J. Plasma Physics (2006), *vol.* 72, *part* 4, *pp.* 457–467. © 2005 Cambridge University Press
doi:10.1017/S0022377805003958 First published online 8 December 2005 Printed in the United Kingdom

MHD stationary symmetric flows with Hall effect

L. J. PALUMBO[1] and A. M. PLATZECK[2]

[1]IPCF-CNR, Via Moruzzi 1, 56124 Pisa, Italy
(ljp@ipcf.cnr.it)

[2]Facultad de Ciencias Astronomicas y Geofisicas, Universidad Nacional de La Plata,
Paseo del Bosque s/n, 1900 La Plata, Argentina
(amp@fcaglp.fcaglp.unlp.edu.ar)

(Received 15 April 2005 and accepted 10 May 2005)

Abstract. Stationary magnetohydrodynamic (MHD) flows with a symmetry are analyzed for very-low-density plasmas in strong magnetic fields, in the infinite conductivity limit. In this particular case the simplified form of Ohm's law usually employed in the ideal MHD case is no longer valid, because Hall's and electron pressure terms in the generalized Ohm's law must be retained. Thus we include them in our equations. The assumption of symmetry makes possible the definition of an ignorable coordinate and of current functions for the divergence free magnitudes. In the case without Hall's effect and electron pressure terms in Ohm's law, there is only one independent current function; in the present case, instead, two Stokes functions are independent. We choose the current function of the magnetic induction field and that of the plasma flux, ψ and χ, to be independent. Then we can express the other Stokes functions of our problem as functions of ψ and χ. We assume helical symmetry, as it is the most general form of spatial symmetry and impose incompressibility, that is $\mathbf{v} \cdot \nabla \rho = 0$. We obtain a system of two differential equations for ψ and χ, and solve them for the particular case of constant density. Then we calculate the limit of our solution for Hall's effect that tends to zero, in order to make a comparison with the results obtained for the case without Hall's effect. Finally, we find the necessary conditions to obtain confined plasma columns embedded in a force-free plasma or in a vacuum.

1. Introduction

The magnetohydrodynamic (MHD) stationary equations with symmetry have been studied by many different authors, (Tsinganos 1982a,b; Agim and Tataronis 1985; Del Zanna and Chiuderi 1996). In addition, in the papers by Palumbo and Platzeck (1993, 1998), the existence of magnetically confined plasmas, for the cases of static equilibria and stationary flows, was analyzed. In particular, the helical symmetry was studied showing that confined plasma columns with helical shape can exist.

In all the papers mentioned above, a simplified form of Ohm's law is employed. However, in the case of very low plasma densities, the Hall and electron pressure gradient terms in a generalized Ohm's law (Priest 1982) are important.

https://doi.org/10.1017/S0022377805003958 Published online by Cambridge University Press

In this paper we work in the MHD approximation when the Hall and the electron pressure gradient terms are not neglected; we consider the stationary case in the limit for infinite conductivity and assuming a symmetry.

The existence of a symmetry makes possible the definition of an ignorable variable and of current functions for the magnitudes of null divergence; now, for this case, we find that two of the current functions are independent, when for ordinary MHDs we had that only one was independent. We choose the current functions of the magnetic field induction (ψ) and of the plasma flux (χ) to be independent; in this way, we set the other current functions for the problem to depend on ψ and χ. The particular case in which $\chi =$ constant, that is when the plasma velocity is in the direction of the ignorable coordinate, must be studied separately from the case where $\chi \neq$ constant. In this more general case, it is necessary to make an additional supposition such as polytropy, adiabaticity or incompressibility, to obtain the system of two coupled differential equations for the functions ψ and χ.

We are going to treat the incompressible case when $\chi \neq$ constant and, in more detail, incompressibility with constant density. The differential equations for ψ and χ depend on four arbitrary functions of ψ or χ; if linear dependences are chosen, it is possible to uncouple the differential equations and to solve the problem. Assuming helical symmetry, we find the general solution for this case and calculate the limit when Hall's term is negligible, showing that it coincides with the solutions for the equivalent case without Hall's effect. Finally, we analyze whether these solutions can be employed to construct magnetically confined columns; it results that this is possible for the axisymmetric modes only.

2. Generalized Ohm's law

The relation among the electric current density $\mathbf{J}$, the electric field $\mathbf{E}$, the magnetic induction field $\mathbf{B}$ and the electron pressure P_e for a totally ionized neutral plasma can be obtained using a model of two fluids, electrons and protons (Priest 1982; Rossi and Olbert 1970). The resulting generalized Ohm's law is

$$\frac{\mathbf{J}}{\sigma} = \mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B} - \frac{1}{en^+c}\mathbf{J} \times \mathbf{B} + \frac{1}{en^+}\nabla P_e, \tag{2.1}$$

where σ is the conductivity, e is the proton charge and n^+ is the number density of protons.

The first three terms in (2.1) coincide with the usual form of Ohm's law employed in MHDs (see, for example, Priest (1982)), the other two are the Hall term and the electron pressure term, respectively. Let us now find out under what physical conditions these last two terms cannot be considered to be second order.

The conductivity, σ, can be introduced explicitly in Hall's term

$$\sigma = \frac{n^+e^2\tau}{m^-}, \tag{2.2}$$

where m^- is the electron mass and τ is the electron–proton collision time. So that

$$\frac{1}{en^+c}\mathbf{J} \times \mathbf{B} = \frac{eB\tau}{cm^-}\frac{\mathbf{J}}{\sigma} \times \frac{\mathbf{B}}{B} = \Omega\tau\frac{\mathbf{J}}{\sigma} \times \frac{\mathbf{B}}{B}, \tag{2.3}$$

where Ω is the electron Larmor frequency. Now we can see that, when $\mathbf{J}$ is not parallel to $\mathbf{B}$, for low plasma densities with strong magnetic fields (that is with $\Omega\tau \gg 1$) Hall's term is much larger than J/σ, and it must be preserved in the

https://doi.org/10.1017/S0022377805003958 Published online by Cambridge University Press

high-conductivity limit. We are going to consider that our plasma is composed of ionized H only. As a consequence $P_e = P/2$ and the equation of motion can be used as a means to compare the electron pressure and the Hall terms in Ohm's law. Retaining only the pressure gradient and the Lorentz force in the equation of motion, we have

$$\frac{D\mathbf{v}}{Dt} = -\nabla P + \frac{1}{c}\mathbf{J}\times\mathbf{B}; \tag{2.4}$$

if the inertial term is much smaller than the electromagnetic force, $\mathbf{J}\times\mathbf{B}/c$ is of the same order as ∇P and so as ∇P_e. Then, if we keep Hall's term in Ohm's law, the electron pressure term cannot, in general, be neglected (Cowling 1957). In this particular case, it is necessary (but not sufficient) that the inertial term be of the order of the electromagnetic force to have an ignorable ∇P_e.

To conclude our analysis of Ohm's law, let us remember that when $R_m \gg 1$, the term J/σ can be neglected; so Hall's term will be of the same order as the induction term if $\Omega\tau/R_m$ is of unity order.

3. Induction and motion equations

When the Hall and electron pressure terms in Ohm's law are not discarded, the resulting induction equation gains two extra terms with respect to its usual form in MHDs. Substituting **E** from Ohm's law into Faraday's law we obtain

$$\frac{\partial\mathbf{B}}{\partial t} - \nabla\times(\mathbf{v}\times\mathbf{B}) = \frac{m^+c}{e}\left[\frac{\varepsilon_p}{2}\frac{\nabla P\times\nabla\rho}{\rho^2} - \frac{\varepsilon_h}{c}\nabla\times\left(\frac{\mathbf{J}\times\mathbf{B}}{\rho}\right)\right] + \frac{c^2}{4\pi\sigma}\nabla^2\mathbf{B}, \tag{3.1}$$

the coefficients ε_p and ε_h can take the values 0 or 1 depending on whether the Hall and electron pressure terms can be neglected or not; they will be very useful for tracing the path of these two terms along further developments.

Therefore, if we consider times much smaller than the diffusion time, the following condition must be fulfilled in order to have time-independent fields:

$$\nabla\times(\mathbf{v}\times\mathbf{B}) + \frac{m^+c}{e}\left[\frac{\varepsilon_p}{2}\frac{\nabla P\times\nabla\rho}{\rho^2} - \frac{\varepsilon_h}{c}\nabla\times\left(\frac{\mathbf{J}\times\mathbf{B}}{\rho}\right)\right] = 0. \tag{3.2}$$

The equation of motion, instead, remains unchanged. In the stationary case its curl gives

$$\nabla\times(\mathbf{v}\times\boldsymbol{\omega}) - \frac{\nabla P\times\nabla\rho}{\rho^2} + \frac{1}{c}\nabla\times\left(\frac{\mathbf{J}\times\mathbf{B}}{\rho}\right) = 0. \tag{3.3}$$

Let us now analyze briefly the static case when the Hall and electron pressure terms cannot be ignored ($\Omega\tau \gg 1$). From (3.2) and (3.3) we see that if $\mathbf{v} = 0$, constant magnetic fields are possible only if $\nabla\rho$ is parallel to ∇P. If this condition is fulfilled, the only field affected by the new terms considered in Ohm's law is the electric field, since the magnetic induction field and the pressure obey the same equations as in ideal MHDs.

4. Stationary symmetric flows

Now we assume that the problem has a symmetry. We consider a general curvilinear coordinate system (α, β, γ) where γ is taken as an ignorable coordinate. If the determinant of the metric tensor is also independent of γ, we can define Stokes

https://doi.org/10.1017/S0022377805003958 Published online by Cambridge University Press

functions for all the magnitudes of null divergence, that is

$$\rho\mathbf{v} = \nabla\chi \times \mathbf{e}^\gamma + \rho v^\gamma \mathbf{e}_\gamma, \tag{4.1}$$

$$\mathbf{B} = \nabla\psi \times \mathbf{e}^\gamma + B^\gamma \mathbf{e}_\gamma, \tag{4.2}$$

$$\boldsymbol{\omega} = \nabla v_\gamma \times \mathbf{e}^\gamma + \omega^\gamma \mathbf{e}_\gamma, \tag{4.3}$$

$$\mathbf{J} = \frac{c}{4\pi}\nabla B_\gamma \times \mathbf{e}^\gamma + J^\gamma \mathbf{e}_\gamma. \tag{4.4}$$

Taking into account that $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ and $\mathbf{J} = (c/4\pi)\nabla \times \mathbf{B}$ and making use of the expression in curvilinear coordinates for the curl of a vector and the relations between co- and contravariant components (Santaló 1961), (4.1) and (4.2) yield

$$\omega^\gamma = G(v_\gamma) - \frac{1}{\rho g_{\gamma\gamma}} D^2\chi + \frac{1}{\rho^2 g_{\gamma\gamma}} \nabla\rho \cdot \nabla\chi, \tag{4.5}$$

$$J^\gamma = \frac{c}{4\pi}\left[G(B_\gamma) - \frac{1}{g_{\gamma\gamma}} D^2\psi\right], \tag{4.6}$$

where the elliptic operator D^2 and the operator G are given by

$$\begin{aligned} D^2\psi = \frac{g_{\gamma\gamma}}{\sqrt{g}}\Bigg[&\frac{\partial}{\partial\alpha}\left(\frac{g^{\alpha\alpha}\sqrt{g}}{g_{\gamma\gamma}}\frac{\partial\psi}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{g^{\beta\beta}\sqrt{g}}{g_{\gamma\gamma}}\frac{\partial\psi}{\partial\beta}\right) \\ &+ \frac{\partial}{\partial\alpha}\left(\frac{g^{\alpha\beta}\sqrt{g}}{g_{\gamma\gamma}}\frac{\partial\psi}{\partial\beta}\right) + \frac{\partial}{\partial\beta}\left(\frac{g^{\alpha\beta}\sqrt{g}}{g_{\gamma\gamma}}\frac{\partial\psi}{\partial\alpha}\right)\Bigg], \end{aligned} \tag{4.7}$$

$$G(f) = \frac{1}{g_{\gamma\gamma}}\Gamma f + \frac{1}{g_{\gamma\gamma}\sqrt{g}}\left(g_{\beta\gamma}\frac{\partial f}{\partial\alpha} - g_{\alpha\gamma}\frac{\partial f}{\partial\beta}\right), \tag{4.8}$$

with

$$\Gamma = \frac{g_{\gamma\gamma}}{\sqrt{g}}\left[\frac{\partial}{\partial\alpha}\left(\frac{g_{\beta\gamma}}{g_{\gamma\gamma}}\right) - \frac{\partial}{\partial\beta}\left(\frac{g_{\alpha\gamma}}{g_{\gamma\gamma}}\right)\right]. \tag{4.9}$$

As for ideal MHDs (see, for example, Palumbo and Platzeck 1998), when stationarity and an ignorable coordinate are assumed, the equations in the magnetohydrodynamical approximation including the Hall and electron pressure terms, can be transformed into a system of relations among the Jacobians of the Stokes functions of the physical magnitudes involved in the problem, the density and the pressure. So we have:

$$[\chi, \psi] - \frac{\varepsilon_{\mathrm{h}} m^+ c}{e 4\pi}[B_\gamma, \psi] = 0, \tag{4.10}$$

$$[\chi, v_\gamma] + \frac{1}{4\pi}[B_\gamma, \psi] = 0, \tag{4.11}$$

$$[\psi, v^\gamma] + \left[\frac{B^\gamma}{\rho}, \chi\right] + \frac{\varepsilon_{\mathrm{h}} m^+}{e}\left(\left[\frac{J^\gamma}{\rho}, \psi\right] - \left[\frac{c}{4\pi\rho}B^\gamma, B_\gamma\right]\right) + \frac{\varepsilon_{\mathrm{p}} m^+ c}{2e}\left[P, \frac{1}{\rho}\right] = 0, \tag{4.12}$$

$$[v_\gamma, v^\gamma] - \left[\chi, \frac{\omega^\gamma}{\rho}\right] - \frac{1}{c}\left[\frac{J^\gamma}{\rho}, \psi\right] + \left[\frac{B^\gamma}{4\pi\rho}, B_\gamma\right] - \left[P, \frac{1}{\rho}\right] = 0; \tag{4.13}$$

these equations can be obtained in an analogous way as done in the work cited above.

Let us now make a comparison with the case in which the Hall and electron pressure terms are neglected (Agim and Tataronis 1985; Palumbo and Platzeck 1998); in this case we had $\chi = \chi(\psi)$ and a single differential equation for ψ was

https://doi.org/10.1017/S0022377805003958 Published online by Cambridge University Press

obtained together with the relations among v_γ, B_γ, ψ, ρ and $g_{\gamma\gamma}$. When the Hall and electron pressure terms are taken into account, it results that $[\chi,\psi]\neq 0$ (see (4.10)), so that the current functions χ and ψ become independent of each other. Then from (4.11)–(4.13) we obtain, in general, a system of two differential equations for ψ and χ. From now onwards we are going to assume that neither χ nor ψ is a constant, because such cases must be analyzed separately.

Given that (4.10) can be written as a single Jacobian with ψ, it can be integrated immediately, giving

$$\chi-\frac{\varepsilon_{\rm h}m^{+}c}{4\pi e}B_\gamma=G_1(\psi). \tag{4.14}$$

An elimination of B_γ between (4.14) and (4.11) yields

$$\psi+\frac{\varepsilon_{\rm h}m^{+}c}{e}v_\gamma=G_2(\chi). \tag{4.15}$$

From (4.14) and (4.15) we can see that there are only two independent current functions in this problem.

Let us choose ψ and χ, so that

$$B_\gamma=\frac{4\pi e}{\varepsilon_{\rm h}m^{+}c}(\chi-G_1(\psi)) \tag{4.16}$$

and

$$v_\gamma=\frac{e}{\varepsilon_{\rm h}m^{+}c}(G_2(\chi)-\psi). \tag{4.17}$$

From (4.12) and (4.13) we obtain the differential equations for ψ and χ. Substituting (4.16) and (4.17) into (4.12) and (4.13) we arrive at

$$\left[-v^\gamma+G_1'(\psi)\frac{B^\gamma}{\rho}+\frac{\varepsilon_{\rm h}m^{+}}{e}\frac{J^\gamma}{\rho},\psi\right]+\frac{\varepsilon_{\rm p}m^{+}c}{2e}\left[P,\frac{1}{\rho}\right]=0, \tag{4.18}$$

$$\left[-\frac{e}{\varepsilon_{\rm h}m^{+}c}v^\gamma G_2'(\chi)+\frac{\omega^\gamma}{\rho}+\frac{e}{\varepsilon_{\rm h}m^{+}c}\frac{B^\gamma}{\rho},\chi\right]+\left(\frac{\varepsilon_{\rm p}}{2\varepsilon_{\rm h}}-1\right)\left[P,\frac{1}{\rho}\right]=0. \tag{4.19}$$

Given that J^γ and ω^γ are second-order differential expressions, it would be useful to perform a first integration of (4.18) and (4.19) for the case in which each one of these equations reduces to a single Jacobian set to zero. To do this, the Jacobian of P with ρ must be transformed by imposing an additional condition. In this paper we are going to analyze the case when incompressibility is assumed.

4.1. Incompressibility

In the stationary case, incompressibility means that

$$\mathbf{v}\cdot\nabla\rho=0, \tag{4.20}$$

which implies

$$[\rho,\chi]=0 \tag{4.21}$$

and

$$\rho=\rho(\chi), \tag{4.22}$$

so that the density is constant in the surfaces where χ is a constant.

Let us now see how the Jacobian of the pressure with density transforms into Jacobians with ψ or with χ.

https://doi.org/10.1017/S0022377805003958 Published online by Cambridge University Press

Introducing (4.22) into the Jacobian of P with ρ we obtain

$$\left[P, \frac{1}{\rho}\right] = \left[-\frac{P\rho'}{\rho^2}, \chi\right]. \tag{4.23}$$

On the other hand, given that ψ and χ are independent functions, we can consider the pressure as a function of ψ and χ. Then, employing the definition of the Jacobian operator, we find

$$\left[P, \frac{1}{\rho}\right] = \frac{\rho'}{\rho^2}\frac{\partial P}{\partial \psi}[\chi, \psi]; \tag{4.24}$$

let us call

$$p_{\text{inc}} = \int \frac{\rho'}{\rho^2}\frac{\partial P}{\partial \psi}\, d\chi, \tag{4.25}$$

so

$$\left[P, \frac{1}{\rho}\right] = [p_{\text{inc}}, \psi]. \tag{4.26}$$

Now, introducing (4.26) and (4.23) into (4.18) and (4.19), and performing a first integration, we finally obtain

$$-\frac{e}{\varepsilon_{\text{h}} m^+} v^\gamma + \frac{e}{\varepsilon_{\text{h}} m^+} G_1'(\psi)\frac{B^\gamma}{\rho} + \frac{J^\gamma}{\rho} + \frac{\varepsilon_{\text{p}}}{2\varepsilon_{\text{h}}} p_{\text{inc}} = G_3(\psi), \tag{4.27}$$

$$-\frac{e}{\varepsilon_{\text{h}} m^+ c} v^\gamma G_2'(\chi) + \frac{\omega^\gamma}{\rho} + \frac{e}{\varepsilon_{\text{h}} m^+ c}\frac{B^\gamma}{\rho} + \left(1 - \frac{\varepsilon_{\text{p}}}{2\varepsilon_{\text{h}}}\right)\frac{P\rho'}{\rho^2} = G_4(\chi). \tag{4.28}$$

Then, employing the relations among the co- and the contravariant components of a vector, replacing the expressions for ω^γ and J^γ given by (4.5) and (4.6), using (4.16) and (4.17) again and after some algebra we arrive at the following differential equations for ψ and χ:

$$D^2\psi - \frac{4\pi e}{\varepsilon_{\text{h}} m^+ c}\Gamma(\chi - G_1(\psi)) - \left(\frac{4\pi e}{\varepsilon_{\text{h}} m^+ c}\right)^2 G_1'(\psi)(\chi - G_1(\psi))$$
$$+ \frac{4\pi e^2}{\varepsilon_{\text{h}}^2 m^{+2} c^2}\rho(G_2(\chi) - \psi) - \frac{\varepsilon_{\text{p}}}{\varepsilon_{\text{h}}}\frac{2\pi}{c}\rho g_{\gamma\gamma} p_{\text{inc}} = -\frac{4\pi}{c}\rho g_{\gamma\gamma} G_3(\psi), \tag{4.29}$$

$$D^2\chi - \frac{e}{\varepsilon_{\text{h}} m^+ c}\rho\Gamma(G_2(\chi) - \psi) - \frac{1}{\rho}\nabla\rho\cdot\nabla\chi + \frac{e^2}{\varepsilon_{\text{h}}^2 m^{+2} c^2}\rho^2 G_2'(\chi)(G_2(\chi) - \psi)$$
$$- \frac{4\pi e^2}{\varepsilon_{\text{h}}^2 m^{+2} c^2}\rho(\chi - G_1(\psi)) - g_{\gamma\gamma}\left(1 - \frac{\varepsilon_{\text{p}}}{2\varepsilon_{\text{h}}}\right)P\rho' = -\rho^2 g_{\gamma\gamma} G_4(\chi). \tag{4.30}$$

From the equation of motion (2.4) we can obtain the expression for P as a function of ψ and χ; taking into account (4.11), replacing ω^γ and J^γ from (4.27) and (4.28) and making use of (4.14) and (4.15) we find

$$\frac{\nabla v^2}{2} + \nabla\left(\frac{P}{\rho}\right) = \frac{1}{c}\left(G_3(\psi) - \frac{\varepsilon_{\text{p}}}{2\varepsilon_{\text{h}}} p_{\text{inc}}\right)\nabla\psi - \left(G_4(\chi) + \frac{\varepsilon_{\text{p}}}{2\varepsilon_{\text{h}}}\frac{P\rho'}{\rho^2}\right)\nabla\chi. \tag{4.31}$$

The integration of (4.31) yields

$$P = -\frac{\rho v^2}{2} + \frac{\rho}{c} g_3(\psi) - \frac{\rho\varepsilon_{\text{p}}}{2\varepsilon_{\text{h}}}\int \frac{P\rho'}{\rho^2}\, d\chi - \rho g_4(\chi), \tag{4.32}$$

https://doi.org/10.1017/S0022377805003958 Published online by Cambridge University Press

where g_3 and g_4 are the primitives of G_3 and G_4 and (see (4.1) and (4.17))

$$v^2 = \frac{|\nabla\chi|^2}{\rho^2 g_{\gamma\gamma}} + \frac{v_\gamma^2}{g_{\gamma\gamma}} = \frac{|\nabla\chi|^2}{\rho^2 g_{\gamma\gamma}} + \frac{1}{g_{\gamma\gamma}}\frac{e^2}{\varepsilon_{\rm h}^2 m^{+2} c^2}(G_2(\chi)-\psi)^2. \tag{4.33}$$

Taking into account that, when the Hall and electron pressure terms are considered, $\varepsilon_{\rm h} = \varepsilon_{\rm p} = 1$, the solution of the integral equation (4.32) is

$$P(\psi,\chi) = -\sqrt{\rho}\int \sqrt{\rho}\frac{\partial}{\partial\chi}\left(\frac{v^2}{2} + g_4(\chi)\right) d\chi, \tag{4.34}$$

from this equation we can obtain $p_{\rm inc}$ by employing (4.25).

4.1.1. Constant density. This case can be studied as a particular case of incompressibility. Taking $\rho = \rho_0 =$ constant, automatically the terms with $p_{\rm inc}$, $\nabla\rho$ and ρ', in the differential equations for ψ and χ, vanish and (4.32) reduces to

$$P = -\rho_0\frac{v^2}{2} + \frac{\rho_0}{c}g_3(\psi) - \rho_0 g_4(\chi). \tag{4.35}$$

In addition, we can see that the term $\nabla P_{\rm e}$ coming from Ohm's law does not appear in the equations for ψ and χ.

To solve (4.29) and (4.30) with constant density we adopt the following linear model:

$$\begin{aligned} G_1(\psi) &= G_1^{(1)}\psi, \\ G_2(\chi) &= G_2^{(1)}\chi, \\ G_3(\psi) &= G_3^{(0)}, \\ G_4(\chi) &= G_4^{(0)}. \end{aligned} \tag{4.36}$$

Now, introducing this model into the differential equations system, we finally get two second-order coupled linear differential equations in ψ and χ,

$$\begin{aligned} D^2\psi &- \frac{4\pi e}{\varepsilon_{\rm h} m^+ c}\Gamma\big(\chi - G_1^{(1)}\psi\big) \\ &- \left(\frac{4\pi e}{\varepsilon_{\rm h} m^+ c}\right)^2 G_1^{(1)}\big(\chi - G_1^{(1)}\psi\big) + \frac{4\pi e^2}{\varepsilon_{\rm h}^2 m^{+2} c^2}\rho_0\big(G_2^{(1)}\chi - \psi\big) \\ &= -\frac{4\pi}{c}\rho_0 g_{\gamma\gamma} G_3^{(0)}, \end{aligned} \tag{4.37}$$

$$\begin{aligned} D^2\chi &- \frac{e}{\varepsilon_{\rm h} m^+ c}\rho_0\Gamma\big(G_2^{(1)}\chi - \psi\big) \\ &+ \frac{e^2}{\varepsilon_{\rm h}^2 m^{+2} c^2}\rho_0^2 G_2^{(1)}\big(G_2^{(1)}\chi - \psi\big) - \frac{4\pi e^2}{\varepsilon_{\rm h}^2 m^{+2} c^2}\rho_0\big(\chi - G_1^{(1)}\psi\big) \\ &= -\rho_0^2 g_{\gamma\gamma} G_4^{(0)}. \end{aligned} \tag{4.38}$$

As a first step to solving the above equation system we propose a change of the unknown functions to remove the equation coupling, so we define

$$u = \psi + q\,\chi, \tag{4.39}$$

where q is a constant that makes the uncoupling possible. Given that the function u must be the solution of a differential equation of the type

$$D^2 u = A\Gamma u + Bu + C g_{\gamma\gamma}, \tag{4.40}$$

https://doi.org/10.1017/S0022377805003958 Published online by Cambridge University Press

applying the definition of u and employing (4.37) and (4.38) we obtain a system of equations for q, A, B and C that yields two solutions for each parameter,

$$q^{\pm} = \frac{-\left(4\pi G_1^{(1)} + \rho_0 G_2^{(1)}\right) \pm \sqrt{\left(4\pi G_1^{(1)} + \rho_0 G_2^{(1)}\right)^2 - 16\pi\rho_0}}{2\rho_0}, \tag{4.41}$$

$$A^{\pm} = \frac{1}{2}\frac{e}{\varepsilon_{\mathrm{h}} m^{+} c}\left(-4\pi G_1^{(1)} + \rho_0 G_2^{(1)} \mp \sqrt{\left(4\pi G_1^{(1)} + \rho_0 G_2^{(1)}\right)^2 - 16\pi\rho_0}\right), \tag{4.42}$$

$$B^{\pm} = -(A^{\pm})^2, \tag{4.43}$$

$$C^{\pm} = -\rho_0\left(\frac{4\pi}{c} G_3^{(0)} + q^{\pm}\rho_0 G_4^{(0)}\right). \tag{4.44}$$

As q, in general, is a two-valued uncoupling constant, we automatically obtain two functions: u^{+} and u^{-} from which we finally obtain (for $q^{+} \neq q^{-}$)

$$\psi = \frac{q^{+}u^{-} - q^{-}u^{+}}{q^{+} - q^{-}}, \tag{4.45}$$

$$\chi = \frac{u^{+} - u^{-}}{q^{+} - q^{-}}. \tag{4.46}$$

We are going to solve this model explicitly in helical coordinates and for helical symmetry. Given that the existence of a symmetry conditions the metric tensor of the chosen coordinates system, it can be demonstrated (Solov'ev 1967) that the helical symmetry is the most general one.

Using adimensional helical coordinates $x = r/|a|$, $y = (a\theta - z)/|a|$, where (r, θ, z) are cylindrical coordinates, the differential equation for u^{+} and u^{-} is

$$x\frac{\partial}{\partial x}\left(\frac{x}{1+x^2}\frac{\partial u^{\pm}}{\partial x}\right) + \frac{\partial^2 u^{\pm}}{\partial y^2} - \frac{2ax^2}{(1+x^2)^2}A^{\pm}u^{\pm} + \frac{a^2x^2}{1+x^2}(A^{\pm})^2 u^{\pm} = a^2x^2C^{\pm}. \tag{4.47}$$

We can see that the above equation has a structure similar to that of the equation we found for the case without Hall's effect (Palumbo and Platzeck 1998; Agim and Tataronis 1985) and its solution is

$$u^{\pm} = u_{\mathrm{h}}^{\pm}(x, y) + u_{\mathrm{p}}^{\pm}(x) \tag{4.48}$$

with

$$u_{\mathrm{h}}^{\pm}(x, y) = \sum_{m=-\infty}^{m=\infty}\left[A_m^{\pm} f_m^{\pm(x)} + B_m^{\pm} g_m^{\pm}(x)\right]\exp(imy), \tag{4.49}$$

where

$$f_m^{\pm}(x) = (aA^{\pm} + m)J_m\left(\sqrt{(A^{\pm})^2a^2 - m^2}x\right) - \sqrt{(A^{\pm})^2a^2 - m^2}\,xJ_{m-1}\left(\sqrt{(A^{\pm})^2a^2 - m^2}x\right), \tag{4.50}$$

$$g_m^{\pm}(x) = (aA^{\pm} + m)N_m\left(\sqrt{(A^{\pm})^2a^2 - m^2}x\right) - \sqrt{(A^{\pm})^2a^2 - m^2}\,xN_{m-1}\left(\sqrt{(A^{\pm})^2a^2 - m^2}x\right), \tag{4.51}$$

and

$$u_{\mathrm{p}}^{\pm} = \frac{C^{\pm}}{(A^{\pm})^2}\left(1 + \frac{2}{aA^{\pm}} + x^2\right). \tag{4.52}$$

Note that, as $A^{+} \neq A^{-}$, the arguments of the Bessel functions present in u^{+} are different from those in u^{-}. This implies that the expressions for both ψ and χ will

https://doi.org/10.1017/S0022377805003958 Published online by Cambridge University Press

contain Bessel functions of different arguments, in contrast to what happens when Hall's effect is not taken into account; in this case, and with similar assumptions, there is a unique argument for the Bessel functions (Palumbo and Platzeck 1998).

In addition, when the density is constant and linear models are given for the arbitrary functions of the problem, we can make an interesting comparison between the cases with and without Hall's effect, by calculating the limit when ε_{h} tends to zero in our present solution. When Hall's term was not taken into account we had, for the linear model, $\chi = \chi^{(1)}\psi$. In the present case, in the limit when Hall's effect is negligible (i.e. $\varepsilon_{\mathrm{h}} \to 0$), from (4.14) and (4.15) we find

$$\begin{aligned} \chi &= G_1(\psi), \\ \psi &= G_2(\chi), \end{aligned} \tag{4.53}$$

which means that $G_2(\chi)$ is the inverse function of $G_1(\psi)$. For the linear model this means that $G_1^{(1)} = \chi^{(1)}$ and $G_2^{(1)} = 1/\chi^{(1)}$. We propose a first-order development in ε_{h} for $G_1^{(1)}$ and $G_2^{(2)}$:

$$\begin{aligned} G_1^{(1)} &= \chi^{(1)} + \frac{\varepsilon_{\mathrm{h}} m^+ c}{e} H_1, \\ G_2^{(1)} &= \frac{1}{\chi^{(1)}} + \frac{\varepsilon_{\mathrm{h}} m^+ c}{e} H_2, \end{aligned} \tag{4.54}$$

where H_1 and H_2 are two constants that must be determined. Introducing (4.54) into (4.14) and (4.15), together with the equation that gives B_γ and v_γ for the case of variable χ without Hall's effect (see Palumbo and Platzeck 1998, (19)), we obtain the following relation between H_1 and H_2:

$$H_1 + (\chi^{(1)})^2 H_2 = -F_1^{(1)}. \tag{4.55}$$

Now, introducing (4.54) into (4.42) and employing (4.55), if $(\chi^{(1)})^2 \neq \rho_0/4\pi$ it can be shown that when ε_{h} tends to zero, A^+ diverges and

$$\begin{aligned} A^- &\cong \frac{F_1^{(1)}}{1/4\pi - (\chi^{(1)})^2/\rho_0} = k, \\ q^+ &\cong -\frac{1}{\chi^{(1)}}, \\ q^- &\cong -4\pi \frac{\chi^{(1)}}{\rho_0}, \end{aligned} \tag{4.56}$$

where k is a parameter we employed in the case without Hall's effect.

As A^+ diverges when ε_{h} tends to zero, the coefficients A_m^+ and B_m^+ in u^+ must be null. In this way we can see that, for ε_{h} tending to zero, the homogeneous part of our solution coincides with that corresponding to the case without Hall's effect.

To obtain the inhomogeneous part of the solution in the limit for ε_{h} that tends to zero, it is necessary to calculate $C^\pm$, see (4.44), to zero order. So we work on (4.27) and (4.28), in a similar way as done for the homogeneous solution, and with (26) and (41) from Palumbo and Platzeck (1998) in the case without Hall's effect. Then we obtain

$$\begin{aligned} &C^+ \text{ diverges when } \varepsilon_{\mathrm{h}} \to 0 \text{ in the same way as } A^+, \\ &C^- \simeq -4\pi\rho_0 F_4^{(0)} \\ &u_{\mathrm{p}}^+ \to 0, \end{aligned} \tag{4.57}$$

https://doi.org/10.1017/S0022377805003958 Published online by Cambridge University Press

$$u_{\mathrm{p}}^{-} \cong -\frac{4\pi\rho_0 F_4^{(0)}}{k^2}\left(1+\frac{2}{ak}+x^2\right). \tag{4.58}$$

From the relations that link ψ and χ with the functions $u^{\pm}$ (see (4.45) and (4.46)) and employing the limit equations for $q^{\pm}$, results in

$$\begin{aligned}\psi &\cong \frac{u^{-}}{1-4\pi(\chi^{(1)})^2/\rho_0},\\ \chi &\cong \frac{u^{-}\chi^{(1)}}{1-4\pi(\chi^{(1)})^2/\rho_0}.\end{aligned} \tag{4.59}$$

The expression we obtained for ψ coincides with the one we found for the case without Hall's effect; besides $\chi = \chi^{(1)}\psi$.

If Hall's effect is not fully neglectable, A^{+} will not be divergent but it will always obey $|A^{+}| \gg |A^{-}|$; the inhomogeneous part of u^{+} will be a parabola with little curvature, while the homogeneous part will depend on Bessel functions that vary much more rapidly than those which appear in u^{-}.

5. Magnetically confined plasma columns

A highly interesting question is to find out whether magnetically confined plasma columns can be constructed from the helically symmetric solutions obtained above, in a similar way as was done for ideal MHDs (Palumbo and Platzeck 1998). In the first place we are going to analyze the required general conditions for the existence of such confined regions. These conditions are the same as those for the case without Hall's effect (Palumbo and Platzeck 1998); that is, χ and P must be constant along the surface S that surrounds the column and, in the case where the column is in vacuum, B_γ must be constant over all S, so that the currents are tangential to S.

The condition of constant χ, which implies that the plasma velocity is a tangent to the surface of the column, defines S. Now we must see whether it is possible to have $P =$ constant on S too. As ψ and χ have been defined as two independent variables, it results that $P = P(\psi, \chi)$, then it will be enough to impose that ψ will also be invariant on S to obtain constant pressure.

For our particular model, with constant density and a linear model for the arbitrary functions, we can see, from (4.45) and (4.46), that constant ψ and χ implies constant u^{+} and u^{-}; thus, from (4.48)–(4.52), it is evident that the only solution that allows invariant ψ and χ on the same surface is the axisymmetrical one; that is, the confined plasma columns will be z-axis cylinders. In this case B_γ will also be a constant.

6. Matching of confined plasma columns with external media

In this section we are going to study the magnetic fields in the medium surrounding a confined plasma column.

To match a confined plasma column with an external medium, it is necessary to impose boundary conditions on the magnetic field. We are going to consider two cases.

(i) The surrounding medium is a vacuum.

(ii) The surrounding medium is a linear force-free plasma.

https://doi.org/10.1017/S0022377805003958 Published online by Cambridge University Press

To ensure the continuity of the component of the magnetic induction field normal to S, the same conditions hold that were imposed in the case without Hall's effect (Palumbo and Platzeck 1998), i.e. the continuity of ψ and $\nabla\psi$ on S; from this, using the second Green's identity (Palumbo and Platzeck 1993), it is possible to obtain the expression for ψ in the surrounding medium.

The continuity of B_γ, when the surrounding medium is a vacuum, is found by an appropriate choice of the constant B_γ^v, that is, its value in vacuum. If the external medium is a force-free plasma, for our particular case of constant density and axial symmetry, the continuity of B_γ is ensured by an appropriate choice of the constants of the force-free plasma.

7. Conclusions

In many well-known circumstances the plasma physical conditions make it necessary to take into account the Hall and electron pressure gradient terms in Ohm's law. Consequently, two extra terms appear in the induction equation, this fact carries important changes in the final solutions. Even in the stationary case, where a symmetry and constant density are assumed, the Hall term strongly shapes the solutions: two independent current functions are required. Thus, in general, the velocity is no longer tangential to the magnetic surfaces and so the only possible magnetically confined plasma columns are cylinders.

References

Agim, Y. Z. and Tataronis, J. A. 1985 *J. Plasma Phys.* **34**, 337.

Cowling, T. G. 1957 *Magnetohydrodynamics*. New York: Interscience.

Del Zanna, L. and Chiuderi, C. 1996 *Astron. Astrophys.* **310**, 341.

Palumbo, L. J. and Platzeck, A. M. 1993 *Astrophys. J.* **416**, 656.

Palumbo, L. J. and Platzeck, A. M. 1998 *J. Plasma Phys.* **60**, 449–467.

Priest, E. R. 1982 *Solar Magnetohydrodynamics*. Dordrecht: Reidel.

Rossi, B. and Olbert, S. 1970 *Introduction to the Physics of Space*. New York: McGraw-Hill.

Santaló, L. A. 1961 *Vectores y Tensores con sus Aplicaciones*. Buenos Aires: EUDEBA.

Solov'ev, L. S. 1967 *Reviews of Plasma Physics*, Vol. 3. (ed M. A. Leontovitch). New York: Consultants Bureau, p. 277.

Tsinganos, K. C. 1982a *Astrophys. J.* **252**, 775.

Tsinganos, K. C. 1982b *Astrophys. J.* **259**, 820.

https://doi.org/10.1017/S0022377805003958 Published online by Cambridge University Press