THE PARSEVAL FORMULAE FOR MONOTONIC FUNCTIONS. II

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1. Introduction

In the first paper of this sequence†, and in an earlier paper‡, it was shown that the Parseval formulae for Fourier cosine and sine transforms

$$\left.\begin{aligned}\int_0^\infty F_c(x)\,G_c(x)\,dx &= \int_0^\infty f(t)\,g(t)\,dt, \quad (1{\cdot}1)\\ \int_0^\infty F_s(x)\,G_s(x)\,dx &= \int_0^\infty f(t)\,g(t)\,dt, \quad (1{\cdot}2)\end{aligned}\right\} \tag{1}$$

are usually true when two of the functions appearing in them are monotonic and one of the integrals exists. There are corresponding results in the theory of Fourier series; these are discussed in the present paper and in the following one§.

The most obvious course is to consider the well-known formula

$$\tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}^{\infty}(a_n\alpha_n + b_n\beta_n) = \frac{1}{\pi}\int_0^{2\pi} f(t)\,g(t)\,dt, \tag{2}$$

$$\left(\begin{matrix}a_n\\ b_n\end{matrix} = \frac{1}{\pi}\int_0^{2\pi} f(t)\begin{matrix}\cos\\ \sin\end{matrix} nt\,dt, \quad \begin{matrix}\alpha_n\\ \beta_n\end{matrix} = \frac{1}{\pi}\int_0^{2\pi} g(t)\begin{matrix}\cos\\ \sin\end{matrix} nt\,dt\right)$$

when f and g are monotonic. However, it is easy to see that (2) is analogous, not to (1), but to the Parseval formula for 'complex' Fourier transforms‖,

$$\int_{-\infty}^{\infty} F(x)\,G(x)\,dx = \int_{-\infty}^{\infty} f(t)\,g(-t)\,dt.$$

In order to obtain a genuine counterpart of (1) for Fourier series, we must replace the transforms $F_c(x)$ and $F_s(x)$ by the coefficients in the half-range cosine and sine series for $f(t)$; this gives the formulae

$$\left.\begin{aligned}\tfrac{1}{2}a_0'\alpha_0' + \sum_{n=1}^{\infty} a_n'\alpha_n' &= \frac{2}{\pi}\int_0^{\pi} f(t)\,g(t)\,dt, \quad (3{\cdot}1)\\ \sum_{n=1}^{\infty} b_n'\beta_n' &= \frac{2}{\pi}\int_0^{\pi} f(t)\,g(t)\,dt, \quad (3{\cdot}2)\end{aligned}\right\} \tag{3}$$

where

$$\begin{matrix}a_n'\\ b_n'\end{matrix} = \frac{2}{\pi}\int_0^{\pi} f(t)\begin{matrix}\cos\\ \sin\end{matrix} nt\,dt, \quad \begin{matrix}\alpha_n'\\ \beta_n'\end{matrix} = \frac{2}{\pi}\int_0^{\pi} g(t)\begin{matrix}\cos\\ \sin\end{matrix} nt\,dt.$$

† 'The Parseval formulae for monotonic functions. I', *Proc. Cambridge Phil. Soc.* **43** (1947), 289–306. This paper will be referred to in future as M.F. (I).

‡ 'On the Parseval formulae for Fourier transforms', *Proc. Cambridge Phil. Soc.* **38** (1942), 1–19. This paper will be referred to as P.F.

§ 'The Parseval formulae for monotonic functions. III', *Proc. Cambridge Phil. Soc.* **46** (1950), 249–267. This paper will be referred to as M.F. (III).

‖ I take this opportunity of mentioning a correction needed in P.F.; the minus sign on the right-hand side of this formula was accidentally omitted there. This slip did not affect the working in P.F. since none of the proofs dealt with the 'complex' formula.

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We naturally expect that the theory of (3) for monotonic functions f and g will follow more or less the same lines as that of (1); and we may reasonably look for results of the same kind for (2), although here we should not expect too close a resemblance to the theory of (1).

We have proceeded so far by correlating the functions f and g of the transform theorem with their namesakes in the series theorem. We can obtain results of a rather different kind if we correlate them instead with the sequences of Fourier coefficients. Thus several theorems on series may arise from a single theorem on transforms. This process is of course well known. Consider, for instance, the results corresponding to Theorem 1 of M.F. (I), which states that the equations (1) hold if f and g decrease to zero in $(0, \infty)$ and fg belongs to $L(0, \infty)$. The first method of correlation suggests (a) that (2) holds if f and g decrease in $(0, 2\pi)$ and fg belong to $L(0, 2\pi)$, and (b) that (3) holds if f and g decrease in $(0, \pi)$ and fg belongs to $L(0, \pi)$; while the second leads us to expect (c) that (3·1) holds if a_n', α_n' decrease to zero and $\Sigma a_n' \alpha_n'$ converges, $f(t)$ and $g(t)$ being defined by

$$f(t) = \tfrac{1}{2}a_0' + \sum_{n=1}^{\infty} a_n' \cos nt, \quad g(t) = \tfrac{1}{2}\alpha_0' + \sum_{n=1}^{\infty} \alpha_n' \cos nt;$$

with a similar result for (3·2). Of these three statements, (a) is false (although it is true for positive functions f and g), while (b) and (c) are true; and we shall find that (c) is more closely analogous to Theorem 1 than either (a) or (b).

Thus the series theorems, taken as a whole, form a more complicated system than the transform ones do. In § 3, the various sets of conditions are tabulated, with the results obtained, both for series and for transforms. The three theorems corresponding to Theorem 1 are proved in § 4; a fairly full discussion of these is given, as they illustrate rather well the points of likeness and of difference between the series and transform cases. The main purpose of the paper is to prove this group of theorems, but it is convenient to include, in addition, two more results which are proved in a similar way. The remaining theorems on series will be proved in M.F. (III).

2. NOTATION

We continue to use the notation set out in M.F. (I), § 2; there are one or two further points to add.

We suppose throughout this paper that the functions which are not known to be monotonic are defined in terms of those which are. For instance, in the sine entry in [C, 2] of the table on p. 235, the monotonic functions are b_n and $g(x)$, and the functions $f(x)$ and β_n are defined by the equations

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \beta_n = \frac{1}{\pi}\int_0^{2\pi} g(x) \sin nx\, dx.$$

This corresponds to a convention about transforms which has already been assumed throughout M.F. (I); for instance, in section [C] of the transform table, F_s and g are defined as the sine transforms of the monotonic functions f and G_s. The notation is completely standardized as regards the various functions and their associated transforms and coefficients (half-range and full-range), and definitions such as those above

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will sometimes be taken for granted. The intervals used in forming the full-range and half-range Fourier coefficients are always $(0, 2\pi)$ and $(0, \pi)$ respectively.

In series theorems, the functions f and g are assumed to be periodic, with period 2π. Thus 'f is even' means the same as '$f(2\pi - x) = f(x)$ for all x', and it is convenient to use the former phrase even when we are really using the property expressed by the latter.

It may be helpful to recall from M.F. (I) that, when we impose conditions of monotony on functions, the intervals in which these hold are always open. Thus the monotonic functions need not be bounded near the end-points of such intervals.

We define Δa_n, $\Delta^2 a_n$ by

$$\Delta a_n = a_n - a_{n+1}, \quad \Delta^2 a_n = \Delta(\Delta a_n) = a_n - 2a_{n+1} + a_{n+2}.$$

Thus Δa_n is positive if a_n decreases, and $\Delta^2 a_n$ is positive if a_n is convex.

3. Summary of results

It is clear from § 1 that we shall have quite a large number of theorems to consider, as we may distribute the conditions of monotony and existence in a good many different ways. It is therefore convenient to collect together the various results for series and for transforms in tabular form, for reference and comparison; this is done at the end of this section.

The transform table is very simple. The theorems contained in it have been discussed at length in M.F. (I); they may be summarized by saying that (1) holds in all the cases considered, except perhaps in the cosine case of section [**B**] where the truth is not known. The series results are apparently more complicated as well as more numerous, since there are extra conditions to be imposed in [**A**, **1**], [**B**, **1**], [**C′**, **1**] and [**C**, **2**]. However, these are certainly satisfied if we suppose f and g to be positive whenever they are given to be monotonic (a condition which automatically holds in the transform theorems, since in these the monotonic functions tend to zero at infinity). If we confine our attention to this case, the series results are what we should expect from the transform ones, except that the difficulty in section [**B**] for transforms is not reproduced for series in [**B**, **1**] or [**B′**, **1**]; it is reproduced in [**B**, **2**].

It is not only in connexion with series that the question of sign enters into the problem. There is a much stronger reason why this point is important both for series and for transforms. It will be observed from the tables that the Parseval formulae hold wherever the conditions ensure that all the functions concerned are positive (the transforms, coefficients or sums of trigonometrical series, as well as the monotonic functions from which they are derived). Thus, for transforms, the sine formula holds throughout, and also the cosine formula if convexity conditions are added. It was shown in M.F. (I) that for such cases there is a very simple line of attack, which does not require us to assume the existence of one of the integrals in the ordinary finite sense; equations (1) hold in the sense that either both sides are finite and equal, or both sides are $+\infty$. Two of the cosine theorems were also proved in the finite case without the extra condition of convexity, and for these, quite different methods were needed. We were naturally led to ask whether the cosine formula (1·1) necessarily holds in the 'finite or

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infinite' sense when f and g, or f and G_c, decrease to zero. It was stated in M.F. (I) that it does not, and this will be proved in a short paper† which will be referred to as M.F. (IV). For transforms, the position is therefore that (1) holds in the 'finite or infinite' sense in the cases where all the functions must be positive, and not in the remaining cases. For series, the results cannot be stated so simply. For instance, the half-range cosine formula (3·1) must hold in the 'finite or infinite' sense if f and g decrease, but it need not do so if a_n and α_n decrease‡. A note on the 'finite or infinite' problem will be found at the end of the tables.

These 'finite or infinite' results are rather interesting because they provide the only cases in which (1) has been proved to fail for monotonic functions. If we had proved (1) to be true instead of false in such cases, we should thereby have disposed of the one 'finite' problem for transforms which is at present unsolved—the [**B**] case, in which f and g decrease and $\int F_c G_c$ exists; this was, in fact, the main reason for investigating the 'finite or infinite' problem. As it is, our results do not even help us to guess the probable truth in case [**B**]; the fact that (1·1) can be false when one side is infinite gives no grounds for supposing that it might also be false in the 'finite' case. For the [**C**] results show that the answers to the 'finite' and the 'infinite' problems may be quite different; if f and G_c decrease, (1·1) can fail when one side is infinite (the other side need

Tables of results

Some remarks on the interpretation of the tables will be found on p. 236.

Transform theorems

	Conditions		Result
	Decreasing functions	Convergent integral	
[**A**]	f, g	$\int_0^\infty fg$	COSINE. (1·1) holds (Theorem 1) SINE. (1·2) holds (Theorem 1)
[**B**]	f, g	$\int_{\to 0}^{\to \infty} F_c G_c$ $\int_0^\infty F_s G_s$	COSINE. Problem unsolved in general. (1·1) holds if f and g are convex (Theorem 3) or if fg belongs to $L(1, \infty)$ (Theorem 7) SINE. (1·2) holds (Theorem 2)
[**C**]	f, G_c	$\int_{\to 0}^{\to \infty} fg$	COSINE. (1·1) holds (Theorem 6; special case in Theorem 5)
	f, G_s	$\int_0^\infty fg$	SINE. (1·2) holds (Theorem 4)

† 'The Parseval formulae for monotonic functions. IV' (to be published). In M.F. (I) a reference to my dissertation was given for this point, as I did not at that time intend to publish the proof.

‡ There is also the trivial complication that the sum of a sine series with decreasing coefficients b_n is 'not quite positive' in $(0, \pi)$; we can only show that it is not less than $-\frac{1}{2}b_1 \tan \frac{1}{4}x$. For the purposes of the above discussion such series should be classed with the 'positive' cases.

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Series theorems

	Conditions		Result
	Decreasing functions	Convergent series or integral	
[A, 1]	f, g	$\int_0^{2\pi} fg$	(2) holds if $f(t)\,g(2\pi-t)$ belongs to $L(0, 2\pi)$, but may fail otherwise (Theorem 11, and M.F. (II), p. 239)
[A′, 1]	f, g	$\int_0^{\pi} fg$	COSINE. (3·1) holds (Theorem 12) SINE. (3·2) holds (Theorem 12)
[A, 2]	a_n, α_n b_n, β_n	$\Sigma a_n\alpha_n$ $\Sigma b_n\beta_n$	COSINE. (2) holds† (Theorem 13) SINE. (2) holds (Theorem 13)
[B, 1]	f, g	$\Sigma(a_n\alpha_n+b_n\beta_n)$	(2) holds if $f(t)\,g(2\pi-t)$ belongs to $L(0, 2\pi)$, but may fail otherwise (Theorem 14, and M.F. (II), p. 239)
[B′, 1]	f, g	$\Sigma a'_n\alpha'_n$ $\Sigma b'_n\beta'_n$	COSINE. (3·1) holds (Theorem 15) SINE. (3·2) holds (Theorem 17)
[B, 2]	a_n, α_n b_n, β_n	$\int_{\to 0}^{\to 2\pi} fg$ $\int_0^{2\pi} fg$	COSINE. Problem unsolved in general. (2) holds if a_n and α_n are convex (Theorem 21) SINE. (2) holds (Theorem 16)
[C, 1]	a_n, g b_n, g	$\int_{\to 0}^{\to 2\pi} fg$ $\lim_{\delta\to+0}\int_{\delta}^{2\pi-\delta} fg$	COSINE. (2) holds (Theorem 25; special case in Theorem 22) SINE. (2) holds (Theorem 18)
[C′, 1]	a'_n, g b'_n, g	$\int_{\to 0}^{\pi} fg$ $\int_0^{\pi} fg$	COSINE. (3·1) holds (Theorem 26; special case in Theorem 23) SINE. (3·2) holds if g is positive near the origin, or if $\Sigma b'_n/n$ converges (Theorem 19)
[C, 2]	a_n, g b_n, g	$\Sigma a_n\alpha_n$ $\Sigma b_n\beta_n$	COSINE. (2) holds if g is bounded below (Theorem 25; special case in Theorem 22) SINE. (2) holds, provided the integral is interpreted as $\lim_{\delta\to+0}\int_{\delta}^{2\pi-\delta} fg$ (Theorem 18)
[C′, 2]	a'_n, g b'_n, g	$\Sigma a'_n\alpha'_n$ $\Sigma b'_n\beta'_n$	COSINE. (3·1) holds (Theorem 26; special case in Theorem 23) SINE. (3·2) holds. Theorem 27 shows that this is so for any integrable g which has constant sign near the origin; g need not be monotonic. See also Theorem 19

† We note that in cases [A, 2], [B, 2], [C, 1] and [C, 2], the series (2) in fact consists entirely of cosine coefficients or entirely of sine coefficients (since at least one of f, g is defined by a cosine or sine series); thus (2) reduces to a form similar to (3). It is easy to pick out such cases in the tables as the cosine and sine results appear separately.

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not exist in either sense), although this equation must hold if either side exists finitely. Thus the results in the 'infinite' case are not directly useful in the 'finite' one. What they do suggest is that the whole problem is not quite so amenable as one might suppose from the easier theorems on sine transforms. Similar remarks of course hold for Fourier series.

The 'finite or infinite' problem for transforms. The equation (1·1) does not always hold in the 'finite or infinite' sense when f and g decrease, or when f and G_c decrease (Theorems 29, 30). It does hold in this sense if the decreasing functions are also convex (Theorems 3, 5).

The equation (1·2) holds in the 'finite or infinite' sense if f and g decrease, or if f and G_s decrease (Theorems 2, 4).

The 'finite or infinite' problem for series. The equation (2) holds in the 'finite or infinite' sense if f and g are positive and decreasing, or if b_n and β_n decrease, or if b_n and g decrease (with a special interpretation of the integral in the last case). It does not always hold in this sense when a_n and α_n decrease, or when a_n and g decrease (even if g is positive); but it does hold in these two cases if the decreasing functions are convex (Theorems 14, 16, 18, 31, 32, 21, 22).

The equation (3·1) holds in the 'finite or infinite' sense when f and g decrease. It does not always do so when a_n' and α_n' decrease, or when a_n' and g decrease (even if g is positive); but it does hold in these two cases if the decreasing functions are convex (Theorems 15, 31, 32, 21, 23).

The equation (3·2) holds in the 'finite or infinite' sense if f and g decrease, or if b_n' and β_n' decrease, or if b_n' and g decrease and g is positive near the origin (Theorems 17, 16, 19).

Notes on the tables

Arrangement. The system of arrangement of the transform table is based on the use of the letters [A], [B], [C] in M.F. (I) to designate the various sets of conditions. The series table is more complicated, as there are three methods of classification involved. The letters, of course, indicate the corresponding transform theorem. Dashed letters indicate half-range series theorems. There are no entries [**A′, 2**] and [**B′, 2**], since these would be exactly the same as [**A, 2**] and [**B, 2**]. The system of enumeration is to give the number 1 or 2 to a series theorem, according as the functions or the coefficients in it correspond to the functions f, g of the associated transform theorem†.

Interpretation. It is of course assumed in the tables that the monotonic functions tend to zero at infinity whenever the ranges associated with them are infinite.

Phrases such as '(1) holds' in the tables do not necessarily mean that the integrals concerned are to be taken in the Lebesgue sense. For the integral assumed to exist, the correct sense is indicated in the third column; for the other, reference must be made to the theorem quoted in the last column.

References. Of the theorems quoted in the tables, Theorem 1 is given in P.F. (where it is called Theorem IV), Theorems 2–10 in M.F. (I), and Theorems 11–15 in the present paper. Theorems 16–28 will be proved in M.F. (III), and Theorems 29–32 in M.F. (IV).

† This system is slightly complicated by the fact that, for later convenience, we have interchanged f and g in passing from the transform to the series theorem. Thus f, G_c in [C] correspond to g, a_n in [**C, 1**] for instance.

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4. The series theorems of class [A]

We have at our disposal both of the methods used in P.F. to prove Theorem 1, and we shall give examples to show how each can be adapted to deal with the series problem. The crux of the first method was to be found in certain properties of functions of the 'resultant' or 'Faltung' type, and it is convenient to collect here, for reference, all the results of this kind which we shall need.

Lemma 7. *Suppose that $f(t)$ and $g(t)$ decrease to zero in $(0,\infty)$, and that $f(t)g(t)$ belongs to $L(1,\infty)$. Then the following results hold:*

(i) *The integral $\int_0^\infty f(y+t)g(t)\,dt$ decreases to zero in $(0,\infty)$, and tends to $\int_0^\infty f(t)g(t)\,dt$, finite or infinite, as $y\to 0$ from above.*

(ii) *If $f(t)$ and $g(t)$ are even functions, the integral $h(y)=\int_{-\infty}^{\infty} f(y+t)g(t)\,dt$ decreases to zero in $(0,\infty)$, belongs to $L(0,1)$, and tends to $\int_{-\infty}^{\infty} f(t)g(t)\,dt$, finite or infinite, as $y\to 0$.*

(iii) *If $f(t)g(t)$ belongs to $L(0,\infty)$, the integral $\int_0^y f(y-t)g(t)\,dt$ is a function of bounded variation in $(0,\infty)$, and tends to 0 as $y\to 0$ from above and as $y\to\infty$.*

We include also a result on series corresponding to (iii).

(iv) *Suppose that a_n and α_n decrease and tend to zero as $n\to\infty$, and that $\sum_{n=0}^{\infty} a_n\alpha_n$ is convergent. Let $c_n=\sum_{\nu=0}^{n} a_{n-\nu}\alpha_\nu$, where $n\geqslant 0$. Then the sequence c_n is of bounded variation, and tends to zero as $n\to\infty$.*

For proofs of (i), (ii) and (iii), see P.F. pp. 12–14 and M.F. (I), p. 300 (some of the results are stated in a slightly different form there, but the same working may be used). Plainly, (i) is trivial; and (iii) is a corollary of (i) and (ii). We can deduce (iv) from (iii) by applying this result to the appropriate step-functions; or we can, of course, prove (iv) by adapting the proof of (iii) given in P.F. so that it deals with series instead of integrals throughout. I take this opportunity of stating that, some time after the publication of P.F., I found the essence of this proof in *Inequalities* (by Hardy, Littlewood and Polya), p. 273, where a result for series corresponding to (ii) is obtained.

There is a slightly different proof of (iv) which is perhaps worth remark. If n is even,

$$c_n=a_n\alpha_0+a_{n-1}\alpha_1+\ldots+a_{\frac12 n}\alpha_{\frac12 n}+\ldots+a_0\alpha_n,$$

$$c_{n+1}=a_{n+1}\alpha_0+a_n\alpha_1+\ldots+a_{\frac12 n+1}\alpha_{\frac12 n}+a_{\frac12 n}\alpha_{\frac12 n+1}+\ldots+a_0\alpha_{n+1},$$

and so
$$c_n-c_{n+1}=\sum_{\nu=0}^{\frac12 n}\alpha_\nu(a_{n-\nu}-a_{n+1-\nu})+\sum_{\nu=0}^{\frac12 n}a_\nu(\alpha_{n-\nu}-\alpha_{n+1-\nu})-a_{\frac12 n}\alpha_{\frac12 n}.$$

Since the first two terms on the right-hand side are positive and the last negative,

$$|c_n-c_{n+1}|\leqslant\sum_{\nu=0}^{\frac12 n}\alpha_\nu(a_{n-\nu}-a_{n+1-\nu})+\sum_{\nu=0}^{\frac12 n}a_\nu(\alpha_{n-\nu}-\alpha_{n+1-\nu})+a_{\frac12 n}\alpha_{\frac12 n}=c_n-c_{n+1}+2a_{\frac12 n}\alpha_{\frac12 n}.$$

Similarly, if n is odd,

$$|c_n-c_{n+1}|\leqslant c_n-c_{n+1}+2a_{\frac12(n+1)}\alpha_{\frac12(n+1)}.$$

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Summing, we have $$\sum_{n=0}^{\infty} |c_n - c_{n+1}| \leqslant c_0 + 4\sum_{n=0}^{\infty} a_n\alpha_n,$$

since all the c's are positive.

We begin with the theorem tabulated as [A, 1], which is the most obvious analogue of Theorem 1, although not the closest.

THEOREM 11. *Suppose that $f(t)$ and $g(t)$ are decreasing functions of L in $(0, 2\pi)$, that their Fourier coefficients are a_n, b_n and α_n, β_n respectively, and that $f(t)g(t)$ belongs to $L(0, 2\pi)$. Suppose also that $f(t)$ and $g(t)$ are positive, or, more generally, that $f(t)g(2\pi - t)$ belongs to $L(0, 2\pi)$. Then*

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\alpha_n + b_n\beta_n) = \frac{1}{\pi}\int_0^{2\pi} f(t)g(t)\,dt.$$

Proof. We have

$$\tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}^{N}(a_n\cos nt + b_n\sin nt) = \frac{1}{\pi}\int_0^{2\pi} f(x)\frac{\sin(N+\frac{1}{2})(x-t)}{2\sin\frac{1}{2}(x-t)}\,dx$$
$$= \frac{1}{\pi}\int_{-t}^{2\pi-t} f(y+t)\frac{\sin(N+\frac{1}{2})y}{2\sin\frac{1}{2}y}\,dy.$$

Hence

$$\int_0^{2\pi} g(t)\left\{\tfrac{1}{2}a_0 + \sum_{n=1}^{N}(a_n\cos nt + b_n\sin nt)\right\}dt$$
$$= \frac{1}{\pi}\int_0^{2\pi} g(t)\,dt\left\{\int_{-t}^{0} + \int_0^{2\pi-t}\right\} f(y+t)\frac{\sin(N+\frac{1}{2})y}{2\sin\frac{1}{2}y}\,dy$$
$$= \frac{1}{\pi}\int_{-2\pi}^{0}\frac{\sin(N+\frac{1}{2})y}{2\sin\frac{1}{2}y}\,dy\int_{-y}^{2\pi} f(y+t)g(t)\,dt + \frac{1}{\pi}\int_0^{2\pi}\frac{\sin(N+\frac{1}{2})y}{2\sin\frac{1}{2}y}\,dy\int_0^{2\pi-y} f(y+t)g(t)\,dt,$$

the area of integration in the (y, t) plane being a parallelogram with vertices at the points $(0,0)$, $(2\pi, 0)$, $(0, 2\pi)$, $(-2\pi, 2\pi)$. This equation can be written in the form

$$\pi\left\{\tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}^{N}(a_n\alpha_n + b_n\beta_n)\right\} = \frac{1}{\pi}\int_0^{2\pi}\frac{\sin(N+\frac{1}{2})y}{2\sin\frac{1}{2}y}\,dy\int_0^{2\pi-y} f(t)g(y+t) + f(y+t)g(t)\,dt. \quad (4)$$

The inner integral in (4) is obviously a decreasing function of y if f and g are positive. It is a function of bounded variation if we suppose instead that $f(t)g(2\pi - t)$ belongs to $L(0, 2\pi)$. For let $$f_+(t) = \max\{f(t), 0\}, \quad f_-(t) = \min\{f(t), 0\},$$

g_+ and g_- being similarly defined. Then

$$\int_0^{2\pi-y} f(t)g(y+t)\,dt = \int_0^{2\pi-y}\{f_+(t) + f_-(t)\}\{g_+(y+t) + g_-(y+t)\}\,dt$$
$$= I_{++} - I_{-+} - I_{+-} + I_{--},$$

where $$I_{ij} = \int_0^{2\pi-y} |f_i(t)\,g_j(y+t)|\,dt \quad (i, j = + \text{ or } -).$$

The behaviour of these integrands is most easily visualized by drawing, in the same diagram, graphs of $f(t)$ and $g(y+t)$, and noting that the effect of increasing y is to slide the 'g' graph in the negative direction along the t axis. It is then obvious that I_{++} and I_{--} decrease as y increases, and indeed this is easily obtained from Lemma 7 (i); I_{-+}

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also decreases, since both the range and the integrand do so. It remains to consider I_{+-}; and this is a function of bounded variation by Lemma 7 (iii), applied to functions which are equal to $f_+(t)$ and $-g_-(2\pi-t)$ in $(0, 2\pi)$ and zero elsewhere; we know that $f_+(t)\,g_-(2\pi-t)$ belongs to $L(0, 2\pi)$, since $f(t)\,g(2\pi-t)$ does. Similarly for the term involving $f(y+t)\,g(t)$.

The inner integral in (4) is a continuous function of y (this is partly trivial, partly an application of Lemma 7), and so it tends to 0 as $y \to 2\pi$ from below, and to $2\int_0^{2\pi} f(t)\,g(t)\,dt$ as $y \to 0$ from above. Thus, as $N \to \infty$, the right-hand side tends to the arithmetic mean of these limits, namely $\int_0^{2\pi} f(t)\,g(t)\,dt$, as in the proof of the Dirichlet-Jordan test for convergence of Fourier series. The result follows at once†.

It is natural to ask whether we might not be able to simplify Theorem 11 by getting rid of the condition that $f(t)\,g(2\pi-t)$ belongs to $L(0, 2\pi)$. In order to show that this cannot be done, we return to the equation (4), and consider the special case in which

$$f(t) = t^{-\frac{1}{2}}, \quad g(t) = -(2\pi-t)^{-\frac{1}{2}} \quad (0 < t < 2\pi).$$

Now it is elementary that $\int_0^{2\pi-y} \frac{-1}{t^{\frac{1}{2}}(2\pi-y-t)^{\frac{1}{2}}}\,dt = -\pi$

for all y in $(0, 2\pi)$. This function is certainly of bounded variation, but does not tend to 0 as $y \to 2\pi$. The term involving $\int f(y+t)\,g(t)\,dt$ gives no trouble; it may be dealt with as in the proof of the theorem. The right-hand side of (4) therefore tends to

$$\int_0^{2\pi} f(t)\,g(t)\,dt - \tfrac{1}{2}\pi$$

as $N \to \infty$, and we have

$$\tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}^{\infty}(a_n\alpha_n + b_n\beta_n) = \frac{1}{\pi}\int_0^{2\pi} f(t)\,g(t)\,dt - \tfrac{1}{2}.$$

By taking $f(t) = t^{-\alpha}$, $g(t) = -(2\pi-t)^{-\alpha}$, where $\frac{1}{2} < \alpha < 1$, we can obtain an example in which the series on the left-hand side diverges. The extra condition must therefore be retained. It is quite natural that something of the kind should be required; for the behaviour of the Fourier series of f near 2π, for instance, depends on that of f to the right as well as to the left of 2π. Nevertheless, it is perhaps a little surprising that the Parseval formula should fail for functions of such a simple kind, which so narrowly escape belonging to the class L^2.

† The method is really a disguised form of a technique developed by Young (*Proc. Roy. Soc.* A, 85 (1911), 401–14), and now familiar. It is first proved that, if f and g are any periodic integrable functions, then the function $h(y) = \int_0^{2\pi} f(y+t)\,g(t)\,dt$ has the Fourier coefficients $\pi(a_n\alpha_n + b_n\beta_n)$, $\pi(\alpha_n b_n - a_n\beta_n)$. We have then only to consider the convergence of the Fourier series of h at the origin; in the present case, this is ensured by the Dirichlet-Jordan test, since the methods used above show that h is a continuous function of bounded variation. We could thus have omitted the working down to (4) by quoting Young's result; but we have preferred to follow the lines of the transform proof.

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For the sake of its elegance, we give a second example proving the same point; this was constructed by Hardy in order to obtain a case in which both sides of (2) could be evaluated directly and thus shown to be unequal without the use of any general theory. He took

$$f(t) = \frac{\cos \frac{1}{4}t}{(\sin \frac{1}{2}t)^{\frac{1}{2}}} = (\tfrac{1}{2}\cot \tfrac{1}{4}t)^{\frac{1}{2}},$$

which certainly decreases in $(0, 2\pi)$. The Fourier coefficients of f may be found from the equation

$$e^{\frac{1}{2}it}\left(\frac{2i}{e^{\frac{1}{2}it} - e^{-\frac{1}{2}it}}\right)^{\frac{1}{2}} = \frac{1+i}{(1-e^{-it})^{\frac{1}{2}}} = (1+i)\left(1 + \tfrac{1}{2}e^{-it} + \frac{1.3}{2.4}e^{-2it} + \ldots\right)$$

by taking real parts, since this process gives

$$f(t) = 1 + \tfrac{1}{2}(\cos t + \sin t) + \frac{1.3}{2.4}(\cos 2t + \sin 2t) + \ldots.$$

If $g(t) = -f(2\pi - t)$, then $\alpha_n = -a_n$, $\beta_n = b_n$, and since $a_n = b_n$ for $n \geqslant 1$,

$$\tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}^{\infty}(a_n\alpha_n + b_n\beta_n) = \tfrac{1}{2}a_0\alpha_0 = -2.$$

On the other hand,

$$f(t)\,g(t) = -(\tfrac{1}{2}\cot \tfrac{1}{4}t)^{\frac{1}{2}}(\tfrac{1}{2}\cot(\tfrac{1}{2}\pi - \tfrac{1}{4}t))^{\frac{1}{2}} = -\tfrac{1}{2},$$

so that

$$\frac{1}{\pi}\int_0^{2\pi} f(t)\,g(t)\,dt = -1.$$

We could have foreseen from the previous example that the two sides of (2) would differ by 1 here, since f and g are asymptotically equal to $2^{\frac{1}{2}}t^{-\frac{1}{2}}$ and $2^{\frac{1}{2}}(2\pi - t)^{-\frac{1}{2}}$ near the points 0 and 2π respectively.

It is quite easy to generalize Theorem 11 so that it deals with functions f and g which are 'piecewise monotonic' in $(0, 2\pi)$; we have to introduce the condition that, if both f and g are unbounded near a point t_0, then $\int_{-h}^{h} f(t_0+t)\,g(t_0-t)\,dt$ exists for sufficiently small h. When this extension has been made, we can at once deduce the equations (3) for monotonic functions f and g. However, we shall not discuss this method in detail, but shall obtain our results otherwise, so as to bring out as clearly as possible the relation between the series and transform proofs.

We recall a well-known theorem on Parseval's formula.

(III) *If $f(t)$ is of bounded variation in $(0, 2\pi)$, and $g(t)$ belongs to $L(0, 2\pi)$, then* (2) *holds*†.

It is quite easy to deduce that (2) holds if f and g belong to $L(0, 2\pi)$, and f is of bounded variation in the neighbourhood of all but a finite number of points of the closed interval $0 \leqslant t \leqslant 2\pi$, while g is of bounded variation in the neighbourhood of each of these exceptional points‡. Indeed, we can go further, and show that (2) holds if f and g are functions of $L(0, 2\pi)$ such that each is of bounded variation in the neighbourhood of every point where the other is not. The proof is very much like that of

† Zygmund, A., *Trigonometrical series* (Warsaw, 1935), p. 91.

‡ Compare Hobson, E. W., *The theory of functions of a real variable*, 2 (Cambridge, 1926), p. 591.

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Theorem I of P.F. (p. 3), and it is hardly necessary to give details here. It is important to notice that 'a neighbourhood of the origin' means 'some interval $(-\delta, \delta)$'; that is, we are not interpreting the condition of bounded variation one-sidedly at the end-points 0 and 2π of the interval concerned. Thus one of the functions f and g must be of bounded variation near both end-points. This excludes functions such as those discussed on p. 239.

Similar results of course hold for the half-range formulae (3), and may be proved by applying the full-range result to functions which are both even, or both odd. It is easy to see that the end-points of the interval of integration do not have to be treated as in the full-range case. We can ignore points outside the closed interval $0 \leqslant t \leqslant \pi$ in applying the condition of bounded variation; and we can allow the variation of one function to be unbounded near the origin while that of the other is unbounded near the point π. It is for this reason that the half-range result is useful for our purpose, while the full-range one is not. We shall appeal to the following special case.

If $f(t)$ and $g(t)$ are decreasing functions of L in $(0, \pi)$, and if $f(t)$ is bounded near the origin and $g(t)$ bounded near the point π, then equations (3) *hold.*

This is generalized in the theorem which we now go on to prove.

THEOREM 12. *If $f(t)$ and $g(t)$ are decreasing functions of L in $(0, \pi)$, with half-range cosine (sine) coefficients a_n', α_n' (b_n', β_n'), and if $f(t)\,g(t)$ belongs to $L(0, \pi)$, then*

$$\tfrac{1}{2}a_0'\alpha_0' + \sum_{n=1}^{\infty} a_n'\alpha_n' = \frac{2}{\pi}\int_0^{\pi} f(t)\,g(t)\,dt, \quad \sum_{n=1}^{\infty} b_n'\beta_n' = \frac{2}{\pi}\int_0^{\pi} f(t)\,g(t)\,dt.$$

Proof. Let $f_+(t), f_-(t), g_+(t), g_-(t)$ be defined as in Theorem 11. Since $f_-(t)$ is bounded near 0 and $g_+(t)$ is bounded near π, the half-range Parseval formulae hold for f_- and g_+; similarly they hold for f_+ and g_-. It remains to consider f_+ and g_+, and f_- and g_-; we may therefore confine our attention to functions which are either both positive throughout $(0, \pi)$, or both negative in this interval.

Suppose first that $f(t)$ and $g(t)$ are positive. Since

$$\tfrac{1}{2}a_0' + \sum_{n=1}^{N} a_n' \cos nt = \frac{1}{\pi}\int_0^{\pi} f(x)\left\{\frac{\sin(N+\frac{1}{2})(x-t)}{2\sin\frac{1}{2}(x-t)} + \frac{\sin(N+\frac{1}{2})(x+t)}{2\sin\frac{1}{2}(x+t)}\right\}dx,$$

we have

$$\int_0^{\pi} g(t)\left\{\tfrac{1}{2}a_0' + \sum_{n=1}^{N} a_n' \cos nt\right\}dt$$

$$= \frac{1}{\pi}\int_0^{\pi} g(t)\,dt \int_{-t}^{\pi-t} f(y+t)\frac{\sin(N+\frac{1}{2})y}{2\sin\frac{1}{2}y}\,dy + \frac{1}{\pi}\int_0^{\pi} g(t)\,dt \int_t^{\pi+t} f(y-t)\frac{\sin(N+\frac{1}{2})y}{2\sin\frac{1}{2}y}\,dy. \quad (5)$$

The first term is similar to that obtained in Theorem 11; the area of integration for the second is a parallelogram in the (y, t) plane with vertices at $(0, 0)$, $(\pi, 0)$, $(2\pi, \pi)$, (π, π). After a little calculation, we find that (5) may be written in the form

$$\frac{\pi}{2}\left\{\tfrac{1}{2}a_0'\alpha_0' + \sum_{n=1}^{N} a_n'\alpha_n'\right\} = \frac{1}{\pi}\int_0^{\pi} \frac{\sin(N+\frac{1}{2})y}{2\sin\frac{1}{2}y}\,dy\left\{\int_0^{\pi-y} f(t)\,g(y+t) + g(t)f(y+t)\,dt + \int_0^{y} f(t)\,g(y-t) + f^*(y-t)\,g^*(t)\,dt\right\}; \quad (6)$$

where $f^*(t) = f(\pi - t)$, $g^*(t) = g(\pi - t)$.

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By Lemma 7, the inner integrals are functions of bounded variation†, and are continuous at the origin. The right-hand side of (6) therefore tends to $\int_0^\pi f(t)\,g(t)\,dt$ as $N \to \infty$; and the equation (3·1) holds in this case.

If $f(t)$ and $g(t)$ are both negative, we apply the result just proved to the positive decreasing functions $-f(\pi-t)$, $-g(\pi-t)$, observing that

$$\left.\begin{aligned} a_n' &= \frac{2}{\pi}\int_0^\pi f(t)\cos nt\,dt = (-1)^{n+1}\frac{2}{\pi}\int_0^\pi -f(\pi-t)\cos nt\,dt, \\ \alpha_n' &= (-1)^{n+1}\frac{2}{\pi}\int_0^\pi -g(\pi-t)\cos nt\,dt. \end{aligned}\right\} \tag{7}$$

This completes the proof of the cosine formula (3·1); the sine formula (3·2) may be proved similarly, the equations (7) being replaced by

$$b_n' = (-1)^n\frac{2}{\pi}\int_0^\pi -f(\pi-t)\sin nt\,dt, \quad \beta_n' = (-1)^n\frac{2}{\pi}\int_0^\pi -g(\pi-t)\sin nt\,dt. \tag{8}$$

There is also a proof by means of Stieltjes's integrals, corresponding to the second proof of Theorem 1 (P.F. pp. 18, 19; it is called Theorem IV there).

If we compare the first proof of Theorem 1 with those of Theorems 11 and 12, we see at once that neither of the latter theorems gives us a really close analogue of the former. The reason is that the series and transform problems differ in the conditions which are forced upon us at the outset in order to ensure that all the relevant functions exist. In the series case, we have to suppose the functions f and g integrable over the whole interval concerned. This need not be so for transforms, and as much of the first proof of Theorem 1 is designed to meet this difficulty, there is no corresponding section in the series proof. On the other hand, the functions f and g of the transform theorem must be positive, since they decrease to zero; this condition is not inevitable in the series case, and it is natural to discard it. The effect is to create some trivial complications in the proof of Theorem 12, which are easily disposed of at the beginning. Theorem 11 is further removed from Theorem 1, and the question of sign assumes more importance in this case. If f and g are supposed positive, then Theorem 11 is particularly easy to prove, since in the application of Lemma 7 we require only the trivial part (i), instead of part (iii). We can easily extend this result to the case in which f and g are bounded below (the role of the sign condition is really to limit the points near which f and g can be unbounded), and the theorem is in fact proved under more general conditions still. On the other hand, if f or g tends to $-\infty$ as t tends to 2π from below, we have seen that the right-hand side of (2) can exist without the left-hand side converging, and that both sides can exist and have different values.

Questions of this kind do not arise in the next theorem, which corresponds very closely to Theorem 1. The functions f and g are now defined as the sums of cosine (or sine) series with coefficients a_n, α_n decreasing to zero; these functions f and g therefore take the place of the transforms F_c and G_c of Theorem 1. In that theorem, the monotonic

† The integral involving f^* and g^* does not come under Lemma 7, but is obviously a bounded increasing function of y.

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functions were not necessarily integrable over an infinite interval; similarly here, neither of the series Σa_n, $\Sigma\alpha_n$ need converge.

THEOREM 13. (i) *Suppose that a_n, α_n are decreasing sequences, tending to zero as $n\to\infty$, and such that $\sum_1^\infty a_n\alpha_n$ is convergent. Let*

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \quad g(x) = \tfrac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos nx.$$

Then

$$\frac{1}{\pi}\int_{\to 0}^{\to 2\pi} f(x)\, g(x)\, dx = \tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}^{\infty} a_n\alpha_n.$$

(ii) *Suppose that b_n, β_n are decreasing sequences, tending to zero as $n\to\infty$, and such that $\sum_1^\infty b_n\beta_n$ is convergent. Let*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad g(x) = \sum_{n=1}^{\infty} \beta_n \sin nx.$$

Then

$$\frac{1}{\pi}\int_0^{2\pi} f(x)\, g(x)\, dx = \sum_{n=1}^{\infty} b_n\beta_n.$$

To establish these results, we have only to work through either proof of Theorem 1, step by step, replacing integrals by series at the appropriate points. We select the 'Stieltjes integral' method, which is the simpler; the other method is tedious and inelegant in this theorem. However, it is perhaps worth while to indicate very briefly the main outlines of the 'Faltung' proof, in order to show how Theorem 13 reproduces certain features of Theorem 1 which do not appear in Theorems 11 and 12.

We define

$$f_N(x) = \sum_{n=0}^{N} a_n \cos nx = \sum_{n=0}^{\infty} a_n(N) \cos nx,$$

$$g_N(x) = \sum_{n=0}^{N} \alpha_n \cos nx = \sum_{n=0}^{\infty} \alpha_n(N) \cos nx,$$

where $a_n(N)$ and $\alpha_n(N)$ are obtained from a_n and α_n by 'truncating' these sequences at $n = N$. The first stage is to prove the equation

$$\int_\delta^{2\pi-\delta} f_N(x)\, g_N(x)\, dx = (\pi-\delta)\left(2a_0\alpha_0 + \sum_{n=1}^{N} a_n\alpha_n\right) - \sum_{\nu=1}^{\infty} \frac{\sin\nu\delta}{\nu}\left\{\sum_{n=0}^{\infty}[a_{n+\nu}(N)\,\alpha_n(N) + a_n(N)\,\alpha_{n+\nu}(N)] + \sum_{n=0}^{\nu} a_{\nu-n}(N)\,\alpha_n(N)\right\},$$

which corresponds to P.F. (13). The details are a little tiresome, but the method is essentially that of Theorem 1; we substitute for f_N and g_N on the left-hand side, and rearrange the terms obtained on integration. We then use Lemma 7 to prove that the series on the right-hand side converges uniformly in N and δ; this enables us to take limits term by term as $N\to\infty$ and then $\delta\to 0$. It is this stage of the proof which has no counterpart in Theorems 11 and 12. Since

$$f_N(x)\to f(x) + \tfrac{1}{2}a_0, \quad g_N(x)\to g(x) + \tfrac{1}{2}\alpha_0,$$

and since it is known that

$$\int_{\to 0}^{\to 2\pi} f(x)\, dx = \pi a_0, \quad \int_{\to 0}^{\to 2\pi} g(x)\, dx = \pi\alpha_0,$$

the required result follows.

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Turning to the second proof, we give a more detailed discussion.

The following lemmas correspond to Lemmas (iii) and (iv) of P.F.

LEMMA 8. *There is a constant C, such that*

$$\left|\int_{\xi}^{2\pi-\xi'} \frac{\sin(m+\frac{1}{2})x \sin(n+\frac{1}{2})x}{4\sin^2\frac{1}{2}x}\,dx\right| \leqslant C\{\tfrac{1}{2}+\min(m,n)\}$$

for all positive integers m, n and all ξ, ξ' satisfying $0<\xi<\pi$, $0<\xi'<\pi$.

LEMMA 9. *If a_n and α_n decrease and tend to zero as $n\to\infty$, then*

$$\sum_{n=0}^{\infty}\Delta a_n \sum_{m=0}^{\infty}\Delta\alpha_m\{\tfrac{1}{2}+\min(m,n)\} = \tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}^{\infty} a_n\alpha_n,$$

finite or infinite.

In Lemma 9, only the finite case will be needed, but we give the result in the 'finite or infinite' form since this is just as easy to prove. The corresponding lemma for Stieltjes's integrals, Lemma (iv) of P.F., dealt with the finite case only, but this result is also true in the 'finite or infinite' form. The proof which we give for series will indicate how, by slightly modifying the argument in Lemma (iv), we could have included the 'infinite' case; if this had been done in P.F., we could have deduced Lemma 9 from its counterpart for integrals by applying the latter to appropriate step-functions.

Apart from this point, the proofs of Lemmas 8 and 9 are similar to those of the lemmas in P.F. In Lemma 8, we suppose that $n\leqslant m$, express the integrand in the form

$$\left\{\tfrac{1}{2}+\sum_{\nu=1}^{n}\cos\nu x\right\}\frac{\sin(m+\frac{1}{2})x}{2\sin\frac{1}{2}x},$$

and observe that, for all integers ν and m,

$$\begin{aligned}\int_{\xi}^{2\pi-\xi'}\frac{\cos\nu x\sin(m+\frac{1}{2})x}{2\sin\frac{1}{2}x}\,dx &= \left\{\int_{\xi}^{\pi}+\int_{\xi'}^{\pi}\right\}\frac{\sin(m+\frac{1}{2}+\nu)x+\sin(m+\frac{1}{2}-\nu)x}{4\sin\frac{1}{2}x}\,dx\\ &= \left\{\int_{\xi}^{\pi}+\int_{\xi'}^{\pi}\right\}\frac{\sin(m+\frac{1}{2}+\nu)x+\sin(m+\frac{1}{2}-\nu)x}{2x}\,dx+O(1)\\ &= O(1).\end{aligned}$$

Lemma 9 is proved by partial summation. Since

$$\begin{aligned}\sum_{m=1}^{\infty}\min(m,n)\,\Delta\alpha_m &= \sum_{m=1}^{n} m\Delta\alpha_m + n\sum_{m=n+1}^{\infty}\Delta\alpha_m\\ &= \left\{\sum_{m=1}^{n}\alpha_m - n\alpha_{n+1}\right\}+n\alpha_{n+1}\quad(n\geqslant 1),\end{aligned}$$

we have (in the 'finite or infinite' sense)

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta a_n\sum_{m=0}^{\infty}\min(m,n)\,\Delta\alpha_m &= \sum_{n=1}^{\infty}\Delta a_n\sum_{m=1}^{\infty}\min(m,n)\,\Delta\alpha_m\\ &= \lim_{N\to\infty}\sum_{n=1}^{N}\Delta a_n\sum_{m=1}^{n}\alpha_m\\ &= \lim_{N\to\infty}\sum_{n=1}^{N}(a_n-a_{N+1})\alpha_n\\ &= \sum_{n=1}^{\infty}a_n\alpha_n,\end{aligned}$$

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by the theorem of monotonic convergence. Also

$$\sum_{n=0}^{\infty} \Delta a_n \sum_{m=0}^{\infty} \tfrac{1}{2}\Delta\alpha_m = \tfrac{1}{2}a_0\alpha_0.$$

This completes the proof.

A result generalizing the finite case of Lemma 9 will be given in Lemma 11 (M.F. (III)).

Proof of Theorem 13. (i) We have, by partial summation,

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \sum_{n=0}^{\infty} \Delta a_n \frac{\sin (n+\frac{1}{2})x}{2\sin \frac{1}{2}x},$$

$$g(x) = \tfrac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos nx = \sum_{n=0}^{\infty} \Delta\alpha_n \frac{\sin (n+\frac{1}{2})x}{2\sin \frac{1}{2}x},$$

if $0 < x < 2\pi$. Let

$$f(x, N) = \sum_{n=0}^{N} \Delta a_n \frac{\sin (n+\frac{1}{2})x}{2\sin \frac{1}{2}x}, \quad g(x, N) = \sum_{n=0}^{N} \Delta\alpha_n \frac{\sin (n+\frac{1}{2})x}{2\sin \frac{1}{2}x}.$$

Then it is immediate that

$$\int_{\xi}^{2\pi-\xi'} f(x, N)\, g(x, N)\, dx = \sum_{n=0}^{N} \Delta a_n \sum_{m=0}^{N} \Delta\alpha_m \int_{\xi}^{2\pi-\xi'} \frac{\sin (n+\frac{1}{2})x \sin (m+\frac{1}{2})x}{4\sin^2 \frac{1}{2}x}\, dx$$

(where $0 < \xi < \pi$, $0 < \xi' < \pi$). This is the equation corresponding to P.F. (14). It is easy to see that, as $N \to \infty$, $f(x, N)$ and $g(x, N)$ tend to $f(x)$ and $g(x)$ uniformly in the interval $(\xi, 2\pi - \xi')$. Thus

$$\int_{\xi}^{2\pi-\xi'} f(x)\, g(x)\, dx = \sum_{n=0}^{\infty} \Delta a_n \sum_{m=0}^{\infty} \Delta\alpha_m \int_{\xi}^{2\pi-\xi'} \frac{\sin (n+\frac{1}{2})x \sin (m+\frac{1}{2})x}{4\sin^2 \frac{1}{2}x}\, dx,$$

the double series on the right-hand side being absolutely convergent in virtue of Lemmas 8 and 9. The same lemmas show that this series converges uniformly in ξ and ξ'; and, since it is well known† that

$$\int_{\xi}^{2\pi-\xi'} \frac{\sin (n+\frac{1}{2})x \sin (m+\frac{1}{2})x}{4\sin^2 \frac{1}{2}x}\, dx \to \pi\{\tfrac{1}{2} + \min (m, n)\}$$

as ξ and ξ' tend to 0, we have

$$\int_{-0}^{\to 2\pi} f(x)\, g(x)\, dx = \pi \sum_{n=0}^{\infty} \Delta a_n \sum_{m=0}^{\infty} \Delta\alpha_m \{\tfrac{1}{2} + \min (m, n)\}$$

$$= \pi \left\{ \tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}^{\infty} a_n\alpha_n \right\}$$

by Lemma 9.

A similar method may be used in part (ii), where f and g are defined by sine series, and indeed the proof can be somewhat simplified in this case‡. The function

$$\tfrac{1}{2}\sin (n+\tfrac{1}{2})x / \sin \tfrac{1}{2}x$$

† To prove this, we have only to substitute $\frac{1}{2} + \sum_{\nu=1}^{n} \cos \nu x$ for $\sin (n+\frac{1}{2})x / 2\sin \frac{1}{2}x$, with a similar substitution for $\sin (m+\frac{1}{2})x / 2\sin \frac{1}{2}x$, and then integrate from 0 to 2π.

‡ There is, of course, a corresponding simplification of the transform proof in the sine case of Theorem 1; this proof is even easier than for series, and we give it here to serve as a model for the series proof, since the latter is only outlined above. [*Note continued overleaf*

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is now replaced by $\frac{1}{2}\{\cos\frac{1}{2}x - \cos(n+\frac{1}{2})x\}/\sin\frac{1}{2}x$;

and we obtain $$f(x) = \sum_{n=1}^{\infty} \Delta b_n \frac{1-\cos(n+\frac{1}{2})x}{2\sin\frac{1}{2}x} - \tfrac{1}{2}b_1 \tan\tfrac{1}{4}x,$$

with a similar equation for g. The '$\tan\frac{1}{4}x$' terms are quite easily dealt with separately; the remaining terms are positive. Consequently, we can integrate the double series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Delta b_n \Delta\beta_m \frac{(1-\cos(n+\frac{1}{2})x)(1-\cos(m+\frac{1}{2})x)}{4\sin^2\frac{1}{2}x}$$

term by term over the interval $(0, 2\pi)$, appealing to the theorem of monotonic convergence; we must at first interpret our results in the 'finite or infinite' sense, but it is clear at the end that we are dealing with the finite case. This means that there is no need to introduce ξ and ξ', and we need nothing corresponding to Lemma 8. Moreover, the integral $\int_0^{2\pi} fg\,dx$ may be taken in the Lebesgue sense.

5. TWO THEOREMS OF CLASS [B]

This paper is chiefly concerned with the results of class [A] which have just been proved, and any general discussion of the [B] results would be out of place. Nevertheless, there are two theorems of class [B] which are so closely related to Theorems 11 and 12 that it is natural to prove them here. They correspond to Theorem 7 of M.F. (I).

The work of § 4 has illustrated the two methods by which series theorems may be obtained from transform ones; namely, the correlation of the functions f and g of the transform theorem, in the first place with the functions f, g of the series theorem, and in the second place with the Fourier coefficients. It is easy to see, in a general kind of way, that any difficulties in a transform proof which can be traced to the behaviour of f and g at infinity are likely to appear in the series theorem of the second type but not in that of the first type; and that the reverse will probably hold for complications at the origin. Now it was shown in M.F. (I) that, while the general [B] problem is unsolved for cosine transforms, the trouble does not arise from the character of f and g at the origin. Accordingly, we need not expect much difficulty in series theorems of class [B] in which f and g are monotonic; but the same cannot be said when it is the coefficients which decrease.

We use the Stieltjes integral formulae for F_n and G_n, valid when f and g decrease to zero (M.F. (I), p. 298). Since the integrands are positive, we have, by Fubini's theorem,

$$\begin{aligned}\int_0^{\infty} F_n(x)\,G_n(x)\,dx &= \frac{2}{\pi}\int_0^{\infty} dx \int_0^{\infty} \frac{1-\cos xt}{x}\,df(t) \int_0^{\infty} \frac{1-\cos xu}{x}\,dg(u)\\ &= \frac{2}{\pi}\int_0^{\infty} df(t)\,dg(u) \int_0^{\infty} \frac{(1-\cos xt)(1-\cos xu)}{x^2}\,dx\\ &= \int_0^{\infty}\int_0^{\infty} \min(t,u)\,df(t)\,dg(u) = \int_0^{\infty} f(t)\,g(t)\,dt\end{aligned}$$

by P.F. Lemma (iv). We do not require any result similar to Lemma (iii).

This proof is one of those referred to in M.F. (I), p. 296.

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The following theorems are of the easier type. We state the results in the 'finite or infinite' form (the (B^*) form, in the terminology of M.F. (I)); this includes both the [A] and the [B] theorems of this kind.

THEOREM 14. *If $f(t)$ and $g(t)$ are decreasing functions of L in $(0, 2\pi)$, and if $f(t)\,g(2\pi-t)$ belongs to $L(0, 2\pi)$, then*

$$\tfrac{1}{2}a_0\alpha_0+\sum_{n=1}^{\infty}(a_n\alpha_n+b_n\beta_n)=\frac{1}{\pi}\int_0^{2\pi}f(t)\,g(t)\,dt,$$

finite or infinite. In particular, the result holds if $f(t)$ and $g(t)$ are positive decreasing functions of L in $(0, 2\pi)$.

THEOREM 15. *If $f(t)$ and $g(t)$ are decreasing functions of L in $(0, \pi)$, then*

$$\tfrac{1}{2}a_0'\alpha_0'+\sum_{n=1}^{\infty}a_n'\alpha_n'=\frac{2}{\pi}\int_0^{\pi}f(t)\,g(t)\,dt,$$

finite or infinite.

We have only to make slight modifications in the proofs of Theorems 11 and 12 in order to obtain these more general results. In view of the former theorems, we may suppose that the right-hand sides are infinite here.

Proof of Theorem 14. We start from the equation (4) of Theorem 11, which is, in the notation introduced later in that theorem,

$$\pi\left\{\tfrac{1}{2}a_0\alpha_0+\sum_{n=1}^{N}(a_n\alpha_n+b_n\beta_n)\right\}=\frac{1}{\pi}\int_0^{2\pi}\frac{\sin(N+\frac{1}{2})y}{2\sin\frac{1}{2}y}\{I_{++}-I_{-+}-I_{+-}+I_{--}\}\,dy$$
$$+\text{similar terms with } f \text{ and } g \text{ interchanged.}$$

The terms involving I_{-+} and I_{+-} may be dealt with as in Theorem 11; they tend to finite limits as $N\to\infty$. On the other hand, $\{I_{++}+I_{--}\}$ is a decreasing function of y, which tends to infinity as $y\to 0$ and to zero as $y\to 2\pi$; consequently, as $N\to\infty$,

$$\int_0^{2\pi}\frac{\sin(N+\frac{1}{2})y}{2\sin\frac{1}{2}y}\{I_{++}+I_{--}\}\,dy\to\infty$$

by the 'infinite case of the Dirichlet-Jordan theorem'†. A similar result of course holds when f and g are interchanged; and the theorem follows at once.

Proof of Theorem 15. We consider first the case in which f and g are positive, and follow the proof of Theorem 12 as far as (6); this equation can be written in the form

$$\frac{\pi}{2}\left\{\tfrac{1}{2}a_0'\alpha_0'+\sum_{n=1}^{N}a_n'\alpha_n'\right\}=\frac{1}{\pi}\int_0^{\pi}\frac{\sin(N+\frac{1}{2})y}{2\sin\frac{1}{2}y}\,dy\left\{\int_{-\infty}^{\infty}f_1(y+t)\,g_1(t)\,dt+\int_0^{y}f^*(y-t)\,g^*(t)\,dt\right\},\tag{9}$$

where $f_1(t)$ and $g_1(t)$ are equal to $f(t)$ and $g(t)$ in $(0, \pi)$, are zero in (π, ∞), and are even functions. Since f^* and g^* are positive, increasing functions (and are therefore bounded near the point 0), the integral $\int_0^{y}f^*(y-t)\,g^*(t)\,dt$ increases as y increases, and tends to zero as $y\to 0$; thus the contribution of this term to the right-hand side of (9) tends to

† See M.F. (I), pp. 301, 302.

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zero as $N \to \infty$. By Lemma 7 (ii), $\int_{-\infty}^{\infty} f_1(y+t)\, g_1(t)\, dt$ is a decreasing function for positive y, and tends to infinity as $y \to 0$; and so

$$\int_0^{\pi} \frac{\sin (N+\frac{1}{2})y}{2 \sin \frac{1}{2}y}\, dy \int_{-\infty}^{\infty} f_1(y+t)\, g_1(t)\, dt \to \infty$$

as $N \to \infty$.

If f and g are not positive, then (3·1) holds in the finite sense for f_+ and g_-, and for f_- and g_+, as in Theorem 12. It holds in the 'finite or infinite' sense for f_+ and g_+, and also for f_- and g_-, since we can use equations (7) as in Theorem 12; 'infinity' means '$+\infty$' in both cases. The result follows.

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