Combinatorics, Probability and Computing (2004) 13, 115–117. © 2004 Cambridge University Press
DOI: 10.1017/S0963548303006060 Printed in the United Kingdom

PROBLEM SECTION

How Sharp is the Concentration of the Chromatic Number?

BÉLA BOLLOBÁS†

As usual, let us write $G_{n,p}$ for a random graph with vertex set $[n] = \{1, 2, \ldots, n\}$, in which the edges are chosen independently, with probability p. Similarly, $G_{n,m}$ is a random graph on $[n]$ with m edges. For $p = m/\binom{n}{2}$, in many respects, the random graphs $G_{n,m}$ and $G_{n,p}$ are practically indistinguishable. (See [4] for an introduction to random graphs.) When in the late 1950s and early 1960s Erdős and Rényi founded the theory of random graphs, one of the most important problems they left open was the determination of the chromatic number of a random graph $G_{n,m}$. The original question concerned the case $m = O(n)$, but when in the 1970s Erdős popularized the problem, he was asking for good estimates on the chromatic number of $G_{n,m}$ for $m \sim n^2/4$ (equivalently, the chromatic number of $G_{n,1/2}$).

It is easy to show that, w.h.p. (with high probability, *i.e.*, with probability tending to 1), for $0 < p < 1$ constant we have

$$\frac{n}{2\log_{1/q} n} \leqslant \chi(G_{n,p}) \leqslant (1 + o(1))\frac{n}{\log_{1/q} n}, \tag{1}$$

where $q = 1 - p$ (see Grimmett and McDiarmid [6] and Bollobás and Erdős [5]).

In 1987 Shamir and Spencer [10] proved that for $0 < p < 1$ constant, the chromatic number of $G_{n,p}$ is sharply concentrated: if $\omega(n) \to \infty$ then, w.h.p., the chromatic number of $G_{n,p}$ takes one of $\omega(n)\sqrt{n}$ consecutive values. They showed also that if the probability p is small then the concentration of the chromatic number is even sharper; in particular, for $\varepsilon > 0$ fixed and $0 < p < n^{-5/6-\varepsilon}$, w.h.p., the chromatic number is confined to at most five consecutive integers. Somewhat strangely, these concentration results did not improve at all the known bounds on the chromatic number $\chi(G_{n,p})$, in particular, the factor 2 in (1).

For p constant, the asymptotic value of $\chi(G_{n,p})$ was eventually determined in [3] – it turned out that, w.h.p., $\chi(G_{n,p})$ is close to the lower bound in (1):

$$\frac{n}{s_0} \leqslant \chi(G_{n,p}) \leqslant \frac{n}{s_0}\left(1 + \frac{3\log\log n}{\log n}\right) \tag{2}$$

† Department of Mathematical Sciences, University of Memphis, Memphis TN 38152, USA *and* Trinity College, Cambridge CB2 1TQ, UK.

https://doi.org/10.1017/S0963548303006060 Published online by Cambridge University Press

holds w.h.p., where

$$s_0 = \lceil 2\log_{1/q} n - \log_{1/q}\log_{1/q} n + \log_{1/q}(e/2) + 1 \rceil.$$

Similar bounds were proved in the case p is not too small, namely when $p \geqslant n^{-2/3}$.

Over the years, better and better concentration results have been proved, some locating the chromatic number, others not. Łuczak [7] proved that a slight modification of the bounds in (2) holds for a much wider range of p. To state these bounds, it is convenient to use the parameter $d = d(n) = p(n-1)$, the expectation of the degree of a vertex. There is a constant d_0 such that if $d > d_0$ and $p = o(1)$ then

$$\frac{d}{2\log d}\left(1 + \frac{\log\log d - 1}{\log d}\right) \leqslant \chi(G_{n,p}) \leqslant \frac{d}{2\log d}\left(1 + \frac{30\log\log d - 1}{\log d}\right). \tag{3}$$

Concerning concentration, McDiarmid [9] improved the $\log\log n$ term in the upper bound in (2) to $O(1)$, and Łuczak [8] proved that if $\varepsilon > 0$ is fixed and $0 < p < n^{-5/6-\varepsilon}$ then, w.h.p., the chromatic number is maximally concentrated: it takes one of at most *two* consecutive integers. Alon and Krivelevich [2] proved that this maximal concentration holds in a considerably wider range, namely whenever $p < n^{-1/2-\varepsilon}$ for some fixed $\varepsilon > 0$. Analogously to the Shamir–Spencer result, this concentration result of Alon and Krivelevich gives no information about the two values $\chi(G_{n,p})$ is likely to take. In a recent *tour de force* paper, Achlioptas and Naor [1] have done exactly that: they have proved that for every $d > 0$, w.h.p. the chromatic number of $G_{n,d/n}$ is either k or $k+1$, where k is the smallest integer such that $2k\log k > d$. It is easily seen that this k is essentially the lower bound in (3):

$$k = \frac{d}{2\log d}\left(1 + \frac{\log\log d + O(1)}{\log d}\right).$$

As this brief and incomplete survey shows, we have come full circle from the days of the original Erdős–Rényi question about the chromatic number of a random graph: we have numerous impressive results stating that the chromatic number is very sharply concentrated about a known value, but we have no results claiming that we do not always have extremely high concentration in all non-trivial cases. (For example, it is trivial that if $q = o(n^{-3/2})$ and $qn^2 \to \infty$ then for $p = 1 - q$ the chromatic number $\chi(G_{n,p})$ is not confined to $n\sqrt{q}$ values w.h.p.) Thus the following problem I discussed frequently with Paul Erdős in the late 1980s seems more and more timely and less and less trivial.

Problem. *Is there a function $\gamma(n) \to \infty$ such that if $I = I(n)$ is any set of at most $\gamma(n)$ integers then for $m = \lfloor n^2/4 \rfloor$ we have*

$$\mathbb{P}(\chi(G_{n,m}) \in I) < \frac{1}{2}?$$

Needless to say, we would like to find as large a function $\gamma(n)$ as possible. It seems likely that $\gamma(n) = \log n$ would do, and it is not impossible that one can take $\gamma(n) = n^{\varepsilon}$ for some sufficiently small ε.

I have no doubt that, eventually, for most functions $m = m(n)$ we shall be able to determine the size of the smallest set containing $\chi(G_{n,m})$ w.h.p., and maybe even the

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distribution of $\chi(G_{n,m})$ will be found, although that is likely to be very difficult. Perhaps the following task, a little less ambitious, could be attempted.

Problem. *Given integer-valued functions $m = m(n)$ and $\gamma(n)$, set*

$$p(m(n), \gamma(n)) = \max_{|I|=\gamma(n)} \mathbb{P}(\chi(G_{n,m}) \in I).$$

Fixing $m = m(n)$, is there a $\gamma_0(n)$ such that if $\gamma/\gamma_0 \to 0$ then $p(m, \gamma) \to 0$ and if $\gamma/\gamma_0 \to \infty$ then $p(m, \gamma) \to 1$? If there is such a γ_0, find one.

To start with, even the weakest results claiming lack of concentration would be of interest. As a warm-up to the problems above, it would be worth proving that for $m = \lfloor n^2/4 \rfloor$ the chromatic number of $G_{n,m}$ is not concentrated w.h.p. on a set of 100 integers.

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