma^{(n)} \setminus \{\delta_i, \delta_j\}$, Lemma 6.6 yields

$$\mathfrak{B}_i \models \Delta_{i,j} \cup \{\bar{\delta}_i, \delta_j\} \tag{39}$$

$$\mathfrak{B}_j \models \Delta_{i,j} \cup \{\delta_i, \bar{\delta}_j\}. \tag{40}$$

Certainly, then, $\Delta_{i,j}$ is satisfiable; hence, the only θ we need to consider are those such that neither θ nor $\bar{\theta}$ is in $\Delta_{i,j}$. By the completeness of $\Gamma^{(n)}$, this entails $\theta \in \{\delta_i, \bar{\delta}_j, \bar{\delta}_i, \delta_j\}$. In the first two cases, (39) shows that $\Delta_{i,j} \not\models \theta$; in the third and fourth cases, (40) shows the same. □

LEMMA 6.11. *If X is a finite set of sound syllogistic rules in $\mathcal{R}^\dagger$, then there exists n such that, for any absurdity $\perp$ in $\mathcal{R}^\dagger$, $\Gamma^{(n)} \not\vdash_{\mathsf{X}} \perp$. If X is a finite set of sound syllogistic rules in $\mathcal{R}^{*\dagger}$, then there exists n such that, for any absurdity $\perp$ in $\mathcal{R}^{*\dagger}$, $\Gamma^{*(n)} \not\vdash_{\mathsf{X}} \perp$.*

Proof. We prove the lemma for $\mathcal{R}^\dagger$; the proof for $\mathcal{R}^{*\dagger}$ is similar. Since the number of rules in X is finite, let k be the largest number of antecedents in any of these rules, and fix $n = k + 2$. Since no instance of a rule of X has more than $n - 2$ antecedents, any such set of antecedents included in $\Gamma^{(n)}$ must also be included in $\Delta^{(n)}_{i,j}$ for some $1 < i < j < n$. Let θ be the consequent of this rule instance. Since $\vdash_{\mathsf{X}}$ is sound, $\Delta^{(n)}_{i,j} \models \theta$, whence $\theta \in \Delta^{(n)}_{i,j} \subseteq \Gamma^{(n)}$, by Lemma 6.10. A simple induction on the lengths of proofs then shows that any formula derived from $\Gamma^{(n)}$ is in $\Gamma^{(n)}$. But $\Gamma^{(n)}$ contains no absurdity. □

We now have the promised strengthening of Corollary 6.4. In some sense, Lemma 6.11 has done all the work; for (RAA) is, in the context of an $\mathcal{R}^\dagger$-complete (or $\mathcal{R}^{*\dagger}$-complete) set of premises, essentially redundant.

THEOREM 6.12. *There exists no finite set X of syllogistic rules in either $\mathcal{R}^\dagger$ or $\mathcal{R}^{*\dagger}$ such that $\Vdash_{\mathsf{X}}$ is both sound and complete.*

Proof. We prove the lemma for $\mathcal{R}^\dagger$; the proof for $\mathcal{R}^{*\dagger}$ is almost identical. Suppose X is a finite set of syllogistic rules in $\mathcal{R}^\dagger$, with $\Vdash_{\mathsf{X}}$ sound. Let n and $\Gamma^{(n)}$ be as in Lemma 6.11. We show that, for any absurdity $\perp$ in $\mathcal{R}^\dagger$, $\Gamma^{(n)} \not\Vdash_{\mathsf{X}} \perp$. Since, by Lemma 6.5, $\Gamma^{(n)} \models \perp$, this proves the theorem.

Suppose then, for contradiction, that there is an indirect derivation of some absurdity $\perp$ from $\Gamma^{(n)}$, using the rules X. Let the number of applications of (RAA) employed in this derivation be k; and assume without loss of generality that $\perp$ is chosen so that this number k is minimal. Since, by Lemma 6.11, $\Gamma^{(n)} \not\vdash_{\mathsf{X}} \perp$, we know that $k > 0$. Consider the last application of (RAA) in this derivation, which derives an $\mathcal{R}^\dagger$-formula, say, $\bar{\theta}$, discharging a premise θ. That is, there is an (indirect) derivation of some absurdity $\perp'$ from $\Gamma^{(n)} \cup \{\theta\}$, employing fewer than k applications of (RAA). By minimality of k, $\theta \notin \Gamma^{(n)}$, and so, by the $\mathcal{R}^\dagger$-completeness of $\Gamma^{(n)}$, $\bar{\theta} \in \Gamma^{(n)}$. But then we can replace our original derivation of $\bar{\theta}$ with a trivial derivation, so obtaining a derivation of $\perp$ from $\Gamma^{(n)}$ with fewer than k applications of (RAA), a contradiction. □

The upshot: unlike in the case of $\mathcal{R}^*$, the full power of *reductio ad absurdum* does not help with $\mathcal{R}^\dagger$ and $\mathcal{R}^{*\dagger}$.

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§7. Relation to other work.

7.1. Systems related to $\mathcal{S}$ and $\mathcal{S}^\dagger$. Modern treatments of the syllogistic can all be traced back to Łukasiewicz (1957), where a logic is presented in which formulas of the forms (2) are treated as atoms in a propositional calculus. Łukasiewicz provides a collection of axiom schemata which, together with the usual axioms of propositional logic, yields a complete proof system for the resulting language (a strict superset of $\mathcal{S}$). The completeness of this system was shown independently by Westerståhl (1989). Note that it differs from the syllogistic fragments considered in this paper, in that it is embedded within propositional logic. Other commentators, for example, Smiley (1973), Corcoran (1972) and Martin (1997), objecting to Łukasiewicz' exegesis of Aristotle, provide proof systems in the form of syllogistic rules similar to those of S, again proving completeness of their respective systems. These systems are not *direct* syllogistic systems, since they all employ *reductio ad absurdum*. Our emphasis on direct syllogistic systems, and especially the formulation of *refutation-completeness* is new, and motivated by the results we have obtained on relational extensions of the syllogistic. In addition, the authors mentioned above do not treat the fragment $\mathcal{S}^\dagger$. The completeness of $\mathcal{S}$ itself appears as Theorem 6.2 in Moss (2008), where various fragments of $\mathcal{S}$ are also axiomatized. The completeness of $\mathsf{S}^\dagger$ is proved in Moss (2007), using a somewhat different treatment.

7.2. Systems related to $\mathcal{R}$. The first presentation of a complete proof system for a fragment close to the relational syllogistic seems to be Nishihara *et al.* (1990). This logic is in effect a relational version of Łukasiewicz', in that formulas roughly similar to those of the forms (6) are treated as atoms of a propositional calculus. The authors provide axiom schemata which, together with the usual axioms of propositional logic, yield a complete proof system for the language in question. Actually, the propositional atoms in this language are allowed to feature n-ary predicates for all $n \geq 1$. However, the rather strange restrictions on quantifier scope (existentials must always outscope universals) mean that this language is primarily of interest for atoms featuring only unary and binary predicates; these atoms (and their negations) then essentially correspond to the formulas of our fragment $\mathcal{R}$. We mention in passing that the language of Nishihara *et al.* (1990) includes individual constants; but in practice, this leads to no useful increase in expressive power, and we ignore this feature. Because the fragment includes full propositional logic, the results from Nishihara *et al.* (1990) cannot yield a complexity upper bound comparable to that of Theorem 4.11.

A logic inspired by the system of Nishihara *et al.* (1990) may be found in Moss (to appear). Roughly speaking, that logic is the negation-free fragment of $\mathcal{R}$, corresponding to sentence forms involving the words *some* and *all*, but not *no*. However, it extends $\mathcal{R}$ in that it allows both readings of scope-ambiguous sentences. For example, a sentence like "Every painter admires some artist" has both a subject-wide-scope reading and a subject-narrow-scope reading. The fact that the subject-wide-scope reading logically implies the subject-narrow-scope reading is then a rule of inference in Moss' system. No contradictions are possible in the system.

Moss (to appear) also analyzes a syllogistic logic with negated *nouns* (not verbs) and only using *All*. So in our notation, its formulas are $\forall(l, m)$ and $\forall(l, \forall(m, r))$, with l and m unary literals. In addition to rules we have seen, it uses the following additional rules which might be taken to be forms of the law of the excluded middle:

$$\frac{\forall(n, \forall(q, r)) \qquad \forall(\bar{n}, \forall(q, r))}{\forall(p, \forall(q, r))} \qquad \frac{\forall(p, \forall(n, r)) \qquad \forall(p, \forall(\bar{n}, r))}{\forall(p, \forall(q, r)).}$$

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The system is completed by a rule with three premises:

$$\frac{\forall(p, \forall(n, r)) \qquad \forall(o, \forall(q, r)) \qquad \forall(\bar{o}, \forall(\bar{n}, r))}{\forall(p, \forall(q, r)).}$$

7.3. McAllester and Givan's fragment. We have already had occasion to mention the results of McAllester & Givan (1992) in connection with our fragment $\mathcal{R}^*$. McAllester and Givan present a 'Montagovian syntax' for a first-order language over a signature of unary and binary predicates and individual constants, together with (what they call) its 'quantifier-free' fragment. In fact, this latter fragment is like our fragment $\mathcal{R}^*$, except that its 'class-terms' (the equivalent of our c-terms) can be nested to arbitrary depth. Thus, in McAllester and Givan's language, the formula

$$\forall(\exists(\exists(\text{man}, \text{kill}), \text{kill}), \exists(\exists(\text{animal}, \text{kill}), \text{kill}))$$

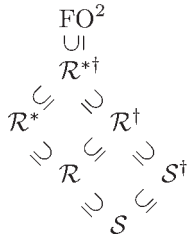
expresses the proposition that, as De Morgan might have put it, he who kills one who kills a man kills one who kills an animal. However, the ability to embed c-terms to arbitrary depth is easily seen not to confer any essential increase in expressive power of sets of formulas, since deeply nested class-terms can always be 'defined out' by introducing new unary predicates. (This is reflected in the fact that the complexity of the satisfiability problem for $\mathcal{R}^*$ is no different from that of McAllester and Givan's fragment.) Interestingly, McAllester and Givan show that the satisfiability of a set Γ of $\mathcal{R}^*$-formulas can be determined in polynomial time if, for each class-term c occurring in Γ, Γ contains either the formula $\exists(c, c)$ or the formula $\forall(c, \bar{c})$. (Such formula sets are said to *determine existentials*.) McAllester and Givan provide a syllogistic-like system for this fragment, which is complete for sets of formulas which determine existentials.

A subfragment of the McAllester and Givan fragment was considered in Moss (to appear). In our terms, the formulas would be of the forms $\forall(c, d)$ and $\exists(c, d)$, where c and d are class-terms of the forms p (an atom), $\exists(c, r)$, or $\forall(d, r)$. Thus, the system lacks negation. The rules are versions of rules we have seen in Section 5: (T), (I), (B), (D), (J), (K), (L), and (II); it also employs a rule allowing for reasoning-by-cases, which is not a syllogistic rule in the sense of this paper. An informal example shows that the system allows nontrivial inferences, and inferences with more than one verb.

$$\frac{\dfrac{\text{All porcupines are mammals}}{\text{All who respect all mammals respect all porcupines}}}{\text{All who dislike all who respect all porcupines dislike all who respect all mammals}}.$$

7.4. Modern revivals of term logic. In recent decades, various logicians have challenged the dominant paradigm of quantification theory by seeking to rehabilitate *term logic*—essentially, the extension of the traditional presentation of the syllogistic to the case of polyadic relations (see, e.g., Sommers, 1982; Englebretsen, 1981). Broadly, the strategy adopted by these term logicians has been to stress the *expressiveness* of the new term-logical syntax, and its ability to represent all that is represented in quantificational logic. (For a very clear account, see Michael Lockwood's Appendix G to Sommers, 1982, pp. 426–456). By adopting the new, term-logical framework—so goes the argument—we obtain a formal system with all the expressive power of first-order logic, but with the added (alleged) advantage of greater fidelity to the structure of natural language. The outlook of the present paper is rather different, however. True, the fragments considered here could be simply and elegantly expressed using the syntax of term logic. The resulting

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$\mathcal{S}$	direct, complete	NLogSpace
$\mathcal{S}^\dagger$	direct, complete	NLogSpace
$\mathcal{R}$	direct, refutation-complete	NLogSpace
$\mathcal{R}^\dagger$	not even indirect	ExpTime
$\mathcal{R}^*$	indirect, complete	Co-NPTime
$\mathcal{R}^{*\dagger}$	not even indirect	ExpTime
FO^2		Co-NExpTime

Fig. 5. The six fragments studied in this paper together with the two-variable fragment FO^2 of first-order logic. The table shows *strongest possible* results on the existence of syllogistic systems, together with tight complexity bounds on the associated validity problem. See Section 8 for an explanation.

formulas would exhibit only cosmetic differences to the syntax introduced in Section 2; and, of course, the associated proof systems would be, modulo these cosmetic differences, unaffected. However, we have been at pains to stress the *inexpressiveness* of the fragments studied above, because with inexpressiveness comes low computational complexity. And from the point of view of computational complexity, the issue of the ruling syntactic régime—predicate logic or term logic—is immaterial.

§8. Conclusions. In this paper, we have investigated the availability of syllogism-like proof systems for various extensions of the traditional syllogistic, with special emphasis on the need for the rule of *reductio ad absurdum*; in addition, we have derived tight complexity bounds for all the logics investigated.

These logics are: (i) $\mathcal{S}$, which corresponds to the traditional syllogistic; (ii) $\mathcal{S}^\dagger$, which extends $\mathcal{S}$ with negated nouns; (iii) $\mathcal{R}$, which extends $\mathcal{S}$ with transitive verbs; (iv) $\mathcal{R}^\dagger$, which extends $\mathcal{S}$ with both these constructions; (v) $\mathcal{R}^*$, which extends $\mathcal{R}$ by allowing subject noun phrases to contain relative clauses; and (vi) $\mathcal{R}^{*\dagger}$, which extends $\mathcal{R}^*$ with negated nouns. The inclusion relations between these systems, together with the familiar *two-variable fragment* of first-order logic (Gr̈adel *et al*. 1997), are shown in Figure 5.

The associated table lists these fragments together with the results we obtained on the existence of syllogistic systems and the complexity of determining validity of sequents. Regarding the existence of syllogistic systems, we showed that: (i) $\mathcal{S}$ and $\mathcal{S}^\dagger$ both have sound and complete direct syllogistic systems (i.e., systems containing no rule of *reductio ad absurdum*); (ii) $\mathcal{R}$ has a sound and *refutation-complete* direct syllogistic system (i.e., one which becomes complete if *reductio ad absurdum* is allowed as a single, final step), but no sound and complete direct syllogistic system; (iii) $\mathcal{R}^*$ has a sound and complete *indirect* syllogistic system (i.e., one allowing unrestricted use of *reductio ad absurdum*), but—unless PTime=NPTime—no sound and refutation-complete direct syllogistic system; (iv) neither $\mathcal{R}^\dagger$ nor $\mathcal{R}^{*\dagger}$ has even an indirect syllogistic system that is sound and complete. Regarding complexity, we showed that: (i) the problem of determining the validity of a sequent in any of $\mathcal{S}$, $\mathcal{S}^\dagger$ or $\mathcal{R}$ is NLogSpace-complete; (ii) the problem of determining the validity of a sequent in $\mathcal{R}^*$ is co-NPTime-complete; (iii) the problem of determining the validity of a sequent in either of $\mathcal{R}^\dagger$ or $\mathcal{R}^{*\dagger}$ is ExpTime-complete.

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