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Transnormal hypersurfaces

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Let M be a C^∞ manifold smoothly embedded as a convex hypersurface of Euclidean n-space $\mathbb{E}^n$. M has *constant width* if the distance between each pair of parallel supporting hyperplanes in $\mathbb{E}^n$ is equal. This is equivalent to the condition that any chord of M normal to M at one end-point be normal to M at both end-points.

In (3) and (4), Robertson generalizes this to consider submanifolds of $\mathbb{E}^n$ of arbitrary codimension. If M^m is a submanifold of $\mathbb{E}^n$ which is complete in its induced metric, then M is a *transnormal submanifold* of $\mathbb{E}^n$ if for each pair $q, r \in M$, the relation $q \in \nu(r)$ implies that $\nu(q) = \nu(r)$, where $\nu(s)$ is the $(n-m)$-plane in $\mathbb{E}^n$ normal to M at s.

Robertson shows that ν is a covering projection onto a submanifold Q of the open Grassmannian of $(n-m)$-planes in $\mathbb{E}^n$ and $\nu^{-1}(q)$ is called the *generating frame* $\Phi(q)$ of M at $q \in Q$. The generating frames at any two points of Q are isometric and the isometry group of each generating frame acts transitively. When M is of codimension one, either each $\Phi(q)$ contains one element and M is homeomorphic to $\mathbb{E}^{n-1}$, or each $\Phi(q)$ contains two elements and M is homeomorphic to a Euclidean cylinder or sphere (3).

Now assume that M is a connected C^∞ submanifold of a connected complete C^∞ Riemannian manifold W^n and assume that M is complete in its induced metric. M is a *transnormal hypersurface* of W if M is of codimension one and if each geodesic of W which cuts M orthogonally at some point cuts M orthogonally at all points of intersection. Examples of such include any orbit of codimension one of the action of a Lie subgroup of the group of isometries of W, or any leaf of codimension one of a foliated manifold with bundle-like metric (2).

Since geodesics of W can have self-intersections, the *generating frame* $\Phi(x)$ at $x \in M$ is defined to be those vectors at x which are orthogonal to M and which are mapped into M by the exponential map of W. In section 1 we generalize the results of (3) by showing that the generating frames at any two points of M are isometric and that the isometry group of each generating frame acts transitively.

In this note, we consider the structure of M when the manifold topology of M is not that induced by its inclusion in W, i.e. M is *not embedded* in W. An example of this is provided by a leaf of an irrational flow on a flat torus T^2. The main result, which is proved in section 2, is

THEOREM. *If M is a non-embedded transnormal hypersurface of W, then M is dense in W and M is a leaf of a foliation of W. In fact, W is a foliated manifold with a bundle-like metric all of whose leaves are dense.*

1. In this section, we generalize some results of (3) concerning the generating frame and the focal set of a transnormal hypersurface.

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Let M be a transnormal hypersurface of W and let $p\colon NM \to M$ be the normal bundle of M in W. The restriction of the exponential map of W to NM will be written $e\colon NM \to W$.

Since M has codimension one, e is transversal to M so $e^{-1}(M)$ is the underlying set of a submanifold A of NM of codimension one. The Gauss Lemma shows that any component A_0 of A remains a constant distance from the zero cross-section of NM, i.e. A_0 is an open submanifold of $N_t M = \{\xi \in NM : \|\xi\| = t\}$ for some $t \in \mathbb{R}$. In fact, since M is complete, it is easy to show that A_0 must be equal to a component of $N_t M$. Thus $\Phi(x) = p^{-1}(x) \cap A$ is independent of $x \in M$ upto isometry. An isometric copy of $\Phi(x)$ in $\mathbb{E}^1$ is called the *generating frame* $\Phi(M)$ of M.

LEMMA 1. *$\Phi(M)$ admits a transitive group of isometries.*

Proof. Let $x \in M$ and let $\xi \in p^{-1}(x) \cap A$. If $y = e(\xi)$, then x may be joined to y by a smooth path $\lambda\colon [0, 1] \to M$. For each $\zeta \in p^{-1}(x)$, let λ_ζ be the unique path in $N_{\|\zeta\|} M$ with $\lambda_\zeta(0) = \zeta$ and $p \circ \lambda_\zeta = \lambda$. The map $\lambda^*\colon p^{-1}(x) \to p^{-1}(y)$ given by $\lambda^*(\zeta) = \lambda_\zeta(1)$ is a linear isometry, as is the map $\eta\colon p^{-1}(x) \to p^{-1}(y)$ which is obtained by parallel translation from x to y along the geodesic of W with initial vector ξ. Thus $\lambda^* = \pm\eta$. If $\lambda^* = \eta$ then the map $\zeta \mapsto \zeta + \xi$ is an isometry of $p^{-1}(x)$ which maps $\Phi(x)$ onto $\Phi(x)$ and maps the zero vector at x to ξ. If $\lambda^* = -\eta$ then the map $\zeta \mapsto \xi - \zeta$ has these properties. The proof now follows.

If $\xi \in NM$, we write $T_\xi e$ for the differential of e at ξ. $T_\xi e$ is a linear map from $T_\xi NM$, the tangent space of NM at ξ, to $T_{e(\xi)} W$. We say that ξ is a *critical point* of e if the rank of $T_\xi e$ is less than the dimension of W. The set C of critical points of e is the *focal locus* of M. We wish to investigate the structure of C and for this purpose we introduce some elementary variation theory. For further details the reader is referred to (1).

Let $x \in M$ and let τ be a geodesic of W intersecting M orthogonally at x (τ is unique upto parameterization). An *(M, x)-Jacobi field* is a vector field along τ which is orthogonal to τ and which is associated with a variation through geodesics intersecting M orthogonally. The set of such vector fields is closed under addition and scalar multiplication, and forms a vector space of dimension $n-1$, where n is the dimension of W. For each $\xi \in T_x M$ there is a unique (M, x)-Jacobi field containing ξ.

If $\zeta \in p^{-1}(x) \cap C$, there is a non-trivial (M, x)-Jacobi field which vanishes at $e(\zeta)$. It is apparent that if $y \in \tau \cap M$, then each (M, x)-Jacobi field is also an (M, y)-Jacobi field. Hence we have

LEMMA 2. *$A \cap C = \phi$, i.e. M does not intersect the image of its focal locus.*

LEMMA 3. *Let $\xi \in C$ and let $\eta \in NM$ be such that $e(\xi) = e(\eta)$. If η is tangential to the geodesic of W with initial vector ξ, then $\eta \in C$.*

2. We study the structure of a *non-embedded* transnormal hypersurface M^{n-1} of W^n. We have already seen that a leaf of an irrational flow on a flat torus is an example of such a submanifold, as indeed is any dense leaf of codimension one of a foliated manifold with bundle-like metric. We show that all non-embedded transnormal hypersurfaces are of this type.

LEMMA 4. *A is not embedded in NM.*

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Proof. We assume A is embedded in NM and show that M is embedded in W. For convenience, we identify M with its image under the zero cross-section of NM.

Let V be an open subset of M. Then there is an open subset U of NM such that $e|U$ is a local diffeomorphism and $U \cap A = V$. The image of U under e gives an open subset of W intersecting M in V.

THEOREM 1. *A non-embedded transnormal hypersurface is dense.*

Proof. Since A is not embedded in NM, $\Phi(M)$ has a limit point. Hence, by Lemma 1, $\Phi(M)$ is a dense subset of $\mathbb{E}^1$. This implies that A is dense in NM. However, the image of e is dense in W so that M is also dense in W.

THEOREM 2. *A non-embedded transnormal hypersurface has no focal points.*

Proof. Assume that C is non-empty. Since A is dense in NM, Lemma 3 shows that C contains vectors of arbitrarily small length. However, by Lemma 2, each component of C is equal to a component of $N_t M$ for some $t \in \mathbb{R}$. This is a contradiction of the fact that e is a local diffeomorphism on a neighbourhood of the zero cross-section of NM.

By combining Rauch's Comparison Theorem for submanifolds (6) with the above, we may obtain restrictions on the sectional curvature K of W. For instance, if K is bounded below by a positive constant, then W does not admit a non-embedded transnormal hypersurface; while if K is everywhere non-negative, any such submanifold must be totally geodesic.

Of more importance to us, however, is the use of the Comparison Theorem to prove the following corollary to Theorem 2. If S is the second fundamental form of M, we regard S as a mapping which assigns to each $x \in M$ a symmetric bilinear map

$$S_x : T_x M \times T_x M \to \mathbb{R}.$$

COROLLARY 1. *If V is a bounded subset of W, then there exists $C > 0$ such that all the eigenvalues of S_x lie in the interval $]-C, C[$ for all $x \in V \cap M$.*

Proof. The result of Cor. 4·2 (*b*) of (6) may be improved by use of the Morse Index Theorem (1) to state (in the teminology of (6)) that if *any* eigenvalue of $S_{\sigma_*(0)}$ is $\leqslant \delta$ and the sectional curvatures of all 2-planes along σ containing σ_* are all $\geqslant c$, then there is a focal point of K on $\sigma|[0, t_0]$.

Let $x \in W$, $r > 0$ be such that V is contained in the closed ball $B(x; r)$ consisting of all points of W distant not more than r from x. Let t be a lower bound for the sectional curvatures of all 2-planes of W with base point in $B(x; 2r)$. By comparing M in W with a submanifold of a space of constant curvature t which has a focal point within distance r, we see the existence of the required C.

If τ is a geodesic of W intersecting M orthogonally, then $\tau \cap M$ is dense in τ. Using Theorem 2, one may show that any geodesic of W intersecting M orthogonally is either disjoint from τ or a reparametrisation of τ. We now have a 1-distribution on the image of e whose value at any point is the tangent space to a geodesic of W through that point which intersects M orthogonally. The orthogonal complementary $(n-1)$-distribution Ψ has M as a dense integral manifold, and hence Ψ is integrable. We show

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below that e is surjective, so that Ψ is an integrable distribution on W. The foliation obtained by integrating Ψ is easily seen to give a bundle-like structure to W.

PROPOSITION 1. *e: $NM \to W$ is surjective.*

Proof. Since M is dense in W, it is enough to show that the image of e contains $\bar{M}$, the closure of M in W.

For any two points $p, q \in W$, let $d(p,q)$ be their distance apart in W. Let $y \in \bar{M} \backslash M$ and for each $x \in W$ let B_x be the closed ball in W with centre y and radius $d(x,y)$. G_x will be the boundary of B_x. For x near enough to y, G_x is a submanifold of W and we write A_x for its second fundamental form at x with respect to the inward unit normal.

Let $x_0 \in W$ and let $C > 0$ be such that all the eigenvalues of the second fundamental form S of M lie between $-C$ and C on $M \cap B_{x_0}$. By choosing x_0 nearer y if necessary, we may assume that all the eigenvalues of A_z are bounded below by $2C$ for each $z \in B_{x_0}$. Define a smooth map h: $M \cap B_{x_0} \to \mathbb{R}$ by $h(q) = d(x_0, y) - d(q, y)$ and let λ be a maximal integral curve of the gradient vector field of h. If we assume that $\{0\}$ is in the domain of λ, then so is $[0, \infty[$. For $t \in [0, \infty[$, $\theta(t)$ will be the angle between the normal to M at $\lambda(t)$ and the normal to $G_{\lambda(t)}$ at $\lambda(t)$. If λ' is the tangent vector field to λ, then $\|\lambda'\| = \sin\theta$ and so θ must take arbitrarily small values on $]0, \infty[$. We write $\dot{\theta}(t)$ for $d\theta/ds$ where $s(t)$ is the arc length from $\lambda(0)$ to $\lambda(t)$. Then $\dot{\theta}\|\lambda'\| = d\theta/dt$. We will show that for small enough θ, $\dot{\theta} < -\frac{1}{2}C$. This will prove the proposition, for if y is not in the image of e, it is clear that $\lambda|[0, \infty[$ must have infinite length. However, this is impossible if $\dot{\theta} < -\frac{1}{2}C$.

Let $t_0 \in]0, \infty[$ and assume that $\theta(t_0) \neq \pi/2$. We work in the 'geodesic polar' co-ordinates defined below. In the following we let α range from 1 to $n-1$, and $i, j, k, \ldots$ from 1 to n. Let $x^1, \ldots, x^{n-1}$ be normal geodesic co-ordinates valid on an open neighbourhood U of $\lambda(t_0)$ in $G_{\lambda(t_0)}$, and let $K(U)$ be the 'solid cone' of points of B_{x_0} which lie on geodesics of W through y and U. Define coordinates $y^1, \ldots, y^n$ on $K(U) \backslash y$ by setting $y^\alpha(q) = x^\alpha(p(q))$ where $p(q)$ is the point of intersection with U of the geodesic through y and q, and set $y^n(q) = h(q)$. We define co-ordinates $z^1, \ldots, z^{n-1}$ on a neighbourhood of $\lambda(t_0)$ in M by setting $z^\alpha = y^\alpha | M$. We may assume, by re-defining $x^1, \ldots, x^{n-1}$ if necessary, that $\partial/\partial z^1|_{\lambda(t_0)}$ is a positive multiple of $\lambda'(t_0)$.

Let $\{\Gamma^i_{jk}\}$ be the Christoffel symbols with respect to $(y^1, \ldots, y^n)$ of the Riemannian connexion ∇ on W. Since $x^\alpha = y^\alpha | U$ for all α, we have that

$$A_{\lambda(t_0)}\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}\right) = \Gamma^n_{11}(\lambda(t_0)) \geqslant 2C. \tag{I}$$

To calculate $S(\partial/\partial z^1, \partial/\partial z^1)$ at $\lambda(t_0)$ we need to know the Γ^i_{jk}'s and we also need the expression for $\partial/\partial z^1$ in terms of the $\partial/\partial y^i$'s. From choice of coordinates, we see that $\Gamma^\alpha_{11}(\lambda(t_0)) = 0$ for all α, and also $\Gamma^i_{nn} = 0$ for all i. Each z^α is a restriction of y^α, so that $\partial/\partial z^\alpha$ and $\partial/\partial y^\alpha$ differ only by a vector along $\partial/\partial y^n$. Thus

$$\frac{\partial}{\partial z^\alpha} = \frac{\partial}{\partial y^\alpha} + \tan\theta_\alpha \left\| \frac{\partial}{\partial y^\alpha} \right\| \frac{\partial}{\partial y^n},$$

where θ_α is the angle between $\partial/\partial z^\alpha$ and $\partial/\partial y^\alpha$.

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This shows that

$$S\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^1}\right) = \text{length of normal component of} \left(\Gamma^n_{11} + \frac{\partial^2 y^n}{\partial z^1 \partial z^1}\right) \frac{\partial}{\partial y^n}$$
$$= \left(\Gamma^n_{11} + \frac{\partial^2 y^n}{\partial z^1 \partial z^1}\right) \cos\theta \quad \text{at} \quad \lambda(t_0). \tag{II}$$

However $S(\partial/\partial z^1, \partial/\partial z^1) \leqslant C \sec^2\theta$ at $\lambda(t_0)$. Thus from (I) and (II) we have

$$\frac{\partial^2 y^n}{\partial z^1 \partial z^1} \leqslant C(1 - 2\cos^3\theta)\sec^3\theta \quad \text{at} \quad \lambda(t_0). \tag{III}$$

Now

$$\frac{\partial y^n}{\partial z^1} = \tan\theta_1 \left\| \frac{\partial}{\partial y^1} \right\| \quad \text{so that at} \quad \lambda(t_0),$$
$$\frac{\partial^2 y^n}{\partial z^1 \partial z^1} = \sec^2\theta_1 \frac{\partial \theta_1}{\partial z^1} + \tan\theta_1 \left\langle \nabla_{\partial/\partial z^1} \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^1} \right\rangle.$$

However

$$\theta_1(\lambda(t_0)) = \theta(t_0) \quad \text{and} \quad \left.\frac{\partial \theta_1}{\partial z^1}\right|_{\lambda(t_0)} = \dot{\theta}\sec\theta \quad \text{at} \quad t_0,$$

so that

$$\frac{\partial^2 y^n}{\partial z^1 \partial z^1} = \dot{\theta}\sec^3\theta + \tan\theta \left\langle \nabla_{\partial/\partial z^1} \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^1} \right\rangle \quad \text{at} \quad \lambda(t_0).$$

However, at $\lambda(t_0)$,

$$\frac{\partial}{\partial z^1} = \frac{\partial}{\partial y^1} + \tan\theta \frac{\partial}{\partial y^n} \quad \text{and} \quad \nabla_{\partial/\partial y^1} \frac{\partial}{\partial y^1}$$

is along $\partial/\partial y^n$. Hence

$$\frac{\partial^2 y^n}{\partial z^1 \partial z^1} = \dot{\theta}\sec^3\theta + \tan^2\theta \left\langle \nabla_{\partial/\partial y^n} \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^1} \right\rangle \quad \text{at} \quad \lambda(t_0). \tag{IV}$$

But $\langle \nabla_{\partial/\partial y^n} \partial/\partial y^1, \partial/\partial y^1 \rangle$ is bounded on B_{x_0} because the restriction of $\partial/\partial y^1$ to a geodesic through y is a Jacobi field. Thus by comparing (III) and (IV) and assuming that $\theta(t_0)$ is sufficiently small we obtain the required result.

Remark. Note that the proof of Proposition 1 does not make essential use of the transnormality hypothesis. In fact, the above shows that if M is a complete hypersurface of W whose second fundamental form satisfies the conclusion of Corollary 1, then $\bar{M}$ is contained in the image of e. The proof can be modified to show that if M is a complete submanifold of W of arbitrary codimension such that the second fundamental form of M in any normal direction satisfies the conclusion of Corollary 1, then $\bar{M}$ is contained in the image of e.

From Proposition 1 we see that if M is a non-embedded transnormal hypersurface of W then M is a dense leaf of a foliation $\mathscr{F}$ of W which gives a bundle-like structure to the metric of W. To complete the proof of the theorem stated in the introduction we must show that each leaf of $\mathscr{F}$ is dense in W. However, this follows from the fact that the distance apart of nearby leaves of $\mathscr{F}$ measured along orthogonal geodesics is locally constant.

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In conclusion, we remark that from Theorem 1·1 of (5) it follows that if W admits a non-embedded transnormal hypersurface M, then $\pi_1(W)$ is infinite. If $\mathscr{F}$ admits a nowhere-vanishing normal vector field then W is covered by $\mathbb{R} \times M$, and otherwise W is covered by $\mathbb{R} \times \tilde{M}$ where $\tilde{M}$ is some double cover of M.

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