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GORENSTEIN SILTING COMPLEXES

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Abstract. We introduce and study the notion of Gorenstein silting complexes, which is a generalization of Gorenstein tilting modules in Gorenstein-derived categories. We give the equivalent characterization of Gorenstein silting complexes. We give a sufficient condition for a partial Gorenstein silting complex to have a complement.

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1. Introduction. The tilting theory is well known and plays an important role in the representation theory of Artin algebra. The silting theory seems to be the relative version of tilting theory in the level of derived categories (while the tilting complexes play the role of progenerators). Silting complexes were first introduced by Keller and Vossieck [**16**] to study t-structures in the bounded derived category of representations of Dynkin quivers. These objects were recently shown to have various nice properties [**17**, **14**]. The results in [**21**] showed that silting complexes (i.e., semi-tilting complexes in [**21**]) have similar properties as that tilting modules have in the module categories.

Relative homological algebra originates in the work of Eilenberg and Moore [**11**] in the 1960s. In the 1990s, Auslander and Solberg [**5**, **6**, **7**] initiated a systematic study of relative homological algebra in representation theory of algebras, and in particular they introduced in [**6**] the notion of relative (co)tilting module and generalized many results of the standard (co)tilting theory to the relative setting. Gorenstein homological algebra has been studied by several authors, for example, [**19**, **22**]. Especially, in [**22**], the authors gave the notions of Gorenstein tilting and cotilting modules and they also showed their Bazzoni' characterizations. Gorenstein homological algebra has been developed to an advanced level. Gao and Zhang in [**13**] introduced the concept of the Gorenstein-derived category of an abelian category $\mathcal{A}$, $\mathcal{D}_{gp}(\mathcal{A})$ as the Verdier quotient of the homotopy category $\mathcal{K}(\mathcal{A})$ with respect to the thick triangulated subcategory $\mathcal{K}_{gpac}(\mathcal{A})$ of $\mathcal{GP}$-acyclic complexes. Gorenstein-derived categories make Gorenstein quasi-isomorphisms become isomorphisms which have some advantages in the relative category. So, it is possible for us to generalize Gorenstein (co)tilting modules in Gorenstein-derived categories.

This paper is organized as follows: in Section 2, we give some useful facts in general triangulated categories. In Section 3, we study the notion of Gorenstein silting complexes as a generalization of Gorenstein tilting modules and show its characterization in bounded Gorenstein-derived categories. In Section 4, we give a sufficient condition for a partial Gorenstein silting complex to have a complement.

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2. Preliminaries. Let $\mathcal{C}$ be an idempotent complete triangulated category with [1] the shift functor. Assume that $\mathcal{B}$ is a full subcategory of $\mathcal{C}$. Recall that $\mathcal{B}$ is closed under extension if for any triangle $X \to Y \to Z \to$ in $\mathcal{C}$ with $X, Z \in \mathcal{B}$, we have $Y \in \mathcal{B}$. The subcategory $\mathcal{B}$ is cosuspended (resp., suspended) if it is closed under extension and under the functor $[-1]$ (resp., $[1]$). It is easy to prove that $\mathcal{B}$ is cosuspended (resp., suspended) if and only if for any triangle $X \to Y \to Z \to$ (resp., $Z \to Y \to X \to$) in $\mathcal{C}$ with $Z \in \mathcal{B}$, one has that "$X \in \mathcal{B} \Leftrightarrow Y \in \mathcal{B}$."

We say that an object $M \in \mathcal{C}$ has a $\mathcal{B}$-resolution (resp., $\mathcal{B}$-coresolution) with the length at most m ($m \geqslant 0$), denoted by $\mathcal{B}\text{-res.dim}(M) \leqslant m$ (resp., $\mathcal{B}\text{-cores.dim}(M) \leqslant m$), if there are triangles $M_{i+1} \to X_i \to M_i \to$ (resp., $M_i \to X_i \to M_{i+1} \to$) with $0 \leqslant i \leqslant m$ such that $M_0 = M$, $M_{m+1} = 0$ and each $X_i \in \mathcal{B}$. One may compare such notions with the usual finite resolutions and coresolutions, respectively, in the module category.

Associated with a subcategory $\mathcal{B}$, we have the following notations, where $n \geqslant 0$ and m is an integer.

$$(\widehat{\mathcal{B}})_n = \{L \in \mathcal{C} \mid \mathcal{B}\text{-res.dim}(L) \leqslant n\}.$$

$$(\breve{\mathcal{B}})_n = \{L \in \mathcal{C} \mid \mathcal{B}\text{-cores.dim}(L) \leqslant n\}.$$

$$\widehat{\mathcal{B}} = \{L \in \mathcal{C} \mid L \in (\widehat{\mathcal{B}})_n \text{for some } n\}.$$

$$\breve{\mathcal{B}} = \{L \in \mathcal{C} \mid L \in (\breve{\mathcal{B}})_n \text{for some } n\}.$$

$$\mathcal{B}^{\perp_{\neq 0}} = \{N \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(M, N[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i \neq 0\}.$$

$${}^{\perp_{\neq 0}}\mathcal{B} = \{N \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(N, M[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i \neq 0\}.$$

$$\mathcal{B}^{\perp_{>m}} = \{N \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(M, N[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i > m\}.$$

$${}^{\perp_{>m}}\mathcal{B} = \{N \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(N, M[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i > m\}$$

$$\mathcal{B}^{\perp_{\gg 0}} = \{N \in \mathcal{C} \mid N \in \mathcal{B}^{\perp_{>m}} \text{ for some } m\}.$$

$\langle \mathcal{B} \rangle$ denotes the smallest triangulated subcategory containing $\mathcal{B}$.

Note that $\mathcal{B}^{\perp_{>m}}$ (resp., ${}^{\perp_{>m}}\mathcal{B}$) is suspended (resp., cosuspended) and closed under direct summands and that $\mathcal{B}^{\perp_{\gg 0}}$ is a triangulated subcategory of $\mathcal{C}$.

The subcategory $\mathcal{B}$ is said to be semi-self-orthogonal (resp., self-orthogonal) if $\mathcal{B} \subseteq \mathcal{B}^{\perp_{>0}}$ (resp., $\mathcal{B} \subseteq \mathcal{B}^{\perp_{\neq 0}}$). For instance, both subcategories of all projective and all injective A-modules are self-orthogonal in A-modules, for a ring A.

In the following results of this section, we always assume that $\mathcal{B}$ is semi-self-orthogonal and that $\mathcal{B}$ is additively closed (i.e., $\mathcal{B} = \mathrm{add}_{\mathcal{C}}\mathcal{B}$ where $\mathrm{add}_{\mathcal{C}}\mathcal{B}$ denotes the subcategory of all objects in $\mathcal{C}$ which are isomorphic to a direct summand of finite direct sums of copies of objects in $\mathcal{B}$).

Associated with the subcategory $\mathcal{B}$, we also have the following two useful subcategories.

$$\mathcal{X}_{\mathcal{B}} = \{N \in {}^{\perp_{>0}}\mathcal{B} \mid \text{there are triangles } N_i \to B_i \to N_{i+1} \to \text{ such that } N_0 = N, N_i \in {}^{\perp_{>0}}\mathcal{B}, \text{ and } B_i \in \mathcal{B} \text{ for all } i \geqslant 0\}.$$

$${}_{\mathcal{B}}\mathcal{X} = \{N \in \mathcal{B}^{\perp_{>0}} \mid \text{there are triangles } N_{i+1} \to B_i \to N_i \to \text{ such that } N_0 = N, N_i \in \mathcal{B}^{\perp_{>0}}, \text{ and } B_i \in \mathcal{B} \text{ for all } i \geqslant 0\}.$$

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Since $\mathcal{B}$ is closed under finite direct sums and direct summands, we could summarize some results on subcategories associated with $\mathcal{B}$ in [21].

LEMMA 2.1. *Let $\mathcal{B}$ be a semi-self-orthogonal subcategory of a triangulated category $\mathcal{C}$ such that $\mathcal{B}$ is additively closed. Then:*

(1) *The three subcategories $\breve{\mathcal{B}} \subseteq \mathcal{X}_{\mathcal{B}} \subseteq {}^{\perp_{>0}}\mathcal{B}$ are cosuspended and closed under direct summands.*
(2) *The three subcategories $\widehat{\mathcal{B}} \subseteq {}_{\mathcal{B}}\mathcal{X} \subseteq \mathcal{B}^{\perp_{>0}}$ are suspended and closed under direct summands.*
(3) $\mathcal{B} = \breve{\mathcal{B}} \bigcap \mathcal{B}^{\perp_{>0}} = \widehat{\mathcal{B}} \bigcap {}^{\perp_{>0}}\mathcal{B}$.
(4) $(\breve{\mathcal{B}})_n = \mathcal{X}_{\mathcal{B}} \bigcap (\mathcal{X}_{\mathcal{B}})^{\perp_{>n}} = \mathcal{X}_{\mathcal{B}} \bigcap ({}^{\perp_{>0}}\mathcal{B})^{\perp_{>n}}$. *In particular, it is closed under extensions and direct summands.*
(5) $(\widehat{\mathcal{B}})_n = {}_{\mathcal{B}}\mathcal{X} \bigcap {}^{\perp_{>n}}({}_{\mathcal{B}}\mathcal{X}) = {}_{\mathcal{B}}\mathcal{X} \bigcap {}^{\perp_{>n}}(\mathcal{B}^{\perp_{>0}})$. *In particular, it is closed under extensions and direct summands.*
(6) *The following three subcategories coincide with each other.*
  (i) $\langle \mathcal{B} \rangle$*: the smallest triangulated subcategories containing* $\mathcal{B}$.
  (ii) $(\widehat{\mathcal{B}})_- := \{X \in \mathcal{C} |$ there exists some $Y \in \widehat{\mathcal{B}}$ and some $i \leqslant 0$ such that $X = Y[i]\}$.
  (iii) $(\breve{\mathcal{B}})_+ := \{X \in \mathcal{C} |$ there exists some $Y \in \breve{\mathcal{B}}$ and some $i \geqslant 0$ such that $X = Y[i]\}$.
(7) $\widehat{\mathcal{B}} = \mathcal{B}^{\perp_{>0}} \bigcap \langle \mathcal{B} \rangle$.
(8) $\breve{\mathcal{B}} = {}^{\perp_{>0}}\mathcal{B} \bigcap \langle \mathcal{B} \rangle$.

Proof. (1) and (2) follow from [21, Lemma 2.1(3), Corrollary 2.4] and their definitions. (3) follows [21, Lemma 2.1(2)]. (4) and (5) follow from [21, Proposition 2.3]. (6) follows from [21, Proposition 2.5]. (7) and (8) follow from [21, Corrollary 2.6]. □

A ring A is Gorenstein if A is two-sided Noetherian and A has finite injective dimension, both as left and right A-modules. A Gorenstein ring A is l-Gorenstein, if the injective dimension of A as a left A-module is at most l. In this case, the injective dimension of A as a right A-module is at most l [12, Definition 9.1.9]. In this paper, we fix that A is an l-Gorenstein ring. Denote by A-Mod (A-mod) the category of (finitely generated) A-modules, and A-$\mathcal{P}$ (resp., A-proj) the full subcategory of (resp., finitely generated) projective A-modules.

An A-module M is Gorenstein-projective in A-Mod (resp., A-mod), if there is an exact sequence:

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

in A-$\mathcal{P}$ (resp., A-proj), which stays exact after applying $\mathrm{Hom}_A(-, P)$ for each projective A-module P in A-$\mathcal{P}$ (resp., A-proj), such that $M \simeq \mathrm{Im}(P_0 \to P^0)$ [12]. Denote by A-$\mathcal{GP}$ (resp., A-$\mathcal{G}proj$), or simply $\mathcal{GP}$ (resp., $\mathcal{G}proj$), the full subcategory of A-modules consisting of Gorenstein-projective modules in A-Mod (resp., A-mod). One has the dual concept of Gorenstein-injective modules. Denote by $\mathcal{GI}$ (resp., $\mathcal{G}inj$) the full subcategory of A-modules consisting of Gorenstein-injective modules in A-Mod (resp., A-mod).

A complex $C^\bullet$ of (finitely generated) A-modules is $\mathcal{GP}$ (resp., $\mathcal{G}proj$)-acyclic, if $\mathrm{Hom}_A(G, C^\bullet)$ is acyclic for each $G \in \mathcal{GP}$ (resp., $\mathcal{G}proj$). It is also called proper exact. Since $C^\bullet$ is acyclic if and only if $\mathrm{Hom}_A(P, C^\bullet)$ is acyclic for each projective module P, a $\mathcal{GP}$-acyclic complex is acyclic.

An A-module M has a proper Gorenstein-projective resolution if there is a proper exact sequence:

$$\cdots \to G_i \cdots \to G_1 \to G_0 \to M \to 0$$

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with each $G_i \in \mathcal{GP}$. The Gorenstein-projective dimension $\mathcal{GP}$-res.dim$M$ of M is defined to be the smallest integer $s \geqslant 0$ such that there is an exact sequence $0 \to G_s \to \cdots \to G_1 \to G_0 \to M \to 0$ with all $G_i \in \mathcal{GP}$, if it is exists; and $\mathcal{GP}$-res.dim$M = \infty$ if there is no such exact sequence of finite length.

A chain map $f^\bullet : X^\bullet \to Y^\bullet$ is a Gorenstein quasi-isomorphism, if $\mathrm{Hom}_A(G, f^\bullet)$ is a quasi-isomorphism for each $G \in \mathcal{GP}$, that is, there are isomorphisms of abelian groups:

$\mathrm{H}^n\mathrm{Hom}_A(G, f^\bullet) : \mathrm{H}^n\mathrm{Hom}_A(G, X^\bullet) \to \mathrm{H}^n\mathrm{Hom}_A(G, Y^\bullet)$, any integer n and $G \in \mathcal{GP}$.

Since $f^\bullet : X^\bullet \to Y^\bullet$ is a quasi-isomorphism if and only if $\mathrm{Hom}_A(P, f^\bullet)$ is a quasi-isomorphism for each projective module P, a Gorenstein quasi-isomorphism is a quasi-isomorphism.

For $* \in \{\text{blank}, -, b\}$, we write $\mathcal{K}^*(A)$ (resp., $\mathcal{K}^*(A\text{−mod})$) for the (corresponding) homotopy category of A-Mod (resp., A-mod). Denote by $\mathcal{K}^*_{gpac}(A)$ (resp., $\mathcal{K}^*_{gpac}(A\text{−mod})$) the (corresponding) homotopy category of $\mathcal{GP}$ (resp., $\mathcal{G}proj$)-acyclic complexes of A-Mod (resp., A-mod). Then $\mathcal{K}^*_{gpac}(A)$, (resp., $\mathcal{K}^*_{gpac}(A\text{−mod})$) is a thick triangulated subcategory of $\mathcal{K}^*(A)$ (resp., $\mathcal{K}^*(A\text{−mod})$) [13, Lemma 2.3]. Now, by [13, 2.3] , we have the following definitions:

$$\mathcal{D}^*_{gp}(A) := \mathcal{K}^*(A)/\mathcal{K}^*_{gpac}(A),$$

and

$$\mathcal{D}^*_{gp}(A\text{−mod}) := \mathcal{K}^*(A\text{−mod})/\mathcal{K}^*_{gpac}(A\text{−mod}),$$

which are called the (corresponding) Gorenstein-projective-derived categories over A-Mod and A-mod, respectively. We will omit the word "projective."

Define $\mathcal{D}^{\leqslant 0}_{gp} := (\mathcal{GP})^{\perp_{>0}} = \{X \in \mathcal{D}_{gp}(A) |\ \mathrm{Hom}_{\mathrm{D}_{gp}}(G, X^\bullet[i]) = 0,\ \text{for any } G \in \mathcal{GP} \text{ and } i > 0\}$. For any $X \in \mathcal{D}^{\leqslant 0}_{gp}$, it satisfies $\mathrm{H}^i\mathrm{Hom}_A(G, X^\bullet) = \mathrm{Hom}_{\mathcal{K}(A)}(G, X^\bullet[i]) \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(G, X^\bullet[i]) = 0$ by Lemma 3.1, for any $G \in \mathcal{GP}$ and $i > 0$. We can get that $\mathcal{D}^-_{gp}(A) = (\mathcal{GP})^{\perp_{\gg 0}}$. Indeed, $X^\bullet \in \mathcal{GP}^{\perp_{\gg 0}}$ if and only if there is an integer n such that $\mathrm{Hom}_{\mathcal{D}_{gp}}(G, X^\bullet[i]) \simeq \mathrm{Hom}_{\mathcal{K}(A)}(G, X^\bullet[i]) = \mathrm{H}^i\mathrm{Hom}_A(G, X^\bullet) = 0$ for $G \in \mathcal{GP}$ and $i > n$ if and only if $X^\bullet \in \mathcal{D}^-_{gp}(A)$. $\mathcal{GP}$ ($\mathcal{G}proj$) is an additive category and $\mathcal{K}^b(\mathcal{GP})$ ($\mathcal{K}^b(\mathcal{G}proj)$) is a triangulated category [13, Proposition 2.8].

Let $G = \mathrm{Gext}^1(-, -)$(see [22]). Then G is an additive subfunctor of $\mathrm{Ext}^1(-, -)$. A short exact sequence $0 \to M \to N \to L \to 0$ is called G-exact if it is in $\mathrm{Gext}^1(L, M)$. For an A-module T and some $n > 0$, we give the following notations:

$$T^{G\perp_{>0}} = \{M \in A\text{−Mod} |\ \mathrm{Gext}^i(T, M) = 0,\ \text{for all } i > 0\}.$$

$$\mathrm{Pres}^n_G(T) = \{M \in A\text{−Mod} |\ \text{there is a G} - \text{exact sequence } T_n \to T_{n-1} \to \cdots \to T_1 \to M \to 0,\ \text{with each } T_i \in \mathrm{Add}_A T\}.$$

LEMMA 2.2 [22]. *The following statements are equivalent for an exact sequence* $0 \to M \to N \to L \to 0$*:*

(1) *The sequence is G-exact;*
(2) $0 \to \mathrm{Hom}_A(G, M) \to \mathrm{Hom}_A(G, N) \to \mathrm{Hom}_A(G, L) \to 0$ *is exact sequence for all* $G \in \mathcal{GP}$*;*
(3) $0 \to \mathrm{Hom}_A(L, E) \to \mathrm{Hom}_A(N, E) \to \mathrm{Hom}_A(M, E) \to 0$ *is exact sequence for all* $E \in \mathcal{GI}$.

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LEMMA 2.3 [13, Proposition 3.11]. *Assume that the exact sequence* $0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ *is G-exact. Then,* $M \xrightarrow{f} N \xrightarrow{g} L \to$ *is a triangle in* $\mathcal{D}_{gp}(A)$.

LEMMA 2.4 [12, Theorem 11.5.1, Remark 11.5.2]. *If A is an l-Gorenstein ring, then each A-module M admits a proper Gorenstein-projective resolution* $0 \to G_l \to \cdots \to G_0 \to M \to 0$*; and every finitely generated A-module M admits a proper Gorenstein-projective resolution* $0 \to G^l \to \cdots \to G^0 \to M \to 0$ *with all* $G^i \in \mathcal{G}proj$.

3. Gorenstein silting complexes. In this section, we take the general triangulated category $\mathcal{C}$ as the Gorenstein-derived category $\mathcal{D}_{gp}(A)$.

LEMMA 3.1 [13, Proposition 2.8]. *Let* $G^\bullet \in \mathcal{K}^-(\mathcal{GP})$ *and* $Y^\bullet$ *be an arbitrary complex. Then there is an isomorphism of abelian groups* $\mathrm{Hom}_{\mathcal{K}(A)}(G^\bullet, Y^\bullet) \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(G^\bullet, Y^\bullet)$.

LEMMA 3.2 [13, Theorem 3.12]. $\mathrm{Gext}^n_A(M, N) \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(M, N[n])$*, for any* $M, N \in A$-Mod *and any integer n.*

We will give the definition of Gorenstein silting complexes.

DEFINITION 3.3. (1) A complex $T^\bullet$ is said to be partial Gorenstein silting if it satisfies the following two conditions:

(i) $T^\bullet \in \mathcal{K}^b(\mathcal{GP})$,
(ii) $\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \subseteq T^{\bullet\perp_{>0}}$.

(2) A complex $T^\bullet$ is said to be Gorenstein silting if it satisfies (*i*), (*ii*) and

(iii) $\mathcal{K}^b(\mathcal{GP}) = \langle \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \rangle$, that is, $\mathcal{K}^b(\mathcal{GP})$ coincides with the smallest triangulated subcategory containing $\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet$.

Recall an A-module T is n-Gorenstein tilting in [22] provided that

(i) The Gorenstein-projective dimension of T is at most n.
(ii) $Gext^i_A(T, T^{(I)}) = 0$ for any set I and $i > 0$.
(iii) There is a long G-exact sequence $0 \longrightarrow G \longrightarrow T^0 \longrightarrow \cdots \longrightarrow T^r \longrightarrow 0$ with each $T^i \in \mathrm{Add}_A T$ for all $G \in \mathcal{GP}$.

In the following results, we can see that Gorenstein silting complexes are a generalization of Gorenstein tilting modules in Gorenstein-derived categories.

THEOREM 3.4. *Let* $T^\bullet \in \mathcal{D}_{gp}(A)$ *satisfy* $\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \subseteq \mathcal{D}^{\leqslant 0}_{gp}$. *Then* $T^\bullet$ *is a Gorenstein silting complex if and only if it satisfies the following three conditions:*

(i) $T^\bullet \in \widehat{\mathcal{GP}}$,
(ii) $\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \subseteq T^{\bullet\perp_{>0}}$, *and*
(iii) $\mathcal{GP} \subseteq \widetilde{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet}$.

Proof. The only-if part. Since $\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \subseteq \mathcal{D}^{\leqslant 0}_{gp} = (\mathcal{GP})^{\perp_{>0}}$, we have that $\mathcal{GP} \subseteq {}^{\perp_{>0}}(\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet)$. By the Definition 3.3, $\mathcal{K}^b(\mathcal{GP}) = \langle \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \rangle$, then $\mathcal{GP} \subseteq \langle \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \rangle$. Hence, $\mathcal{GP} \subseteq \widetilde{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet}$ by Lemma 2.1 (8). $T^\bullet \in \mathcal{K}^b(\mathcal{GP}) = \langle \mathcal{GP} \rangle$ by assumptions. Then we have $T^\bullet \in \langle \mathcal{GP} \rangle \bigcap (\mathcal{GP})^{\perp_{>0}} = \widehat{\mathcal{GP}}$ by Lemma 2.1.

The if part. By (*i*), $T \in \widehat{\mathcal{GP}}$. Hence, $\mathrm{Add}_{\mathcal{D}_{gp}} T \subseteq \widehat{\mathcal{GP}}$. By Lemma 2.1, $\widehat{\mathcal{GP}} \subseteq \langle \mathcal{GP} \rangle = \mathcal{K}^b(\mathcal{GP})$. By (*iii*) and Lemma 2.1, we have $\mathcal{GP} \subseteq \langle \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \rangle$. Hence, we have $\mathcal{K}^b(\mathcal{GP}) = \langle \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \rangle$. □

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COROLLARY 3.5. *Assume that $T \in A$-Mod. Then T is a Gorenstein tilting module if and only if T is (Gorenstein quasi-isomorphic to) a Gorenstein silting complex.*

Proof. Since T is a Gorenstein tilting module, there is a G-exact sequence $0 \to G_n \xrightarrow{f_n} G_{n-1} \xrightarrow{f_{n-1}} \cdots \to G_0 \xrightarrow{f_0} T \to 0$. By Lemma 2.3, we have a series of triangles in $\mathcal{D}_{gp}(A)$: $T_{i+1} \to G_i \to T_i \to$, where $0 \leqslant i \leqslant n$, such that $T_{n+1} = 0$, $T_0 = T$ and $T_j = \mathrm{Coker} f_{j+1}$ for $0 \leqslant j \leqslant n-1$. So, we have $T \in \widehat{\mathcal{GP}}$. By Lemma 3.2, we have the following equivalence:

$$0 = \mathrm{Gext}^i_A(T, T^{(I)}) \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(T, T^{(I)}[i]),$$

for any set I and $i > 0$. So, for any $X \in \mathrm{Add}_{\mathcal{D}_{gp}} T$, there exists Y such that $X \oplus Y = T^{(J)}$ for some set J. It is easy to see that $\mathrm{Hom}_{\mathcal{D}_{gp}}(T, X[i]) = 0$. For any $G \in \mathcal{GP}$, there is a G-exact sequence $0 \to G \to T_0 \to T_1 \to \cdots \to T_r \to 0$. Similarly, we have a series of triangles in $\mathcal{D}_{gp}(A)$: $G_i \to T_i \to G_{i+1} \to$, where $G_0 = G$ and $G_{r+1} = 0$ for $0 \leqslant i \leqslant r$. Thus, $\mathcal{GP} \subseteq \widehat{\mathrm{Add}_{\mathcal{D}_{gp}} T}$.

Now we show the sufficiency. Let $G^\bullet \in \mathcal{K}^b(\mathcal{GP})$ be a Gorenstein silting complex such that $G^\bullet$ is Gorenstein quasi-isomorphic to T. By Lemma 3.2, we have the following equivalences:

$$0 = \mathrm{Hom}_{\mathcal{D}_{gp}}(G^\bullet, G^{\bullet(I)}[i]) \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(T, T^{(I)}[i]) \simeq \mathrm{Gext}^i_A(T, T^{(I)}),$$

for any set I and $i > 0$. Since $T \in \widehat{\mathcal{GP}}$, there is a series of triangles in $\mathcal{D}_{gp}(A)$: $T^\bullet_{i+1} \to G_i \xrightarrow{f_i} T^\bullet_i \to$ with $G_i \in \mathcal{GP}$, $T^\bullet_0 = T$ and $T^\bullet_{t+1} = 0$, where $0 \leqslant i \leqslant t$. Since $T^\bullet_t \simeq G_t \in \mathcal{GP}$ and $\mathcal{GP} \subseteq \mathcal{GP}^{\perp_{>0}}$, we get $T^\bullet_i \in \mathcal{GP}^{\perp_{>0}}$ for each i. Applying $\mathrm{Hom}_{\mathcal{D}_{gp}}(G, -)$ to the triangle $T^\bullet_1 \to G_0 \xrightarrow{f_0} T \to$ for any $G \in \mathcal{GP}$, we obtain that $\mathrm{Hom}_A(G, G_0) \to \mathrm{Hom}_A(G, T)$ is surjective by Lemma 3.1. Since all projective modules are Gorenstein-projective, we have that f_0 is surjective. And then we have a G-exact sequence $0 \to \mathrm{Ker} f_0 \to G_0 \xrightarrow{f_0} T \to 0$. Hence, we have a triangle in $\mathcal{D}_{gp}(A)$: $\mathrm{Ker} f_0 \to G_0 \xrightarrow{f_0} T \to$. So, $T^\bullet_1 \simeq \mathrm{Ker} f_0$ in $\mathcal{D}_{gp}(A)$. Thus, $T^\bullet_1$ is Gorenstein quasi-isomorphic to $\mathrm{Ker} f_0 \in A$-Mod in $\mathcal{K}(A)$. Repeating the process, we have a G-exact sequence $0 \to G_t \to G_{t-1} \to \cdots \to G_0 \to T \to 0$ in A-Mod.

For any $G \in \mathcal{GP}$, we have a series of triangles in $\mathcal{D}_{gp}(A)$: $G^\bullet_i \xrightarrow{g_i} T_i \to G^\bullet_{i+1} \to$ with $G^\bullet_0 = G$, $T_i \in \mathrm{Add}_{\mathcal{D}_{gp}} T$ and $G^\bullet_{s+1} = 0$ for $0 \leqslant i \leqslant s$. By Lemma 3.2, we have $0 = \mathrm{Gext}^j_A(T, I) \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(T, I[j])$ for any $I \in \mathcal{GI}$ and $j > 0$. It follows that $G^\bullet_i \in {}^{\perp_{>0}}\mathcal{GI}$ for $0 \leqslant i \leqslant s$. Applying $\mathrm{Hom}_{\mathcal{D}_{gp}}(-, I)$ to the triangle $G \xrightarrow{g_0} T_0 \to G^\bullet_1 \to$, we obtain that $\mathrm{Hom}_{\mathcal{D}_{gp}}(T_0, I) \to \mathrm{Hom}_{\mathcal{D}_{gp}}(G, I) \to 0$ is surjective. Then, $\mathrm{Hom}_A(T_0, I) \to \mathrm{Hom}_A(G, I) \to 0$ is surjective by [**13**, Proposition 2.9]. Since an injective cogenerator Q in A-Mod is a Gorenstein-injective module, we obtain $\mathrm{Hom}_A(\mathrm{Ker} g_0, Q) = 0$. Then, $\mathrm{Ker} g_0 = 0$. It follows that g_0 is injective. And then we have a G-exact sequence $0 \to G \xrightarrow{g_0} T_0 \to \mathrm{Coker} g_0 \to 0$. Hence, there is a triangle in $\mathcal{D}_{gp}(A)$: $G \xrightarrow{g_0} T_0 \to \mathrm{Coker} g_0 \to$. So, $G^\bullet_1 \simeq \mathrm{Coker} g_0$ in $\mathcal{D}_{gp}(A)$. Thus, $G^\bullet_1$ is Gorenstein quasi-isomorphic to $\mathrm{Coker} g_0 \in A$-Mod in $\mathcal{K}(A)$. Repeating the process, we have a G-exact sequence for any $G \in \mathcal{GP}$: $0 \to G \to T_0 \to T_1 \to \cdots \to T_s \to 0$. □

PROPOSITION 3.6. *Suppose that $T^\bullet$ is Gorenstein silting. If $N^\bullet \bigoplus T^\bullet$ is also Gorenstein silting for some $N^\bullet$, then $N^\bullet \in \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet$.*

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Proof. Since $T^\bullet \bigoplus N^\bullet$ is a Gorenstein silting complex, we have that $\langle \mathrm{Add}_{\mathcal{D}_{gp}}(T^\bullet \bigoplus N^\bullet)\rangle = \mathcal{K}^b(\mathcal{GP}) = \langle \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet\rangle$. It follows that $N^\bullet \in T^{\bullet\perp_{>0}}$ and $N^\bullet \in {}^{\perp_{>0}}(\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet)$. Then, we have

$$N^\bullet \in T^{\bullet\perp_{>0}} \bigcap \langle \mathrm{Add}_{\mathcal{D}_{gp}}(T^\bullet \bigoplus N^\bullet)\rangle = T^{\bullet\perp_{>0}} \bigcap \langle \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet\rangle = \widehat{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet}.$$

Hence, $N^\bullet \in \widehat{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet} \bigcap {}^{\perp_{>0}}(\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet) = \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet$ by Lemma 2.1. □

PROPOSITION 3.7. *Assume that $T^\bullet$ with $\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \subseteq \mathcal{D}_{gp}^{\leqslant 0}$ is Gorenstein silting. Then, $T^\bullet \in (\widehat{\mathcal{GP}})_n$ if and only if $\mathcal{GP} \subseteq (\widehat{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet})_n$.*

Proof. Take $G \in \mathcal{GP}$. By Theorem 3.4, we have $G \in (\widehat{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet})_t$ for some $t \geqslant 0$. Assume that $t > n$. We have a series of triangles in $\mathcal{D}_{gp}(A)$: $G_i^\bullet \to T_i^\bullet \to G_{i+1}^\bullet \to$ with $G_0^\bullet = G$, $T_i^\bullet \in \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet$ and $G_{t+1}^\bullet = 0$ for $0 \leqslant i \leqslant t$. It follows that $G_t^\bullet \simeq T_t^\bullet \in \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet$. Applying $\mathrm{Hom}_{\mathcal{D}_{gp}}(G_t^\bullet, -)$ to these triangles, we have

$$\mathrm{Hom}_{\mathcal{D}_{gp}}(G_t^\bullet, G_{t-1}^\bullet[1]) \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(G_t^\bullet, G_{t-2}^\bullet[2]) \simeq \cdots \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(G_t^\bullet, G_0^\bullet[t]).$$

Since $T^\bullet \in (\widehat{\mathcal{GP}})_n$, we have $\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \subseteq {}^{\perp_{>n}}(\mathcal{GP}^{\perp_{>0}})$ by Lemma 2.1. It is followed that $\mathrm{Hom}_{\mathcal{D}_{gp}}(G_t^\bullet, G_{t-1}^\bullet[1]) = 0$. Hence, $G_{t-1}^\bullet \to T_{t-1}^\bullet \to G_t^\bullet \to$ splits. Thus, we have $G \in (\widehat{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet})_{t-1}$. Repeating the process, we obtain $\mathcal{GP} \subseteq (\widehat{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet})_n$.

Conversely, it is dual to the necessity. □

LEMMA 3.8. *Assume that $T^\bullet \in \mathcal{D}_{gp}(A)$ with $\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \subseteq T^{\bullet\perp_{>0}}$. Then, $T^{\bullet\perp_{>0}} = {}_{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet}\mathcal{X}$.*

Proof. By the definition of ${}_{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet}\mathcal{X}$, we know that ${}_{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet}\mathcal{X} \subseteq T^{\bullet\perp_{>0}}$. Take any $M^\bullet \in T^{\bullet\perp_{>0}}$. Let $X = \mathrm{Hom}_{\mathcal{D}_{gp}}(T^\bullet, M^\bullet)$. Then, we have a homomorphism $f^\bullet : T^{\bullet(X)} \to M^\bullet$ such that $\mathrm{Hom}_{\mathcal{D}_{gp}}(T^\bullet, f^\bullet)$ is surjective. Thus, we have a triangle $M_1^\bullet \to T^{\bullet(X)} \to M^\bullet \to$. It is easy to obtain $M_1^\bullet \in T^{\bullet\perp_{>0}}$. Repeating the process to $M_1^\bullet$, and so on, we have triangles $M_{i+1}^\bullet \to T_i^\bullet \to M_i^\bullet \to$ for all $i \geqslant 1$, with each $T_i^\bullet \in \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet$ and $M_i^\bullet \in T^{\bullet\perp_{>0}}$. So, we have $M^\bullet \in {}_{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet}\mathcal{X}$. □

We give a sufficient condition for partial Gorenstein silting complexes to be Gorenstein silting complexes.

PROPOSITION 3.9. *Assume that $T^\bullet$ with $\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \subseteq \mathcal{D}_{gp}^{\leqslant 0}$ is partial Gorenstein silting. Then $T^\bullet$ is Gorenstein silting if and only if $T^{\bullet\perp_{>0}} \subseteq \mathcal{D}_{gp}^{\leqslant 0}$.*

Proof. Take $M^\bullet \in T^{\bullet\perp_{>0}}$. Since $T^\bullet$ is Gorenstein silting, we have $G \in (\widehat{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet})_n$ for some n and any $G \in \mathcal{GP}$. Then there is a series of triangles: $G_i^\bullet \to T_i^\bullet \to G_{i+1}^\bullet \to$ with $G_0^\bullet = G$, $T_i \in \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet$ and $G_{n+1}^\bullet = 0$ for $0 \leqslant i \leqslant n$. Applying $\mathrm{Hom}_{\mathcal{D}_{gp}}(-, M^\bullet)$ to these triangles, we have isomorphisms:

$$\mathrm{Hom}_{\mathcal{D}_{gp}}(G, M^\bullet[j]) \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(G_1^\bullet, M^\bullet[j+1]) \simeq \cdots \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(G_n^\bullet, M^\bullet[j+n]) = 0$$

since $T_n^\bullet \simeq G_n^\bullet \in \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet$ for any $j > 0$. Then, $M^\bullet \in \mathcal{GP}^{\perp_{>0}} = \mathcal{D}_{gp}^{\leqslant 0}$.

Conversely, since $T^\bullet \in \mathcal{D}_{gp}^{\leqslant 0}$ is partial Gorenstein silting, $T^\bullet \in \mathcal{K}^b(\mathcal{GP}) = \langle \mathcal{GP}\rangle$. Then, $T^\bullet \in \langle \mathcal{GP}\rangle \bigcap \mathcal{GP}^{\perp_{>0}} = \widehat{\mathcal{GP}}$. There is a series of triangles in $\mathcal{D}_{gp}(A)$: $T_{i+1}^\bullet \to G_i \to T_i^\bullet \to$

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with $T_0^\bullet = T^\bullet$, $G_i \in \mathcal{GP}$, and $T_{n+1}^\bullet = 0$ for $0 \leqslant i \leqslant n$. Then, $T_n^\bullet \simeq G_n \in \mathcal{GP}$. Applying $\mathrm{Hom}_{\mathcal{D}_{gp}}(-, G)$ for any $G \in \mathcal{GP}$ to these triangles, we have isomorphisms:

$$\mathrm{Hom}_{\mathcal{D}_{gp}}(T^\bullet, G[j]) \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(T_1^\bullet, G[j-1]) \simeq \cdots \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(T_n^\bullet, G[j-n]) = 0$$

for $j > n$. Then, $\mathcal{GP} \subseteq T^{\bullet\perp_{>n}}$. We obtain $\mathcal{GP}[n] \subseteq T^{\bullet\perp_{>0}}$. Since $0 \in T^{\bullet\perp_{>0}}$, we obtain that $\mathcal{GP} \subseteq (\widetilde{T^{\bullet\perp_{>0}}})_n$. By Lemma 3.8 and [**21**, Corollary 2.8] we have a triangle in $\mathcal{D}_{gp}(A)$: $U^\bullet \to V^\bullet \to G \to$ for any $G \in \mathcal{GP}$ with $U \in T^{\bullet\perp_{>0}}$ and $V^\bullet \in (\widetilde{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet})_n$. Then the triangle splits by assumption. So, $G \in (\widetilde{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet})_n$ for any $G \in \mathcal{GP}$ by Lemma 2.1. We conclude that $T^\bullet$ is Gorenstein silting by Theorem 3.4. □

We say that a complex $T^\bullet \in \mathcal{D}_{gp}(A)$ is n-Gorenstein silting if it is a Gorenstein silting complex such that $\mathcal{GP} \in (\widetilde{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet})_n$. A characterization of n-Gorenstein tilting modules [**22**, Theorem 3.16] says that an A-module T is n-Gorenstein tilting if and only if $T^{G\perp_{>0}} = \mathrm{Pres}^n_G(T)$.

We need the following subcategory of $\mathcal{D}_{gp}(A)$. Let $T^\bullet \in \mathcal{D}_{gp}(A)$ and $n > 0$. We denote

$$\begin{aligned}\mathrm{Pres}^n_{\mathcal{D}_{gp}^{\leqslant 0}}(T^\bullet) = \{M^\bullet \in \mathcal{D}_{gp}(A) \mid &\text{ there exist some triangles in } \mathcal{D}_{gp}(A)\colon M_{i+1}^\bullet \to T_i^\bullet \to \\ &M_i^\bullet \to \text{ with } T_i^\bullet \in \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \text{ for all } 0 \leqslant i < n, \text{ where } M_n^\bullet \in \mathcal{D}_{gp}^{\leqslant 0} \text{ and } M_0^\bullet = M^\bullet\}.\end{aligned}$$

It is not difficult to verify that $\mathrm{Pres}^n_{\mathcal{D}_{gp}^{\leqslant 0}}(T^\bullet)$ is closed under coproducts. The following result gives more properties about this subcategory.

LEMMA 3.10. (1) $\mathcal{D}_{gp}^{\leqslant 0}[n] \subseteq \mathrm{Pres}^n_{\mathcal{D}_{gp}^{\leqslant 0}}(T^\bullet)$.

(2) *If* $\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \subseteq \mathcal{D}_{gp}^{\leqslant 0}$, *then* $\mathrm{Pres}^n_{\mathcal{D}_{gp}^{\leqslant 0}}(T^\bullet) \subseteq \mathcal{D}_{gp}^{\leqslant 0}$.

Proof. Since $0 \in \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet$ and $\mathcal{D}_{gp}^{\leqslant 0}$ is closed under [1], it is easy to verify the conclusions by the definition. □

PROPOSITION 3.11. *Assume that* $T^\bullet$ *with* $\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \subseteq \mathcal{D}_{gp}^{\leqslant 0}$ *is* n*-Gorenstein silting. Then* $T^{\bullet\perp_{>0}} = \mathrm{Pres}^n_{\mathcal{D}_{gp}^{\leqslant 0}}(T^\bullet)$.

Proof. By Proposition 3.9, we have $T^{\bullet\perp_{>0}} \subseteq \mathcal{D}_{gp}^{\leqslant 0}$. By Lemma 3.8, we have $T^{\bullet\perp_{>0}} = {}_{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet}\mathcal{X}$. Take $M^\bullet \in T^{\bullet\perp_{>0}}$. By the definition of ${}_{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet}\mathcal{X}$, we have a series of triangles in $\mathcal{D}_{gp}(A)$: $M_{i+1}^\bullet \to T_i^\bullet \to M_i^\bullet \to$ with $T_i^\bullet \in \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet$ for all $0 \leqslant i < n$, where $M_n^\bullet \in {}_{\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet}\mathcal{X} \subseteq \mathcal{D}_{gp}^{\leqslant 0}$ and $M_0^\bullet = M^\bullet$. So, we obtain that $T^{\bullet\perp_{>0}} \subseteq \mathrm{Pres}^n_{\mathcal{D}_{gp}^{\leqslant 0}}(T^\bullet)$.

Take any $M^\bullet \in \mathrm{Pres}^n_{\mathcal{D}_{gp}^{\leqslant 0}}(T^\bullet)$. There is a series of triangles in $\mathcal{D}_{gp}(A)$: $M_{i+1}^\bullet \to T_i^\bullet \to M_i^\bullet \to$ with $M_n^\bullet \in \mathcal{D}_{gp}^{\leqslant 0}$, $T_i^\bullet \in \mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet$ and $M_0^\bullet = M^\bullet$ for $0 \leqslant i < n$. Then, we obtain $L^\bullet \in T^{\bullet\perp_{>n}}$ for any $L^\bullet \in \mathcal{D}_{gp}^{\leqslant 0} = \mathcal{GP}^{\perp_{>0}}$.

Indeed, by Theorem 3.7, we have $T^\bullet \in (\widehat{\mathcal{GP}})_n$. Then there is a series of triangles in $\mathcal{D}_{gp}(A)$: $T_{i+1}^\bullet \to G_i \to T_i^\bullet \to$ with $T_{n+1}^\bullet = 0$, $G_i \in \mathcal{GP}$ and $T_0^\bullet = T^\bullet$ for $0 \leqslant i \leqslant n$. Applying $\mathrm{Hom}_{\mathcal{D}_{gp}}(-, L^\bullet)$ to these triangles, we obtain isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}_{gp}}(T^\bullet, L^\bullet[j]) &\simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(T_1^\bullet, L^\bullet[j-1]) \simeq \cdots \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(T_n^\bullet, L^\bullet[j-n]) \\ &\simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(G_n, L^\bullet[j-n]) = 0\end{aligned}$$

since $L^\bullet \in \mathcal{D}_{gp}^{\leqslant 0} = \mathcal{GP}^{\perp_{>0}}$ and $T_n^\bullet \simeq G_n$, for $j > n$.

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Now we apply $\mathrm{Hom}_{\mathcal{D}_{gp}}(T^\bullet, -)$ to triangles $M^\bullet_{i+1} \to T^\bullet_i \to M^\bullet_i \to$ for $0 \leqslant i < n$. Then, we have isomorphisms:

$$\mathrm{Hom}_{\mathcal{D}_{gp}}(T^\bullet, M^\bullet[j]) \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(T^\bullet, M^\bullet_1[j+1]) \simeq \cdots \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(T^\bullet, M^\bullet_n[j+n]) = 0$$

since $M^\bullet_n \in \mathcal{D}^{\leqslant 0}_{gp} = \mathcal{GP}^{\perp_{>0}}$. So, we obtain $\mathrm{Pres}^n_{\mathcal{D}^{\leqslant 0}_{gp}}(T^\bullet) \subseteq T^{\bullet\perp_{>0}}$. □

LEMMA 3.12 [**13**, Corollary 3.8]. *Let A be a Gorenstein ring. Then there is a triangle equivalence $\mathcal{D}^b_{gp}(A) \simeq \mathcal{K}^b(\mathcal{GP})$.*

PROPOSITION 3.13. *Assume that* $\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \subseteq \mathcal{D}^b_{gp}(A) \bigcap \mathcal{D}^{\leqslant 0}_{gp}$. *If* $T^{\bullet\perp_{>0}} = \mathrm{Pres}^n_{\mathcal{D}^{\leqslant 0}_{gp}}(T^\bullet)$, *then $T^\bullet$ is n-Gorenstein silting.*

Proof. Since $\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \subseteq \mathrm{Pres}^n_{\mathcal{D}^{\leqslant 0}_{gp}}(T^\bullet)$, we have $\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \subseteq T^{\bullet\perp_{>0}}$.

By assumption, $T^\bullet \in \mathcal{D}^b_{gp}(A)$, then $T^\bullet \in \mathcal{K}^b(\mathcal{GP}) = \langle \mathcal{GP} \rangle$. So we obtain that $T^\bullet$ is partial Gorenstein silting. By assumption and Lemma 3.10(2), we have $T^{\bullet\perp_{>0}} = \mathrm{Pres}^n_{\mathcal{D}^{\leqslant 0}_{gp}}(T^\bullet) \subseteq \mathcal{D}^{\leqslant 0}_{gp}$. By Proposition 3.9, we have that $T^\bullet$ is Gorenstein silting.

Since $T^\bullet \in \mathcal{D}^{\leqslant 0}_{gp} = \mathcal{GP}^{\perp_{>0}}$, we have $T^\bullet \in \langle \mathcal{GP} \rangle \bigcap \mathcal{GP}^{\perp_{>0}} = \widehat{\mathcal{GP}}$ by Lemma 2.1. Then there is an integer m such that $T^\bullet \in (\widehat{\mathcal{GP}})_m$. Hence, there are triangles in $\mathcal{D}_{gp}(A)$: $T^\bullet_{i+1} \to G_i \to T^\bullet_i \to$ with $G_i \in \mathcal{GP}$ for all $0 \leqslant i \leqslant m$, where $T^\bullet_{m+1} = 0$ and $T^\bullet_0 = T^\bullet$. Applying the functor $\mathrm{Hom}_{\mathcal{D}_{gp}}(-, G)$ to these triangles, for any $G \in \mathcal{GP}$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}_{gp}}(T^\bullet_{m-1}, G) &\simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(T^\bullet_{m-2}, G[2]) \simeq \cdots \simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(T^\bullet_0, G[m]) \\ &\simeq \mathrm{Hom}_{\mathcal{D}_{gp}}(T^\bullet, G[m])\end{aligned}$$

Since $G[n] \in \mathcal{D}^{\leqslant 0}_{gp}[n] \subseteq \mathrm{Pres}^n_{\mathcal{D}^{\leqslant 0}_{gp}}(T^\bullet) = T^{\bullet\perp_{>0}}$, we have $\mathrm{Hom}_{\mathcal{D}_{gp}}(T^\bullet, G[i+n]) = 0$ for any $i > 0$. If $m \leqslant n$, then $T^\bullet$ is n-Gorenstein silting. If $m > n$, then the triangle $T^\bullet_m \to G_{m-1} \to T^\bullet_{m-1} \to$ splits, that is, $T \in (\widehat{\mathcal{GP}})_{m-1}$. Repeating the process, we finally have that $T \in (\widehat{\mathcal{GP}})_n$. Then $T^\bullet$ is n-Gorenstein silting. □

Note that $\mathcal{D}^{[a,b]}_{gp} = \{N^\bullet \in \mathcal{D}_{gp}(A) |\ \mathrm{H}^i\mathrm{Hom}_{\mathcal{D}gp}(G, N^\bullet[i]) = 0,$ where $i > b$ and $i < a$, for $G \in \mathcal{GP}\}$.

Combining Propositions 3.11 and 3.13, we obtain the following characterization of n-Gorenstein silting complexes.

THEOREM 3.14. *Assume that* $\mathrm{Add}_{\mathcal{D}_{gp}} T^\bullet \subseteq \mathcal{D}^{[-t,0]}_{gp}$ *for $t \geqslant 0$. Then the following are equivalent:*

(1) *$T^\bullet$ is n-Gorenstin silting,*
(2) $T^{\bullet\perp_{>0}} = \mathrm{Pres}^n_{\mathcal{D}^{\leqslant 0}_{gp}}(T^\bullet)$.

In the last, we consider the "small" version of Gorenstein silting complexes.

DEFINITION 3.15. (1) A complex $T^\bullet$ is said to be partial Gorenstein silting if it satisfies the following two conditions:
(*i*) $T^\bullet \in \mathcal{K}^b(\mathcal{G}proj)$,
(*ii*) $T^\bullet \in T^{\bullet\perp_{>0}}$.
(2) A complex $T^\bullet$ is said to be Gorenstein silting if it satisfies (*i*), (*ii*), and
(*iii*) $\mathcal{K}^b(\mathcal{G}proj) = \langle \mathrm{add}_{\mathcal{D}_{gp}} T^\bullet \rangle$, that is, $\mathcal{K}^b(\mathcal{G}proj)$ coincides with the smallest triangulated subcategory containing $\mathrm{add}_{\mathcal{D}_{gp}} T^\bullet$.

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REMARK 3.16. All above results hold if we replace $\mathrm{Add}_{\mathcal{D}_{gp}}T^\bullet$ and $\mathcal{GP}$ by $\mathrm{add}_{\mathcal{D}_{gp}}T^\bullet$ and $\mathcal{G}proj$, respectively.

Recall that $\mathcal{C}$ is a triangulated category and a subcategory $\mathcal{M}$ of $\mathcal{C}$ is called a silting subcategory if it is stable for taking direct summands and generates $\mathcal{C}$ (i.e., $\mathcal{C} = \mathrm{thick}(\mathcal{M})$) and if $\mathrm{Hom}_{\mathcal{C}}(M, N[m]) = 0$ for $m > 0$ and $M,\ N \in \mathcal{M}$ [1, 16]. Here, $\mathrm{thick}(\mathcal{M})$ stands for the smallest thick subcategory of $\mathcal{C}$ containing $\mathcal{M}$. An object M of $\mathcal{C}$ is called a silting object if $\mathrm{add}M$ is a silting subcategory of $\mathcal{C}$ [16].

REMARK 3.17. (1) By [13, Corollary 2.5], every Gorenstein-projective module is projective if and only if $\mathcal{D}^*_{gp}(A)$ coincides with the derived category $\mathcal{D}^*(A)$, where $* \in \{\text{blank}, -, +\}$. Under this case, Gorenstein silting complexes coincide with silting complexes, also called semi-tilting complexes in [21]. Then, Gorenstein silting complexes are silting objects in $\mathcal{K}^b(proj)$.

(2) By the definition of silting objects, Gorenstein silting complexes are silting objects in $\mathcal{K}^b(\mathcal{G}proj)$.

4. Partial Gorenstein silting complexes. Assume that A is a finite dimensional algebra over a field k. In this section, we consider finitely generated module category A-mod. We set that A is Cohen–Macaulay finite, or simply, CM-finite, that is, A has only finitely many isomorphism classes of indecomposable finitely generated Gorenstein-projective modules in A-mod. For a CM-finite algebra A, let $G_1, \cdots, G_t$ be all the pairwise non-isomorphic indecomposable finitely generated Gorenstein-projective A-modules, and $E = \oplus_{1 \leqslant i \leqslant t} G_i$. It is easy to see that A is CM-finite if and only if there is a module $E \in A$-mod such that $\mathcal{G}proj = \mathrm{add}_A E$. We will provide a sufficient condition for a partial Gorenstein silting complex to have a complement.

DEFINITION 4.1. For a complex $X^\bullet$, we define the following truncation:

$$\sigma_{\leqslant n}(X^\bullet) : \cdots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \to \mathrm{Ker}d^{n+1} \to 0 \to \cdots .$$

DEFINITION 4.2. For a complex $X^\bullet \in \mathcal{K}^b(\mathcal{G}proj)$, we set

$$\mathcal{R}(X^\bullet) = \sup\{i \in \mathbf{Z} |\ \mathrm{H}^i\mathrm{Hom}_A(E, X^\bullet) \neq 0\}.$$

REMARK 4.3. For any morphism $f^\bullet : X^\bullet \to Y^\bullet$ in $\mathcal{K}^b(\mathcal{G}proj)$, we obtain a triangle $X^\bullet \xrightarrow{f^\bullet} Y^\bullet \to \mathrm{Cone}f^\bullet \to$. If $\mathcal{R}(X^\bullet) = m$ and $\mathcal{R}(Y^\bullet) = n$, then $\mathcal{R}(\mathrm{Cone}f^\bullet) = \max\{m, n\}$.

We give a truncation of a complex in $\mathcal{K}^b(\mathcal{G}proj)$ in term of $\mathcal{R}(X^\bullet)$.

LEMMA 4.4. *Let $X^\bullet \in \mathcal{K}^b(\mathcal{G}proj)$. Then $X^\bullet$ is Gorenstein quasi-isomorphic to $\sigma_{\leqslant \mathcal{R}(X^\bullet)}(X^\bullet)$ in $\mathcal{K}^b(\mathcal{G}proj)$.*

Proof. Without the loss of generality, assume that $X^i = 0$ for $i < 0$ and $i > s$. Then,

$$X^\bullet : \cdots \to 0 \to X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{s-1}} X^s \to 0 \to \cdots .$$

Let $\mathcal{R}(X^\bullet) = m$. If $m = s$ or $m = s - 1$, it is easy to see that $X^\bullet$ is Gorenstein quasi-isomorphic to $\sigma_{\leqslant m}(X^\bullet)$.

Assume $m \leqslant s - 2$. So, we have that the sequence:

$$0 \to \mathrm{Im}\mathrm{Hom}_A(E, d^{s-2}) \to \mathrm{Hom}_A(E, X^{s-1}) \to \mathrm{Hom}_A(E, X^s) \to 0$$

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is exact. Since $\mathrm{add}_A A \subseteq \mathrm{add}_A E$, we have $\mathrm{Im}d^{s-2} \simeq \mathrm{Ker}d^{s-1}$. Applying $\mathrm{Hom}_A(E, -)$ to the exact sequence $0 \to \mathrm{Im}d^{s-2} \to X^{s-1} \to X^s \to 0$, we have $\mathrm{ImHom}_A(E, d^{s-2}) \simeq \mathrm{Hom}_A(E, \mathrm{Im}d^{s-2})$. It follows that the exact sequence $0 \to \mathrm{Ker}d^{s-1} \to X^{s-1} \to X^s \to 0(*)$ is G-exact. Since $X^s \in \mathrm{add}_A E$, we obtain $(*)$ splits. Then, $\mathrm{Ker}d^{s-1} \in \mathcal{G}proj$. So, $X^\bullet$ is Gorenstein quasi-isomorphic to $\sigma_{\leqslant s-2}(X^\bullet) \in \mathcal{K}^b(\mathcal{G}proj)$. Repeating the process to $\sigma_{\leqslant s-2}(X^\bullet)$ and so on, we finally obtain that $X^\bullet$ is Gorenstein quasi-isomorphic to $\sigma_{\leqslant m}(X^\bullet) \in \mathcal{K}^b(\mathcal{G}proj)$. □

LEMMA 4.5. *For any $X^\bullet \in \mathcal{K}^b(\mathcal{G}proj)$ and $Y^\bullet \in \mathcal{K}^b(\mathcal{G}proj)$, we have a morphism:*

$$f^\bullet : X^{\bullet n} \to Y^\bullet$$

with $\mathrm{Hom}_{\mathcal{K}(A)}(X^\bullet, f^\bullet)$ *surjective.*

Proof. Since $\mathrm{Hom}_{\mathcal{K}(A)}(X^\bullet, Y^\bullet) \in \mathrm{mod}k$, $\mathrm{Hom}_{\mathcal{K}(A)}(X^\bullet, Y^\bullet)$ is finite dimensional as a k-vector space. It follows that $\mathrm{Hom}_{\mathcal{K}(A)}(X^\bullet, Y^\bullet)$ is a finitely generated module over $\mathrm{End}_{\mathcal{K}(A)}(X^\bullet)$ [2, **I**.2]. Let $f_1^\bullet, \cdots, f_n^\bullet \in \mathrm{Hom}_{\mathcal{K}(A)}(X^\bullet, Y^\bullet)$ be generators over $\mathrm{End}_{\mathcal{K}(A)}(X^\bullet)$ and set

$$f^\bullet = (f_1^\bullet, \cdots, f_n^\bullet) : X^{\bullet n} \to Y^\bullet,$$

which is a desired cochain map. □

For the following part, we fix a complex $U^\bullet \in \mathcal{K}^b(\mathcal{G}proj)$ satisfying $\mathrm{Hom}_{\mathcal{D}_{gp}}(U^\bullet, U^\bullet[i]) = 0$ for $i > 0$. Up to shift, assume that $\mathcal{R}(U^\bullet) = m \geqslant 0$. Let $V_0^\bullet = E$ and n_0 be the cardinality of a generating set of $\mathrm{Hom}_{\mathcal{K}(A)}(U^\bullet, V_0^\bullet)$. We have the morphism $f_0^\bullet : U^{\bullet n_0} \to V_0^\bullet$ in $\mathcal{K}^b(\mathcal{G}proj)$ satisfying that $\mathrm{Hom}_{\mathcal{K}(A)}(U^\bullet, f_0^\bullet)$ is surjective. Then, we have a triangle in $\mathcal{K}^b(\mathcal{G}proj)$:

$$\triangle_0 : V_1^\bullet \xrightarrow{g_0^\bullet} U^{\bullet n_0} \xrightarrow{f_0^\bullet} V_0^\bullet \to .$$

We construct a series of triangles in $\mathcal{K}^b(\mathcal{G}proj)$:

$$\triangle_k : V_{k+1}^\bullet \xrightarrow{g_k^\bullet} U^{\bullet n_k} \xrightarrow{f_k^\bullet} V_k^\bullet \to$$

with $\mathrm{Hom}_{\mathcal{K}(A)}(U^\bullet, f_k^\bullet)$ surjective for $k \geqslant 1$, where n_k is the cardinality of a generating set of $\mathrm{Hom}_{\mathcal{K}(A)}(U^\bullet, V_k^\bullet)$.

LEMMA 4.6. *For any $k \geqslant 0$, the following hold.*
(1) $\mathcal{R}(V_k^\bullet) \leqslant k + max\{m-1, 0\}$.
(2) $\mathrm{Hom}_{\mathcal{K}(A)}(U^\bullet, V_k^\bullet[i]) = 0$ *for* $i > 0$.
(3) $\mathrm{Hom}_{\mathcal{K}(A)}(V_k^\bullet, V_k^\bullet[i]) \simeq \mathrm{H}^{i+k}\mathrm{Hom}_A(E, V_k^\bullet)$ *for* $i > 0$. *Particularly, we have* $\mathrm{Hom}_{\mathcal{K}(A)}(V_k^\bullet, V_k^\bullet[i]) = 0$ *for* $i > max\{m-1, 0\}$.

Proof. (1) It is easy to see $\mathrm{H}^i\mathrm{Hom}_A(E, V_0^\bullet) = 0$ for $i > \max\{m-1, 0\}$. Assume that $\mathrm{H}^i\mathrm{Hom}_A(E, V_{k-1}^\bullet) = 0$ for $i > k-1+\max\{m-1, 0\}$. Applying $\mathrm{Hom}_A(E, -)$ to $\triangle_{k-1}$, we have the following long exact sequence:

$$\cdots \to \mathrm{H}^{i-1}\mathrm{Hom}_A(E, V_{k-1}^\bullet) \to \mathrm{H}^i\mathrm{Hom}_A(E, V_k^\bullet) \to \mathrm{H}^i\mathrm{Hom}_A(E, U^{\bullet n_{k-1}}) \to \cdots,$$

we have $\mathrm{H}^i\mathrm{Hom}_A(E, V_k^\bullet) = 0$ for $i > k + \max\{m-1, 0\}$.

(2) Applying $\mathrm{Hom}_{\mathcal{K}(A)}(U^\bullet, -)$ to $\triangle_0, \cdots, \triangle_{k-1}$, for $i > k$, we have

$$\mathrm{Hom}_{\mathcal{K}(A)}(U^\bullet, V_k^\bullet[i]) \simeq \mathrm{Hom}_{\mathcal{K}(A)}(U^\bullet, V_{k-1}^\bullet[i-1]) \simeq \cdots \simeq \mathrm{Hom}_{\mathcal{K}(A)}(U^\bullet, V_0^\bullet[i-k]) = 0.$$

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For $0 < i \leqslant k$, since $\mathrm{Hom}_{\mathcal{K}(A)}(U^\bullet, f^\bullet_{k-i})$ is surjective and $\mathrm{Hom}_{\mathcal{K}(A)}(U^\bullet, U^\bullet[1]) = 0$, we have

$$\mathrm{Hom}_{\mathcal{K}(A)}(U^\bullet, V^\bullet_k[i]) \simeq \mathrm{Hom}_{\mathcal{K}(A)}(U^\bullet, V^\bullet_{k-1}[i-1]) \simeq \cdots \simeq \mathrm{Hom}_{\mathcal{K}(A)}(U^\bullet, V^\bullet_{k-i+1}[1]) = 0.$$

(3) Applying $\mathrm{Hom}_{\mathcal{K}(A)}(-, V^\bullet_k)$ to $\triangle_0, \cdots, \triangle_{k-1}$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{K}(A)}(V^\bullet_k, V^\bullet_k[i]) &\simeq \mathrm{Hom}_{\mathcal{K}(A)}(V^\bullet_{k-1}, V^\bullet_k[i+1]) \simeq \cdots \simeq \mathrm{Hom}_{\mathcal{K}(A)}(V^\bullet_0, V^\bullet_k[i+k]) \\ &\simeq \mathrm{H}^{i+k}\mathrm{Hom}_A(E, V^\bullet_k).\end{aligned}$$

In particular, we have $\mathrm{H}^{i+k}\mathrm{Hom}_A(E, V^\bullet_k) = 0$ for $i > \max\{m-1, 0\}$ by (1). □

LEMMA 4.7. *Let $k \geqslant m$. We have $\mathrm{Hom}_{\mathcal{K}(A)}(V^\bullet_k, U^\bullet[i]) = 0$ for $i > 0$.*

Proof. Applying $\mathrm{Hom}_{\mathcal{K}(A)}(-, U^\bullet)$ to $\triangle_j$ for $0 \leqslant j \leqslant k-1$, by assumption, we have $\mathrm{Hom}_{\mathcal{K}(A)}(V^\bullet_k, U^\bullet[i]) \simeq \mathrm{Hom}_{\mathcal{K}(A)}(V^\bullet_{k-1}, U^\bullet[i+1]) \simeq \cdots \simeq \mathrm{Hom}_{\mathcal{K}(A)}(V^\bullet_0, U^\bullet[i+k]) = 0$. □

LEMMA 4.8. *Assume that $m \geqslant 2$. For any $k \geqslant m$, the following are equivalent.*
(1) $\mathrm{Hom}_{\mathcal{K}(A)}(V^\bullet_k, V^\bullet_k[i]) = 0$ *for* $0 < i < m$.
(2) $\mathrm{Hom}_{\mathcal{K}(A)}(E, f^\bullet_i[m])$ *is surjective for* $0 < i < m$.
(3) $\mathcal{R}(V^\bullet_m) \leqslant m$.
(4) $\mathcal{R}(V^\bullet_k) \leqslant k$,

Proof. (1) $\Rightarrow$ (2) Applying $\mathrm{Hom}_{\mathcal{K}(A)}(E, -)$ to $\triangle_i, \cdots, \triangle_{k-1}$, by assumption, we have $\mathrm{CokerHom}_A(E, f^\bullet_i[m]) \simeq \mathrm{Hom}_{\mathcal{K}(A)}(E, V^\bullet_{i+1}[m+1]) \simeq \mathrm{Hom}_{\mathcal{K}(A)}(E, V^\bullet_{i+2}[m+2]) \simeq \cdots \simeq \mathrm{Hom}_{\mathcal{K}(A)}(E, V^\bullet_k[m+k-i]) \simeq \mathrm{Hom}_{\mathcal{K}(A)}(V^\bullet_k, V^\bullet_k[m-i]) = 0$.

(2) $\Rightarrow$ (3) We will show that $\mathcal{R}(V^\bullet_i) \leqslant m$ for $1 \leqslant i \leqslant m$ by induction.

If $i = 1$, applying $\mathrm{Hom}_{\mathcal{K}(A)}(E, -)$ to $\triangle_0$, we have the long exact sequence:

$$\cdots \to \mathrm{Hom}_{\mathcal{K}(A)}(E, V^\bullet_0[j-1]) \to \mathrm{Hom}_{\mathcal{K}(A)}(E, V^\bullet_1[j]) \to \mathrm{Hom}_{\mathcal{K}(A)}(E, U^{\bullet n_0}[j]) \to \cdots .$$

It is easy to see $\mathrm{Hom}_{\mathcal{K}(A)}(E, V^\bullet_1[j]) = 0$ for $j > m$.

Assume that $\mathcal{R}(V^\bullet_i) \leqslant m$ for $0 \leqslant i < m$. Applying $\mathrm{Hom}_{\mathcal{K}(A)}(E, -)$ to $\triangle_{m-1}$, we have the long exact sequence:

$$\cdots \to \mathrm{Hom}_{\mathcal{K}(A)}(E, V^\bullet_{m-1}[m-1]) \to \mathrm{Hom}_{\mathcal{K}(A)}(E, V^\bullet_m[m]) \to \mathrm{Hom}_{\mathcal{K}(A)}(E, U^{\bullet n_{m-1}}[m]) \to \cdots .$$

Since $\mathrm{Hom}_{\mathcal{K}(A)}(E, f^\bullet_{m-1}[m])$ is surjective and $\mathcal{R}(U^\bullet) = m$, we obtain $\mathrm{Hom}_{\mathcal{K}(A)}(E, V^\bullet_m[m+1]) = 0$. By assumption, we obtain $\mathrm{Hom}_{\mathcal{K}(A)}(E, V^\bullet_m[i]) = 0$ for $i > m+1$. So, $\mathcal{R}(V^\bullet_m) \leqslant m$.

(3) $\Rightarrow$ (4) Applying $\mathrm{Hom}_{\mathcal{K}(A)}(E, -)$ to $\triangle_j$ for $0 \leqslant j \leqslant k-1$, we have

$$\mathrm{Hom}_{\mathcal{K}(A)}(E, V^\bullet_k[i]) \simeq \mathrm{Hom}_{\mathcal{K}(A)}(E, V^\bullet_{k-1}[i-1]) \simeq \cdots \simeq \mathrm{Hom}_{\mathcal{K}(A)}(E, V^\bullet_m[i+m-k]) = 0$$

for $i > k$.

(4) $\Rightarrow$ (1) By Lemma 4.6. □

Now, we give a sufficient condition for partial Gorenstein silting complexes to have a complement.

THEOREM 4.9. *Assume that $\mathcal{R}(V^\bullet_m) \leqslant m$. Then $U^\bullet \oplus V^\bullet_k$ is a Gorenstein silting complex for $k \geqslant m$.*

Proof. By the construction, we have that $E \in \langle \mathrm{add}_{\mathcal{D}_{gp}}(U^\bullet \oplus V^\bullet_k)\rangle$. So we get $\langle \mathrm{add}_{\mathcal{D}_{gp}}(U^\bullet \oplus V^\bullet_k)\rangle = \mathcal{K}^b(\mathcal{G}proj)$. By Lemmas 4.6, 4.7, and 4.8, we have that $U^\bullet \oplus V^\bullet_k$ is a Gorenstein silting complex for $k \geqslant m$. □

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Theorem 4.9 always applies for $m = 1$, where it is the analog of Bongartz completion. When $m = 1$, by Lemma 4.6, we obtain $\mathcal{R}(V_1^\bullet) \leqslant 1$. Therefore, $U^\bullet \oplus V_k^\bullet$ is a Gorenstein silting complex for $k \geqslant 1$ by Theorem 4.9.

EXAMPLE 4.10. We give a self-injective algebra A which is a 0-Gorenstein algebra:

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

with the relation: $\{\alpha\beta\alpha,\ \ \beta\alpha\beta\}$. We give its all finitely generated indecomposable Gorenstein-projective A-modules: $\{1,\ 2,\ \begin{smallmatrix}1\\2\end{smallmatrix},\ \begin{smallmatrix}2\\1\end{smallmatrix},\ P_1,\ P_2\}$. So, we have that A is a CM-finite Gorenstein ring.

Take $U^\bullet : P_1 \to P_2$ where P_1 and P_2 are at 0 and 1 terms especially. Then, $\mathrm{Hom}_{\mathcal{D}_{gp}}(U^\bullet, U^\bullet[i]) \simeq \mathrm{Hom}_{\mathcal{K}(A)}(U^\bullet, U^\bullet[i]) = 0$ for $i > 0$. It is easy to see $\mathcal{R}(U^\bullet) = 1$. Let n be the cardinality of a generating set of $\mathrm{Hom}_{\mathcal{D}_{gp}}(U^\bullet, E)$. Then, we have a triangle : $V_1^\bullet \to U^{\bullet n} \to E \to$. Hence, we have $V_1^\bullet \in \mathcal{K}(\mathcal{G}proj)$ and $\mathcal{R}(V_1^\bullet) = 1$. By the above construction, we have triangles in $\mathcal{K}^b(\mathcal{G}proj)$:

$$\triangle_k : V_{k+1}^\bullet \xrightarrow{g_k} U^{\bullet n_k} \xrightarrow{f_k} V_k^\bullet \to$$

for $k \geqslant 1$. Then, we have that $U^\bullet \oplus V_k^\bullet$ is a Gorenstein silting complex for $k \geqslant 1$ by Theorem 4.9.

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